A Spider's Web of Doughnuts

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ABSTRACT

A SPIDER’S WEB OF DOUGHNUTS

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Northern Illinois University, 2020
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This dissertation studies an interplay between the dynamics of iterated quasiregular mappings and certain topological structures. In particular, the relationship between the Julia set of a uniformly quasiregular mapping $f : \mathbb{R}^3 \to \mathbb{R}^3$ and the fast escaping set of its associated Poincaré linearizer is explored. It is shown that, if the former is a Cantor set, then the latter is a spider’s web. A new class of uniformly quasiregular maps is constructed to which this result applies. Toward this, a geometrically self-similar Cantor set of genus 2 is constructed.

It is also shown that for any uniformly quasiregular mapping $f : \mathbb{R}^n \to \mathbb{R}^n, n \geq 2$, if the Julia set of $f$ is a Cantor set, then the Julia set is the closure of the set of repelling periodic points of $f$. Some growth estimates on generalized derivatives are established, as well as a bound on the order of growth of the associated Poincaré linearizer.
A SPIDER’S WEB OF DOUGHNUTS

BY

DANIEL STOERTZ
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My cats, Hank and Reese, whose cuddles and purrs are also an integral part of this work and my life.
DEDICATION

I would like to dedicate this work to my father, David William Stoertz, who showed me more interest and support than I ever realized.
<table>
<thead>
<tr>
<th>LIST OF FIGURES</th>
<th>vi</th>
</tr>
</thead>
<tbody>
<tr>
<td>Chapter</td>
<td></td>
</tr>
<tr>
<td>1 INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td>1.1 Complex Dynamics</td>
<td>1</td>
</tr>
<tr>
<td>1.1.1 Fixed Points and Linearization</td>
<td>10</td>
</tr>
<tr>
<td>1.1.2 The Structure of the Julia Set</td>
<td>13</td>
</tr>
<tr>
<td>1.2 Quasiregular Maps</td>
<td>16</td>
</tr>
<tr>
<td>1.2.1 Quasiregular Dynamics</td>
<td>23</td>
</tr>
<tr>
<td>1.2.2 Quasiregular Linearization</td>
<td>29</td>
</tr>
<tr>
<td>1.3 Cantor Sets</td>
<td>35</td>
</tr>
<tr>
<td>1.3.1 Defining Sequences</td>
<td>44</td>
</tr>
<tr>
<td>1.3.2 The Genus of a Cantor Set</td>
<td>48</td>
</tr>
<tr>
<td>1.3.3 Higher Dimensions</td>
<td>53</td>
</tr>
<tr>
<td>1.4 Knot Theory</td>
<td>54</td>
</tr>
<tr>
<td>2 SPIDER’S WEBS</td>
<td>60</td>
</tr>
<tr>
<td>2.1 Lemmas on Generalized Derivatives</td>
<td>60</td>
</tr>
<tr>
<td>2.2 Density of Repelling Periodic Points</td>
<td>64</td>
</tr>
<tr>
<td>2.3 Positive Lower Order of Growth</td>
<td>67</td>
</tr>
<tr>
<td>2.4 The Proof of Theorem 2.0.1</td>
<td>70</td>
</tr>
<tr>
<td>Chapter</td>
<td>Title</td>
</tr>
<tr>
<td>---------</td>
<td>----------------------------------------------------------------------</td>
</tr>
<tr>
<td>3</td>
<td>A GEOMETRICALLY SELF-SIMILAR CANTOR SET</td>
</tr>
<tr>
<td>3.1</td>
<td>Square Tori.</td>
</tr>
<tr>
<td>3.2</td>
<td>The Four-Way Linking.</td>
</tr>
<tr>
<td>3.2.1</td>
<td>A Bound on $r$.</td>
</tr>
<tr>
<td>3.3</td>
<td>Bounding $m$ and Constructing the Chain</td>
</tr>
<tr>
<td>3.4</td>
<td>The Cantor Set</td>
</tr>
<tr>
<td>4</td>
<td>THE GENUS OF THE CANTOR SET</td>
</tr>
<tr>
<td>4.1</td>
<td>Alternate Defining Sequences</td>
</tr>
<tr>
<td>4.2</td>
<td>Linking Lemmas</td>
</tr>
<tr>
<td>4.3</td>
<td>Getting a Handle on Separation</td>
</tr>
<tr>
<td>4.4</td>
<td>Separating $X_2$.</td>
</tr>
<tr>
<td>4.5</td>
<td>Inductive Proof of Genus</td>
</tr>
<tr>
<td>5</td>
<td>CONSTRUCTING A UQR MAP WITH JULIA SET A GENUS 2 CANTOR SET</td>
</tr>
<tr>
<td>5.1</td>
<td>A Basic Covering Map</td>
</tr>
<tr>
<td>5.2</td>
<td>A Genus 2 Julia Set</td>
</tr>
<tr>
<td>REFERENCES</td>
<td></td>
</tr>
<tr>
<td>Figure</td>
<td>Description</td>
</tr>
<tr>
<td>--------</td>
<td>-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------</td>
</tr>
<tr>
<td>1.1</td>
<td>Some Julia sets. See also page 42 of [38]. Images generated via <a href="https://www.marksmath.org/visualization/julia_sets/">https://www.marksmath.org/visualization/julia_sets/</a></td>
</tr>
<tr>
<td>1.2</td>
<td>Antoine’s necklace construction.</td>
</tr>
<tr>
<td>1.3</td>
<td>The solid torus $D_0$.</td>
</tr>
<tr>
<td>1.4</td>
<td>A chain of 2-holed tori. See also Figure 1 in [3].</td>
</tr>
<tr>
<td>1.5</td>
<td>A defining sequence for a genus $r$ Cantor set. See also Figure 1 in [58].</td>
</tr>
<tr>
<td>1.6</td>
<td>A chain of 2-holed tori. The $C_1$ tori in blue are drawn up to homotopy type for simplicity.</td>
</tr>
<tr>
<td>1.7</td>
<td>The chain $C_1$ with surrounding 1-holed torus $T_0$.</td>
</tr>
<tr>
<td>1.8</td>
<td>The trefoil knot, as tied with string.</td>
</tr>
<tr>
<td>1.9</td>
<td>Bachelor’s unknotting.</td>
</tr>
<tr>
<td>1.10</td>
<td>The knot $K_1$. The section marked by a blue dotted square is repeated $i$ times to form $K_i$.</td>
</tr>
<tr>
<td>1.11</td>
<td>Construction of $K_\infty$.</td>
</tr>
<tr>
<td>3.1</td>
<td>A square 1-holed torus.</td>
</tr>
<tr>
<td>3.2</td>
<td>A square 2-holed torus.</td>
</tr>
<tr>
<td>3.3</td>
<td>A link between three 2-holed tori.</td>
</tr>
<tr>
<td>3.4</td>
<td>A four-way linking between figure-eight core curves. Thickening by a small value $r &gt; 0$ yields a link between 2-holed tori.</td>
</tr>
<tr>
<td>3.5</td>
<td>A square 1-holed torus oriented as a diamond.</td>
</tr>
<tr>
<td>3.6</td>
<td>The four-way linking, before rotation.</td>
</tr>
<tr>
<td>Figure</td>
<td>Description</td>
</tr>
<tr>
<td>--------</td>
<td>-----------------------------------------------------------------------------</td>
</tr>
<tr>
<td>3.7</td>
<td>The distance between cylinders.</td>
</tr>
<tr>
<td>3.8</td>
<td>Lines used in distance estimations.</td>
</tr>
<tr>
<td>3.9</td>
<td>Size of small tori.</td>
</tr>
<tr>
<td>3.10</td>
<td>Position of $X_0$ in the $x_1x_2$-plane.</td>
</tr>
<tr>
<td>3.11</td>
<td>Length and width direction vectors for $X_{1,j}$.</td>
</tr>
<tr>
<td>3.12</td>
<td>$X_{1,j}$ as determined by $p_i$ and $p_{i+1}$.</td>
</tr>
<tr>
<td>3.13</td>
<td>Counterclockwise orientation on $\gamma$, starting at the origin.</td>
</tr>
<tr>
<td>3.14</td>
<td>$X_{1,j}$ with associated disks $D_j$ and $D_j'$.</td>
</tr>
<tr>
<td>3.15</td>
<td>A four-way linking of plane polygons. Approximate locations of intersections with $D$ marked in red.</td>
</tr>
<tr>
<td>3.16</td>
<td>The 3-annulus $U$ with generators of $\pi_1(U)$.</td>
</tr>
<tr>
<td>3.17</td>
<td></td>
</tr>
<tr>
<td>3.18</td>
<td>Left: $A_1, A_{m/2+1}$ Right: $A'_{m/2}, A'_m$.</td>
</tr>
<tr>
<td>4.1</td>
<td>An undesirable intersection of $T$ and $S$.</td>
</tr>
<tr>
<td>4.2</td>
<td>Turning a surface 1-holed torus into a sphere.</td>
</tr>
<tr>
<td>4.3</td>
<td>An arc $Y$ (left) whose complement (right) is not an arc.</td>
</tr>
<tr>
<td>4.4</td>
<td>Left: A subchain $Y$ of $X_1$. Right: A non-maximal and a maximal arc of $Y$.</td>
</tr>
<tr>
<td>4.5</td>
<td>The linking of $Y$ and $X'_0$.</td>
</tr>
<tr>
<td>4.6</td>
<td>A linking between loops of 2-elements.</td>
</tr>
<tr>
<td>4.7</td>
<td>A union of arcs of different types.</td>
</tr>
<tr>
<td>5.1</td>
<td></td>
</tr>
</tbody>
</table>
CHAPTER 1
INTRODUCTION

For \( n \geq 2 \), let \( \mathbb{B}^n \) denote the open Euclidean ball in \( \mathbb{R}^n \), and \( S^{n-1} = \partial \mathbb{B}^n \). In the special case of \( \mathbb{C} \), we may use \( \mathbb{D} \) to denote the open unit disk. Let \( | \cdot | \) denote the Euclidean metric in \( \mathbb{C} \) or \( \mathbb{R}^n \), and for \( A, B \subseteq \mathbb{R}^n \), let \( d(A, B) = \inf\{|x - y| : x \in A, y \in B\} \). For \( x_0 \in \mathbb{R}^n \) and \( r > 0 \), let \( B(x_0, r) \) be the open ball centered at \( x_0 \) of radius \( r \), and let \( S(x_0, r) = \partial B(x_0, r) \).

For \( 0 < r < s \) and \( x_0 \in \mathbb{R}^n \), let \( A(x_0, r, s) = \{ x \in \mathbb{R}^n : r < |x - x_0| < s \} \). If \( x_0 = 0 \), let \( S_r = S(0, r) \), and let \( A(r, s) = A(0, r, s) \). For a subset \( A \) of a topological space \( X \), denote by \( \overline{A} \) and \( \text{Int}(A) \) the closure and interior of \( A \) in \( X \), respectively.

1.1 Complex Dynamics

This section will constitute a standard introduction to complex dynamics, where more background can be found in Milnor’s book [38]. Additional background on complex analysis can be found in Conway’s book [15]. Bergweiler’s paper [4] is also an excellent introduction to dynamics.

