Some Special Cases of the andrews-Bowman Continued Fraction

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ABSTRACT

SOME SPECIAL CASES OF THE ANDREWS-BOWMAN CONTINUED FRACTION

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One of the most famous results from q-series is that of the Rogers-Ramanujan continued fraction [17], given by $\prod_{n=0}^{\infty} \frac{(1-q^{5n+2})(1-q^{5n+3})}{(1-q^{5n+1})(1-q^{5n+4})} = 1 + \frac{q}{1+} \frac{q^2}{1+} \frac{q^3}{1+} \frac{q^4}{1+} \cdots$. G.E. Andrews and D. Bowman [3] gave a full extension of this continued fraction using G.N. Watson's [19] nonterminating very well-poised ${}_8\phi_7$ function. As opposed to Ramanujan's generalization that only used four variables, this generalization is given in seven variables, and certain q-series identities naturally arise from it. As a special case of their theorem, Andrews and Bowman gave the following identity: $\frac{1-q}{\sum_{n=0}^{\infty} q^{n^2+2n}} = 1 - q - q^3 + q^6 + \frac{q^4(1-q^3)(1-q^4)(1-q^5)}{1-q^5-q^6-q^7+q^{10}+q^{14}+} \frac{q^8(1-q^7)(1-q^8)(1-q^9)}{1-q^9-q^{10}-q^{11}+q^{18}+q^{22}+} \cdots$. This thesis will give a full proof of Andrews and Bowman's result, as well as investigate other special cases of their continued fraction that have not been discovered before. Many identities will be verified using an open-source symbolic algebra package called Maxima [16].

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DEDICATION

To my wonderful parents Ken and Penny Zollinger

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CHAPTER 1

DEFINITIONS AND BASIC RESULTS

This chapter will contain the background information about the mathematics being used within this thesis. We will look at various definitions and examples involving q-products, basic hypergeometric series, and continued fractions.

1.1 q-Products

Throughout this thesis, unless stated otherwise, let q and z denote fixed complex numbers such that $|q| < 1$ and $|z| < 1$. We begin with a simple definition.

Definition 1. Define

$$(q)_n := (1-q)(1-q^2)\cdots(1-q^n) = \prod_{k=1}^{n}(1-q^k). \tag{1.1}$$

More generally, define

$$(a;q)_n := (1-a)(1-aq)\cdots(1-aq^{n-1}) = \prod_{k=0}^{n-1}(1-aq^k), \tag{1.2}$$

where $(a;q)_0 := 1$.

This is known as a *q-shifted factorial*, *q-Pochhammer symbol*, or *q-product*.

When q is understood from context, $(a)_n$ may be written instead of $(a;q)_n$. Thus, $(q)_n$ denotes

$$(q;q)_n = (1-q)(1-q^2)\cdots(1-q^n) = \prod_{k=1}^{n}(1-q^k). \tag{1.3}$$

Remark 1. Note that $(a;q)_n$ is extended to $(a;q)_\infty$ by

$$\lim_{n\to\infty}\prod_{k=0}^{n-1}(1-aq^k) = \prod_{k=0}^{\infty}(1-aq^k). \tag{1.4}$$

Also, a q-product is defined in multiple parameters by

$$(a_1, a_2, \ldots, a_j; q)_n = (a_1;q)_n(a_2;q)_n\cdots(a_j;q)_n, \tag{1.5}$$

and in the infinite case,

$$(a_1, a_2, \ldots, a_j; q)_\infty = (a_1;q)_\infty(a_2;q)_\infty\cdots(a_j;q)_\infty. \tag{1.6}$$

Example 1. Consider $(a;q)_n$ once again. Then

$$\begin{aligned}(a;q)_n &= (1-a)(1-aq)\cdots(1-aq^{n-1})\\ &= \frac{(1-a)(1-aq)\cdots(1-aq^{n-1})(1-aq^n)(1-aq^{n+1})\cdots}{(1-aq^n)(1-aq^{n+1})\cdots}\\ &= \frac{(a;q)_\infty}{(aq^n;q)_\infty}.\end{aligned} \tag{1.7}$$

Using this fact, our current notion of $(q)_n$ is extended to the negative integers.

Example 2. Consider $(a;q)_{-n}$. (1.7) gives the following:

$$\begin{aligned}(a;q)_{-n} &= \frac{(a;q)_\infty}{(aq^{-n};q)_\infty}\\ &= \frac{(1-a)(1-aq)(1-aq)^2\cdots}{(1-aq^{-n})(1-aq^{-n+1})\cdots(1-aq^{-1})(1-a)(1-aq)\cdots}\\ &= \frac{1}{(1-aq^{-n})(1-aq^{-n+1})\cdots(1-aq^{-1})}\\ &= \frac{1}{(aq^{-n};q)_n}.\end{aligned} \tag{1.8}$$

By defining

$$(a;q)_x := \frac{(a;q)_\infty}{(aq^x;q)_\infty}, \tag{1.9}$$

then the q-Pochhammer symbol is extended to any $x \in \mathbb{C}$.

1.2 Hypergeometric Series

With q-products in mind, we now turn our attention to hypergeometric series.

Definition 2. A *hypergeometric series* is a series $\sum a_n$ where $\frac{a_{n+1}}{a_n} \in \mathbb{C}(n)$. A hypergeometric series is called *basic* if $\frac{a_{n+1}}{a_n} \in \mathbb{C}(q^n)$ for $|q| < 1$. Here, $\mathbb{C}(n)$ and $\mathbb{C}(q^n)$ respectively denote the fields of rational functions of the variables n and q^n with complex coefficients.

This was first discovered in 1846 by Eduard Heine [11] when he considered the series

$$\begin{aligned}{}_2\phi_1(q^\alpha, q^\beta; q^\gamma; q, z) &= 1 + \frac{(q^\alpha-1)(q^\beta-1)}{(q-1)(q^\gamma-1)}z + \frac{(q^\alpha-1)(q^{\alpha+1}-1)(q^\beta-1)(q^{\beta+1}-1)}{(q-1)(q^2-1)(q^\gamma-1)(q^{\gamma+1}-1)}z^2 + \cdots\\ &= \sum_{n=0}^{\infty} \frac{(q^\alpha, q^\beta; q)_n}{(q^\gamma, q; q)_n} z^n.\end{aligned} \tag{1.10}$$

Taking the limit as $q \to 1$ in (1.10) gives us

$$\begin{aligned} {}_2F_1(\alpha, \beta; \gamma; z) &= 1 + \frac{\alpha \cdot \beta}{1 \cdot \gamma} z + \frac{\alpha(\alpha+1) \cdot \beta(\beta+1)}{1 \cdot 2 \cdot \gamma(\gamma+1)} z^2 + \cdots \\ &= \sum_{n=0}^{\infty} \frac{(\alpha, \beta)_n}{(\gamma)_n} \cdot \frac{z^n}{n!}. \end{aligned} \tag{1.11}$$

This series was studied systematically by Gauss in 1813 [9].

The process used to obtain (1.11) is not unique to that particular problem. In general, given any mathematical object O_q depending on a parameter q, if it tends to an object O as $q \to 1$, then O_q is called a *q-analogue* of O.

Remark 2. The basic hypergeometric function ${}_r\phi_s$ is defined by the following series:

$${}_r\phi_s(a_1, a_2, \ldots, a_r; b_1, b_2, \ldots, b_s; q, z) = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \ldots a_r; q)_n}{(b_1, b_2, \ldots, b_s, q; q)_n} \left[(-1)^n q^{\binom{n}{2}} \right]^{1+s-r} z^n, \tag{1.12}$$

where $\binom{n}{k}$ is the binomial coefficient.

We now establish some fundamental theorems about basic hypergeometric series.

Theorem 1. Cauchy's [6] q-binomial theorem is given by

$$\frac{(az)_\infty}{(z)_\infty} = \sum_{n=0}^{\infty} \frac{(a)_n}{(q)_n} z^n. \tag{1.13}$$

Proof. Set $f(z) = \frac{(az)_\infty}{(z)_\infty}$. It is easy to see that $f(z)$ is a meromorphic function of z with a simple pole at $z = 1$ and is analytic in the unit disk.

Expanding $f(z)$, we have

$$\begin{aligned} f(z) = \frac{(az)_\infty}{(z)_\infty} &= \frac{(1-az)(1-azq)(1-azq^2)\cdots}{(1-z)(1-zq)(1-zq^2)\cdots} \\ &= \frac{1-az}{1-z} \cdot \frac{(1-azq)(1-azq^2)\cdots}{(1-zq)(1-zq^2)\cdots} \\ &= \frac{1-az}{1-z} \cdot \frac{(azq)_\infty}{(zq)_\infty} \\ &= \frac{1-az}{1-z} f(zq). \end{aligned}$$

Rearranging terms gives

$$(1-z)f(z) = (1-az)f(zq). \tag{1.14}$$

Since $f(0) = 1$, put $f(z) = \sum_{n=0}^{\infty} b_n z^n$ with $b_o = 1$.

(1.14) will now be solved by equating coefficients. For the left-hand side of the equation,

$$\begin{aligned} (1-z)f(z) &= (1-z)\sum_{n=0}^{\infty} b_n z^n \\ &= \sum_{n=0}^{\infty} b_n z^n - \sum_{n=0}^{\infty} b_n z^{n+1} \\ &= 1 + \sum_{n=1}^{\infty} b_n z^n - \sum_{n=1}^{\infty} b_{n-1} z^n \\ &= 1 + \sum_{n=1}^{\infty} (b_n - b_{n-1}) z^n. \end{aligned} \tag{1.15}$$

For the right-hand side of (1.14),

$$\begin{aligned}(1-az)f(zq) &= (1-az)\sum_{n=0}^{\infty} b_n (zq)^n \\ &= (1-az)\sum_{n=0}^{\infty} b_n z^n q^n \\ &= \sum_{n=0}^{\infty} b_n z^n q^n - \sum_{n=0}^{\infty} b_n a z^{n+1} q^n \\ &= 1 + \sum_{n=1}^{\infty} b_n z^n q^n - \sum_{n=1}^{\infty} b_{n-1} a z^n q^{n-1} \\ &= 1 + \sum_{n=1}^{\infty} (b_n q^n - b_{n-1} a q^{n-1}) z^n. \end{aligned} \tag{1.16}$$

Combining (1.15) and (1.16) gives

$$1 + \sum_{n=1}^{\infty} (b_n - b_{n-1}) z^n = 1 + \sum_{n=1}^{\infty} (b_n q^n - b_{n-1} a q^{n-1}) z^n.$$

For $n \geq 1$,

$$b_n - b_{n-1} = b_n q^n - b_{n-1} a q^{n-1},$$

which implies

$$(1-q^n) b_n = (1 - a q^{n-1}) b_{n-1},$$

which then implies that

$$b_n = \frac{1 - a q^{n-1}}{1 - q^n} b_{n-1}.$$

Since $b_o = 1$,

$$b_1 = \frac{1-a}{1-q}, b_2 = \frac{(1-a)(1-aq)}{(1-q)(1-q^2)}, \ldots, b_n = \frac{(1-a)(1-aq)\cdots(1-aq^{n-1})}{(1-q)(1-q^2)\cdots(1-q^n)} = \frac{(a)_n}{(q)_n}.$$

Thus, we conclude that

$$\frac{(az)_\infty}{(z)_\infty} = \sum_{n=0}^{\infty} \frac{(a)_n}{(q)_n} z^n,$$

as was to be shown. □

Theorem 2. Jacobi's [13] triple product is given by

$$\sum_{n\in\mathbb{Z}} (-1)^n q^{n^2} z^n = (zq, z^{-1}q, q^2; q^2)_\infty, \tag{1.17}$$

or by

$$\sum_{n\in\mathbb{Z}} (-1)^n q^{\binom{n}{2}} z^n = (zq, z^{-1}, q; q)_\infty. \tag{1.18}$$

We will only prove the version of the triple product formula corresponding with (1.17), but (1.18) is very similar.

Proof. Consider the equations [8, p. xiv, eqns. (18), (19)]

$$(z;q)_\infty = \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n}{2}}}{(q;q)_n} z^n, \tag{1.19}$$

and

$$\frac{1}{(z;q)_\infty} = \sum_{n=0}^{\infty} \frac{z^n}{(q;q)_n}. \tag{1.20}$$

Mapping $q \mapsto q^2$ and $z \mapsto zq$ in (1.19) gives

$$\begin{aligned}(zq;q^2)_\infty &= \sum_{k=0}^{\infty} \frac{(-1)^k (q^2)^{\binom{k}{2}}}{(q^2;q^2)_k} (zq)^k \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k q^{k(k-1)}}{(q^2;q^2)_k} z^k q^k \\ &= \sum_{k=0}^{\infty} \frac{(-z)^k q^{k^2}}{(q^2;q^2)_k}.\end{aligned} \tag{1.21}$$

Using (1.7),

$$\frac{1}{(q^2;q^2)_k} = \frac{(q^{2k+2};q^2)_\infty}{(q^2;q^2)_\infty}.$$

So, (1.21) is rewritten as

$$\begin{aligned}(zq;q^2)_\infty &= \frac{1}{(q^2;q^2)_\infty} \sum_{k=0}^{\infty} (-z)^k q^{k^2} (q^{2k+2};q^2)_\infty \\ &= \frac{1}{(q^2;q^2)_\infty} \sum_{k\in\mathbb{Z}} (-z)^k q^{k^2} (q^{2k+2};q^2)_\infty.\end{aligned}$$

We use (1.19) on the term $(q^{2k+2};q^2)_\infty$, mapping $q \mapsto q^2$, $z \mapsto q^{2k+2}$ to obtain

$$\begin{aligned}(zq;q^2)_\infty &= \frac{1}{(q^2;q^2)_\infty} \sum_{k\in\mathbb{Z}} (-z)^k q^{k^2} \sum_{m=0}^{\infty} \frac{(-1)^m (q^2)^{\binom{m}{2}}}{(q^2;q^2)_m} (q^{2k+2})^m \\ &= \frac{1}{(q^2;q^2)_\infty} \sum_{k\in\mathbb{Z}} (-z)^k q^{k^2} \sum_{m=0}^{\infty} \frac{(-1)^m (q^{2k+2})^m q^{m^2-m}}{(q^2;q^2)_m} \\ &= \frac{1}{(q^2;q^2)_\infty} \sum_{m=0}^{\infty} \frac{(-1)^m q^{2m} q^{m^2-m}}{(q^2;q^2)_m} \sum_{k\in\mathbb{Z}} (-z)^k q^{k^2} q^{2km} \\ &= \frac{1}{(q^2;q^2)_\infty} \sum_{m=0}^{\infty} \frac{(-1)^m q^{m^2+m}}{(q^2;q^2)_m} \sum_{k\in\mathbb{Z}} (-z)^{m+k} (-z^{-1})^m q^{k^2} q^{2km} \\ &= \frac{1}{(q^2;q^2)_\infty} \sum_{m=0}^{\infty} \frac{(-1)^m (-z^{-1})^m q^m}{(q^2;q^2)_m} \sum_{k\in\mathbb{Z}} (-z)^{m+k} q^{(m+k)^2}.\end{aligned}$$

Now,

$$(zq;q^2)_\infty = \frac{1}{(q^2;q^2)_\infty} \sum_{m=0}^{\infty} \frac{(z^{-1}q)^m}{(q^2;q^2)_m} \sum_{n\in\mathbb{Z}} (-z)^n q^{n^2}.$$

For the first sum, (1.20) is applied with the mappings $q \mapsto q^2$ and $z \mapsto z^{-1}q$. This yields

$$(zq;q^2)_\infty = \frac{1}{(q^2;q^2)_\infty} \cdot \frac{1}{(z^{-1}q;q^2)_\infty} \sum_{n\in\mathbb{Z}} (-z)^n q^{n^2}.$$