Informally, complex dynamics is concerned with the behavior of holomorphic maps under iteration. Given a holomorphic map \( f : \mathbb{C} \to \mathbb{C} \), the Julia set of \( f \) is the subset of \( \mathbb{C} \) where the iterates of \( f \) exhibit chaotic behavior. By contrast, the Fatou set of \( f \) is the set of stable behavior of the iterates. These sets are the principal objects of study in complex dynamics. To understand the formal definitions of these sets, it is necessary to understand normal families of functions.
Let $G \subset \mathbb{C}$ be a domain. Denote by $C(G, \mathbb{C})$ the space of continuous complex-valued functions on $G$. We say a sequence $\{f_n\} \subset C(G, \mathbb{C})$ converges *locally uniformly* on $G$ if $\{f_n\}$ converges on $G$, and this convergence is uniform on every compact subset $K$ of $G$. Similarly, we say $\{f_n\}$ *diverges locally uniformly* from $G$ if for every compact $K \subset G$ and $K' \subset \mathbb{C}$, we have that $f_n(K) \cap K' = \emptyset$ for $n$ sufficiently large.

**Definition 1.1.1** (Normal families). Let $G \subset \mathbb{C}$ be a domain. A family $\mathcal{F} \subset C(G, \mathbb{C})$ is called *normal* if every sequence in $\mathcal{F}$ has a subsequence which converges locally uniformly to some continuous function $g : G \to \mathbb{C}$, or diverges locally uniformly to $\infty$.

For $z \in G$, denote by $z : \mathcal{F} \to \mathbb{C}$ the evaluation map given by $z(f) = f(z)$ for $f \in \mathcal{F}$. The Arzela-Ascoli Theorem then shows that the family $\mathcal{F} \subset C(G, \mathbb{C})$ is normal if $\mathcal{F}$ is equicontinuous at each point of $G$ and the image of $\mathcal{F}$ under the evaluation map has compact closure in $\mathbb{C} \cup \{\infty\}$.

Define the *$n$-th iterate* of $f$ by $f^n = f^{n-1} \circ f$ for $n \geq 2$, where $f^1 = f$. The family

$$\mathcal{F} = \{f^n : n \in \mathbb{N}\}$$

is then called the *family of iterates* of $f$. Complex dynamics is concerned with the behavior of the family of iterates for functions in the space $H(G, \mathbb{C})$ of holomorphic maps. A useful tool in the study of normal families is the following theorem.

**Theorem 1.1.2** (Montel’s Theorem). Let $G \subset \mathbb{C}$ be a domain and $\mathcal{F} \subset H(G, \mathbb{C})$. If $\mathcal{F}$ is locally bounded, then $\mathcal{F}$ is normal. That is, if for each $z \in G$, there exist $M > 0$ and $r > 0$ such that for any $f \in \mathcal{F}$, $f(B(z, r)) \subset B(f(z), M)$, then $\mathcal{F}$ is normal.

One interesting result from complex analysis that we will see extended into the quasiregular setting later is the following theorem.
**Theorem 1.1.3** (Little Picard Theorem). *If* $f : \mathbb{C} \to \mathbb{C}$ *is an entire map which omits at least two values, then* $f$ *is constant.\*

The machinery of normal families is actually used in establishing the Little Picard Theorem. The following theorem helps bridge the gap between complex dynamics and the LPT.

**Theorem 1.1.4** (Montel-Carathéodory Theorem). *If* $\mathcal{F}$ *is the family of all holomorphic functions on a domain* $G \subset \mathbb{C}$ *that do not assume the values 0 and 1, then* $\mathcal{F}$ *is normal in* $C(G, \mathbb{C} \setminus \{0, 1\})$.

In general one can apply the study of normality to rational and meromorphic maps as well, but for the purposes of this work we will focus on entire maps, that is families of maps in the space $H(\mathbb{C}, \mathbb{C})$. With this in mind, we are ready to formally define the foundational objects of study in complex dynamics.

**Definition 1.1.5.** Let $f : \mathbb{C} \to \mathbb{C}$ be entire. The *Fatou set* of $f$, denoted by $F(f)$, is the domain of normality for $f$. In other words,

$$
F(f) = \{ z \in \mathbb{C} : \text{there exists a neighborhood } N \text{ of } z \text{ such that } \{f^n\} \text{ is normal on } N \}.
$$

The complement of $F(f)$ is called the *Julia set* of $f$ and is denoted by $J(f)$. Note that $J(f)$ satisfies

$$
J(f) = \{ z \in \mathbb{C} : \text{there is no neighborhood } N \text{ of } z \text{ such that } \{f^n\} \text{ is normal on } N \}.
$$

There is an important aspect of normality for holomorphic maps that is worth considering. If $f \in H(G, \mathbb{C})$ is normal on the open set $U \subset G$, then the sequence of iterates $\{f^n\}$ has a subsequence converging locally uniformly to some map $g : U \to \mathbb{C}$. This map $g$ must itself be holomorphic.
Example 1.1.6. Consider the map $f(z) = z^d$ defined on $\mathbb{C}$, where $d \geq 2$ is an integer. Then $f$ is clearly entire, and $f$ can be extended to be defined on $\mathbb{C} \cup \{\infty\}$ by setting $f(\infty) = \infty$. Now, since $|f^n(z)| = |z|^{dn}$ for all $n \in \mathbb{N}$, it is clear that

$$f^n(z) \to \begin{cases} 
0 & \text{for all } z \in \mathbb{D}, \\
\infty & \text{for all } z \in \mathbb{C} \setminus \mathbb{D}.
\end{cases}$$

Suppose $z \in \mathbb{D}$. Then $|f^n(z)| < 1$ for all $n \in \mathbb{N}$, and so the family $\{f^n\}$ is locally bounded. By Montel’s Theorem, $\{f^n\}$ is normal on $\mathbb{D}$.

For $z \in \mathbb{C} \setminus \overline{\mathbb{D}}$, we can conjugate $f$ by the conformal automorphism $g(w) = 1/w$ of $\mathbb{C} \cup \{\infty\}$. The resulting map is still $z \mapsto z^d$, except now $z \in \mathbb{D}$. Applying Montel’s Theorem again, we see that $\{f^n\}$ is normal on $\mathbb{C} \setminus \overline{\mathbb{D}}$.

Finally, suppose that $z \in \partial \mathbb{D}$, and let $U$ be a neighborhood of $z$. Note that $|f^n(z)| = 1$ for all $n \in \mathbb{N}$. Furthermore, we have that all points in $U \cap \mathbb{D}$ will converge to 0 under iteration of $f$, and similarly that all points in $U \cap (\mathbb{C} \setminus \overline{\mathbb{D}})$ will converge to $\infty$. So the limiting function of $f^n$ must have a jump discontinuity across $U \cap \partial \mathbb{D}$. Hence there is no continuous limit function. We conclude that $F(f) = (\mathbb{C} \cup \{\infty\}) \setminus \partial \mathbb{D}$ and $J(f) = \partial \mathbb{D}$.

Example 1.1.7. Consider the map $g(z) = z^2 - 2$ defined on $\mathbb{C}$. To find $F(g)$ and $J(g)$, it is useful to semiconjugate $g$ by the function $h(z) = z + 1/z$. That is, we find a function $f$ such that

$$g \circ h = h \circ f. \tag{1.1}$$

Observe that $f(z) = z^2$ satisfies (1.1). Since $g^n \circ h = g^{n-1} \circ (g \circ h) = g^{n-1} \circ (h \circ f)$ for all $n \geq 2$, we have that

$$g^n \circ h = h \circ f^n.$$

In other words, semi-conjugation by $h$ relates the iterative behavior between $g$ and $f$. 
From Example 1.1.6, we already know that $F(f) = (\mathbb{C} \cup \{\infty\}) \setminus \mathbb{D}$ and $J(f) = \partial\mathbb{D}$. Let $e^{i\theta} \in \partial\mathbb{D}$. Then
\[ h(e^{i\theta}) = e^{i\theta} + e^{-i\theta} = 2 \cos \theta. \]

As $\theta$ ranges from 0 to $2\pi$, $h(e^{i\theta})$ ranges across the closed segment $L$ of the real axis between $-2$ and 2. Hence $J(g) \supset L$. Now let $z \in F(f)$. Then the sequence $\{f^n(z)\}$ either approaches 0 or $\infty$. Since $h$ maps both $\mathbb{D}$ and $(\mathbb{C} \cup \{\infty\}) \setminus \mathbb{D}$ to $(\mathbb{C} \cup \{\infty\}) \setminus L$, and since $h(0) = h(\infty) = \infty$, the sequence $\{g^n(h(z))\}$ approaches $\infty$. Hence $F(g) = (\mathbb{C} \cup \{\infty\}) \setminus L$ and $J(g) = L$.

From the previous examples, the reader may be lead to believe that Julia sets are often smooth. However, these examples are exceptions to the reality that, even for polynomials, Julia sets are almost never smooth. See Figure 1.1 for more typical examples of Julia sets. Note that these are all based on quadratic functions.

Here we list some basic properties of the Fatou and Julia sets.

**Theorem 1.1.8.** Let $f$ be entire. Then

(i) $F(f)$ is open and $J(f)$ is closed;

(ii) $F(f^n) = F(f)$, and $J(f^n) = J(f)$ for all $n \in \mathbb{N}$;

(iii) $F(f)$ and $J(f)$ are completely invariant, that is $z \in F(f)$ if and only if $f(z) \in F(f)$ (and similarly for $J(f)$).

Suppose further that $\deg f \geq 2$. Then

(iv) $J(f)$ is perfect, that is $J(f) \neq \emptyset$ and $J(f)$ has no isolated points.

From (ii) and (iii), it follows that all the iterates of a given entire $f$ share the same dynamics. More observations about the Julia set will be made in Section 1.1.2.
(a) A “dendrite,”
\[ z \mapsto z^2 + i \]

(b) A Cantor set,
\[ z \mapsto z^2 + (-.765 + .12i) \]

(c) The Douady rabbit,
\[ z \mapsto z^2 + (-.12 + .745i) \]

(d) The “airplane,”
\[ z \mapsto z^2 - 1.74588 \]

Figure 1.1: Some Julia sets. See also page 42 of [38]. Images generated via https://www.marksmath.org/visualization/julia_sets/
Definition 1.1.9. Let $f$ be a holomorphic function, and let $z \in \mathbb{C}$. We call the set

$$O^+(z) := \{f^n(z) : n \geq 0\}$$

the forward orbit of $z$, and we call the set

$$O^-(z) := \{f^{-n}(z) : n \geq 0\} = \bigcup_{n \geq 0} \{w \in \mathbb{C} : f^n(w) = z\}$$

the backward orbit of $z$. The grand orbit, denoted by $GO(z)$ is the set of points whose orbit intersects the orbit of $z$.

Other sets with dynamical significance concern the growth behavior of an entire map under iteration. For $f$ entire and $R > 0$, we define the maximum modulus function $M(R, f) = \max\{|f(z)| : |z| = R\}$. The iteration of the maximum modulus function is given by

$$M^2(R, f) = M(M(R, f), f).$$

In other words, we find the maximum modulus of $f$ on the circle of radius $R$, then draw a new circle with radius $M(R, f)$ and find the maximum modulus of $f$ there.

Definition 1.1.10. Let $f$ be an entire function. The escaping set of $f$, denoted by $I(f)$, satisfies

$$I(f) = \{z \in \mathbb{C} : \lim_{n \to \infty} |f^n(z)| = \infty\}.$$

The fast escaping set of $f$, denoted by $A(f)$, satisfies

$$A(f) = \{z \in \mathbb{C} : \exists N \in \mathbb{N} \text{ such that } |f^{n+N}(z)| \geq M^n(R, f) \text{ for all } n \in \mathbb{N}\},$$
where \( R > 0 \) is sufficiently large so that \( \lim_{n \to \infty} M^n(R, f) = \infty \). This set does not depend on the choice of such an \( R \).