Isolating the sum, it is clear that

$$\begin{aligned}(zq, z^{-1}q, q^2; q^2) &= \sum_{n\in\mathbb{Z}} (-z)^n q^{n^2} \\ &= \sum_{n\in\mathbb{Z}} (-1)^n q^{n^2} z^n,\end{aligned}$$

completing the proof. □

Theorem 3. Heine's ${}_2\phi_1$ transformation [12] is given by

$${}_2\phi_1(a,b;c;q,z) = \frac{(az,b)_\infty}{(c,z)_\infty} {}_2\phi_1\left(\frac{c}{b}, z; az; q, b\right). \tag{1.22}$$

Proof. Using (1.7),

$$\begin{aligned}{}_2\phi_1(a,b;c;q,z) &= \sum_{n=0}^{\infty} \frac{(a,b)_n}{(c,q)_n} z^n \\ &= \sum_{n=0}^{\infty} \frac{(a)_n}{(q)_n} z^n \cdot \frac{(b,cq^n)_\infty}{(c,bq^n)_\infty} \\ &= \frac{(b)_\infty}{(c)_\infty} \sum_{n=0}^{\infty} \frac{(a)_n}{(q)_n} z^n \cdot \frac{(cq^n)_\infty}{(bq^n)_\infty}.\end{aligned}$$

From here, (1.13) is applied to the term $\frac{(cq^n)_\infty}{(bq^n)_\infty}$ with the mappings $a \mapsto \frac{c}{b}, z \mapsto bq^n$. This gives us

$$\begin{aligned} {}_2\phi_1(a,b;c;q,z) &= \frac{(b)_\infty}{(c)_\infty} \sum_{n=0}^{\infty} \frac{(a)_n}{(q)_n} z^n \sum_{k=0}^{\infty} \frac{\left(\frac{c}{b}\right)_k}{(q)_k} (bq^n)^k \\ &= \frac{(b)_\infty}{(c)_\infty} \sum_{n,k=0}^{\infty} \frac{(a)_n \left(\frac{c}{b}\right)_k}{(q)_n (q)_k} z^n b^k q^{kn} \\ &= \frac{(b)_\infty}{(c)_\infty} \sum_{k=0}^{\infty} \frac{\left(\frac{c}{b}\right)_k}{(q)_k} b^k \sum_{n=0}^{\infty} \frac{(a)_n}{(q)_n} (zq^k)^n. \end{aligned}$$

For the rightmost sum, we use (1.13) with the mapping $z \mapsto zq^k$. Thus,

$${}_2\phi_1(a,b;c;q,z) = \frac{(b)_\infty}{(c)_\infty} \sum_{k=0}^{\infty} \frac{\left(\frac{c}{b}\right)_k}{(q)_k} b^k \cdot \frac{(azq^k)_\infty}{(zq^k)_\infty}.$$

Using (1.7) on the term $\frac{(azq^k)_\infty}{(zq^k)_\infty}$,

$$\begin{aligned} {}_2\phi_1(a,b;c;q,z) &= \frac{(b)_\infty}{(c)_\infty} \sum_{k=0}^{\infty} \frac{\left(\frac{c}{b}\right)_k}{(q)_k} b^k \cdot \frac{(az)_\infty (z)_k}{(az)_k (z)_\infty} \\ &= \frac{(az,b)_\infty}{(c,z)_\infty} \sum_{k=0}^{\infty} \frac{\left(\frac{c}{b}, z\right)_k}{(az,q)_k} b^k \\ &= \frac{(az,b)_\infty}{(c,z)_\infty} {}_2\phi_1\left(\frac{c}{b}, z; az; q, b\right), \end{aligned}$$

completing the proof. □

Corollary 1. The next two iterates of (1.22) are given by

$$\begin{aligned} {}_2\phi_1(a,b;c;q,z) &= \frac{\left(bz, \frac{c}{b}\right)_\infty}{(c,z)_\infty} {}_2\phi_1\left(\frac{abz}{c}, b; bz; q, \frac{c}{b}\right) \\ &= \frac{\left(\frac{abz}{c}\right)_\infty}{(z)_\infty} {}_2\phi_1\left(\frac{c}{a}, \frac{c}{b}; c; q, \frac{abz}{c}\right). \end{aligned} \tag{1.23}$$

Proof. We use (1.22) to obtain the following:

$$\begin{aligned}
{}_2\phi_1(a,b;c;q,z) &= \frac{(az,b)_\infty}{(c,z)_\infty}{}_2\phi_1\left(\frac{c}{b},z;az;q,b\right)\\
&= \frac{(az,b)_\infty}{(c,z)_\infty}{}_2\phi_1\left(z,\frac{c}{b};az;q,b\right)\\
&= \frac{(az,b)_\infty}{(c,z)_\infty}\cdot\frac{\left(bz,\frac{c}{b}\right)_\infty}{(az,b)_\infty}{}_2\phi_1\left(\frac{abz}{c},b,bz;q,\frac{c}{b}\right)\\
&= \frac{\left(bz,\frac{c}{b}\right)_\infty}{(c,z)_\infty}{}_2\phi_1\left(\frac{abz}{c},b;bz;q,\frac{c}{b}\right).
\end{aligned}$$

Additionally, by (1.22),

$$\begin{aligned}
\frac{\left(bz,\frac{c}{b}\right)_\infty}{(c,z)_\infty}{}_2\phi_1\left(\frac{abz}{c},b;bz;q,\frac{c}{b}\right) &= \frac{\left(bz,\frac{c}{b}\right)_\infty}{(c,z)_\infty}{}_2\phi_1\left(b,\frac{abz}{c};bz;q,\frac{c}{b}\right)\\
&= \frac{\left(bz,\frac{c}{b}\right)_\infty}{(c,z)_\infty}\cdot\frac{\left(c,\frac{abz}{c}\right)_\infty}{\left(bz,\frac{c}{b}\right)_\infty}{}_2\phi_1\left(\frac{c}{a},\frac{c}{b};c;q,\frac{abz}{c}\right)\\
&= \frac{\left(\frac{abz}{c}\right)_\infty}{(z)_\infty}{}_2\phi_1\left(\frac{c}{a},\frac{c}{b};c;q,\frac{abz}{c}\right),
\end{aligned}$$

completing the proof. □

Corollary 2. The q-Gauss sum is given by

$$ {}_2\phi_1\left(a,b;c;q,\frac{c}{ab}\right) = \frac{\left(\frac{c}{a},\frac{c}{b}\right)_\infty}{\left(c,\frac{c}{ab}\right)_\infty}. \tag{1.24}$$

Proof. This is actually just a special case of (1.22). Mapping $z \mapsto \frac{c}{ab}$ gives

$$\begin{aligned}
{}_2\phi_1\left(a,b,c;q,\frac{c}{ab}\right) &= \frac{\left(\frac{c}{b},b\right)_\infty}{\left(c,\frac{c}{ab}\right)_\infty}{}_2\phi_1\left(\frac{c}{b},\frac{c}{ab};\frac{c}{b};q,b\right)\\
&= \frac{\left(\frac{c}{b},b\right)_\infty}{\left(c,\frac{c}{ab}\right)_\infty}\sum_{n=0}^{\infty}\frac{\left(\frac{c}{b},\frac{c}{ab}\right)_n}{\left(\frac{c}{b},q\right)_n}b^n\\
&= \frac{\left(\frac{c}{b},b\right)_\infty}{\left(c,\frac{c}{ab}\right)_\infty}\sum_{n=0}^{\infty}\frac{\left(\frac{c}{ab}\right)_n}{(q)_n}b^n.
\end{aligned}$$

From here, the sum is simplified using (1.13) with the mapping $a \mapsto \frac{c}{ab}$, $z \mapsto b$. This gives us

$$\begin{aligned} {}_2\phi_1\left(a, b, c; q, \frac{c}{ab}\right) &= \frac{\left(\frac{c}{b}, b\right)_\infty}{\left(c, \frac{c}{ab}\right)_\infty} \cdot \frac{\left(\frac{c}{a}\right)_\infty}{(b)_\infty} \\ &= \frac{\left(\frac{c}{a}, \frac{c}{b}\right)_\infty}{\left(c, \frac{c}{ab}\right)_\infty}, \end{aligned}$$

as was to be shown. □

Though the name is attributed to Gauss, (1.24) was actually first discovered by Heine [12] in 1847. It is called a q-Gauss sum because the theorem itself is a q-analogue of Gauss's sum for a hypergeometric series [5].

Theorem 4. The Bailey-Daum q-Kummer sum [4, 7] is given by

$$ {}_2\phi_1\left(a, b; \frac{aq}{b}; q, -\frac{q}{b}\right) = \frac{(-q;q)_\infty \left(aq, \frac{aq^2}{b^2}; q^2\right)_\infty}{\left(\frac{aq}{b}, -\frac{q}{b}; q\right)_\infty}. \tag{1.25}$$

Proof. Begin by writing

$$ {}_2\phi_1\left(a, b; \frac{aq}{b}; q, -\frac{q}{b}\right) = {}_2\phi_1\left(b, a; \frac{aq}{b}; q, -\frac{q}{b}\right).$$

From here, we use (1.22), mapping $c \mapsto \frac{aq}{b}$, $z \mapsto -\frac{q}{b}$ and swapping a and b to obtain

$$\begin{aligned} {}_2\phi_1\left(b, a; \frac{aq}{b}; q, -\frac{q}{b}\right) &= \frac{(-q, a; q)_\infty}{\left(\frac{aq}{b}, -\frac{q}{b}, q\right)_\infty} {}_2\phi_1\left(\frac{q}{b}, -\frac{q}{b}; -q; q, a\right) \\ &= \frac{(-q, a; q)_\infty}{\left(\frac{aq}{b}, -\frac{q}{b}, q\right)_\infty} \sum_{n=0}^{\infty} \frac{\left(\frac{q}{b}, -\frac{q}{b}; q\right)_n}{(-q, q; q)_n} a^n. \end{aligned}$$

In order to simplify the sum, [8, p. 6, eqn. (1.2.40)] is used twice, mapping $a \mapsto \frac{q}{b}$ in the numerator and mapping $a \mapsto q$ in the denominator. So,

$$\frac{(-q,a;q)_\infty}{\left(\frac{aq}{b},-\frac{q}{b},q\right)_\infty}\sum_{n=0}^{\infty}\frac{\left(\frac{q}{b},-\frac{q}{b};q\right)_n}{(-q,q;q)_n}a^n = \frac{(-q,a;q)_\infty}{\left(\frac{aq}{b},-\frac{q}{b},q\right)_\infty}\sum_{n=0}^{\infty}\frac{\left(\frac{q^2}{b^2};q^2\right)_n}{(q^2;q^2)_n}a^n.$$

In order to get rid of the sum, we use (1.13), mapping $q \mapsto q^2$, $a \mapsto \frac{q^2}{b^2}$, and $z \mapsto a$ to obtain

$$\begin{aligned}\frac{(-q,a;q)_\infty}{\left(\frac{aq}{b},-\frac{q}{b},q\right)_\infty}\sum_{n=0}^{\infty}\frac{\left(\frac{q^2}{b^2};q^2\right)_n}{(q^2;q^2)_n}a^n &= \frac{(-q,a;q)_\infty}{\left(\frac{aq}{b},-\frac{q}{b},q\right)_\infty}\cdot\frac{\left(\frac{aq^2}{b^2};q^2\right)_\infty}{(a;q^2)_\infty} \\ &= \frac{(-q;q)_\infty\left(\frac{aq^2}{b^2};q^2\right)_\infty}{\left(\frac{aq}{b},-\frac{q}{b},q\right)_\infty}\cdot\frac{(a;q)_\infty}{(a;q^2)_\infty}.\end{aligned} \tag{1.26}$$

For the term $(a;q)_\infty$ in (1.26), an alternative version of [8, p. 6, eqn. (1.2.40)] (taking $n \to \infty$) is used to obtain

$$(a;q)_\infty = (a,aq;q^2)_\infty.$$

With this in mind, (1.26) reduces to the following:

$$\begin{aligned}\frac{(-q;q)_\infty\left(\frac{aq^2}{b^2};q^2\right)_\infty}{\left(\frac{aq}{b},-\frac{q}{b},q\right)_\infty}\cdot\frac{(a;q)_\infty}{(a;q^2)_\infty} &= \frac{(-q;q)_\infty\left(\frac{aq^2}{b^2};q^2\right)_\infty}{\left(\frac{aq}{b},-\frac{q}{b},q\right)_\infty}\cdot\frac{(a,aq;q^2)_\infty}{(a;q^2)_\infty} \\ &= \frac{(-q;q)_\infty\left(aq,\frac{aq^2}{b^2};q^2\right)_\infty}{\left(\frac{aq}{b},-\frac{q}{b},q\right)_\infty}.\end{aligned}$$

This completes the proof. □

The theorems and corollaries given comprise only a small fraction of the many hypergeometric identities that have been discovered. However, these tend to be the most important due to their use in many different scenarios. New identities involving q-series come about by substituting different variables into preexisting identities and applying different transformation techniques. When we connect q-series to continued fractions, as discussed in the next section, it allows us to create new identities that we may not have ever considered.

1.3 Continued Fractions

Definition 3. A *continued fraction* is of the form

$$b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{\ddots + \cfrac{a_n}{b_n}}}}. \tag{1.27}$$

n is the *length* of the continued fraction, and the a_i and b_i are called *elements*. If $a_i = 1$ for all i, then the continued fraction is called *simple*. Continued fractions are either *finite* or *infinite* depending on whether or not they respectively have a finite or infinite amount of elements.

Remark 3. We often abbreviate continued fractions with the notation

$$b_0 + \frac{a_1}{b_1 +} \frac{a_2}{b_2 +} \frac{a_3}{b_3 +} \cdots \frac{a_n}{b_n}. \tag{1.28}$$

For compactness, the notation

$$b_0 + \mathop{\mathrm{K}}_{i=1}^{\infty} \frac{a_i}{b_i} \tag{1.29}$$

is also used. The use of the letter K in (1.29) comes from *Kettenbruch*, the German word for continued fraction.

Typically, a continued fraction is used to express any real number through an iterative process involving rational numbers. This process can be done much more generally, however. Much interest lies in the relationship between q-series and continued fractions. In this thesis, we will try to express a particular type of hypergeometric function as a continued fraction through an iterative process involving various functions.

To conclude this section, we will look at an example of a q-continued fraction, which will naturally lead into the next chapter.

Example 3. The Rogers-Ramanujan [17] continued fraction is given by

$$\prod_{n=0}^{\infty} \frac{(1-q^{5n+2})(1-q^{5n+3})}{(1-q^{5n+1})(1-q^{5n+4})} = 1 + \frac{q}{1+} \frac{q^2}{1+} \frac{q^3}{1+} \frac{q^4}{1} \cdots. \tag{1.30}$$

Andrews and Bowman [3] gave a full extension of this continued fraction, which will be explored within this thesis. Several continued fraction identities naturally come about from this extension, and many of them have not been discovered yet.

CHAPTER 2

q-CONTINUED FRACTIONS

2.1 The Andrews-Bowman Continued Fraction

The main purpose of this thesis is to find special cases of the Andrews-Bowman continued fraction, a continued fraction that arises from a natural extension of the Rogers-Ramanujan continued fraction given by (1.30). Before we can consider any of these special cases, we must first detail how the Andrews-Bowman continued fraction is derived. All of the preliminary results in this section come from Andrews and Bowman [3].

In [3], the equations used when proving (1.30) are considered first. If

$$G(z,q) = G(z) = \sum_{n=0}^{\infty} \frac{q^{n^2} z^n}{(q)_n}, \tag{2.1}$$

then

$$G(z) = \frac{1}{(z)_\infty} \left(\sum_{n=0}^{\infty} \frac{(z)_n (1 - zq^{2n})(-1)^n q^{\frac{n(5n-1)}{2}} z^{2n}}{(q)_n} \right). \tag{2.2}$$

(2.1) and (2.2) are equal because they satisfy the following q-difference equation given by Rogers and Ramanujan [18]:

$$G(z) = G(zq) + zqG(zq^2).$$

Additionally, $G(0) = 1$ and $G(z)$ is analytic around the origin.

Let us note the following:

$$\begin{aligned} G(z) &= G(zq) + zqG(zq^2) \\ \frac{G(z)}{G(zq)} &= 1 + \frac{zqG(zq^2)}{G(zq)} \\ \frac{G(z)}{G(zq)} &= 1 + \frac{zq}{\frac{G(zq)}{G(zq^2)}}. \end{aligned}$$

By repeatedly mapping $z \mapsto zq$, it is clear that

$$\frac{G(z)}{G(zq)} = 1 + \frac{zq}{1} + \frac{zq^2}{1} + \frac{zq^3}{1} + \frac{zq^4}{1} + \cdots. \tag{2.3}$$

This is known as the *generalized Rogers-Ramanujan continued fraction.*

Andrews and Bowman [3] consider G.N. Watson's [19] much more general version of (2.2) using five parameters. This equation is:

$$\begin{aligned} &{}_8\phi_7\left(z, q\sqrt{z}, -q\sqrt{z}, a_1, a_2, a_3, a_4, a_5; \sqrt{z}, -\sqrt{z}, \frac{zq}{a_1}, \frac{zq}{a_2}, \frac{zq}{a_3}, \frac{zq}{a_4}, \frac{zq}{a_5}; q, \frac{z^2q^2}{a_1a_2a_3a_4a_5}\right) \\ &= \frac{\left(zq, \frac{zq}{a_3a_4}\right)_N}{\left(\frac{zq}{a_3}, \frac{zq}{a_4}\right)_N} {}_4\phi_3\left(\frac{zq}{a_1a_2}, a_3, a_4, a_5; \frac{zq}{a_1}, \frac{zq}{a_2}, \frac{a_3a_4a_5}{z}; q, q\right), \end{aligned} \tag{2.4}$$

where $a_5 = q^{-N}$ for some nonnegative integer N.

Note that it is more natural to consider (2.4) as a function of $\frac{1}{a_k}$ rather than a_k, since it is continuous in every $\frac{1}{a_k}$ around zero. In other words, $a_k = \infty$. To keep consistent with the notation given in [3], as well as standard notation given by Gasper and Rahman [8], whenever we set any of the $a_k = \infty$ throughout this thesis, we will be doing so by considering their reciprocals, as described above. When $a_k = \infty$ for all k (here $a_5 = \infty$ denotes $\lim_{N\to\infty} q^{-N}$), it then follows that (2.4) reduces term-by-term to (2.2). A full derivation of this result will be given in Chapter 3.

The ultimate question that was considered in [3] is whether or not (2.3) can be extended such that $G(z)$ is replaced by Watson's very well-poised ${}_8\phi_7$ function in (2.4). (For a reference of the full definition of "very well-poised", please see [2].) The answer to this question is "yes", and this extension actually leads into some very natural q-series identities, which will be discussed in the next chapter.

In [1], Andrews defined a generalization of the left-hand side of (2.4). We will define this below, keeping consistent with Andrews' notation. Let

$$\begin{aligned} C_{k,i}(a_1, a_2, \ldots, a_\lambda; x; q) &= C_{k,i}((a); x; q)_\lambda \\ &= \sum_{n=0}^{\infty} (-1)^{n(\lambda+1)} x^{kn} (a_1 a_2 \cdots a_\lambda)^{-n} q^{\frac{(2k-\lambda+1)n^2+(\lambda+1)n-2in}{2}} \\ &\quad \times \frac{(1-x^i q^{2ni})}{(1-x)} \cdot \frac{(x, a_1, a_2, \ldots, a_\lambda)_n}{\left(\frac{xq}{a_1}, \frac{xq}{a_2}, \ldots, \frac{xq}{a_\lambda}, q\right)_n}. \end{aligned} \tag{2.5}$$

In the particular case $i = 1$, $\lambda = 2k+1$, then $C_{k,1}((a); x; q)_{2k+1}$ can be written as [1, p. 434]

$$\begin{aligned} &C_{k,1}((a); x; q)_{2k+1} \\ &= {}_{2k+4}\phi_{2k+3}\left(x, q\sqrt{x}, -q\sqrt{x}, a_1, a_2, \ldots, a_{2k+1}; \sqrt{x}, -\sqrt{x}, \frac{xq}{a_1}, \frac{xq}{a_2}, \ldots, \frac{xq}{a_{2k+1}}; q, \frac{x^k q^k}{a_1 a_2 \cdots a_{2k+1}}\right). \end{aligned} \tag{2.6}$$

Remark 4. Note that for $k = 2$, this is precisely the ${}_8\phi_7$ function described in (2.4).

Additionally, let

$$\begin{aligned} H_{k,i}(a_1, a_2, \ldots, a_\lambda; x, q) &= H_{k,i}((a); x; q)_\lambda \\ &= \frac{\left(\frac{xq}{a_1}, \frac{xq}{a_2}, \ldots, \frac{xq}{a_\lambda}\right)_\infty}{(xq)_\infty} C_{k,i}((a); x; q)_\lambda. \end{aligned} \tag{2.7}$$

Definition 4. The j^{th} *elementary symmetric function* of $x_1, \ldots, x_\lambda$ is defined by

$$\sigma_j(x_1, x_2, \ldots, x_\lambda) = \sum_{1 \le i_1 < i_2 < \cdots < i_j \le \lambda} x_{i_1} \cdots x_{i_j}, \tag{2.8}$$

where

$$\sigma_0(x_1, x_2, \ldots, x_\lambda) := 1.$$

In [1, p. 435], some auxiliary relations between (2.5)–(2.7) are defined. They are given below:

$$\begin{aligned} H_{k,i}((a);x;q)_\lambda - H_{k,i-1}((a);x;q)_\lambda \\ = x^{i-1} \sum_{j=0}^{\lambda} (-1)^j \sigma_j \left(\frac{1}{a_1}, \frac{1}{a_2}, \cdots, \frac{1}{a_\lambda} \right) x^j q^j H_{k,k+1-i-j}((a);xq;q)_\lambda, \end{aligned} \tag{2.9}$$

$$H_{k,-i}((a);x;q)_\lambda = -x^{-i} H_{k,i}((a);x;q)_\lambda, \tag{2.10}$$

and

$$H_{k,0}((a);x;q)_\lambda \equiv 0. \tag{2.11}$$

For the remainder of this thesis, we will only be considering the case $k = 2$, $\lambda = 5$ (due to Remark 4). Additionally, to keep consistent with the notation in [3], we will use the shorthand notation

$$H_i(x) = H_{2,i}(a_1, a_2, a_3, a_4, a_5; x; q), \tag{2.12}$$

and

$$\sigma_j = \sigma_j \left(\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, \frac{1}{a_4}, \frac{1}{a_5} \right). \tag{2.13}$$

The following proposition will be integral in obtaining the necessary q-difference relations used in the Andrews-Bowman continued fraction.

Proposition 1. The four simple cases $i = 2, 1, 0, -1$ of (2.9) respectively reduce to the following:

$$H_2(x) - H_1(x) - \sigma_3 x^2 q H_2(xq) + \sigma_4 x^2 q H_3(xq) - \sigma_5 x^2 q H_4(xq) = (x - \sigma_2 x^2 q) H_1(xq), \quad (2.14)$$

$$H_1(x) - (1 - \sigma_4 x^2 q^2) H_2(xq) - \sigma_5 x^2 q^2 H_3(xq) = (-\sigma_1 xq + \sigma_3 x^2 q^2) H_1(xq), \quad (2.15)$$

$$H_1(x) + (\sigma_1 xq - \sigma_5 x^3 q^3) H_2(xq) - H_3(xq) = (\sigma_2 x^2 q^2 - \sigma_4 x^3 q^3) H_1(xq), \quad (2.16)$$

$$H_2(x) - xH_1(x) - \sigma_2 x^2 q^2 H_2(xq) + \sigma_1 xq H_3(xq) - H_4(xq) = (-\sigma_3 x^3 q^3 + \sigma_5 x^4 q^4) H_1(xq). \quad (2.17)$$

Proof. (2.10) and (2.11) will be used to simplify wherever necessary.

Case 1: $i = 2$

Substituting $i = 2$ into (2.9) gives

$$\begin{aligned} H_2(x) - H_1(x) &= x \sum_{j=0}^{5} (-1)^j \sigma_j x^j q^j H_{1-j}(xq) \\ &= x(\sigma_0 x^0 q^0 H_1(xq) - \sigma_1 xq H_0(xq) + \sigma_2 x^2 q^2 H_{-1}(xq) - \sigma_3 x^3 q^3 H_{-2}(xq) \\ &\qquad + \sigma_4 x^4 q^4 H_{-3}(xq) - \sigma_5 x^5 q^5 H_{-4}(xq)) \\ &= xH_1(xq) - \sigma_2 x^2 q H_1(xq) + \sigma_3 x^2 q H_2(xq) - \sigma_4 x^2 q H_3(xq) + \sigma_5 x^2 q H_4(xq), \end{aligned}$$

which implies that

$$H_2(x) - H_1(x) - \sigma_3 x^2 q H_2(xq) + \sigma_4 x^2 q H_3(xq) - \sigma_5 x^2 q H_4(xq) = (x - \sigma_2 x^2 q) H_1(xq).$$

Case 2: $i = 1$

Substituting $i = 1$ into (2.9) gives

$$H_1(x) - H_0(x) = \sum_{j=0}^{5} (-1)^j \sigma_j x^j q^j H_{2-j}(xq).$$

We use (2.11) on the left-hand side and expand and simplify the right-hand side using (2.10) and (2.11) to obtain

$$\begin{aligned}H_1(x) &= \sigma_0 x^0 q^0 H_2(xq) - \sigma_1 xqH_1(xq) + \sigma_2 x^2 q^2 H_0(xq) - \sigma_3 x^3 q^3 H_{-1}(xq)\\ &\quad + \sigma_4 x^4 q^4 H_{-2}(xq) - \sigma_5 x^5 q^5 H_{-3}(xq)\\ &= H_2(xq) - \sigma_1 xqH_1(xq) + \sigma_3 x^2 q^2 H_1(xq) - \sigma_4 x^2 q^2 H_2(xq) + \sigma_5 x^2 q^2 H_3(xq).\end{aligned}$$

This implies that

$$H_1(x) - (1 - \sigma_4 x^2 q^2)H_2(xq) - \sigma_5 x^2 q^2 H_3(xq) = (-\sigma_1 xq + \sigma_3 x^2 q^2)H_1(xq).$$

Case 3: $i = 0$

Substituting $i = 0$ into (2.9) gives

$$H_0(x) - H_{-1}(x) = x^{-1} \sum_{j=0}^{5} (-1)^j \sigma_j x^j q^j H_{3-j}(xq).$$

Using equations (2.10) and (2.11) on the left-hand side and expanding the right-hand side, we obtain

$$\begin{aligned}x^{-1}H_1(x) = x^{-1}(&\sigma_0 x^0 q^0 H_3(xq) - \sigma_1 xqH_2(xq) + \sigma_2 x^2 q^2 H_1(xq) - \sigma_3 x^3 q^3 H_0(xq)\\ &+ \sigma_4 x^4 q^4 H_{-1}(xq) - \sigma_5 x^5 q^5 H_{-2}(xq)).\end{aligned}$$

Cancelling common terms and simplifying the right-hand side using (2.10) and (2.11) gives

$$H_1(x) = H_3(xq) - \sigma_1 xqH_2(xq) + \sigma_2 x^2 q^2 H_1(xq) - \sigma_4 x^3 q^3 H_1(xq) + \sigma_5 x^3 q^3 H_2(xq),$$

which implies that

$$H_1(x) + (\sigma_1 xq - \sigma_5 x^3 q^3) H_2(xq) - H_3(xq) = (\sigma_2 x^2 q^2 - \sigma_4 x^3 q^3) H_1(xq).$$

Case 4: $i = -1$

Finally, substituting $i = -1$ into (2.9) gives

$$H_{-1}(x) - H_{-2}(x) = x^{-2} \sum_{j=0}^{5} (-1)^j \sigma_j x^j q^j H_{4-j}(xq).$$

Simplifying the left-hand side using (2.10) and (2.11) and expanding the right-hand side,

$$\begin{aligned} x^{-2} H_2(x) - x^{-1} H_1(x) = x^{-2}(&\sigma_0 x^0 q^0 H_4(xq) - \sigma_1 xq H_3(xq) + \sigma_2 x^2 q^2 H_2(xq) - \sigma_3 x^3 q^3 H_1(xq) \\ &+ \sigma_4 x^4 q^4 H_0(xq) - \sigma_5 x^5 q^5 H_{-1}(xq)). \end{aligned}$$

Multiplying each side by x^2 and simplifying the right-hand side using (2.10) and (2.11) yields

$$\begin{aligned} H_2(x) - xH_1(x) = &H_4(xq) - \sigma_1 xq H_3(xq) + \sigma_2 x^2 q^2 H_2(xq) - \sigma_3 x^3 q^3 H_1(xq) \\ &+ \sigma_5 x^4 q^4 H_1(xq), \end{aligned}$$

which implies that

$$H_2(x) - xH_1(x) - \sigma_2 x^2 q^2 H_2(xq) + \sigma_1 xq H_3(xq) - H_4(xq) = (-\sigma_3 x^3 q^3 + \sigma_5 x^4 q^4) H_1(xq).$$

This completes the proof. $\square$

At this point in [3], the mapping $x \mapsto xq$ was applied to (2.14) – (2.17). The resulting equations were rewritten, isolating $H_1(xq)$ and $H_1(xq^2)$ on the right. These new equations are given below.

$$H_2(xq) - \sigma_3 x^2 q^3 H_2(xq^2) + \sigma_4 x^2 q^3 H_3(xq^2) - \sigma_5 x^2 q^3 H_4(xq^2) = H_1(xq) + (xq - \sigma_2 x^2 q^3) H_1(xq^2), \tag{2.18}$$

$$-(1 - \sigma_4 x^2 q^4) H_2(xq^2) - \sigma_5 x^2 q^4 H_3(xq^2) = (-\sigma_1 xq^2 + \sigma_3 x^2 q^4) H_1(xq^2) - H_1(xq), \tag{2.19}$$

$$(\sigma_1 xq^2 - \sigma_5 x^3 q^6) H_2(xq^2) - H_3(xq^2) = (\sigma_2 x^2 q^4 - \sigma_4 x^3 q^6) H_1(xq^2) - H_1(xq), \tag{2.20}$$

$$H_2(xq) - \sigma_2 x^2 q^4 H_2(xq^2) + \sigma_1 xq^2 H_3(xq^2) - H_4(xq^2) = xqH_1(xq) + (-\sigma_3 x^3 q^6 + \sigma_5 x^4 q^8) H_1(xq^2). \tag{2.21}$$

Note that there are two misprints in (2.21) as it appears in [3]. The coefficient of $H_3(xq^2)$ should be $\sigma_1 xq^2$, not xq^2. Additionally, the term $xqH_1(xq)$ should be present, as it is missing.

If we consider (2.14) – (2.17) and (2.18) – (2.21) as a system of linear equations in the unknowns $H_1(x), H_2(x), H_2(xq), H_2(xq^2), H_3(xq), H_3(xq^2), H_4(xq), H_4(xq^2)$, then there exists a linear relation between $H_1(x), H_1(xq)$, and $H_1(xq^2)$. Cramer's Rule and the open-source symbolic algebra package Maxima [16] were used to find this relation in [3]. We wish to make said relation as compact as possible.