Eremenko showed in [19] that \( I(f) \) is nonempty for any entire map \( f \) by explicitly constructing escaping points. Bergweiler and Hinkkanen introduced the fast escaping set in [9], and showed therein that the points constructed by Eremenko are actually fast escaping, and hence that \( A(f) \) is nonempty.

To identify the differences between \( I(f) \) and \( A(f) \), we need to first recall that the space \( H(\mathbb{C}, \mathbb{C}) \) consists of two types of maps, namely polynomials and transcendental entire maps. One key distinction between these maps is whether they can be extended to be holomorphic on \( \mathbb{C} \cup \{ \infty \} \). If \( f \) is a polynomial, then \( f \) can be extended by setting \( f(\infty) = \infty \). However, transcendental entire maps have an essential singularity at \( \infty \), and thus cannot be continuously extended.

We think of the fast escaping set as the set of points that escape to infinity as fast as possible. Clearly, \( A(f) \subset I(f) \) for any entire map \( f \). If \( f \) is a polynomial, then \( A(f) = I(f) \).

To see this, suppose \( f \) has degree \( d \) with leading coefficient \( a_d \). Then outside of a disk of sufficiently large radius \( R \), we have that \( f \) behaves like \( a_d z^d \). So, for any escaping point \( z_0 \), we can iterate \( f \) finitely many, say \( N \), times to get \( f^N(z_0) \) outside of \( B(0, R) \). Any subsequent iterations will satisfy \( |f^{n+N}(z_0)| \geq M^n(R, f) \approx |a_n| |z|^d \), and hence \( I(f) = A(f) \).

For \( f \) transcendental entire, Rippon and Stallard showed in [49] that \( A(f) \subsetneq I(f) \). To illustrate this, consider the following example.

**Example 1.1.11.** Let \( f(z) = z + 1 + e^{-z} \). Let us first consider what happens to 0 under iteration. We have that \( f(0) = 1 \), and it is easy to see from here that \( f^n \) rapidly approaches a shift in the positive real direction by one unit when applied to \( f^{n-1}(0) \). So 0 escapes to \( \infty \) at a polynomial rate.
Now let $R > 0$ and define a square $S_0 = \{ z \in \mathbb{C} : -(R + 1) \leq \text{Re} z \leq -R, -\pi i \leq \text{Im} z \leq \pi i \}$. Then $f(S_0)$ is an annulus $A_1$ having inner radius close to $e^R$ and outer radius close to $e^{R+1}$. If $R$ is sufficiently large, then $A_1$ contains another square $S_1$ in the left half-plane having real width 1 and imaginary width $2\pi$. Additionally, since $e^R > R$, the square $S_1$ is farther down the negative real direction than the square $S_0$. Defining further annuli and squares inductively by iteration of $f$, we obtain a sequence of squares $\{S_n\}$ which escape to $\infty$ at an exponential rate. Choosing any point from $S_0$ that passes through all the $S_n$, in other words choosing

$$z \in \bigcap_{n=1}^{\infty} f^{-n}(S_n),$$

we obtain a point escaping to $\infty$ at an exponential rate. Hence $0 \notin A(f)$.

The fast escaping set of a transcendental entire map will play a very important role in this work. This is because of an additional special dynamical property of the fast escaping set, which will be discussed further in Section 1.1.2. We close out this section with an example of a common topological structure of fast escaping sets.

**Example 1.1.12.** Consider the map $f(z) = \lambda e^z$, where $0 < \lambda < 1/e$. The full details of this example are beyond the scope of this work, so we present an outline. It can be shown that there exists a point $z_0$ on the real axis with $0 < z_0 < 1$ such that $f(z_0) = z_0$. Furthermore, it can be shown that for every point $z \in F(f)$, we have that $f^n(z) \to z_0$. The points that are not attracted to $z_0$ are hence in $J(f)$.

It turns out that $J(f)$ is a Cantor bouquet of hairs. That is, $J(f) = \bigcup_\alpha \gamma_\alpha$, where each $\gamma_\alpha : [0, \infty) \to \mathbb{C}$ is a curve satisfying $\gamma_\alpha(t) \to \infty$ as $t \to \infty$, and such that any transverse cross-section of the collection of curves yields a Cantor set. To be more precise about the escaping tails of the curves, in this example we have that $\text{Re}(\gamma_\alpha(t)) \to \infty$ as $t \to \infty$. The points in the subset of $\gamma_\alpha$ parameterized by the open interval $(0, \infty)$ are fast escaping, so here $A(f)$ is also a Cantor bouquet of hairs. We refer to [18] for the details on $J(f)$, and to [54]
for $A(f)$. Rempe’s thesis ([45]) is also an excellent resource on the dynamics of functions of this type.

1.1.1 Fixed Points and Linearization

So far, we have been considering the global dynamics of entire maps. However, an equally fruitful avenue of study is obtained by focusing on local behavior. We begin by defining periodic points.

**Definition 1.1.13.** Let $f$ be an entire map. We say that $z_0 \in \mathbb{C}$ is periodic if there exists $p \in \mathbb{N}$ such that $f^p(z_0) = z_0$. The smallest $p$ with this property is called the period of $z_0$. If $p = 1$, we say $z_0$ is fixed.

For a periodic point $z_0$ of period $p$, we call $(f^p)'(z_0)$ the multiplier of $z_0$. The multiplier of $z_0$ is often denoted by $\lambda_{z_0}$. We classify $z_0$ as attracting, repelling, or indifferent depending on whether the modulus of the multiplier is less than, greater than, or equal to 1, respectively. The same classification holds for non-transcendental maps if $z_0 = \infty$, where the multiplier is now given by $\lambda_{\infty} = \left. \frac{d}{dz}(1/f(1/z)) \right|_{z=0}$. In the special case when $\lambda_{z_0} = 0$, we call $z_0$ super-attracting.

Given an attracting periodic point $z_0$ of period $p$ of a map $f$, the set

$$\mathcal{A}(z_0) := \{z \in \mathbb{C} \cup \{\infty\} : \lim_{n \to \infty} f^{pn} = z_0\}$$

is called the attracting basin of $z_0$. If $f$ is a polynomial, note that $\infty$ is a super-attracting fixed point, and that $\mathcal{A}(\infty)$ is a neighborhood of $\infty$ on the Riemann sphere.

Though there is much interesting dynamics concerning indifferent periodic points (see for instance [38]), this work will not consider them. We will instead be focused on attracting
and repelling periodic points. In fact, we will often further restrict our attention to fixed points. First, the following theorem justifies the terms ‘attracting’ and ‘repelling.’

**Theorem 1.1.14 (Topological Characterization of Fixed Points).** Let \( z_0 \) be a fixed point of the entire map \( f \).

(i) The point \( z_0 \) is attracting if and only if there is a neighborhood \( U \) of \( z_0 \) such that \( f(U) \subset U \).

(ii) The point \( z_0 \) is repelling if and only if there is a neighborhood \( U \) of \( z_0 \) such that \( f \) is injective on \( U \) and \( f(U) \supset U \).

The proof of this theorem heavily relies on the fact that the multiplier \( \lambda_{z_0} \) is the dominant term in a Taylor expansion of \( f \) near \( z_0 \).

Understanding the local dynamics near a fixed point has been one of the most important themes throughout the history of complex dynamics. As such, finding ways to relate the dynamics of a complicated map to those of a simpler map is highly useful. The Koenigs Linearization Theorem gives us a way to conjugate a map locally in a particularly nice way.

**Theorem 1.1.15 (Koenigs Linearization).** If the multiplier \( \lambda \) of the fixed point \( z_0 \) of the holomorphic map \( f \) satisfies \( |\lambda| \neq 0, 1 \), then there exists a local homeomorphic change of coordinate \( w = L(z) \), with \( L(z_0) = 0 \), so that \( L \circ f \circ L^{-1} \) is the linear map \( w \mapsto \lambda w \) for all \( w \) in some neighborhood of the origin. Furthermore, \( L \) is unique up to multiplication by a nonzero constant.

Even at first glance, this theorem is immensely powerful, allowing us to study the local dynamics of any entire map \( f \) in terms of a scaling and rotating map. But there is more to this theorem that makes it one of the most important results ever to come out of dynamics. Consider the functional equation

\[
L \circ f = T \circ L,
\]  

(1.2)
where $T : \mathbb{C} \to \mathbb{C}$ is the linear map $T(z) = \lambda z_0 z$. From the Kœnigs Linearization theorem, we know that (1.2) holds in $B(z_0, \epsilon)$ for some $\epsilon > 0$ with $L$ injective. Suppose for now that $z_0$ is an attracting fixed point, and hence that $|\lambda z_0| < 1$. If $z \in \mathcal{A}(z_0)$, then we can iterate the functional equation to define $L$ at $z$. Observe that $L \circ f \circ f = T \circ L \circ f = T \circ T \circ L$. Since $z \in \mathcal{A}(z_0)$, we can iterate $f$ a finite number, say $n$, times to get $f^n(z) \in B(z_0, \epsilon)$. We can then apply $L$ followed by $T^{-n}$, in other words

$$L(z) = T^{-n}(L(f^n(z))).$$

This allows us to extend $L$ to be holomorphic on all of $\mathcal{A}(z_0)$. 

Now suppose that $z_0$ is a repelling fixed point. Then $|\lambda z_0| > 1$. Let us regard (1.2) as a commutative diagram.

$$\begin{array}{ccc}
B(z_0, \epsilon) & \xrightarrow{f} & f(B(z_0, \epsilon)) \\
\uparrow L & & \uparrow L \\
\mathbb{C} & \xrightarrow{T} & \mathbb{C}
\end{array}$$

Note that we have reversed the direction of $L$ compared to (1.2). This is not a problem, since $L$ is injective on the appropriate neighborhoods, so for this example we have simply replaced $L$ with $L^{-1}$. Now let $z$ be any point outside the domain of injectivity for $L$. Since $T^{-1}$ has the effect of shrinking the modulus of $z$, we need only apply $T^{-1}$ a finite number, say $n$, times to obtain that $T^{-n}(z) \in L^{-1}(B(z_0, \epsilon))$. Applying $L$ and then $f^n$, we can thus define $L$ on any $z \in \mathbb{C}$, extending $L$ to an entire map. We then call $L$ a Poincaré linearizer of $f$ at the repelling fixed point $z_0$. Additionally, if $L'(0) = 1$, $L$ is then called the normalized linearizer.
An important fact about Poincaré linearizers is that they are always transcendental entire. To see why this is true, let \( d = \deg f \), and assume that \( d \geq 2 \). Choose \( z \in \mathbb{C} \) so that \( O_f^-(z) \) does not contain any critical points. Then there exist distinct points

\[
w_1, \ldots, w_d \in O_f^-(z)
\]

such that \( z = f(L(w_i)) = L(\lambda_{z_0} w_i) \) for all \( i \). So \( \deg L \geq d \). Now, since \( f^n \circ L = L \circ T^n \) holds on all of \( \mathbb{C} \), applying the same logic (and relabelling if necessary) there exist distinct points

\[
w_1, \ldots, w_d, \ldots \in O_f^-(z)
\]

such that \( z = f^n(L(w_i)) = L(\lambda_{z_0}^n w_i) \) for all \( i \). So \( \deg L \geq d^n \) for all \( n \). Hence \( \deg L = \infty \), showing that \( L \) is transcendental entire.

**Example 1.1.16.** Let \( f(z) = z^d \) for \( d \geq 2 \). Then \( z_0 = 1 \) is a repelling fixed point with multiplier \( \lambda_1 = d \). Letting \( T(z) = dz \), observe that

\[
f(e^z) = (e^z)^d = e^{dz} = e^{T(z)}.
\]

Hence \( L(z) = e^z \) is a Poincaré linearizer of \( f \) at the repelling fixed point 1. Since \( L'(0) = 1 \), it is in fact the normalized linearizer.

### 1.1.2 The Structure of the Julia Set

Having considered global and local dynamics separately in the previous sections, we will soon turn our attention to an interesting interplay between these ideas. In particular, linearization will prove a useful tool in determining the structure of the Julia set of complicated
maps. Before seeing this play out, it is important to understand what makes the Julia set so difficult to study on its own.

**Theorem 1.1.17.** Let $f$ be entire of degree at least 2. Then

(i) if $z_0$ is an attracting fixed point, then $A(z_0) \subset F(f)$, and $\partial A(z_0) = J(f)$;

(ii) if $z_0 \in J(f)$, then $J(f) = \overline{O^-(z_0)}$;

(iii) $J(f)$ is the closure of the set of repelling periodic points of $f$.

These properties, together with some from Theorem 1.1.8, illustrate why Julia sets are often incredibly intricate (and why high-resolution colorful images of them are so appealing to the general public). Consider for instance property (i) in the case when $f$ has three distinct attracting fixed points $z_1, z_2,$ and $z_3$. Then $J(f)$ is the common boundary of three distinct attracting basins. This is the case when $f$ is the rational Newton’s method function for the cubic map $z \mapsto z^3 - 1$. Additionally, we highlight another very special property of Julia sets in the following theorem.

**Theorem 1.1.18** (Explosion Property). Let $f$ be entire of degree at least 2. If $U$ is open and $U \cap J(f) \neq \emptyset$, then $O^+(U)$ is all of $\mathbb{C}$ minus at most one point.

This theorem is important for several reasons. First, it is equivalent to the definition of the Julia given in Definition 1.1.5, and so it is taken as the definition by many authors. Second, if we combine the explosion property with the complete invariance of the Julia set from Theorem 1.1.8, we see why Julia sets are often self-similar. Indeed, if $U$ is an open set intersecting $J(f)$, then $O^+(U \cap J(f)) = J(f)$.

Let us connect Julia sets with the escaping sets from the previous section. Clearly if $f$ is a polynomial, then, since $\infty$ is a super-attracting fixed point, we have from property (i) of Theorem 1.1.17 that $J(f) = \partial I(f)$. Since here the escaping set and the fast escaping set coincide, we immediately have that $J(f) = \partial A(f)$.
Suppose now that $f$ is transcendental entire. Then $\infty$ is an essential singularity, and the reasoning for polynomials no longer applies. However, Eremenko showed in [19] that $J(f) = \partial I(f)$ still holds. Bergweiler and Hinkkanen then showed in [9] that $J(f) = \partial A(f)$ also holds. Other subsets of $I(f)$ characterized by different rates of escape have been studied for transcendental entire maps, and have also been shown to have their boundary equal the Julia set. The fast escaping set is however in some sense the smallest known subset of $I(f)$ to have this property.

In light of the relationship to the Julia set, much work has been done recently to understand what possible structure $A(f)$ can have when $f$ is transcendental entire. So far, there have been two different topological structures that have emerged. The first we saw in Example 1.1.12, that of a Cantor bouquet of hairs. The second was defined by Rippon and Stallard in [50], and is as follows.

**Definition 1.1.19.** A connected set $E \subset \mathbb{C}$ is called an *(infinite) spider’s web* if there exists a sequence of simply-connected domains $(G_n)$ whose union is all of $\mathbb{C}$ such that $\partial G_n \subset E$ for all $n$.

Rippon and Stallard give $f(z) = \frac{1}{2} \left( \cos \frac{z}{4} + \cosh \frac{z}{4} \right)$ as a map having a spider’s web structure for $A(f)$.

Recall that, if $L$ is a Poincaré linearizer for a polynomial $p$ at a repelling fixed point, then $L$ is transcendental entire. Studying the dynamics of $L$ is in general very difficult, so it would be helpful to relate the dynamics of $L$ to the more understandable dynamics of $p$. A recent result to that effect is the following, due to Mihaljević-Brandt and Peter.

**Theorem 1.1.20** ([37], Theorem 1.1). *Let $p$ be a polynomial of degree $d \geq 2$, let $z_0$ be a repelling fixed point of $p$ and let $L$ be a linearizer of $p$ at $z_0$. Then $A(L)$ is a spider’s web if the component of $J(p)$ which contains $z_0$ equals $\{z_0\}$.***
Note that Theorem 1.1.20 holds in particular when $J(p)$ is a Cantor set. This is for instance the case if $p = z^2 + c$, where $c \in \mathbb{C}$ is outside the Mandelbrot set.

### 1.2 Quasiregular Maps

A natural question to ask is whether the methods and results from complex dynamics can be extended into higher dimensions. If we attempt to do this in a straightforward way, defining holomorphic maps on $\mathbb{R}^n$ for $n \geq 3$ in terms of Taylor series or else using some higher-dimensional analogue of the Cauchy-Riemann equations, we run into a significant problem. Due to a result known as the generalized Liouville Theorem proved by Reshetnyak (see [46]), any non-constant holomorphic map on $\mathbb{R}^n$, $n \geq 3$, must be a restriction of a Möbius map on $\mathbb{R}^n \cup \{\infty\}$. This drastically reduces the class of functions to which the dynamical results could apply compared to the complex plane.

This reduction of scope is circumvented by relaxing the condition that the considered maps be holomorphic. If we instead study so-called quasiregular maps, we find a much broader class of maps to which the theory of dynamics applies. Many of the results in this section can be found in Rickman’s monograph [46]. Bergweiler’s paper [4] also gives a good introduction to the material of this section.

Before discussing higher-dimensional maps, we remain in the complex plane to discuss some basic properties of holomorphic maps. Let $f : \mathbb{C} \to \mathbb{C}$ be entire, and write $f(x + iy) = u(x, y) + iv(x, y)$. Then $f$ satisfies the Cauchy-Riemann equations on $\mathbb{C}$, that is

$$u_x = v_y \quad \text{and} \quad u_y = -v_x.$$
For any $z_0 \in \mathbb{C}$, we then have by Taylor’s Theorem that $f$ is well approximated near $z_0$ by the map

$$z \mapsto f(z_0) + f'(z_0)(z - z_0).$$

Suppose for the sake of simplicity that $z_0 = f(z_0) = 0$. Then we can rewrite the previous map as

$$z \mapsto f'(z_0)z.$$

Regarding this as a map from $\mathbb{R}^2$ to $\mathbb{R}^2$, it is left multiplication by the matrix

$$f'(z_0) = \begin{pmatrix} u_x(x_0, y_0) & u_y(x_0, y_0) \\ -u_y(x_0, y_0) & u_x(x_0, y_0) \end{pmatrix}.$$ 

Finally, since $f' = u_x + iu_y$, we can decompose this matrix as

$$f'(z_0) = \begin{pmatrix} |f'(z_0)| & 0 \\ 0 & |f'(z_0)| \end{pmatrix} \begin{pmatrix} \cos \arg(f'(z_0)) & -\sin \arg(f'(z_0)) \\ \sin \arg(f'(z_0)) & \cos \arg(f'(z_0)) \end{pmatrix}.$$ 

So $f$ is well approximated near $z_0$ by the composition of a rotation and a scaling. From this we can see that holomorphic mappings map infinitesimal circles to infinitesimal circles.

This is the condition that is relaxed to broaden the class of function to which dynamics can apply. Informally, a map $f : \mathbb{C} \to \mathbb{C}$ is called quasiregular if it maps infinitesimal circles to infinitesimal ellipses of uniformly bounded eccentricity. By uniformly bounded, we mean that the eccentricity of the resulting ellipses has a finite bound that is valid for all inputs to $f$. In this sense, quasiregular mappings allow more distortion than holomorphic mappings.
Let us make these notions precise. Suppose \( f : \mathbb{C} \to \mathbb{C} \) is \( \mathbb{R} \)-differentiable at \( z_0 \). Then \( f \) is well approximated near \( z_0 \) by the affine map

\[
A(z) = f_z(z_0)(z - z_0) + f_{\bar{z}}(z_0)(\bar{z} - \bar{z}_0) + f(z_0),
\]

where \( f_z = (f_x - if_y)/2 \) and \( f_{\bar{z}} = (f_x + if_y)/2 \). We then define the complex dilatation of a map \( f \) at \( z_0 \) by \( \mu_f(z_0) = f_{\bar{z}}(z_0)/f_z(z_0) \). In some sense \( \mu_f(z_0) \) measures the failure of \( f \) to be \( \mathbb{C} \)-differentiable, since \( \mu_f = 0 \) if and only if \( f \) is holomorphic at \( z_0 \). Now we define the distortion of \( f \) at \( z_0 \) by

\[
D_f(z_0) = \frac{1 + |\mu_f(z_0)|}{1 - |\mu_f(z_0)|}.
\]

The dilatation and distortion encode information about the image of an infinitesimal circle about \( z_0 \) under \( f \) in the following way.

(i) The distortion \( D_f \) is the ratio of the major and minor axes of the ellipse that is the image of the circle.

(ii) \( \arg \mu_f(z_0) = 2\theta \), where \( \theta \) is the direction of maximal distortion, that is the direction of the major axis.

Note that, if \( f_{\bar{z}} \equiv 0 \), then \( D_f \equiv 1 \), and \( f \) is holomorphic. Finally, define \( K_f = \sup D_f(z) \) to be the distortion of \( f \).

**Example 1.2.1.** Consider the map \( f(x + iy) = Kx + iy \), where \( K > 1 \). Before running any calculations, we can see that \( f \) maps circles centered at the origin to ellipses with major axis on the real axis, and with distortion equal to \( K \). Let us confirm this intuition globally. With \( f_x = K \) and \( f_y = i \), we have that

\[
\mu_f(x_0 + iy_0) = \frac{K - 1}{K + 1},
\]
and hence
\[ D_f(x_0 + iy_0) = \frac{1 + \frac{K-1}{K+1}}{1 - \frac{K-1}{K+1}} = K. \]

This is independent of \(x_0\) and \(y_0\), and so the distortion of \(f\) is \(K_f = K\).

**Definition 1.2.2.** If \(f : U \to \mathbb{C}\) is a \(C^1(U)\) homeomorphism and \(D_f(z)\) is uniformly bounded in \(U\), then we call \(f\) *quasiconformal* (abbreviated *qc*). Further, if \(K_f \leq K\), then we call \(f\) *\(K\)-quasiconformal* (\(K\)-\(qc\)).

The assumption that quasiconformal maps be differentiable everywhere is rather strong. It can be dropped by employing the so-called analytic definition of quasiconformal maps. To that end, a map \(f : \mathbb{C} \to \mathbb{C}\) is called *ACL* if \(f\) is absolutely continuous on almost every line parallel to the standard basis vectors of \(\mathbb{C}\). In contrast to the geometric definition, it turns out that *ACL* maps are differentiable almost everywhere. If the first-order partial derivatives of an *ACL* map \(f\) are additionally in \(L^2(\mathbb{C})\), we say \(f\) is *ACL*\(^2\).

**Definition 1.2.3.** A map \(f : \mathbb{C} \to \mathbb{C}\) is called *quasiconformal* if \(f\) is a homeomorphism, *ACL*\(^2\), and there exists \(0 \leq k < 1\) so that \(|f\bar{z}| < k|fz|\) almost everywhere.

If we drop the requirement that \(f\) be injective, we arrive at the definition of a *quasiregular* (abbreviated *qr*) mapping on \(\mathbb{C}\). The study of such maps is relatively limited on \(\mathbb{C}\), as illustrated by the following theorem.