Keeping consistent with the notation in [3], the polynomial

$$p(x) = p(x; \sigma_1, \sigma_4, \sigma_5, q) = 1 - \sigma_4 x^2 q^2 + \sigma_1 \sigma_5 x^3 q^3 - \sigma_5^2 x^5 q^5 \tag{2.22}$$

is introduced. Using this polynomial $p(x)$, the relation simplifies into

$$Q(x) H_1(x) = P(x) H_1(xq) + R(x) H_1(xq^2), \tag{2.23}$$

where

$$Q(x) = (1 - \sigma_5 x^2 q^2)(1 - \sigma_5 x^2 q^3) p(xq), \tag{2.24}$$

$$\begin{aligned} P(x) = &- xq(1 - \sigma_5 x^2 q^3) p(xq)(\sigma_1 - \sigma_3 xq + \sigma_2 \sigma_5 x^3 q^3 - \sigma_4 \sigma_5 x^4 q^4) \\ &- p(x)((- 1 - \sigma_1 xq + \sigma_5 x^3 q^4) p(xq) - \sigma_2 \sigma_5^2 x^6 q^{11} + \sigma_1 \sigma_4 \sigma_5 x^5 q^9 \\ &- \sigma_4 \sigma_5 x^5 q^9 + \sigma_3 \sigma_5 x^4 q^7 + \sigma_2 \sigma_5 x^4 q^7 - \sigma_4^2 x^4 q^7 \\ &- \sigma_1 \sigma_5 x^3 q^5 + \sigma_4 x^2 q^3 - \sigma_3 x^2 q^3 + \sigma_1 xq), \end{aligned} \tag{2.25}$$

and

$$R(x) = xqp(x) \prod_{1 \le i < j \le 5} \left(1 - \frac{xq^2}{a_i a_j}\right). \tag{2.26}$$

Note that there is a misprint in (2.25) as it appears in [3]. The coefficient of $x^6 q^{11}$ should be $-\sigma_2 \sigma_5^2$, not $-\sigma_2 \sigma_5$. This was corrected in [10, p. 177].

With the definitions of $P(x), Q(x)$, and $R(x)$, Andrews and Bowman proved the following theorem.

Theorem 5.

$$\frac{H_1(x)}{H_1(xq)} = \frac{P(x)}{Q(x)} + \frac{\frac{R(x)}{Q(x)}}{\frac{P(xq)}{Q(xq)} +} \frac{\frac{R(xq)}{Q(xq)}}{\frac{P(xq^2)}{Q(xq^2)} +} \frac{\frac{R(xq^2)}{Q(xq^2)}}{\frac{P(xq^3)}{Q(xq^3)} + \cdots}. \tag{2.27}$$

A short proof of this theorem was given in [3], which will be paraphrased below for clarity.

Proof. Let us consider (2.23). If we divide through by $Q(x)$ and $H_1(xq)$, we obtain

$$\frac{H_1(x)}{H_1(xq)} = \frac{P(x)}{Q(x)} + \frac{\frac{R(x)}{Q(x)}}{\frac{H_1(xq)}{H_1(xq^2)}}.$$

By repeatedly mapping $x \mapsto xq$, this process can be iterated with $\frac{H_1(xq)}{H_1(xq^2)}$, $\frac{H_1(xq^2)}{H_1(xq^3)}$, etc. From this, it is clear that this will simplify into (2.27). For convergence, Auric's Theorem [14, p. 170, thm. (5.10)] applies because of the factor x at the front of the right-hand side of (2.26).

This completes the proof. □

Theorem 5 is the main focal point of this thesis. In the next chapter, we will look at special cases of this continued fraction, including one that Andrews and Bowman proved in [3].

CHAPTER 3

MAIN RESULTS

We will now take a look at some of the special cases that arise from equation (2.27). The simplest case involves only three finite parameters and is what Andrews and Bowman [3] observed in their paper. In any case, all final results were verified using the Maxima code described in Appendix A, and each verification is presented in Appendix B.

3.1 Special Case 1: $a_1, a_2, a_3, a_4, a_5 \to \infty$

With (2.27) in mind, let $a_1, a_2, a_3, a_4, a_5 \to \infty$ (meaning that $\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5 = 0$). It is to be noted that this special case will ultimately reduce down to (2.3). Keeping consistent with the notation given in [3], denote the new $p(x), P(x), Q(x)$, and $R(x)$ by $\overline{p}(x), \overline{P}(x)$, $\overline{Q}(x)$, and $\overline{R}(x)$. By (2.22), it is clear that

$$\overline{p}(x) = \overline{p}(xq) = 1. \tag{3.1}$$

Using (2.24)–(2.26) and (3.1),

$$\overline{Q}(x) = \overline{P}(x) = 1, \tag{3.2}$$

and

$$\overline{R}(x) = xq. \tag{3.3}$$

So, we obtain the following corollary to Theorem 5.

Corollary 3.

$$\frac{H_{2,1}(x;q)}{H_{2,1}(xq;q)} = 1 + \frac{xq}{1\,+}\,\frac{xq^2}{1\,+}\,+\cdots. \tag{3.4}$$

Consider the following proposition.

Proposition 2.

$$\begin{aligned} H_{2,1}(x;q) &= G(x) \\ &= \sum_{n=0}^{\infty} \frac{q^{n^2} x^n}{(q)_n} \\ &= \frac{1}{(x)_\infty}\left(\sum_{n=0}^{\infty} \frac{(x)_n(1-xq^{2n})(-1)^n q^{\frac{n(5n-1)}{2}} x^{2n}}{(q)_n}\right). \end{aligned} \tag{3.5}$$

Note that in the proof of this proposition, we will also prove that (2.1) and (2.2) are equal.

Proof. By (2.7),

$$H_{2,1}(a_1,a_2,a_3,a_4,a_5;x;q) = \frac{\left(\frac{xq}{a_1},\frac{xq}{a_2},\frac{xq}{a_3},\frac{xq}{a_4},\frac{xq}{a_5}\right)_\infty}{(xq)_\infty} C_{2,1}((a);x;q)_5. \tag{3.6}$$

Using (2.6) and (2.4),

$$\begin{aligned} &C_{2,1}((a);x;q)_5 \\ &= {}_8\phi_7\left(x, q\sqrt{x}, -q\sqrt{x}, a_1,a_2,a_3,a_4,a_5; \sqrt{x}, -\sqrt{x}, \frac{xq}{a_1},\frac{xq}{a_2},\frac{xq}{a_3},\frac{xq}{a_4},\frac{xq}{a_5}; q, \frac{x^2q^2}{a_1a_2a_3a_4a_5}\right) \\ &= \frac{\left(xq, \frac{xq}{a_3a_4}\right)_N}{\left(\frac{xq}{a_3},\frac{xq}{a_4}\right)_N} {}_4\phi_3\left(\frac{xq}{a_1a_2}a_3,a_4,a_5; \frac{xq}{a_1},\frac{xq}{a_2},\frac{a_3a_4a_5}{z}; q, q\right). \end{aligned}$$

With this in mind, it is clear that (3.6) reduces to

$$
\begin{aligned}
&H_{2,1}(a_1,a_2,a_3,a_4,a_5;x;q)\\
&=\frac{\left(\frac{xq}{a_1},\frac{xq}{a_2},\frac{xq}{a_3},\frac{xq}{a_4},\frac{xq}{a_5}\right)_\infty}{(xq)_\infty}\\
&\quad\times {}_8\phi_7\left(x,q\sqrt{x},-q\sqrt{x},a_1,a_2,a_3,a_4,a_5;\sqrt{x},-\sqrt{x},\frac{xq}{a_1},\frac{xq}{a_2},\frac{xq}{a_3},\frac{xq}{a_4},\frac{xq}{a_5};q,\frac{x^2q^2}{a_1a_2a_3a_4a_5}\right)\\
&=\frac{\left(\frac{xq}{a_1},\frac{xq}{a_2},\frac{xq}{a_3},\frac{xq}{a_4},\frac{xq}{a_5}\right)_\infty}{(xq)_\infty}\\
&\quad\times\frac{\left(xq,\frac{xq}{a_3a_4}\right)_N}{\left(\frac{xq}{a_3},\frac{xq}{a_4}\right)_N}{}_4\phi_3\left(\frac{xq}{a_1a_2},a_3,a_4,a_5;\frac{xq}{a_1},\frac{xq}{a_2},\frac{a_3a_4a_5}{x};q,q\right).
\end{aligned}
\tag{3.7}
$$

Consider the ${}_8\phi_7$ side of (3.7). Keeping in mind that $a_5=q^{-N}$, the sum is simplified into the following:

$$
\begin{aligned}
&{}_8\phi_7\left(x,q\sqrt{x},-q\sqrt{x},a_1,a_2,a_3,a_4,q^{-N};\sqrt{x},-\sqrt{x},\frac{xq}{a_1},\frac{xq}{a_2},\frac{xq}{a_3},\frac{xq}{a_4},xq^{N+1};q,\frac{x^2q^{N+2}}{a_1a_2a_3a_4}\right)\\
&=\sum_{n=0}^{\infty}\frac{(x,q\sqrt{x},-q\sqrt{x},a_1,a_2,a_3,a_4,q^{-N})_n\left(\frac{x^2q^{N+2}}{a_1a_2a_3a_4}\right)^n}{\left(\sqrt{x},-\sqrt{x},\frac{xq}{a_1},\frac{xq}{a_2},\frac{xq}{a_3},\frac{xq}{a_4},xq^{N+1},q\right)_n}\\
&=\sum_{n=0}^{\infty}\frac{(1-q\sqrt{x})(1-q^2\sqrt{x})\cdots(1-q^n\sqrt{x})(1+q\sqrt{x})(1+q^2\sqrt{x})\cdots(1+q^n\sqrt{x})}{(1-\sqrt{x})(1-q\sqrt{x})\cdots(1-q^{n-1}\sqrt{x})(1+\sqrt{x})(1+q\sqrt{x})\cdots(1+q^{n-1}\sqrt{x})}\\
&\qquad\times\frac{(1-x)(1-xq)\cdots(1-xq^{n-1})(a_1,a_2,a_3,a_4,q^{-N})_n\left(\frac{x^2q^{N+2}}{a_1a_2a_3a_4}\right)^n}{\left(\frac{xq}{a_1},\frac{xq}{a_2},\frac{xq}{a_3},\frac{xq}{a_4},xq^{N+1},q\right)_n}\\
&=\sum_{n=0}^{\infty}\frac{(xq)_{n-1}(1-xq^{2n})(a_1,a_2,a_3,a_4,q^{-N})_n\left(\frac{x^2q^{N+2}}{a_1a_2a_3a_4}\right)^n}{\left(\frac{xq}{a_1},\frac{xq}{a_2},\frac{xq}{a_3},\frac{xq}{a_4},xq^{N+1},q\right)_n}.
\end{aligned}
$$

Remark 5. Note that for any a_i, we can write

$$(a_i)_n \left(\frac{1}{a_i}\right)^n = (1-a_i)(1-a_iq)\cdots(1-a_iq^{n-1})\left(\frac{1}{a_i}\right)^n$$
$$= \left(\frac{1}{a_i}-1\right)\left(\frac{1}{a_i}-q\right)\cdots\left(\frac{1}{a_i}-q^{n-1}\right).$$

Letting $a_i \to \infty$, we obtain

$$(-1)(-q)\cdots(-q^{n-1}) = (-1)^n q^{\frac{n(n-1)}{2}}.$$

Let $a_1, a_2, a_3, a_4, a_5 \to \infty$ on the ${}_8\phi_7$ side of (3.7). (Here $a_5 \to \infty$ means $N \to \infty$ with $a_5 = q^{-N}$.) Then by Remark 5,

$$\begin{aligned}
H_{2,1}(x;q) &= \frac{1}{(xq)_\infty}\left(\sum_{n=0}^{\infty} \frac{(xq)_{n-1}(1-xq^{2n})\left((-1)^n q^{\frac{n(n-1)}{2}}\right)^5 x^{2n}q^{2n}}{(q)_n}\right) \\
&= \frac{1}{(xq)_\infty}\left(\sum_{n=0}^{\infty} \frac{(xq)_{n-1}(1-xq^{2n})(-1)^n q^{\frac{5n(n-1)}{2}} x^{2n}q^{2n}}{(q)_n}\right) \\
&= \frac{1}{(xq)_\infty}\left(\sum_{n=0}^{\infty} \frac{(xq)_{n-1}(1-xq^{2n})(-1)^n q^{\frac{n(5n-1)}{2}} x^{2n}}{(q)_n}\right) \\
&= \frac{1}{(1-x)(xq)_\infty}\left(\sum_{n=0}^{\infty} \frac{(1-x)(xq)_{n-1}(1-xq^{2n})(-1)^n q^{\frac{n(5n-1)}{2}} x^{2n}}{(q)_n}\right) \\
&= \frac{1}{(x)_\infty}\left(\sum_{n=0}^{\infty} \frac{(x)_n(1-xq^{2n})(-1)^n q^{\frac{n(5n-1)}{2}} x^{2n}}{(q)_n}\right) \\
&= G(x).
\end{aligned}$$

Now consider the ${}_4\phi_3$ side of (3.7). Set $a_5 = q^{-N}$, and let $N \to \infty$. Then for the outside coefficient,

$$\lim_{N\to\infty} \frac{\left(\frac{xq}{a_1}, \frac{xq}{a_2}, \frac{xq}{a_3}, \frac{xq}{a_4}, xq^{N+1}\right)_\infty}{(xq)_\infty} \cdot \frac{\left(xq, \frac{xq}{a_3a_4}\right)_N}{\left(\frac{xq}{a_3}, \frac{xq}{a_4}\right)_N} = \frac{\left(\frac{xq}{a_1}, \frac{xq}{a_2}, \frac{xq}{a_3}, \frac{xq}{a_4}\right)_\infty}{(xq)_\infty} \cdot \frac{\left(xq, \frac{xq}{a_3a_4}\right)_\infty}{\left(\frac{xq}{a_3}, \frac{xq}{a_4}\right)_\infty} = \left(\frac{xq}{a_1}, \frac{xq}{a_2}, \frac{xq}{a_3a_4}\right)_\infty . \tag{3.8}$$

Using L'Hôpital's Rule and the Dominated Convergence Theorem for the sum gives

$$\begin{aligned} &\lim_{N\to\infty} \sum_{n=0}^{\infty} \frac{\left(\frac{xq}{a_1a_2}, a_3, a_4, q^{-N}\right)_n q^n}{\left(\frac{xq}{a_1}, \frac{xq}{a_2}, \frac{a_3a_4q^{-N}}{x}, q\right)_n} \\ &= \sum_{n=0}^{\infty} \lim_{N\to\infty} \frac{\left(\frac{xq}{a_1a_2}, a_3, a_4\right)_n (1-q^{-N})(1-q^{-N+1})\cdots(1-q^{-N+n})q^n}{\left(\frac{xq}{a_1}, \frac{xq}{a_2}, q\right)_n (1-\frac{a_3a_4}{x}q^{-N})(1-\frac{a_3a_4}{x}q^{-N+1})\cdots(1-\frac{a_3a_4}{x}q^{-N+n})} \\ &= \sum_{n=0}^{\infty} \frac{\left(\frac{xq}{a_1a_2}, a_3, a_4\right)_n q^n}{\left(\frac{xq}{a_1}, \frac{xq}{a_2}, q\right)_n \left(\frac{a_3a_4}{x}\right)^n} \\ &= \sum_{n=0}^{\infty} \frac{\left(\frac{xq}{a_1a_2}, a_3, a_4\right)_n \left(\frac{xq}{a_3a_4}\right)^n}{\left(\frac{xq}{a_1}, \frac{xq}{a_2}, q\right)_n} . \end{aligned} \tag{3.9}$$