**Theorem 1.2.4** (Stoilow Decomposition Theorem). *Let \(f : \mathbb{C} \to \mathbb{C}\) be quasiregular. Then there exist a quasiconformal map \(h\) and a holomorphic map \(g\) such that \(f = g \circ h\).*

In this sense, the study of quasiregular maps on \(\mathbb{C}\) boils down to a combination of holomorphic and quasiconformal maps. However, due to the generalized Liouville Theorem mentioned at the beginning of this section, there is much to gain by extending the definition of *qr* maps into higher dimensions.
It is the analytic definition of quasiregular maps that is typically given in generalized form. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a map, with $n \geq 2$. We say $f$ is $ACL^n$ if $f$ is absolutely continuous on almost every line parallel to the standard basis vectors in $\mathbb{R}^n$, and the first-order partial derivatives of $f$ lie in $L^n(\mathbb{R}^n)$.

**Definition 1.2.5.** A continuous map $f : U \to \mathbb{R}^n$ is quasiregular (abbreviated qr) if $f$ is $ACL^n$, and if there exists $K \geq 1$ such that

$$|f'(x)| = \sup_{|h|=1} |f'(x)h|^n \leq K J_f(x) \text{ a.e.},$$

where $J_f(x)$ is the Jacobian determinant of $f$. The infimum of $K$ satisfying the given inequality it called the *outer dilatation* of $f$, and is denoted by $K_O(f)$. It also follows that $f$ satisfies

$$J_f(x) \leq K' \inf_{|h|=1} |f'(x)h|^n \text{ a.e.},$$

where the infimum of $K'$ is called the *inner dilatation*, and is denoted by $K_I(f)$. The *distortion* of $f$ is $K = \max\{K_O, K_I\}$.

From this definition, quasiregular maps have many desirable properties. They are open, discrete, orientation-preserving maps with continuous first-order partial derivatives a.e., all properties that holomorphic maps have. They are also locally Hölder continuous and map infinitesimal spheres to infinitesimal ellipsoids of uniformly bounded eccentricity where differentiable.
To illustrate the latter point, consider the map \( f \) on \( \mathbb{R}^n \) given by left-multiplication by the matrix

\[
A = \begin{pmatrix}
K_1 & 0 & \cdots & 0 \\
0 & K_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & K_n
\end{pmatrix}.
\]

Here the \( K_i \) correspond directly to the lengths of the axes of the solid ellipsoid \( E \) having \( \partial E = f(S^{n-1}) \). For simplicity, suppose that \( \max\{K_i\} = K_1 \) and \( \min\{K_i\} = K_n \). Then we have that

\[
K_O = \frac{\prod_{i=1}^{n-1} K_1}{\prod_{i=2}^{n} K_i} \quad \text{and} \quad K_I = \frac{\prod_{i=1}^{n-1} K_i}{\prod_{i=1}^{n-1} K_n}.
\]

These values also have a geometric interpretation. Let \( B_I \) be the maximal ball that can be embedded in \( E \), and let \( B_O \) be the minimal ball that contains \( E \). Then

\[
K_I = \frac{\text{Vol}(E)}{\text{Vol}(B_I)} \quad \text{and} \quad K_O = \frac{\text{Vol}(B_O)}{\text{Vol}(E)}.
\]

So through an analytic definition of quasiregular mappings, we were able to retain the geometric properties we hoped they would have.

Constructing examples of qr maps in higher dimensions is in general quite difficult. We give a classical example here, originally introduced by Zorich in [60].

**Example 1.2.6** (The Zorich Map, see [4], Example 3.2). *Consider the square*

\[
Q = \{(x_1, x_2) \in \mathbb{R}^2 : |x_1| \leq 1, |x_2| \leq 1\}
\]

*and the upper hemisphere*

\[
H = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1, x_3 \geq 0\}.
\]
Let \( h : Q \to H \) be a bi-Lipschitz map and define a map

\[
Z : Q \times \mathbb{R} \to \mathbb{R}^3, \quad Z(x_1, x_2, x_3) = e^{x_3}h(x_1, x_2).
\]

Then \( Z \) maps \( Q \times \mathbb{R} \), which may be regarded as an “infinite square beam,” to the upper half-space of \( \mathbb{R}^3 \). Repeated reflection along the faces of the square beam and the \( x_1x_2 \)-plane then yields a map \( Z : \mathbb{R}^3 \to \mathbb{R}^3 \). This map turns out to be quasiregular, with distortion bounded in terms of the bi-Lipschitz constant of \( h \). We call this map a Zorich map.

Note first that

\[
Z(x_1 + 4, x_2, x_3) = Z(x_1, x_2 + 4, x_3) = Z(x_1, x_2, x_3)
\]

for all \( (x_1, x_2, x_3) \in \mathbb{R}^3 \). So \( Z \) is doubly periodic. Note also that \( Z \) omits the origin. Moreover, observe the effect of changing the \( x_3 \) value on \( Z(x) \). For \( x_3 = 0 \), \( |Z(x)| = 1 \). As \( x_3 \to \infty \), \( |Z(x)| \to \infty \), and as \( x_3 \to -\infty \), \( |Z(x)| \to 0 \). This is similar to the effect of varying the real input to the complex exponential map. In fact, all of these listed properties are analogous to properties of the complex exponential. For these reasons, the map \( Z \) is regarded as a higher-dimensional version of the exponential map.

Another important property, which stands as an analogy to the Little Picard Theorem (Theorem 1.1.3), is the following.

**Theorem 1.2.7** (Rickman’s Theorem). Let \( n \geq 2 \), and \( K \geq 1 \). There exists \( p = p(n, K) \in \mathbb{N} \) such that if \( f : \mathbb{R}^n \to \mathbb{R}^n \) is a \( K \)-qr map omitting at least \( p \) points in \( \mathbb{R}^n \), then \( f \) is constant.

The following result says that global quasiconformal mappings are global quasisymmetries, which we will need in Chapter 2.

**Theorem 1.2.8** ([27], Theorem 11.14). Let \( n \geq 2 \) and \( K \geq 1 \). There exists an increasing homeomorphism \( \eta : [0, \infty) \to [0, \infty) \) depending only on \( n \) and \( K \) so that if \( f : \mathbb{R}^n \to \mathbb{R}^n \) is
$K$-quasiconformal, then

$$\frac{|f(x) - f(y)|}{|f(x) - f(z)|} \leq \eta \left( \frac{|x - y|}{|x - z|} \right),$$

for all $x, y, z \in \mathbb{R}^n$.

We end this section with the quick observation that holomorphic maps are 1-qr.

### 1.2.1 Quasiregular Dynamics

Before getting into the definitions of the Fatou and Julia sets for quasiregular mappings, it is worth establishing a fundamental dichotomy for such mappings.

**Definition 1.2.9.** Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be quasiregular. If $\lim_{|x| \to \infty} f(x)$ exists (in $\mathbb{R}^n$), then we say $f$ is of polynomial type. If this limit does not exist, then we say $f$ is of transcendental type.

Note that, since quasiregular mappings of polynomial type have a limit at $\infty$, they can be extended to quasiregular mappings from $\overline{\mathbb{R}^n}$ to itself. This is not possible for transcendental type qr mappings, due to the essential singularity at $\infty$. Additionally, qr mappings of polynomial type have finite degree, and qr mappings of transcendental type have infinite degree. Another important property of qr mappings of polynomial type is that their iterates grow in some sense similarly to higher-dimensional power maps, as made precise in the following Lemma.
Lemma 1.2.10 ([20], Lemma 2.3). Let \( h : \mathbb{R}^n \to \mathbb{R}^n \) be a \( K \)-quasiregular mapping of polynomial type of degree \( d > K \). Then there exists \( R_0 > 0 \) and positive constants \( C_1, C_2 \) such that

\[
C_1 q_j((d/K)^{1/(n-1)}) |x|^{(d/K)^{1/(n-1)}} \leq |h^j(x)| \leq C_2 q_j((dK)^{1/(n-1)}) |x|^{(dK)^{1/(n-1)}},
\]

for \( |x| > R_0 \), where \( q_j \) is the polynomial \( q_j(y) = y^{j-1} + y^{j-1} + \cdots + y + 1 \).

Now we wish to define the principal objects of the study of dynamics, the Fatou and Julia sets. The definition of a normal family remains the same, that is a family \( \mathcal{F} \) is normal if every sequence of functions from \( \mathcal{F} \) has a subsequence converging locally uniformly to a continuous function. With this, we then define the Fatou set of a qr map \( f \) as the domain of normality for the sequence \( \{f^k\} \) of iterates, just as before. It is then tempting to define the Julia set as the complement of the Fatou set, but this is not the right way to go. As the following example illustrates, such a definition would in general fail to capture the chaotic nature of the Julia set.

Example 1.2.11. Consider the quasiregular map \( f(re^{i\theta}) = re^{i2\theta} \) on \( \mathbb{C} \). Let \( U \subset \mathbb{C} \) be a connected open set. Observe that iterating \( f \) on \( U \) has the effect of smearing \( U \) onto either a disk or an annulus centered at the origin, depending on whether \( 0 \in U \) or not. In fact, for large \( k \), \( f^k(U) \) wraps \( U \) around the origin multiple times. Taking \( k \) to \( \infty \), the smearing never slows down, and so no subsequence of \( \{f^k\} \) can approach a continuous map. Hence \( F(f) = \emptyset \).

If we defined \( J(f) \) as the complement of \( F(f) \), then \( J(f) \) would be all of \( \mathbb{C} \). However, the behavior of \( f^k \) on \( \mathbb{C} \) fails to have certain properties we would expect \( J(f) \) to exhibit. To be more precise, let \( z_0 \in \mathbb{C} \) and let \( N \) be a small neighborhood of \( z_0 \). We know that all points in \( N \) will behave similarly to \( z_0 \) under iteration of \( f \). In particular, there exists \( R > 0 \) such that \( |f^k(z) - f^k(z_0)| < R \) for all \( z \in \mathbb{C} \) and all \( k \in \mathbb{N} \), where \( R \) depends on \( \max_{z \in N} |z| \). This
means that $J(f)$ fails to have the explosion property (see Theorem 1.1.18) at every point in $\mathbb{C}$. This property is one of the key reasons why $J(f)$ is considered the set of chaotic behavior, so we conclude that defining $J(f)$ as the complement of $F(f)$ is not dynamically satisfying for quasiregular mappings.

To properly define the Julia set, we must define sets of capacity zero. Let $U \subset \mathbb{R}^n$ be open, and $C \subset U$ a non-empty compact set. We call the pair $(U, C)$ a condenser and define its capacity $\text{cap}(U, C)$ by

$$
\text{cap}(U, C) = \inf_u \int_U |\nabla u|^n \, dm,
$$

where the infimum is taken over all $u \in C_0^\infty(U)$ having $u(x) \geq 1$ for all $x \in C$. If $U$ is bounded and $\text{cap}(U, C) = 0$ for some compact $C \subset U$, then $\text{cap}(V, C) = 0$ for any open set $V \supset C$. Such a compact set $C$ is said to be of capacity zero, and we write $\text{cap} C = 0$. Sets of capacity zero are totally disconnected, and, by [46, Corollary VII.1.15], have Hausdorff dimension 0.

Bergweiler extended the definition of the Julia set to the setting of quasiregular mappings of polynomial type in [5] with the help of the following theorem.

**Theorem 1.2.12** ([5], Theorem 1.1). Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be quasiregular of polynomial type. Suppose that $\deg f > K_I(f)$. Then there exists $x \in \mathbb{R}^n$ such that

$$
\text{cap}(\mathbb{R}^n \setminus O^+(U)) = 0 \quad (1.3)
$$

for every neighborhood $U$ of $x$.

**Definition 1.2.13.** Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be quasiregular of polynomial type. Then the Julia set of $f$, denoted $J(f)$, is the set of points $x \in \mathbb{R}^n$ satisfying (1.3) in every neighborhood $U$ of $x$. 
This definition was extended to transcendental type qr mappings by Bergweiler and Nicks in [10]. It was further shown in the same paper that, if $f$ is a transcendental type qr map on $\mathbb{R}^n$, then $J(f) \neq \emptyset$ and $\text{card}(J(f)) = \infty$.

With this definition, we can now see that it is proper regarding the map $f$ in Example 1.2.11 to say that $J(f) = \emptyset$. In fact, in this example we have that $\deg f = K_I(f)$, and hence we cannot weaken the inequality in Theorem 1.2.12.

Before engaging further with the Julia set, let us examine iteration of quasiregular mappings in itself. We have already had iterates appear underlying in the definition of the Fatou set. It is important, however, to observe a fundamental difference between general quasiregular mappings and holomorphic mappings in particular. The iteration of a $K$-qr map has a noteworthy relationship with distortion, as the following example illustrates.

**Example 1.2.14.** Let $f(x + iy) = Kx + iy$ with $K > 1$, as in Example 1.2.1. Recall that $f$ takes circles centered at the origin and maps them to ellipses stretched by a factor of $K$ along the real axis. Note that $f^n(x + iy) = K^n x + iy$. So as $n$ grows, $f^n$ stretches the aforementioned ellipses without bound in the real direction. We can confirm this intuition with the same calculations as before, showing that the distortion of $f^n$ really equals $K^n$. Hence the distortion of a qr-map typically goes up under iteration, and in this case even tends to $\infty$.

In general, if $f$ and $g$ are quasiregular maps with $K(f) = K_1$ and $K(g) = K_2$, then $K(f \circ g) \leq K_1K_2$. Equality is possible, as the previous example illustrates. In fact, for a general $K$-qr map $f$, it is likely that the distortion of the iterates tends to infinity. Quasiregular maps without this property appeared first in [32] due to Iwaniec and Martin, and are of particular dynamical interest. Hence they get their own name.
**Definition 1.2.15.** Let $f$ be a qr-map. We say $f$ is *uniformly quasiregular* (abbreviated *uqr*) if the distortion of the iterates of $f$ is bounded above. If $f$ is injective, we say $f$ is *uniformly quasiconformal* (*uqc*).

Since holomorphic maps are 1-qr, they are also uqr. Conformal mapping are also 1-qr, and hence uqr. However, if $f$ is uqr with distortion $K > 1$, then it is possible to find a Möbius mappings $A$ such that $A \circ f$ is non-uniformly quasiregular. To see this, find a point $x_0$ where $f$ achieves its maximal dilatation $K$. If $A$ is a rotation and scaling map such that $A \circ f(x_0) = x_0$, and $A$ shifts the direction of maximal dilatation to, say, the $x_1$-direction, then iteration of $A \circ f$ at $x_0$ has dilatation that grows without bound. So uqr maps are in some sense unstable, since the composition of two uqr maps can be non-uniformly quasiregular.

One reason uniformly quasiregular maps are dynamically important is the qr analogue of the Montel-Carathéodory Theorem (Theorem 1.1.4), due to Miniowitz ([39]).

**Theorem 1.2.16.** Let $n \in \mathbb{N}$, $n \geq 2$, and $K \geq 1$. Let $a_1, \ldots, a_p \in \mathbb{R}^n$ be distinct point, where $p = p(n, K)$ is the constant from Rickman’s Theorem. Then the family of all $K$-quasiregular maps $f : \mathbb{R}^n \to \mathbb{R}^n \setminus \{a_1, \ldots, a_p\}$ is normal.

If we wish to apply this theorem to a family $\mathcal{F}$ of iterates of a quasiregular map $f$, it is essential that $f$ be uniformly quasiregular to ensure that all maps in $\mathcal{F}$ are $K$-quasiregular with the same $K$.

An important implication of the previous theorem is that uqr maps allow for the original definition of the Julia set. In other words, if $f$ is a uqr map, then defining $J(f)$ as the complement of the domain of normality gives us that $J(f)$ is nonempty and perfect, and that $F(f) \cup J(f) = \mathbb{R}^n$. So studying the dynamics of uqr maps is in general significantly easier than studying the dynamics of non-uniformly qr maps.
As in the complex setting, we can define the escaping and fast escaping sets for a qr map $f$ by

$$I(f) = \{ x \in \mathbb{R}^n : \lim_{k \to \infty} |f^k(x)| = \infty \}$$

and

$$A(f) = \{ x \in \mathbb{R}^n : \text{there exists } P \in \mathbb{N} \text{ such that for all } k \in \mathbb{N}, |f^{k+P}(x)| \geq M^k(R,f) \},$$

respectively, where $R > 0$ is sufficiently large so that $M^k(R,f) \to \infty$ as $k \to \infty$. If $\deg f > K_I$, then the escaping set is well-defined and nonempty ([7], Theorem 1.1). As with holomorphic mappings, we have that $J(f) \subset \partial I(f)$. If $f$ is uqr, then $J(f) = \partial I(f)$. However, there exist qr mappings for which this containment is strict (see Example 7.3 in [10]). In fact, if $f$ is non-uniformly qr and of transcendental type, then $J(f) \subsetneq I(f)$. But, if $f$ has positive lower order, that is if

$$(n - 1) \liminf_{r \to \infty} \frac{\log \log M(r, f)}{\log r} > 0,$$

then $J(f) = \partial A(f)$ ([8], Theorem 1.2). Though it remains an open question whether $J(f) = \partial A(f)$ for a general qr map $f$, we can see that studying the structure of $A(f)$ has important applications to the study of the structure of $J(f)$. In particular, this equality will hold when $f$ is the quasiregular version of a Poincaré Linearizer.
1.2.2 Quasiregular Linearization

A facet of complex dynamics that has been extended into the quasiregular setting is the study of local dynamics and linearization near fixed points. Although the definition of periodic points is identical to Definition 1.1.13, we state it here again for easier reference.

**Definition 1.2.17.** Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a quasiregular map. We say that $x_0 \in \mathbb{R}^n$ is periodic if there exists $p \in \mathbb{N}$ such that $f^p(x_0) = x_0$. The smallest $p$ with this property is called the period of $x_0$. If $p = 1$, we say $x_0$ is fixed.

To classify the behavior of a qr map $f$ near a periodic point $x_0$, it is not in general possible to define the multiplier of $x_0$ using the derivative of $f$. This is because qr maps are only in general differentiable almost everywhere, so in particular $f$ may not be differentiable at $x_0$. One option is to employ the topological classification of fixed points. Note that, for a periodic point of period $p$, we may replace $f$ with $f^p$ so that $x_0$ is a fixed point of $f^p$.

**Definition 1.2.18** (Topological Classification of Fixed Points). Let $n \geq 2$ and $f : \mathbb{R}^n \to \mathbb{R}^n$ be a quasiregular mapping with fixed point $x_0$. Then $x_0$ is said to be

- **topologically attracting** if there is a neighborhood $U$ of $x_0$ such that $f$ is injective on $U$ and $U \supseteq \overline{f(U)}$;

- **topologically repelling** if there is a neighborhood $U$ of $x_0$ such that $f$ is injective on $U$ and $\overline{U} \subset f(U)$.

There are two main drawbacks to this definition. First, there is so far no equivalent to an indifferent fixed point as given in Section 1.1.1. This is not important to the present work, but it is still worth noting. More importantly, this definition lacks the convenience of being able to determine the local behavior of a map by simply checking its derivative. A
workaround to this problem was introduced in [26] in the form of the generalized derivative. For our purposes, we will only need a restricted version of this concept given by Hinkkanen, Martin, and Mayer in [30]. This is because we will be linearizing only uniformly quasiregular maps near their fixed points.

Let $f$ be a uniformly quasiregular map which is locally injective near a fixed point $x_0 \in \mathbb{R}^n$. Define the set $Df(x_0)$ of generalized derivatives of $f$ at $x_0$ by

$$Df(x_0) = \left\{ \lim_{k \to \infty} \frac{f(x_0 + \rho_k x) - f(x_0)}{\rho_k} \right\} \text{ as } \rho_k \to 0 \text{ as } k \to \infty.$$

These limits make sense to discuss since, by [30, Lemma 4.1], $f$ is locally bi-Lipschitz near $x_0$. Moreover, since $f$ is locally injective near $x_0$, the limit function $\varphi \in Df(x_0)$ is uniformly quasiconformal (see [30, Section 4.2]). If $f$ is differentiable at $x_0$, then $Df(x_0)$ contains only the linear map $x \mapsto f'(x_0)x$.

If we can classify the behavior of uniformly quasiconformal maps with fixed points, we can then use this to classify fixed points for uniformly quasiregular maps.

**Definition 1.2.19.** Suppose $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ is a uqc map fixing 0 and $\infty$. Then $\varphi$ is called loxodromic repelling or loxodromic attracting if $\varphi^k(x) \to \infty$ locally uniformly on $\mathbb{R}^n \setminus \{0\}$ or $\varphi^k(x) \to 0$ locally uniformly on $\mathbb{R}^n$, respectively. Otherwise $\varphi$ is called elliptic.

Hinkkanen and Martin established with [29, Theorem 1.2] that loxodromic repelling uqc maps are quasiconformally conjugate to $x \mapsto 2x$. So, if $Df(x_0)$ contains only a loxodromic repelling uqc map, then this is in analogy with $f$ having a multiplier with modulus greater than 1 at $x_0$. For cases when $Df(x_0)$ contains more than one element, we refer to the following Theorem.

**Theorem 1.2.20 ([30], Lemma 4.4).** If one element of $Df(x_0)$ is loxodromic repelling or attracting, then all elements of $Df(x_0)$ are loxodromic repelling or attracting, respectively.
Definition 1.2.21 (Classification of Fixed Points). Let \( x_0 \) be a fixed point of a uqr map \( f \) at which \( f \) is locally injective. Then we call \( x_0 \)

(i) attracting (repelling) if one, and therefore every, element \( \varphi \in Df(x_0) \) is loxodromic attracting (repelling);

(ii) neutral if the elements of \( Df(x_0) \) are elliptic.

If \( f \) fails to be locally injective at \( x_0 \), then we call \( x_0 \) super-attracting.

Case (i) of this definition is equivalent to the topological classification of fixed points in Definition 1.2.18. This new classification then gives way to the quasiregular version of Koenigs linearization (Theorem 1.1.15). Note that here we restrict our attention to repelling fixed points, since then the linearizer is entire.

Theorem 1.2.22 ([30], Theorem 6.3). Let \( f \) be uqr, \( x_0 \) be a repelling fixed point of \( f \), and \( \psi \in Df(x_0) \). Then there exists a qr map of transcendental type \( L : \mathbb{R}^n \rightarrow \mathbb{R}^n \) such that \( f \circ L = L \circ \psi \) with \( L(0) = x_0 \) and \( L \) is locally injective near 0.

The fact that \( L \) is of transcendental type follows by the same argument as in the holomorphic case (see Section 1.1.1). As before, we call \( L \) a Poincaré linearizer of \( f \) at \( x_0 \).