From (3.8) and (3.9),

$$H_{2,1}(a_1, a_2, a_3, a_4; x; q) = \left(\frac{xq}{a_1}, \frac{xq}{a_2}, \frac{xq}{a_3a_4}\right)_\infty \sum_{n=0}^{\infty} \frac{\left(\frac{xq}{a_1a_2}, a_3, a_4\right)_n \left(\frac{xq}{a_3a_4}\right)^n}{\left(\frac{xq}{a_1}, \frac{xq}{a_2}, q\right)_n} .$$

Next, let $a_1 \to \infty$. Then

$$H_{2,1}(a_2, a_3, a_4; x; q) = \left(\frac{xq}{a_2}, \frac{xq}{a_3a_4}\right)_\infty \sum_{n=0}^{\infty} \frac{(a_3, a_4)_n \left(\frac{xq}{a_3a_4}\right)^n}{\left(\frac{xq}{a_2}, q\right)_n} . \tag{3.10}$$

Now let $a_2 \to \infty$. Then

$$H_{2,1}(a_3, a_4; x; q) = \left(\frac{xq}{a_3 a_4}\right)_\infty \sum_{n=0}^{\infty} \frac{(a_3, a_4)_n \left(\frac{xq}{a_3 a_4}\right)^n}{(q)_n}.$$

Finally, let $a_3, a_4 \to \infty$. With Remark 5 in mind, we obtain

$$\begin{aligned} H_{2,1}(x; q) &= \sum_{n=0}^{\infty} \frac{((-1)^n q^{\frac{n(n-1)}{2}})^2 x^n q^n}{(q)_n} \\ &= \sum_{n=0}^{\infty} \frac{q^{n(n-1)} x^n q^n}{(q)_n} \\ &= \sum_{n=0}^{\infty} \frac{q^{n^2} x^n}{(q)_n} \\ &= G(x), \end{aligned}$$

which completes the proof of (3.5) and also proves the equality of (2.1) and (2.2). □

From this, it is clear that Corollary 3 reduces to

$$\frac{G(x)}{G(xq)} = 1 + \frac{xq}{1\ +}\,\frac{xq^2}{1\ +}\,\frac{xq^3}{1\ +}\,\frac{xq^4}{1\ +}\cdots,$$

which is precisely (2.3). This is verified using the Maxima program described in Appendix A, and the results are given in Appendix B.

3.2 Special Case 2: $a_4, a_5 \to \infty$, $q \mapsto q^4$, $a_1 = q$, $a_2 = q^2$, $a_3 = q^3$

For this special case, the main results are due to Andrews and Bowman [3]. However, many of the smaller details were left out of the original paper, so we will complete any missing proofs, as well as provide further detail wherever necessary.

Let $a_4, a_5 \to \infty$ (meaning that $\sigma_4, \sigma_5 = 0$). Keeping with the same notation used for Special Case 1, then it is clear that by (2.22),

$$\overline{p}(x) = \overline{p}(xq) = 1. \tag{3.11}$$

Using (2.24)–(2.26) and (3.11),

$$\overline{Q}(x) = 1, \tag{3.12}$$

$$\begin{aligned}
\overline{P}(x) &= -xq(\sigma_1 - \sigma_3 xq) - (-1 - \sigma_1 xq - \sigma_3 x^2 q^3 + \sigma_1 xq) \\
&= 1 - \sigma_1 xq + \sigma_3 x^2 q^2 + \sigma_3 x^2 q^3 \\
&= 1 - \sigma_1 xq + \sigma_3 x^2 q^2 (1+q) \\
&= 1 - \left(\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3}\right) xq + \left(\frac{1}{a_1 a_2 a_3}\right) x^2 q^2 (1+q) \\
&= 1 - \frac{xq}{a_1} - \frac{xq}{a_2} - \frac{xq}{a_3} + \frac{x^2 q^2}{a_1 a_2 a_3}(1+q),
\end{aligned} \tag{3.13}$$

and

$$\begin{aligned}
\overline{R}(x) &= xq \prod_{1 \le i < j \le 3} \left(1 - \frac{xq^2}{a_i a_j}\right) \\
&= xq \left(1 - \frac{xq^2}{a_1 a_2}\right) \left(1 - \frac{xq^2}{a_1 a_3}\right) \left(1 - \frac{xq^2}{a_2 a_3}\right).
\end{aligned} \tag{3.14}$$

So, we obtain the following corollary to Theorem 5.

Corollary 4.

$$\frac{H_{2,1}(a_1,a_2,a_3;x;q)}{H_{2,1}(a_1,a_2,a_3;xq;q)} = \overline{P}(x) + \frac{\overline{R}(x)}{\overline{P}(xq)+}\,\frac{\overline{R}(xq)}{\overline{P}(xq^2)+\cdots}. \tag{3.15}$$

Note that there is a typo in (3.15) as it appears in [3]. The denominator of the fraction on the left-hand side of the equation should read $H_{2,1}(a_1,a_2,a_3;xq;q)$, not $H_{2,1}(a_1,a_2,a_3;q;q)$.

Consider the following proposition.

Proposition 3.

$$H_{2,1}(a_1,a_2,a_3;x;q) = \left(\frac{xq}{a_1a_3}, \frac{xq}{a_2}\right)_\infty \sum_{n=0}^{\infty} \frac{(a_1,a_3)_n \left(\frac{xq}{a_1a_3}\right)^n}{\left(\frac{xq}{a_2}, q\right)_n}. \tag{3.16}$$

Proof. This follows directly from (3.10), substituting $a_4 = a_1$. □

We claim that

$$\lim_{x\to 1^-} H_{2,1}(q,q^2,q^3;x;q^4) = (q;q^2)_\infty. \tag{3.17}$$

Proof. Using (3.16), if we map $q \mapsto q^4$ and substitute q, q^2, q^3 for a_1, a_2, a_3, respectively, then

$$\begin{aligned}
\lim_{x\to 1^-} H_{2,1}(q,q^2,q^3;x;q^4) &= \lim_{x\to 1^-} \left((x,xq^2;q^4)_\infty \sum_{n=0}^{\infty} \frac{(q,q^3;q^4)_n}{(xq^2,q^4;q^4)_n} x^n\right) \\
&= \lim_{x\to 1^-} \left((xq^4,xq^2;q^4)_\infty \cdot (1-x) \sum_{n=0}^{\infty} \frac{(q,q^3;q^4)_n}{(xq^2,q^4;q^4)_n} x^n\right) \\
&= \lim_{x\to 1^-} (xq^4,xq^2;q^4)_\infty \cdot \lim_{x\to 1^-} (1-x) \sum_{n=0}^{\infty} \frac{(q,q^3;q^4)_n}{(xq^2,q^4;q^4)_n} x^n.
\end{aligned}$$

For the outside coefficient, it is clear that

$$\lim_{x\to 1^-} (xq^4,xq^2;q^4)_\infty = (q^4,q^2;q^4)_\infty. \tag{3.18}$$

For the sum, Abel's Limit Theorem [15, p. 177] is used to obtain the following:

$$\lim_{x\to 1^-}(1-x)\sum_{n=0}^{\infty}\frac{(q,q^3;q^4)_n}{(xq^2,q^4;q^4)_n}x^n=\frac{(q,q^3;q^4)_\infty}{(q^2,q^4;q^4)_\infty}. \tag{3.19}$$

Combining (3.18) and (3.19),

$$\begin{aligned}\lim_{x\to 1^-}H_{2,1}(q,q^2,q^3;x;q^4)&=(q^4,q^2;q^4)_\infty\cdot\frac{(q,q^3;q^4)_\infty}{(q^2,q^4;q^4)_\infty}\\&=(q,q^3;q^4)_\infty\\&=(1-q)(1-q^5)(1-q^9)\cdots(1-q^3)(1-q^7)(1-q^{11})\cdots\\&=(q;q^2)_\infty,\end{aligned}$$

completing the proof. □

Set

$$f(q)=H_{2,1}(q,q^2,q^3;q^4;q^4). \tag{3.20}$$

In [3], it is claimed that

$$\begin{aligned}f(q)&=\frac{(q^2;q^4)_\infty^2}{q(1-q)}\sum_{n=0}^{\infty}\frac{(q;q^2)_{2n+1}q^{2n+1}}{(q^2;q^2)_{2n+1}}\\&=\frac{(q^2;q^4)_\infty^2}{q(1-q)}\sum_{n=0}^{\infty}\frac{(q;q^2)_nq^n}{(q^2;q^2)_n}\frac{(1-(-1)^n)}{2}\\&=\frac{(q^2;q^4)_\infty^2}{2q(1-q)}\left(\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}-\frac{(-q^2;q^2)_\infty}{(-q;q^2)_\infty}\right)\\&=\frac{(q^2;q^4)_\infty}{2q(1-q)}((-q;-q)_\infty-(q;-q)_\infty).\end{aligned}$$

Instead of proving this, we will prove a different set of equalities for $f(q)$ that are much simpler to derive. They are given below.

$$\begin{aligned} f(q) &= -\frac{(q^2;q^2)_\infty}{q(1-q)}\sum_{n=0}^{\infty}\frac{(q^{-1};q^2)_{2n+1}(q^2)^{2n+1}}{(q^2;q^2)_{2n+1}} \\ &= -\frac{(q^2;q^2)_\infty}{q(1-q)}\sum_{n=0}^{\infty}\frac{(q^{-1};q^2)_n(q^2)^n}{(q^2;q^2)_n}\frac{1-(-1)^n}{2} \\ &= -\frac{(q^2;q^2)_\infty}{2q(1-q)}\left(\frac{(q;q^2)_\infty}{(q^2;q^2)_\infty}-\frac{(-q;q^2)_\infty}{(-q^2;q^2)_\infty}\right) \\ &= -\frac{1}{2q(1-q)}\left((q;q^2)_\infty-(-q;q^2)_\infty^2(q;q)_\infty\right) \\ &= -\frac{(q;q^2)_\infty}{2q(1-q)}\left(1-\sum_{n\in\mathbb{Z}}q^{n^2}\right). \end{aligned} \tag{3.21}$$

Interestingly enough, although the two sets of equations look strikingly different, they will eventually reduce into the same result from Corollary 4. A proof of (3.21) is provided below and is given in a completely different manner than what is presented in [3].

Proof. Using (3.16), if we map $q \mapsto q^4$ and substitute q, q^2, q^3, q^4 for a_1, a_2, a_3, x, respectively, then

$$f(q) = (q^4, q^6; q^4)_\infty \sum_{n=0}^{\infty} \frac{(q, q^3; q^4)_n}{(q^6, q^4; q^4)_n} q^{4n}.$$

For the outside coefficient,

$$\begin{aligned} (q^4, q^6; q^4)_\infty &= (1-q^4)(1-q^8)(1-q^{12})\cdots(1-q^6)(1-q^{10})(1-q^{14})\cdots \\ &= \frac{(1-q^2)(1-q^4)(1-q^6)(1-q^8)(1-q^{10})(1-q^{12})(1-q^{14})\cdots}{1-q^2} \\ &= \frac{(q^2;q^2)_\infty}{1-q^2}. \end{aligned}$$

For the numerator of the fraction inside the sum, [8, p. 6, eqn. (1.2.39)] is used with the mapping $q \mapsto q^2$, $a \mapsto q$. This gives

$$(q, q^3; q^4)_n = (q; q^2)_{2n}.$$

For the denominator of the fraction inside the sum, [8, p. 6, eqn. (1.2.39)] is used once again with the mapping $q \mapsto q^2$, $a \mapsto q^4$. This gives

$$(q^6, q^4; q^4)_n = (q^4; q^2)_{2n}.$$

Thus, the sum is written as

$$\sum_{n=0}^{\infty} \frac{(q, q^3; q^4)_n}{(q^4, q^6; q^4)_n} q^{4n} = \sum_{n=0}^{\infty} \frac{(q; q^2)_{2n}}{(q^4; q^2)_{2n}} (q^2)^{2n}.$$

Also,

$$\begin{aligned}
\frac{(q^2; q^2)_\infty}{1 - q^2} \sum_{n=0}^{\infty} \frac{(q; q^2)_{2n}}{(q^4; q^2)_{2n}} (q^2)^{2n} &= (q^2; q^2)_\infty \sum_{n=0}^{\infty} \frac{(q; q^2)_{2n}}{(1 - q^2)(q^4; q^2)_{2n}} (q^2)^{2n} \\
&= (q^2; q^2)_\infty \sum_{n=0}^{\infty} \frac{\frac{(q^{-1}; q^2)_{2n+1}}{1 - q^{-1}}}{(q^2; q^2)_{2n+1}} (q^2)^{2n} \\
&= \frac{(q^2; q^2)_\infty}{1 - q^{-1}} \sum_{n=0}^{\infty} \frac{(q^{-1}; q^2)_{2n+1}}{(q^2; q^2)_{2n+1}} (q^2)^{2n} \\
&= \frac{(q^2; q^2)_\infty}{(1 - q^{-1}) q^2} \sum_{n=0}^{\infty} \frac{(q^{-1}; q^2)_{2n+1}}{(q^2; q^2)_{2n+1}} (q^2)^{2n+1} \\
&= -\frac{(q^2; q^2)_\infty}{q(1 - q)} \sum_{n=0}^{\infty} \frac{(q^{-1}; q^2)_{2n+1}}{(q^2; q^2)_{2n+1}} (q^2)^{2n+1}.
\end{aligned}$$

Now, we have

$$-\frac{(q^2;q^2)_\infty}{q(1-q)}\sum_{n=0}^{\infty}\frac{(q^{-1};q^2)_{2n+1}}{(q^2;q^2)_{2n+1}}(q^2)^{2n+1} = -\frac{(q^2;q^2)_\infty}{q(1-q)}\sum_{\substack{n\in\mathbb{N}\\ n \text{ odd}}}\frac{(q^{-1};q^2)_n}{(q^2;q^2)_n}(q^2)^n$$
$$= -\frac{(q^2;q^2)_\infty}{q(1-q)}\sum_{n=0}^{\infty}\frac{(q^{-1};q^2)_n(q^2)^n}{(q^2;q^2)_n}\left(\frac{1-(-1)^n}{2}\right).$$

Next,

$$-\frac{(q^2;q^2)_\infty}{q(1-q)}\sum_{n=0}^{\infty}\frac{(q^{-1};q^2)_n(q^2)^n}{(q^2;q^2)_n}\left(\frac{1-(-1)^n}{2}\right)$$
$$= -\frac{(q^2;q^2)_\infty}{2q(1-q)}\sum_{n=0}^{\infty}\frac{(q^{-1};q^2)_n(q^2)^n}{(q^2;q^2)_n}(1-(-1)^n)$$
$$= -\frac{(q^2;q^2)_\infty}{2q(1-q)}\left(\sum_{n=0}^{\infty}\frac{(q^{-1};q^2)_n(q^2)^n}{(q^2;q^2)_n} - \sum_{n=0}^{\infty}\frac{(-1)^n(q^{-1};q^2)_n(q^2)^n}{(q^2;q^2)_n}\right)$$
$$= -\frac{(q^2;q^2)_\infty}{2q(1-q)}\left(\sum_{n=0}^{\infty}\frac{(q^{-1};q^2)_n(q^2)^n}{(q^2;q^2)_n} - \sum_{n=0}^{\infty}\frac{(q^{-1};q^2)_n(-q^2)^n}{(q^2;q^2)_n}\right).$$

For the first summand inside the parentheses, (1.13) is used with the mapping $q \mapsto q^2, a \mapsto q^{-1}, z \mapsto q^2$ to obtain

$$\sum_{n=0}^{\infty}\frac{(q^{-1};q^2)_n(q^2)^n}{(q^2;q^2)_n} = \frac{(q;q^2)_\infty}{(q^2;q^2)_\infty}.$$

For the second summand inside the parentheses, (1.13) is used once again with the mapping $q \mapsto q^2$, $a \mapsto q^{-1}$, $z \mapsto -q^2$. This gives

$$\sum_{n=0}^{\infty}\frac{(q^{-1};q^2)_n(-q^2)^n}{(q^2;q^2)_n} = \frac{(-q;q^2)_\infty}{(-q^2;q^2)_\infty}.$$

Additionally,

$$-\frac{(q^2;q^2)_\infty}{2q(1-q)}\left(\frac{(q;q^2)_\infty}{(q^2;q^2)_\infty}-\frac{(-q;q^2)_\infty}{(-q^2;q^2)_\infty}\right)=-\frac{1}{2q(1-q)}\left(\frac{(q;q^2)_\infty(q^2;q^2)_\infty}{(q^2;q^2)_\infty}-\frac{(-q;q^2)_\infty(q^2;q^2)_\infty}{(-q^2;q^2)_\infty}\right)$$
$$=-\frac{1}{2q(1-q)}\left((q;q^2)_\infty-\frac{(-q;q^2)_\infty(q^2;q^2)_\infty}{(-q^2;q^2)_\infty}\right).$$

For the second term inside the parentheses,

$$\frac{(-q;q^2)_\infty(q^2;q^2)_\infty}{(-q^2;q^2)_\infty}$$
$$=\frac{(1+q)(1+q^3)(1+q^5)(1+q^7)\cdots(1-q^2)(1-q^4)(1-q^6)(1-q^8)\cdots}{(1+q^2)(1+q^4)(1+q^6)(1+q^8)\cdots}$$
$$=\frac{(1+q)(1+q^3)(1+q^5)\cdots(1-q)(1+q)(1-q^2)(1+q^2)(1-q^3)(1+q^3)(1-q^4)(1+q^4)\cdots}{(1+q^2)(1+q^4)(1+q^6)(1+q^8)\cdots}$$
$$=(1+q)^2(1-q)(1-q^2)(1+q^3)^2(1-q^3)(1-q^4)(1+q^5)^2\cdots$$
$$=(1+q)^2(1+q^3)^2(1+q^5)^2\cdots(1-q)(1-q^2)(1-q^3)\cdots$$
$$=(-q;q^2)_\infty^2(q;q)_\infty.$$

Finally,

$$-\frac{1}{2q(1-q)}((q;q^2)_\infty-(-q;q^2)_\infty^2(q;q)_\infty)=-\frac{(q;q^2)_\infty}{2q(1-q)}\left(1-\frac{(-q;q^2)_\infty^2(q;q)_\infty}{(q;q^2)_\infty}\right).$$

For the second term inside the parentheses,

$$\frac{(-q;q^2)_\infty^2(q;q)_\infty}{(q;q^2)_\infty}=\frac{(1+q)^2(1+q^3)^2(1+q^5)^2\cdots(1-q)(1-q^2)(1-q^3)\cdots}{(1-q)(1-q^3)(1-q^5)\cdots}$$
$$=(1+q)^2(1-q^2)(1+q^3)^2(1-q^4)(1+q^5)^2\cdots$$
$$=(-q;q^2)_\infty^2(q^2;q^2)_\infty.$$

From here, we use (1.17) with the mapping $z \mapsto -1$ to obtain

$$(-q;q^2)_\infty^2(q^2;q^2)_\infty = \sum_{n\in\mathbb{Z}} q^{n^2}.$$

This completes the proof. □

We now claim that Corollary 4 reduces to the following.

Corollary 5.

$$\frac{1-q}{\sum_{n=0}^{\infty} q^{n^2+2n}} = 1-q-q^3+q^6+\operatorname*{K}_{i=1}^{\infty}\frac{q^{4i}(1-q^{4i-1})(1-q^{4i})(1-q^{4i+1})}{1-q^{4i+1}-q^{4i+2}-q^{4i+3}+q^{8i+2}+q^{8i+6}}. \tag{3.22}$$

Proof. For the left-hand side of the equation, (3.17) and (3.21) are used to obtain

$$\begin{aligned}
\frac{H_{2,1}(q,q^2,q^3;1;q^4)}{H_{2,1}(q,q^2,q^3;q^4;q^4)} &= -\frac{(q;q^2)_\infty}{\frac{(q;q^2)_\infty\left(1-\sum_{n\in\mathbb{Z}} q^{n^2}\right)}{2q(1-q)}} \\
&= -\frac{2q(1-q)}{1-\sum_{n\in\mathbb{Z}} q^{n^2}} \\
&= -\frac{2q(1-q)}{1-(2\sum_{n=1}^{\infty} q^{n^2}+1)} \\
&= -\frac{2q(1-q)}{-2\sum_{n=1}^{\infty} q^{n^2}} \\
&= \frac{q(1-q)}{\sum_{n=1}^{\infty} q^{n^2}} \\
&= \frac{1-q}{\sum_{n=1}^{\infty} q^{n^2-1}}.
\end{aligned}$$

Reindexing the sum gives

$$\begin{aligned}\frac{H_{2,1}(q,q^2,q^3;1;q^4)}{H_{2,1}(q,q^2,q^3;q^4;q^4)} &= \frac{1-q}{\sum_{n=0}^{\infty} q^{(n+1)^2-1}}\\ &= \frac{1-q}{\sum_{n=0}^{\infty} q^{n^2+2n}}.\end{aligned}$$

For the right-hand side of the equation, consider (3.13) and (3.14), mapping $q \mapsto q^4$ and substituting q, q^2, q^3 for a_1, a_2, a_3, respectively. Then

$$\begin{aligned}\lim_{x\to 1^-} \overline{P}(x) &= \lim_{x\to 1^-} (1 - xq^3 - xq^2 - xq + x^2q^2(1+q^4))\\ &= \lim_{x\to 1^-} (1 - xq^3 - xq^2 - xq + x^2q^2 + x^2q^6)\\ &= 1 - q^3 - q^2 - q + q^2 + q^6\\ &= 1 - q - q^3 + q^6.\end{aligned} \tag{3.23}$$

More generally, for any $k \in \mathbb{N}$,

$$\begin{aligned}\lim_{x\to 1^-} \overline{P}(xq^k) &= \lim_{x\to 1^-} \left(1 - \frac{xq^{4k+4}}{q} - \frac{xq^{4k+4}}{q^2} - \frac{xq^{4k+4}}{q^3} + \frac{x^2q^{8k+8}}{q^6}(1+q^4)\right)\\ &= \lim_{x\to 1^-} (1 - xq^{4k+3} - xq^{4k+2} - xq^{4k+1} + x^2q^{8k+2}(1+q^4))\\ &= \lim_{x\to 1^-} (1 - xq^{4k+3} - xq^{4k+2} - xq^{4k+1} + x^2q^{8k+2} + x^2q^{8k+6})\\ &= 1 - q^{4k+1} - q^{4k+2} - q^{4k+3} + q^{8k+2} + q^{8k+6}.\end{aligned} \tag{3.24}$$

Additionally,

$$\begin{aligned}\lim_{x\to 1^-} \overline{R}(x) &= \lim_{x\to 1^-} (xq^4(1-xq^5)(1-xq^4)(1-xq^3))\\ &= q^4(1-q^3)(1-q^4)(1-q^5).\end{aligned} \tag{3.25}$$

More generally, for any $k \in \mathbb{N}$,

$$\begin{aligned}\lim_{x\to 1^-} \overline{R}(xq^k) &= \lim_{x\to 1^-} \left(xq^{4k+4}\left(1-\frac{xq^{4k+8}}{q^3}\right)\left(1-\frac{xq^{4k+8}}{q^4}\right)\left(1-\frac{xq^{4k+8}}{q^5}\right)\right) \\ &= \lim_{x\to 1^-} (xq^{4k+4}(1-xq^{4k+5})(1-xq^{4k+4})(1-xq^{4k+3})) \\ &= q^{4k+4}(1-q^{4k+3})(1-q^{4k+4})(1-q^{4k+5}).\end{aligned} \tag{3.26}$$

By (3.23) – (3.26), the right-hand side of the equation simplifies into

$$\begin{aligned}&\overline{P}(x)+\frac{\overline{R}(x)}{\overline{P}(xq)+\frac{R(xq)}{\overline{P}(xq^2)+\cdots}} \\ &= 1-q-q^3+q^6+\mathop{\mathrm{K}}_{i=1}^{\infty}\frac{q^{4(i-1)+4}(1-q^{4(i-1)+3})(1-q^{4(i-1)+4})(1-q^{4(i-1)+5})}{1-q^{4i+1}-q^{4i+2}-q^{4i+3}+q^{8i+2}+q^{8i+6}} \\ &= 1-q-q^3+q^6+\mathop{\mathrm{K}}_{i=1}^{\infty}\frac{q^{4i}(1-q^{4i-1})(1-q^{4i})(1-q^{4i+1})}{1-q^{4i+1}-q^{4i+2}-q^{4i+3}+q^{8i+2}+q^{8i+6}},\end{aligned}$$

completing the proof of (3.22). □

Corollary 5 is verified using the Maxima program described in Appendix A, and the results are given in Appendix B.

3.3 Special Case 3: $a_4, a_5 \to \infty$, $a_2 = 1$

Let $a_4, a_5 \to \infty$ as in Special Case 2. For this special case, we will consider what happens when $a_2 = 1$. We claim that

$$H_{2,1}(a_1, 1, a_3; x; q) = \left(\frac{xq}{a_1}, \frac{xq}{a_3}\right)_\infty . \tag{3.27}$$

Proof. Substituting 1 for a_2 in (3.16) gives

$$H_{2,1}\left(a_1, 1, a_3; x; q\right) = \left(\frac{xq}{a_1a_3}, xq\right)_\infty \sum_{n=0}^{\infty} \frac{(a_1, a_3)_n \left(\frac{xq}{a_1a_3}\right)^n}{(xq, q)_n}. \tag{3.28}$$

For the sum, we apply (1.24) with the mapping $a \mapsto a_1, b \mapsto a_3$, $c \mapsto xq$. Then

$$\sum_{n=0}^{\infty} \frac{(a_1, a_3)_n \left(\frac{xq}{a_1a_3}\right)^n}{(xq, q)_n} = \frac{\left(\frac{xq}{a_1}, \frac{xq}{a_3}\right)_\infty}{\left(xq, \frac{xq}{a_1a_3}\right)_\infty}. \tag{3.29}$$

Combining (3.28) and (3.29),

$$\begin{aligned} H_{2,1}\left(a_1, 1, a_3; x; q\right) &= \left(\frac{xq}{a_1a_3}, xq\right)_\infty \cdot \frac{\left(\frac{xq}{a_1}, \frac{xq}{a_3}\right)_\infty}{\left(xq, \frac{xq}{a_1a_3}\right)_\infty} \\ &= \left(\frac{xq}{a_1}, \frac{xq}{a_3}\right)_\infty, \end{aligned}$$

as was claimed. □

Mapping $x \mapsto xq$, it is clear that

$$H_{2,1}\left(a_1, 1, a_3; xq; q\right) = \left(\frac{xq^2}{a_1}, \frac{xq^2}{a_3}\right)_\infty. \tag{3.30}$$

We now claim that Corollary 4 reduces to the following.

Corollary 6.

$$xq - \frac{x^2q^3}{a_1a_3} = \operatorname*{K}_{i=1}^{\infty} \frac{xq^i\left(1 - \frac{xq^{i+1}}{a_1}\right)\left(1 - \frac{xq^{i+1}}{a_1a_3}\right)\left(1 - \frac{xq^{i+1}}{a_3}\right)}{1 - \frac{xq^{i+1}}{a_1} - \frac{xq^{i+1}}{a_3} - xq^{i+1} + \frac{xq^{2i+2}}{a_1a_3} + \frac{xq^{2i+3}}{a_1a_3}}. \tag{3.31}$$

Proof. For the left-hand side of the equation, (3.27) and (3.30) are used to obtain

$$\begin{aligned}
\frac{H_{2,1}(a_1,1,a_3;x;q)}{H_{2,1}(a_1,1,a_3;xq;q)} &= \frac{\left(\frac{xq}{a_1},\frac{xq}{a_3}\right)_\infty}{\left(\frac{xq^2}{a_1},\frac{xq^2}{a_3}\right)_\infty} \\
&= \frac{\left(1-\frac{xq}{a_1}\right)\left(1-\frac{xq^2}{a_1}\right)\left(1-\frac{xq^3}{a_1}\right)\cdots\left(1-\frac{xq}{a_3}\right)\left(1-\frac{xq^2}{a_3}\right)\left(1-\frac{xq^3}{a_3}\right)\cdots}{\left(1-\frac{xq^2}{a_1}\right)\left(1-\frac{xq^3}{a_1}\right)\left(1-\frac{xq^4}{a_1}\right)\cdots\left(1-\frac{xq^2}{a_3}\right)\left(1-\frac{xq^3}{a_3}\right)\left(1-\frac{xq^4}{a_3}\right)\cdots} \\
&= \left(1-\frac{xq}{a_1}\right)\left(1-\frac{xq}{a_3}\right).
\end{aligned} \tag{3.32}$$

For the right-hand side of the equation, (3.13) and (3.14) are used, substituting 1 for a_2. This gives us the following:

$$\begin{aligned}
\overline{P}(x) &= 1-\frac{xq}{a_1}-xq-\frac{xq}{a_3}+\frac{x^2q^2}{a_1a_3}(1+q) \\
&= 1-\frac{xq}{a_1}-\frac{xq}{a_3}-xq+\frac{x^2q^2}{a_1a_3}+\frac{x^2q^3}{a_1a_3}.
\end{aligned} \tag{3.33}$$

More generally, for any $k \in \mathbb{N}$,

$$\overline{P}(xq^k) = 1-\frac{xq^{k+1}}{a_1}-\frac{xq^{k+1}}{a_3}-xq^{k+1}+\frac{xq^{2k+2}}{a_1a_3}+\frac{xq^{2k+3}}{a_1a_3}. \tag{3.34}$$

Additionally,

$$\overline{R}(x) = xq\left(1-\frac{xq^2}{a_1}\right)\left(1-\frac{xq^2}{a_1a_3}\right)\left(1-\frac{xq^2}{a_3}\right). \tag{3.35}$$

More generally, for any $k \in \mathbb{N}$,

$$\overline{R}(xq^k) = xq^{k+1}\left(1-\frac{xq^{k+2}}{a_1}\right)\left(1-\frac{xq^{k+2}}{a_1a_3}\right)\left(1-\frac{xq^{k+2}}{a_3}\right) \tag{3.36}$$