Example 1.2.23. This example is due to Mayer [36], and is well summarized by Bergweiler in [4]. Recall the Zorich map \( Z : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) constructed in Example 1.2.6 as a higher-dimensional version of the exponential map. Let \( p \in \mathbb{N} \) with \( p \geq 2 \). Recall that \( Z(x) = Z(y) \) if and only if there exists integers \( m \) and \( n \) such that \( x - y = (4m, 4n, 0) \). Hence, if \( Z(x) = Z(y) \), then \( Z(px) = Z(py) \). So it is possible to define \( f : Z(\mathbb{R}^3) \rightarrow \mathbb{R}^3 \) by \( f(Z(x)) := Z(px) \). So far, \( f \) is defined on \( \mathbb{R}^3 \setminus \{0\} \), since \( Z \) omits the origin. However, noting that \( Z(x_1, x_2, x_3) \rightarrow 0 \) as \( x_3 \rightarrow -\infty \), and \( Z(x_1, x_2, x_3) \rightarrow \infty \) as \( x_3 \rightarrow \infty \), we can extend \( f \) to a self-map of \( \mathbb{R}^3 \) by setting \( f(0) = 0 \) and \( f(\infty) = \infty \).
For \( k \in \mathbb{N} \), we have that \( f^k(Z(x)) = Z(p^k(x)) \). So, whenever a branch of \( Z^{-1} \) can be defined, we have \( f^k(x) = Z(p^k(Z^{-1}(x))) \). Hence \( f \) is uniformly quasiregular, with \( K(f^k) \leq (K(Z))^2 \).

Considering again the behavior of \( Z \) with regard to the \( x_3 \)-coordinate, we can see that \( f^k(Z(x_1, x_2, x_3)) \to \infty \) if \( x_3 > 0 \), and that \( f^k(Z(x_1, x_2, x_3)) \to 0 \) if \( x_3 < 0 \). Hence \( f^k(x) \to \infty \) if \( |x| > 1 \) and \( f^k(x) \to 0 \) if \( |x| < 1 \). Additionally, if \( |x| = 1 \), then \( |f^k(x)| = 1 \) for all \( k \in \mathbb{N} \). Finally, notice that, by the periodicity of \( Z \), we have that \( \deg f = p^2 \). All this behavior is consistent with the behavior of power maps in the plane. For this reason, maps like \( f \) are called uqr power-type maps.

Note that \( f \) fixes the point \( Z(0) \). It can be shown (see Section 1.9.1 of [35]) that \( Z(0) \) is a repelling fixed point, and, for an appropriate construction of \( Z \), that a valid generalized derivative of \( f \) at \( Z(0) \) is the linear map \( x \mapsto px \). By construction of \( f \), we then have that \( Z \) is a Poincaré linearizer of \( f \) at \( Z(0) \). This is an analogue of the relationship in Example 1.1.16 between complex power maps and the complex exponential map.

The power-type map \( f \) constructed in the previous example is so important to constructing further examples of qr maps that we highlight its existence and properties in the following theorem.

**Theorem 1.2.24** ([36], Theorem 2). For every \( d \in \mathbb{N} \) with \( d > 1 \), there is a uqr map \( f : \mathbb{R}^3 \to \mathbb{R}^3 \) of polynomial type of degree \( d^2 \), with Julia set \( J(f) = S(0,1) \) and whose Fatou set consists of \( B(0,1) \) and \( \mathbb{R}^3 \setminus \overline{B(0,1)} \).

In general, we do not expect a Poincaré linearizer \( L \) to be uqr. In fact, we expect \( L \) to be non-uniformly quasiregular. This, together with being of transcendental type, makes the dynamics of \( L \) quite difficult to study. Even studying \( J(L) \) is not easy. However, recall that
there is a relationship between the Julia set and the fast escaping set for certain qr maps of transcendental type. In particular, if \( L \) has positive lower order, then

\[
J(L) = \partial A(L).
\]

(1.4)

A particular scenario where a Poincaré linearizer has positive lower was given by Fletcher in [20]. Let \( f \) be a \( K \)-uqr mapping of polynomial type with degree \( d > K \), and let \( L \) be a linearizer conjugating \( f \) to the linear map \( T : x \mapsto 2x \). It was shown that \( L \) has positive lower order, and hence (1.4) holds ([20, Theorem 1.6]). But let us examine how the specific choice of \( L \) is limiting. Via a quasiconformal conjugacy described previously, we can take an arbitrary linearizer \( L' \), satisfying \( f \circ L' = L' \circ \psi \) for some \( \psi \in \mathcal{D}f(x_0) \), and replace \( L' \) with \( L = L' \circ g \), where \( g \) is the conjugating qc map. Note that studying the dynamics of \( L \) is not necessarily equivalent to studying the dynamics of \( L' \), since quasiregular dynamics do not in general play well with pre- or post-composition. One aim of this work is to generalize this result of Fletcher to arbitrary Poincaré linearizers.

Establishing that (1.4) holds for linearizers opens the door to studying their fast escaping sets. Recall from Section 1.1.2 that fast escaping sets of transcendental entire maps have had two observed structures, that of hairs, and the spider’s web. The same dichotomy has persisted into higher dimensions with quasiregular mappings, though the definition of a spider’s web needs to be updated.

**Definition 1.2.25** ([6], Definition 1.5). A set \( E \subset \mathbb{R}^n \) is called a spider’s web if \( E \) is connected and there exists a sequence of bounded topologically convex domains \( G_k \) with \( G_k \subset G_{k+1} \) for each \( k \in \mathbb{N} \), \( \partial G_k \subset E \), and \( \bigcup G_k = \mathbb{R}^n \).

Recall that a domain is topologically convex if the only components of its complement are unbounded. Note that \( \mathbb{R}^n \) is itself a spider’s web. It is however impossible for a linearizer
$L$ to have $A(L) = \mathbb{R}^n$, since by [52, Satz 2.5.6], quasiregular maps of transcendental type have infinitely many periodic points.

In [20], Fletcher proved the following partial generalization of the result of Mihaljević-Brandt and Peter (Theorem 1.1.20).

**Theorem 1.2.26** ([20], Theorem 1.13). *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a uqr map of polynomial type whose Julia set $J(f)$ is a tame Cantor set and let $x_0 \in J(f)$ be a repelling fixed point. If $L$ is a linearizer of $f$ at $x_0$, then $A(L)$ is a spider’s web.*

A central aim of this work is to generalize this result as far as possible. To be precise, we will relax the condition of tameness of $J(f)$. Recall that a Cantor set $C \subset \mathbb{R}^n$ is called *tame* if there exists a homeomorphism $h$ of $\mathbb{R}^n$ such that $h(C)$ is the standard Cantor ternary set on the $x_1$-axis of $\mathbb{R}^n$. A Cantor set that is not tame is called *wild*. We will generalize the given result to hold if $J(f)$ is any Cantor set with a defining sequence. This is in particular a property that all Cantor sets in $\mathbb{R}^3$ have. See Section 1.3 for more on Cantor sets.

A useful tool toward generalizing Theorem 1.2.26 is the following, established by Bergweiler, Drasin, and Fletcher.

**Theorem 1.2.27** (see [6], Proposition 6.5). *Let $L$ be a quasiregular mapping of transcendental type. Then $A(L)$ is a spider’s web if there exists a sequence $G_k$ of bounded topologically convex domains such that for all $k \geq 0$,

$$B(0, M^k(R, L)) \subset G_k,$$

and $G_{k+1}$ is contained in a bounded component of $\mathbb{R}^n \setminus L(\partial G_k)$.*

To close out this section, recall from Theorem 1.1.17 that the Julia set of an entire map on $\mathbb{C}$ is the closure of the set of repelling periodic points. It remains an open question
whether this is true for general quasiregular maps. The broadest result toward answering this question is the following.

**Theorem 1.2.28** ([52], Satz 4.3.5). \( \text{Let } f : \mathbb{R}^n \to \mathbb{R}^n \text{ be uniformly quasiregular, and suppose that } F(f) \text{ has at least two components. Then the repelling periodic points of } f \text{ are dense in } J(f). \)

In pursuing a generalization of Theorem 1.2.26, we will be working with uqr maps whose Julia sets are Cantor sets. Such maps do not meet the conditions of Theorem 1.2.28, since Cantor sets have connected complement (see Theorem 1.3.6 for more details). A partial result toward extending Theorem 1.2.28 to uqr maps with Cantor sets as Julia sets was given by Fletcher.

**Theorem 1.2.29** ([20], Theorem 1.11). \( \text{Let } f : \mathbb{R}^n \to \mathbb{R}^n \text{ be a uniformly quasiregular mapping of polynomial type and suppose that } J(f) \text{ is a tame Cantor set. Then the repelling periodic points of } f \text{ are dense in } J(f). \)

One further aim of this work is to relax the condition of tameness of \( J(f) \) from the previous theorem.

### 1.3 Cantor Sets

One of the main results of this work concerns the role that Cantor sets play in the study of dynamics. The goal of this section is to give an overview of the topological and dynamical properties of Cantor sets. For a more detailed discussion of the topological properties, we refer to Moise’s book [40].
Let $\mathcal{C}$ be the standard Cantor ternary set, that is

$$\mathcal{C} = \left\{ \sum_{i=1}^{\infty} \frac{a_i}{3^i} : a_i \in \{0, 2\} \right\} \subset [0, 1].$$

This set is typically studied in introductory analysis courses, as it has many interesting analytical properties (see [48]). It is for instance non-empty, uncountable, and has Lebesgue measure 0. Some notable topological properties are that $\mathcal{C}$ is a compact totally disconnected metric space with no isolated points. By totally disconnected, we mean that every component of $\mathcal{C}$ is a singleton.

The Cantor ternary set $\mathcal{C}$ is often thought of as arising by removing open middle thirds from the interval $[0, 1]$ in an iterative process. Alternatively, this can be regarded as taking the intersection of closed “outer thirds” in the interval $[0, 1]$. To be more precise, let $C_1 = [0, 1]$, $C_2 = [0, 1/3] \cup [2/3, 1]$, $C_3 = [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1]$, and so on. Then $\mathcal{C} = \bigcap C_i$.

Consider taking a similar iterative intersection, but instead of using unions of disjoint closed intervals in $\mathbb{R}$, consider taking unions of disjoint solid squares in $\mathbb{R}^2$. Let

$$D_1 = [0, 1] \times [0, 1]$$
$$D_2 = ([0, 1/3] \times [0, 1/3]) \cup ([0, 1/3] \times [2/3, 1]) \cup ([2/3, 1] \times [0, 1/3]) \cup ([2/3, 1] \times [2/3, 1])$$
$$:$$

In other words, each $D_i$ is the union of all the possible squares that result from crossing the corresponding line segments in the definition of $\mathcal{C}$. We then let $\mathcal{D} = \bigcap D_i$. This resulting set is sometimes called Cantor dust. Similar to $\mathcal{C}$, the set $\mathcal{D}$ is a non-empty compact totally disconnected metric space with no isolated points. These properties are used to topologically
characterize sets of this type. This was first done by Brouwer in [14], and is given in modern
terminology for example in Moise’s book [40].

**Theorem 1.3.1** (see [40], Theorem 12.8). *Every non-empty, compact, totally disconnected
metric space without isolated points is homeomorphic to \( C \), the standard Cantor middle thirds
set.*

Any topological space having the properties listed in Theorem 1.3.1 is called a *Cantor set*,
or in some literature a *Cantor space*.

The previous theorem means that, as topological spaces, Cantor sets are incredibly rigid.
It is the embedding of Cantor sets into Euclidean space where all the interesting topology
happens.

**Definition 1.3.2.** Let \( n \geq 1 \), and let \( C \) and \( D \) be Cantor sets in \( \mathbb{R}^n \), with \( f : C \to D \) a
homeomorphism. If \( f \) can be extended to a homeomorphism \( F : \mathbb{R}^n \to \mathbb{R}^n \), then we say
that \( C \) and \( D \) are *equivalently embedded* in \( \mathbb{R}^n \). Alternately we may say that \( C \) and \( D \) are
*ambiently homeomorphic* in \( \mathbb{R}^n \).