By (3.33) – (3.36), the right-hand side of the equation simplifies into

$$\begin{aligned}\overline{P}(x) + \frac{\overline{R}(x)}{\overline{P}(xq) + \frac{R(xq)}{\overline{P}(xq^2) + \cdots}} &= 1 - \frac{xq}{a_1} - \frac{xq}{a_3} - xq + \frac{x^2q^2}{a_1a_3} + \frac{x^2q^3}{a_1a_3} \\ &\quad + \operatorname*{K}_{i=1}^{\infty} \frac{xq^{(i-1)+1}\left(1 - \frac{xq^{(i-1)+2}}{a_1}\right)\left(1 - \frac{xq^{(i-1)+2}}{a_1a_3}\right)\left(1 - \frac{xq^{(i-1)+2}}{a_3}\right)}{1 - \frac{xq^{i+1}}{a_1} - \frac{xq^{i+1}}{a_3} - xq^{i+1} + \frac{xq^{2i+2}}{a_1a_3} + \frac{xq^{2i+3}}{a_1a_3}} \\ &= 1 - \frac{xq}{a_1} - \frac{xq}{a_3} - xq + \frac{x^2q^2}{a_1a_3} + \frac{x^2q^3}{a_1a_3} \\ &\quad + \operatorname*{K}_{i=1}^{\infty} \frac{xq^{i}\left(1 - \frac{xq^{i+1}}{a_1}\right)\left(1 - \frac{xq^{i+1}}{a_1a_3}\right)\left(1 - \frac{xq^{i+1}}{a_3}\right)}{1 - \frac{xq^{i+1}}{a_1} - \frac{xq^{i+1}}{a_3} - xq^{i+1} + \frac{xq^{2i+2}}{a_1a_3} + \frac{xq^{2i+3}}{a_1a_3}}.\end{aligned} \tag{3.37}$$

We can use (3.32) and (3.37) to simplify further. Thus,

$$\begin{aligned}\left(1 - \frac{xq}{a_1}\right)\left(1 - \frac{xq}{a_3}\right) &= 1 - \frac{xq}{a_1} - \frac{xq}{a_3} - xq + \frac{x^2q^2}{a_1a_3} + \frac{x^2q^3}{a_1a_3} \\ &\quad + \operatorname*{K}_{i=1}^{\infty} \frac{xq^{i}\left(1 - \frac{xq^{i+1}}{a_1}\right)\left(1 - \frac{xq^{i+1}}{a_1a_3}\right)\left(1 - \frac{xq^{i+1}}{a_3}\right)}{1 - \frac{xq^{i+1}}{a_1} - \frac{xq^{i+1}}{a_3} - xq^{i+1} + \frac{xq^{2i+2}}{a_1a_3} + \frac{xq^{2i+3}}{a_1a_3}}.\end{aligned}$$

Expanding the left-hand side gives

$$\begin{aligned}1 - \frac{xq}{a_1} - \frac{xq}{a_3} + \frac{x^2q^2}{a_1a_3} &= 1 - \frac{xq}{a_1} - \frac{xq}{a_3} - xq + \frac{x^2q^2}{a_1a_3} + \frac{x^2q^3}{a_1a_3} \\ &\quad + \operatorname*{K}_{i=1}^{\infty} \frac{xq^{i}\left(1 - \frac{xq^{i+1}}{a_1}\right)\left(1 - \frac{xq^{i+1}}{a_1a_3}\right)\left(1 - \frac{xq^{i+1}}{a_3}\right)}{1 - \frac{xq^{i+1}}{a_1} - \frac{xq^{i+1}}{a_3} - xq^{i+1} + \frac{xq^{2i+2}}{a_1a_3} + \frac{xq^{2i+3}}{a_1a_3}}.\end{aligned}$$

Cancelling common terms on both sides, we obtain

$$xq - \frac{x^2q^3}{a_1a_3} = \operatorname*{K}_{i=1}^{\infty} \frac{xq^{i}\left(1 - \frac{xq^{i+1}}{a_1}\right)\left(1 - \frac{xq^{i+1}}{a_1a_3}\right)\left(1 - \frac{xq^{i+1}}{a_3}\right)}{1 - \frac{xq^{i+1}}{a_1} - \frac{xq^{i+1}}{a_3} - xq^{i+1} + \frac{xq^{2i+2}}{a_1a_3} + \frac{xq^{2i+3}}{a_1a_3}},$$

completing the proof of (3.31). □

Corollary 6 is verified using the Maxima program described in Appendix A, and the results are given in Appendix B. Since a_1 and a_3 can vary, only two instances of (3.31) are given in Appendix B.

3.4 Special Case 4: $a_4, a_5 \to \infty$, $a_1 = -x$, $a_2 = -a$, $a_3 = a$

Let $a_4, a_5 \to \infty$ again, and set $a_1 = -x, a_2 = -a, a_3 = a$. We claim that

$$H_{2,1}(-x,-a,a;x;q) = (-q;q)_\infty \left(-xq, -\frac{xq^2}{a^2}; q^2\right)_\infty . \tag{3.38}$$

Proof. We use (3.16), substituting $-x, -a, a$ for a_1, a_2, a_3, respectively. Then

$$H_{2,1}(-x,-a,a;x;q) = \left(-\frac{q}{a}, -\frac{xq}{a}; q\right)_\infty \sum_{n=0}^{\infty} \frac{(-x,a;q)_n \left(-\frac{q}{a}\right)^n}{\left(-\frac{xq}{a}, q; q\right)_n}.$$

For the sum, (1.25) is used with the mapping $a \mapsto -x$, $b \mapsto a$. This gives

$$\begin{aligned}\left(-\frac{q}{a}, -\frac{xq}{a}; q\right)_\infty \sum_{n=0}^{\infty} \frac{(-x,a;q)_n \left(-\frac{q}{a}\right)^n}{\left(-\frac{xq}{a}, q; q\right)_n} &= \left(-\frac{q}{a}, -\frac{xq}{a}; q\right)_\infty \cdot \frac{(-q;q)_\infty \left(-xq, \frac{-xq^2}{a^2}; q^2\right)_\infty}{\left(-\frac{xq}{a}, -\frac{q}{a}; q\right)_\infty} \\ &= (-q;q)_\infty \left(-xq, -\frac{xq^2}{a^2}; q^2\right)_\infty ,\end{aligned}$$

as was claimed. □

We now claim that

$$H_{2,1}(-x,-a,a;xq;q) = \left(-\frac{q^2}{a}, -\frac{xq^2}{a}; q\right)_\infty \sum_{n=0}^{\infty} \frac{(-x,a;q)_n \left(-\frac{q^2}{a}\right)^n}{\left(-\frac{xq^2}{a}, q; q\right)_n}. \tag{3.39}$$

Proof. Using (3.16), if we map $x \mapsto xq$ and substitute $-x, -a, a$ for a_1, a_2, a_3, respectively, then

$$H_{2,1}(-x,-a,a;xq;q) = \left(-\frac{q^2}{a}, -\frac{xq^2}{a}; q\right)_\infty \sum_{n=0}^{\infty} \frac{(-x,a;q)_n \left(-\frac{q^2}{a}\right)^n}{\left(-\frac{xq^2}{a}, q; q\right)_n},$$

as was claimed. ☐

Interestingly enough, unlike Special Cases 1, 2, and 3, the "$H_1(xq)$" term cannot be simplified any further than this. (1.25) can either be used on the numerator term $H_{2,1}(-x,-a,a;x;q)$ or the denominator term $H_{2,1}(-x,-a,a;xq;q)$, but not both. If we ultimately wanted the "$H_1(xq)$" term to simplify instead of the "$H_1(x)$" term, then the substitutions $a_1 = -xq$, $a_2 = -a$, $a_3 = a$ could have been applied.

We claim that Corollary 4 reduces to the following.

Corollary 7.

$$\begin{aligned}(-q;q)_\infty \left(-xq, -\frac{xq^2}{a^2}; q^2\right)_\infty \times \left(\left(-\frac{q^2}{a}, -\frac{xq^2}{a}; q\right)_\infty \sum_{n=0}^{\infty} \frac{(-x,a;q)_n \left(-\frac{q^2}{a}\right)^n}{\left(-\frac{xq^2}{a}, q; q\right)_n}\right)^{-1} \\ = 1 + q + \frac{xq^2}{a^2} + \frac{xq^3}{a^3} + \operatorname*{K}_{i=1}^{\infty} \frac{xq^i\left(1-\frac{q^{i+1}}{a}\right)\left(1+\frac{q^{i+1}}{a}\right)\left(1-\frac{xq^{i+1}}{a^2}\right)}{1+q^{i+1}+\frac{xq^{2i+2}}{a^2}+\frac{xq^{2i+3}}{a^2}}.\end{aligned} \tag{3.40}$$

Proof. The left-hand side of the equation immediately follows from (3.38) and (3.39). For the right-hand side of the equation, (3.13) and (3.14) are used, substituting $-x, -a, a$ for a_1, a_2, a_3, respectively. This gives us the following:

$$\begin{aligned}\overline{P}(x) &= 1 + q + \frac{xq}{a} - \frac{xq}{a} + \frac{xq^2}{a^2}(1+q) \\ &= 1 + q + \frac{xq^2}{a^2} + \frac{xq^3}{a^3}.\end{aligned} \tag{3.41}$$

More generally, for any $k \in \mathbb{N}$,

$$\begin{aligned}\overline{P}(xq^k) &= 1 + q^{k+1} + \frac{xq^k}{a} - \frac{xq^k}{a} + \frac{xq^{2k+2}}{a^2}(1+q) \\ &= 1 + q^{k+1} + \frac{xq^{2k+2}}{a^2} + \frac{xq^{2k+3}}{a^2}.\end{aligned} \tag{3.42}$$

Additionally,

$$\overline{R}(x) = xq\left(1 - \frac{q^2}{a}\right)\left(1 + \frac{q^2}{a}\right)\left(1 + \frac{xq^2}{a^2}\right). \tag{3.43}$$

More generally, for any $k \in \mathbb{N}$,

$$\overline{R}(xq^k) = xq^{k+1}\left(1 - \frac{q^{k+2}}{a}\right)\left(1 + \frac{q^{k+2}}{a}\right)\left(1 - \frac{xq^{k+2}}{a^2}\right) \tag{3.44}$$

By (3.41) – (3.44), the right-hand side of the equation simplifies into

$$\begin{aligned}\overline{P}(x) + \cfrac{\overline{R}(x)}{\overline{P}(xq) + \cfrac{R(xq)}{\overline{P}(xq^2) + \cdots}} &= 1 + q + \frac{xq^2}{a^2} + \frac{xq^3}{a^3} \\ &\quad + \operatorname*{K}_{i=1}^{\infty} \frac{xq^{(i-1)+1}\left(1 - \frac{q^{(i-1)+2}}{a}\right)\left(1 + \frac{q^{(i-1)+2}}{a}\right)\left(1 - \frac{xq^{(i-1)+2}}{a^2}\right)}{1 + q^{i+1} + \frac{xq^{2i+2}}{a^2} + \frac{xq^{2i+3}}{a^2}} \\ &= 1 + q + \frac{xq^2}{a^2} + \frac{xq^3}{a^3} \\ &\quad + \operatorname*{K}_{i=1}^{\infty} \frac{xq^{i}\left(1 - \frac{q^{i+1}}{a}\right)\left(1 + \frac{q^{i+1}}{a}\right)\left(1 - \frac{xq^{i+1}}{a^2}\right)}{1 + q^{i+1} + \frac{xq^{2i+2}}{a^2} + \frac{xq^{2i+3}}{a^2}},\end{aligned}$$

completing the proof of (3.40). □

Corollary 7 is verified using the Maxima program described in Appendix A, and the results are given in Appendix B. Since a_1, a_2, and a_3 can vary, only two instances of (3.40) are given in Appendix B.

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APPENDIX A

MAXIMA CODE EXAMPLES

A.1 Taylor Series Expansion

In order to verify that the q-series identities presented for each special case in Chapter 3 are in fact true, we use a simple Maxima program involving the Taylor series expansions of both sides of each equation. For any function $f(q)$, we create a Taylor series in the variable q centered about 0 with 75 terms. Since this is an approximation, then for any continued fractions that appear in the identities, only a few numerators and denominators of each are used. The Maxima code that will be used for verification is given by the following.

```
taylor(f(q),q,0,75);
```

For an example of this, consider (1.30). Then for the left-hand side of the equation, the above code is used to obtain

```
taylor(product(((1-q^(5*n+2))*(1-q^(5*n+3)))/((1-q^(5*n+1))*(1-q^(5*n+4))),n,0,inf),
q,0,75);
```

$$1 + q - q^{3} + q^{5} + q^{6} - q^{7} - 2q^{8} + 2q^{10} + 2q^{11} - q^{12} - 3q^{13} - q^{14} + 3q^{15} + 3q^{16} - 2q^{17} - 5q^{18} - q^{19} + 6q^{20} + 5q^{21} - 3q^{22} - 8q^{23} - 2q^{24} + 8q^{25} + 7q^{26} - 5q^{27} - 12q^{28} - 2q^{29} + 13q^{30} + 12q^{31} - 7q^{32} - 18q^{33} - 4q^{34} + 18q^{35} + 16q^{36} - 11q^{37} - 26q^{38} - 5q^{39} + 27q^{40} + 24q^{41} - 14q^{42} - 37q^{43} - 8q^{44} + 37q^{45} + 33q^{46} - 21q^{47} - 52q^{48} - 10q^{49} + 53q^{50} + 47q^{51} - 29q^{52} - 72q^{53} - 15q^{54} + 71q^{55} + 63q^{56} - 40q^{57} - 98q^{58} - 19q^{59} + 99q^{60} + 88q^{61} - 53q^{62} - 133q^{63} - 27q^{64} + 131q^{65} + 115q^{66} - 73q^{67} - 178q^{68} - 35q^{69} + 177q^{70} + 156q^{71} - 95q^{72} - 236q^{73} - 48q^{74} + 232q^{75} + \cdots .$$

For the right-hand side of the equation, the above code is used again to obtain

```
taylor(1+q/(1+q^2/(1+q^3/(1+q^4/(1+q^5/(1+q^6/(1+q^7/(1+q^8/(1+q^9/
(1+q^10/(1+q^11/(1+q^12)))))))))))),q,0,75);
```

$1 + q - q^3 + q^5 + q^6 - q^7 - 2q^8 + 2q^{10} + 2q^{11} - q^{12} - 3q^{13} - q^{14} + 3q^{15} + 3q^{16} - 2q^{17} - 5q^{18} - q^{19} + 6q^{20} + 5q^{21} - 3q^{22} - 8q^{23} - 2q^{24} + 8q^{25} + 7q^{26} - 5q^{27} - 12q^{28} - 2q^{29} + 13q^{30} + 12q^{31} - 7q^{32} - 18q^{33} - 4q^{34} + 18q^{35} + 16q^{36} - 11q^{37} - 26q^{38} - 5q^{39} + 27q^{40} + 24q^{41} - 14q^{42} - 37q^{43} - 8q^{44} + 37q^{45} + 33q^{46} - 21q^{47} - 52q^{48} - 10q^{49} + 53q^{50} + 47q^{51} - 29q^{52} - 72q^{53} - 15q^{54} + 71q^{55} + 63q^{56} - 40q^{57} - 98q^{58} - 19q^{59} + 99q^{60} + 88q^{61} - 53q^{62} - 133q^{63} - 27q^{64} + 131q^{65} + 115q^{66} - 73q^{67} - 178q^{68} - 35q^{69} + 177q^{70} + 156q^{71} - 95q^{72} - 236q^{73} - 48q^{74} + 232q^{75} + \cdots.$

A.2 q-Pochhammer Symbol, $\overline{P}(x)$, $\overline{R}(x)$

For some of the identities presented in Chapter 3, q-Pochhammer symbols remain in the final results. In order to verify such identities in Maxima, define the q-Pochhammer symbol with the following code.

```
qpoch(a,q,n):=block(if n=0 then return(1) else return(product(1-a*q^k,k,0,n-1)));
```

Additionally, define the functions

```
Pbar(x,a_1,a_2,a_3,q):=1-(x*q)/a_1-(x*q)/a_2-(x*q)/a_3+(x^2*q^2)/(a_1*a_2*a_3)*(1+q);
```

and

```
Rbar(x,a_1,a_2,a_3,q):= x*q*(1-(x*q^2)/(a_1*a_2))*(1-(x*q^2)/(a_1*a_3))
*(1-(x*q^2)/(a_2*a_3));
```

to represent the polynomials $\overline{P}(x)$ and $\overline{R}(x)$ given by (3.13) and (3.14).

APPENDIX B

VERIFICATION OF IDENTITIES

B.1 Verification of Special Case 1

Define the functions

```
G_1(x):=prod((1-x*q^n)^-1,n,1,inf)*(1+sum((prod((1-x*q^k),k,1,n-1)*(1-x*q^(2*n))*(-1)^n*q^((n*(5*n-1)/2))*x^(2*n))/(qpoch(q,q,n)),n,1,inf));
```

and

```
G_2(x):=1+sum((q^(n^2)*x^n)/(qpoch(q,q,n)),n,1,inf);
```

to represent the two versions of $G(x)$ given in (3.5). For this special case, only 25 terms will be listed for each Taylor series due to space limitations. The verification is given below.

```
taylor(G_1(x)/G_1(x*q),q,0,25);
```

$$1+xq-x^2\,q^3+x^3\,q^5+x^3\,q^6-x^4\,q^7-2x^4\,q^8+\left(x^5-x^4\right)q^9+\left(3x^5-x^4\right)q^{10}+\left(-x^6+3x^5\right)q^{11}+\left(-4x^6+3x^5\right)q^{12}+\left(x^7-6x^6+2x^5\right)q^{13}+\left(5x^7-7x^6+x^5\right)q^{14}+\left(-x^8+10x^7-7x^6+x^5\right)q^{15}+\left(-6x^8+14x^7-5x^6\right)q^{16}+\left(x^9-15x^8+17x^7-5x^6\right)q^{17}+\left(7x^9-25x^8+16x^7-3x^6\right)q^{18}+\left(-x^{10}+21x^9-35x^8+16x^7-2x^6\right)q^{19}+\left(-8x^{10}+41x^9-40x^8+14x^7-x^6\right)q^{20}+\left(x^{11}-28x^{10}+65x^9-43x^8+11x^7-x^6\right)q^{21}+\left(9x^{11}-63x^{10}+86x^9-44x^8+9x^7\right)q^{22}+\left(-x^{12}+36x^{11}-112x^{10}+102x^9-40x^8+7x^7\right)q^{23}+\left(-10x^{12}+92x^{11}-167x^{10}+115x^9-37x^8+5x^7\right)q^{24}+\left(x^{13}-45x^{12}+182x^{11}-219x^{10}+118x^9-32x^8+3x^7\right)q^{25}+\cdots$$

taylor(G_2(x)/G_2(x*q),q,0,25);

$$1+xq-x^2\,q^3+x^3\,q^5+x^3\,q^6-x^4\,q^7-2x^4\,q^8+\left(x^5-x^4\right)\,q^9+\left(3x^5-x^4\right)\,q^{10}+\left(-x^6+3x^5\right)\,q^{11}+\left(-4x^6+3x^5\right)\,q^{12}+\left(x^7-6x^6+2x^5\right)\,q^{13}+\left(5x^7-7x^6+x^5\right)\,q^{14}+\left(-x^8+10x^7-7x^6+x^5\right)\,q^{15}+\left(-6x^8+14x^7-5x^6\right)\,q^{16}+\left(x^9-15x^8+17x^7-5x^6\right)\,q^{17}+\left(7x^9-25x^8+16x^7-3x^6\right)\,q^{18}+\left(-x^{10}+21x^9-35x^8+16x^7-2x^6\right)\,q^{19}+\left(-8x^{10}+41x^9-40x^8+14x^7-x^6\right)\,q^{20}+\left(x^{11}-28x^{10}+65x^9-43x^8+11x^7-x^6\right)\,q^{21}+\left(9x^{11}-63x^{10}+86x^9-44x^8+9x^7\right)\,q^{22}+\left(-x^{12}+36x^{11}-112x^{10}+102x^9-40x^8+7x^7\right)\,q^{23}+\left(-10x^{12}+92x^{11}-167x^{10}+115x^9-37x^8+5x^7\right)\,q^{24}+\left(x^{13}-45x^{12}+182x^{11}-219x^{10}+118x^9-32x^8+3x^7\right)\,q^{25}+\cdots$$

taylor(1+(x*q)/(1+(x*q^2)/(1+(x*q^3)/(1+(x*q^4)/(1+(x*q^5)/(1+(x*q^6)))))),q,0,25);

$$1+xq-x^2\,q^3+x^3\,q^5+x^3\,q^6-x^4\,q^7-2x^4\,q^8+\left(x^5-x^4\right)\,q^9+\left(3x^5-x^4\right)\,q^{10}+\left(-x^6+3x^5\right)\,q^{11}+\left(-4x^6+3x^5\right)\,q^{12}+\left(x^7-6x^6+2x^5\right)\,q^{13}+\left(5x^7-7x^6+x^5\right)\,q^{14}+\left(-x^8+10x^7-7x^6+x^5\right)\,q^{15}+\left(-6x^8+14x^7-5x^6\right)\,q^{16}+\left(x^9-15x^8+17x^7-5x^6\right)\,q^{17}+\left(7x^9-25x^8+16x^7-3x^6\right)\,q^{18}+\left(-x^{10}+21x^9-35x^8+16x^7-2x^6\right)\,q^{19}+\left(-8x^{10}+41x^9-40x^8+14x^7-x^6\right)\,q^{20}+\left(x^{11}-28x^{10}+65x^9-43x^8+11x^7-x^6\right)\,q^{21}+\left(9x^{11}-63x^{10}+86x^9-44x^8+9x^7\right)\,q^{22}+\left(-x^{12}+36x^{11}-112x^{10}+102x^9-40x^8+7x^7\right)\,q^{23}+\left(-10x^{12}+92x^{11}-167x^{10}+115x^9-37x^8+5x^7\right)\,q^{24}+\left(x^{13}-45x^{12}+182x^{11}-219x^{10}+118x^9-32x^8+3x^7\right)\,q^{25}+\cdots$$

B.2 Verification of Special Case 2

```
taylor((((1-q))/(sum(q^(n^2+2*n),n,0,inf))),q,0,75);
```

$$1-q-q^3+q^4+q^6-q^7-q^8+q^{10}+2q^{11}-q^{12}-q^{13}-3q^{14}+q^{15}+3q^{16}+3q^{17}-q^{18}-6q^{19}-2q^{20}+q^{21}+10q^{22}+2q^{23}-5q^{24}-13q^{25}-3q^{26}+13q^{27}+15q^{28}+5q^{29}-25q^{30}-19q^{31}-2q^{32}+39q^{33}+27q^{34}-12q^{35}-53q^{36}-41q^{37}+39q^{38}+75q^{39}+53q^{40}-79q^{41}-113q^{42}-49q^{43}+129q^{44}+177q^{45}+17q^{46}-203q^{47}-264q^{48}+52q^{49}+330q^{50}+351q^{51}-159q^{52}-550q^{53}-413q^{54}+331q^{55}+895q^{56}+425q^{57}-642q^{58}-1360q^{59}-376q^{60}+1220q^{61}+1927q^{62}+217q^{63}-2227q^{64}-2581q^{65}+208q^{66}+3802q^{67}+3341q^{68}-1221q^{69}-6074q^{70}-4198q^{71}+3333q^{72}+9218q^{73}+4946q^{74}-7175q^{75}+\cdots$$

```
taylor(Pbar(1,q,q^2,q^3,q^4)+(Rbar(1,q,q^2,q^3,q^4))/(Pbar(q^4,q,q^2,q^3,q^4)
+(Rbar(q^4,q,q^2,q^3,q^4))/(Pbar(q^8,q,q^2,q^3,q^4)
+(Rbar(q^8,q,q^2,q^3,q^4))/(Pbar(q^12,q,q^2,q^3,q^4)
+(Rbar(q^12,q,q^2,q^3,q^4))/(Pbar(q^16,q,q^2,q^3,q^4)
+(Rbar(q^16,q,q^2,q^3,q^4))/(Pbar(q^20,q,q^2,q^3,q^4)))))),q,0,75);
```

$$1-q-q^3+q^4+q^6-q^7-q^8+q^{10}+2q^{11}-q^{12}-q^{13}-3q^{14}+q^{15}+3q^{16}+3q^{17}-q^{18}-6q^{19}-2q^{20}+q^{21}+10q^{22}+2q^{23}-5q^{24}-13q^{25}-3q^{26}+13q^{27}+15q^{28}+5q^{29}-25q^{30}-19q^{31}-2q^{32}+39q^{33}+27q^{34}-12q^{35}-53q^{36}-41q^{37}+39q^{38}+75q^{39}+53q^{40}-79q^{41}-113q^{42}-49q^{43}+129q^{44}+177q^{45}+17q^{46}-203q^{47}-264q^{48}+52q^{49}+330q^{50}+351q^{51}-159q^{52}-550q^{53}-413q^{54}+331q^{55}+895q^{56}+425q^{57}-642q^{58}-1360q^{59}-376q^{60}+1220q^{61}+1927q^{62}+217q^{63}-2227q^{64}-2581q^{65}+208q^{66}+3802q^{67}+3341q^{68}-1221q^{69}-6074q^{70}-4198q^{71}+3333q^{72}+9218q^{73}+4946q^{74}-7175q^{75}+\cdots$$

B.3 Verification of Special Case 3

Two specific instances of Special Case 3 are verified below. The first instance is when we map $q \mapsto q^4$ and set $a_1 = q$, $a_3 = q^3$. The second instance is when we map $q \mapsto q^6$ and set $a_1 = q, a_3 = q^5$. In both instances, $x \to 1^-$, though this is not necessary in general for this special case.

```
taylor(q^4-q^8,q,0,75);
```

$q^4 - q^8 + \ldots$

```
taylor((Rbar(1,q,1,q^3,q^4))/(Pbar(q^4,q,1,q^3,q^4)
+(Rbar(q^4,q,1,q^3,q^4))/(Pbar(q^8,q,1,q^3,q^4)
+(Rbar(q^8,q,1,q^3,q^4))/(Pbar(q^12,q,1,q^3,q^4)
+(Rbar(q^12,q,1,q^3,q^4))/(Pbar(q^16,q,1,q^3,q^4)
+(Rbar(q^16,q,1,q^3,q^4))/(Pbar(q^20,q,1,q^3,q^4)))))),q,0,75);
```

$q^4 - q^8 + \cdots$

```
taylor(q^6-q^12,q,0,75);
```

$q^6 - q^{12} + \ldots$

```
taylor((Rbar(1,q,1,q^5,q^6))/(Pbar(q^6,q,1,q^5,q^6)
+(Rbar(q^6,q,1,q^5,q^6))/(Pbar(q^12,q,1,q^5,q^6)
+(Rbar(q^12,q,1,q^5,q^6))/(Pbar(q^18,q,1,q^5,q^6)
+(Rbar(q^18,q,1,q^5,q^6))/(Pbar(q^24,q,1,q^5,q^6)
+(Rbar(q^24,q,1,q^5,q^6))/(Pbar(q^30,q,1,q^5,q^6)))))),q,0,75);
```

$q^6 - q^{12} + \ldots$

B.4 Verification of Special Case 4

Two specific instances of Special Case 4 are verified below. The first instance is when we map $q \mapsto q^4$ and set $a = q$. The second instance is when we map $q \mapsto q^6$ and set $a = q$. In both instances, $x \to 1^-$. However, as with Special Case 3, this is not necessary in general.

```
taylor((qpoch(-q^4,q^4,inf)*qpoch(-q^4,q^8,inf)*qpoch(-q^6,q^8,inf))/(qpoch(-q^7,q^4,inf)^2*sum((qpoch(-1,q^4,n)*qpoch(q,q^4,n)*(-q^7)^n)/(qpoch(-q^7,q^4,n)*qpoch(q^4,q^4,n)),n,0,inf)),q,0,75);
```

$1+2q^4+q^6+2q^{10}-2q^{12}-4q^{18}+4q^{20}-2q^{22}+10q^{26}-10q^{28}+10q^{30}-20q^{34}+26q^{36}-34q^{38}+4q^{40}+34q^{42}-70q^{44}+92q^{46}-30q^{48}-50q^{50}+176q^{52}-230q^{54}+140q^{56}+60q^{58}-404q^{60}+576q^{62}-496q^{64}-6q^{66}+842q^{68}-1446q^{70}+1514q^{72}-380q^{74}+\cdots$

```
taylor(Pbar(1,-1,-q,q,q^4)+(Rbar(1,-1,-q,q,q^4))/(Pbar(q^4,-1,-q,q,q^4)+(Rbar(q^4,-1,-q,q,q^4))/(Pbar(q^8,-1,-q,q,q^4)+(Rbar(q^8,-1,-q,q,q^4))/(Pbar(q^12,-1,-q,q,q^4)+(Rbar(q^12,-1,-q,q,q^4))/(Pbar(q^16,-1,-q,q,q^4)+(Rbar(q^16,-1,-q,q,q^4))/(Pbar(q^20,-1,-q,q,q^4)+(Rbar(q^20,-1,-q,q,q^4))/(Pbar(q^24,-1,-q,q,q^4))))))),q,0,75);
```

$1+2q^4+q^6+2q^{10}-2q^{12}-4q^{18}+4q^{20}-2q^{22}+10q^{26}-10q^{28}+10q^{30}-20q^{34}+26q^{36}-34q^{38}+4q^{40}+34q^{42}-70q^{44}+92q^{46}-30q^{48}-50q^{50}+176q^{52}-230q^{54}+140q^{56}+60q^{58}-404q^{60}+576q^{62}-496q^{64}-6q^{66}+842q^{68}-1446q^{70}+1514q^{72}-380q^{74}+\cdots$

taylor((qpoch(-q^6,q^6,inf)*qpoch(-q^6,q^12,inf)*qpoch(-q^10,q^12,inf))/(qpoch(-q^11,q^6,inf)^2*sum((qpoch(-1,q^6,n)*qpoch(q,q^6,n)*(-q^11)^n)/(qpoch(-q^11,q^6,n)*qpoch(q^6,q^6,n)),n,0,inf)),q,0,75);

$1+2q^{6}+q^{10}+2q^{16}-2q^{18}-4q^{28}+4q^{30}-2q^{34}+2q^{36}-2q^{38}+10q^{40}-8q^{42}-2q^{44}+10q^{46}-8q^{48}+8q^{50}-20q^{52}+12q^{54}+14q^{56}-36q^{58}+24q^{60}-18q^{62}+28q^{64}-6q^{66}-58q^{68}+100q^{70}-56q^{72}+18q^{74}+\cdots$

taylor(Pbar(1,-1,-q,q,q^6)+(Rbar(1,-1,-q,q,q^6))/(Pbar(q^6,-1,-q,q,q^6)+(Rbar(q^6,-1,-q,q,q^6))/(Pbar(q^12,-1,-q,q,q^6)+(Rbar(q^12,-1,-q,q,q^6))/(Pbar(q^18,-1,-q,q,q^6)+(Rbar(q^18,-1,-q,q,q^6))/(Pbar(q^24,-1,-q,q,q^6)+(Rbar(q^24,-1,-q,q,q^6))/(Pbar(q^30,-1,-q,q,q^6)+(Rbar(q^30,-1,-q,q,q^6))/(Pbar(q^36,-1,-q,q,q^6)+(Rbar(q^36,-1,-q,q,q^6))/(Pbar(q^42,-1,-q,q,q^6)))))))),q,0,75);

$1+2q^{6}+q^{10}+2q^{16}-2q^{18}-4q^{28}+4q^{30}-2q^{34}+2q^{36}-2q^{38}+10q^{40}-8q^{42}-2q^{44}+10q^{46}-8q^{48}+8q^{50}-20q^{52}+12q^{54}+14q^{56}-36q^{58}+24q^{60}-18q^{62}+28q^{64}-6q^{66}-58q^{68}+100q^{70}-56q^{72}+18q^{74}+\cdots$