**Theorem 1.3.3** (see [40], Theorem 13.7). *Let \( C \) and \( D \) be Cantor sets in \( \mathbb{R}^2 \). Then \( C \) and
\( D \) are ambiently homeomorphic.*

A Cantor set \( C \subset \mathbb{R}^n \) that is ambiently homeomorphic to the standard Cantor ternary
set in \( \mathbb{R}^n \) is called *tame*. The previous theorem can thus be reformulated by saying that all
Cantor sets in \( \mathbb{R}^2 \) (and hence in \( \mathbb{R} \)) are tame. It is in \( \mathbb{R}^3 \) that we find our first example of a
so-called *wild* Cantor set. This example is due to Antoine [1].

**Example 1.3.4.** *Let \( A_0 \subset \mathbb{R}^3 \) be a solid torus, and let \( m \geq 4 \) be a positive even integer.
Then choose mutually disjoint solid tori \( A_{1,1}, \ldots, A_{1,m} \) contained in the interior of \( A_0 \) with
the following properties.*
• The tori $A_{1,i}$ and $A_{1,j}$ are linked if and only if $i - j \equiv \pm 1 \pmod{m}$.

• When linked, the tori $A_{1,i}$ and $A_{1,j}$ form a Hopf link.

Now fix homeomorphisms $\phi_j : A_0 \to A_{1,j}$ for $j \in \{1, \ldots, m\}$, and define

$$A_1 = \bigcup_{j=1}^{m} A_{1,j} = \bigcup_{j=1}^{m} \phi_j(A_0).$$

In other words, $A_1$ is a ‘chain’ of linked solid tori. The winding number of $A_1$ with respect to the torus $A_0$ is assumed to equal 1. We then inductively define

$$A_{k+1} = \bigcup_{j=1}^{m} \phi_j(A_k),$$

for $k \geq 1$. Note that $A_k$ consists of $m^k$ tori $A_{k,1}, \ldots, A_{k,m^k}$. See Figure 1.2 for an illustration.

![Figure 1.2: Antoine's necklace construction.](image)

Finally, an Antoine’s necklace of multiplicity $m$ is defined as

$$\mathcal{A} = \bigcap_{k=1}^{\infty} A_k.$$
In order for \( A \) to be Cantor set, we require that the maximum diameter of any torus in \( A_k \), call it \( d_k \), satisfy \( d_k \to 0 \) as \( k \to \infty \).

**Theorem 1.3.5.** The Cantor set \( A \subset \mathbb{R}^3 \) from Example 1.3.4, also known as Antoine’s necklace, is a wild Cantor set. That is, \( A \) is not ambiently homeomorphic to the standard Cantor ternary set in \( \mathbb{R}^3 \).

Antoine himself proved this result in [1] by showing that no points in \( A \) can be separated by a sphere. A more modern proof using homotopy theory can be found in Rolfsen’s book [47]. The basic idea is that the complement of \( A \) is not simply connected.

To intuitively see this, let \( L \) be a simple closed meridional curve on \( \partial A_0 \). If \( h : \mathbb{R}^3 \to \mathbb{R}^3 \) is a homeomorphism satisfying \( h(A) = C \), then \( h(L) \) is a loop in \( \mathbb{R}^3 \setminus C \), which is clearly contractible in \( \mathbb{R}^3 \setminus C \). Hence \( L \) must be contractible in \( \mathbb{R}^3 \setminus A \). However, to contract \( L \), it must pass through \( A_1 \). To pass through \( A_1 \), \( L \) must in turn pass through \( A_2 \). Continuing this process, we see that \( L \) cannot be contracted in \( \mathbb{R}^3 \setminus A \), contradicting the assumption that \( A \) and \( C \) are ambiently homeomorphic.

It is important to note something about the complexity of Cantor sets. If \( C \) is a Cantor set in \( \mathbb{R}^n \) such that \( \mathbb{R}^n \setminus C \) is not simply connected, then \( C \) is wild. The converse however is not true. For a classical example of a wild Cantor set in \( \mathbb{R}^3 \) with simply connected complement, see [53].

It is also worth noting the following property of Cantor sets. We were unable to find a direct reference for this result, but Thurston [55] gives an argument involving the Alexander Duality Theorem and Čech cohomology.

**Theorem 1.3.6.** Let \( n \geq 2 \) and let \( C \subset S^n \cong \mathbb{R}^n \cup \{\infty\} \) be a Cantor set. Then \( S^n \setminus C \) is connected.

By tweaking the construction in Example 1.3.4, it is possible to generate many more inequivalently embedded Cantor sets in \( \mathbb{R}^3 \). Sher established in [51] that there are uncountably
many inequivalently embedded Antoine’s necklaces in $\mathbb{R}^3$. One need only vary the number of tori in a given chain, or else the winding number of a given chain, and one obtains a new embedding. Hence the Antoine construction represents an entire class of Cantor sets.

In addition to being topologically very interesting, wild Cantor sets have made an appearance in quasiregular dynamics. Fletcher and Wu constructed in [23] a uqr map $f : \mathbb{R}^3 \to \mathbb{R}^3$ of polynomial type having an Antoine’s necklace for $J(f)$. We will go over this example in detail, but first we need to tweak the construction of $\mathcal{A}$ in Example 1.3.4. We require here that $\mathcal{A}$ be geometrically self-similar, that is there must exist a finite number of similarities, call them $\varphi_j$, such that $\mathcal{A} = \bigcup \varphi_j(\mathcal{A})$.

**Example 1.3.7.** Let $m$ be a sufficiently large even integer. Let $p_1, \ldots, p_m$ be $m$ equally spaced points on the unit circle $\tau_0 = \{(x_1, x_2, 0) : x_1^2 + x_2^2 = 1\}$, and let $T_0$ be the solid torus $\{x \in \mathbb{R}^3 : d(x, \tau_0) \leq 1/100\}$ with core $\tau_0$. Fix circles $\tau_j$, $j \in \{1, \ldots, m\}$, in $\mathbb{R}^3$ with centers $p_j$ and radius $4/m$ so that

- $\tau_i$ and $\tau_j$ are linked in $\mathbb{R}^3$ if and only if $i - j \equiv \pm 1 \pmod{m}$;
- $\rho(\tau_j) = \tau_{j+2}$ for $j \in \{1, \ldots, m - 2\}$, $\rho(\tau_{m-1}) = \tau_1$, and $\rho(\tau_m) = \tau_2$, where $\rho$ is the rotation about the $x_3$-axis by an angle $4\pi/m$;
- $\tau_1$ and $\tau_m$ are linked with the $x_1$-axis such that a rotation by the angle $\pi$ about the $x_1$-axis sends $\tau_1$ onto $\tau_m$ and vice versa, and the planes containing $\tau_1$ and $\tau_m$ are orthogonal.

Now we can fix $m$ sense preserving similarities $\varphi_j$, $j \in \{1, \ldots, m\}$, of $\mathbb{R}^3$ such that $\varphi_j(\tau_0) = \tau_j$. Define $T_j = \varphi_j(T_0)$. Then the $T_j$ are mutually disjoint solid tori contained in the interior of $T_0$, satisfying the linking properties of Example 1.3.4. Set $A_0 = T_0$ and $A_{1,j} = T_j$ for all $j$, and repeat the same inductive process as before. We then obtain a geometrically self-similar Antoine’s necklace $\mathcal{A}$. 
A quick note on the phrase ‘sufficiently large’ regarding the number $m$. The larger the value of $m$, the smaller the tori $A_{1,1}, \ldots, A_{1,m}$ are, since they have to fit inside the interior of $A_0$. It had long been known that $m$ could not be smaller than 20 in order to construct a geometrically self-similar Antoine’s necklace. Željko gave in [59] an explicit construction of Antoine’s necklace where $m = 20$.

To construct a uqr map $f$ having $J(f) = A$, let $m = d^2$ be the square of an even integer sufficiently large so that the construction of $A$ from Example 1.3.7 is accomplished with conformal similarities $\varphi_1, \ldots, \varphi_m$. Set $B_0 = B(0, 2)$, $B_{-1} = B(0, 2^d)$, and write $\mathbb{R}^3$ as two disjoint unions, one for the domain and codomain of $f$, respectively:

$$\mathbb{R}^3 = A_1 \cup (A_0 \setminus A_1) \cup (B_0 \setminus A_0) \cup (\mathbb{R}^3 \setminus B_0)$$

and

$$\mathbb{R}^3 = A_0 \cup (B_0 \setminus A_0) \cup (B_{-1} \setminus B_0) \cup (\mathbb{R}^3 \setminus B_{-1}).$$

Define $f$ in four pieces.

- Set $f : \overline{A_0 \setminus A_1} \to \overline{B_0 \setminus A_0}$ to be a particular degree $m$ branched covering map $F$, detailed in Example 1.3.9 below.

- Extend $f$ to $A_1$ by defining $f|_{A_1}$ to be $\varphi_j^{-1} : A_{1,j} \to A_0$ for every $j \in \{1, \ldots, m\}$. Again, see Example 1.3.9 for details.

- Define $f : \mathbb{R}^3 \setminus \text{Int}(B_0) \to \mathbb{R}^3 \setminus \text{Int}(B_{-1})$ to be the restriction of $g$ to $\mathbb{R}^3 \setminus \text{Int}(B_0)$, where $g$ is the uqr power-type map of degree $m$ from Theorem 1.2.24. Note that $g$ does indeed map $S(0, 2)$ onto $S(0, 2^d)$. 
• Extend $f$ to $B_0 \setminus \text{Int}(A_0)$ via the Berstein and Edmonds extension theorem, Theorem 1.3.8 below. It is understood here that $C^1$-triangulation has been carried out on $B_0 \setminus \text{Int}(A_0)$ and $B_{-1} \setminus \text{Int}(B_0)$ before applying Theorem 1.3.8.

Fletcher and Wu then prove that the map $f$ has the following properties.

(i) The map $f$ is uqr of polynomial type [23, Lemma 5.1].

(ii) The Julia set of $f$ is equal to $A$ [23, Lemma 5.2].

(iii) The Julia set $J(f)$ is the closure of the set of repelling periodic points [23, Lemma 5.3].

Before elaborating on the construction of $f$, we highlight the consequences of its properties. Property (iii) in particular means that $f$ has a repelling periodic point. Hence $f$ can be linearized there via a linearizer $L$. Property (ii) then shows that an extension of Theorem 1.2.26 would apply to the map $f$ (and all maps constructed by varying the value of $m$).

Here we give the Berstein and Edmonds extension theorem.

**Theorem 1.3.8** ([11], Theorem 6.2). *Let $W$ be a connected, compact, oriented PL 3-manifold in some $\mathbb{R}^n$ whose boundary $\partial W$ consists of two components $M_0$ and $M_1$ with the induced orientation. Let $W' = N \setminus (\text{int}(B_0) \cup \text{int}(B_1))$ be an oriented PL 3-sphere $N$ in $\mathbb{R}^4$ with two disjoint closed polyhedral 3-balls removed, and have the induced orientation on its boundary. Suppose that $\phi_i : M_i^2 \rightarrow \partial B_i$ is a sense-preserving oriented branched covering of degree $n \geq 3$, for each $i = 0, 1$. Then there exists a sense-preserving PL branched cover $\phi : W \rightarrow W'$ of degree $n$ that extend $\phi_0$ and $\phi_1$.*

Fletcher and Wu use this theorem in an extended form. Specifically, through the work of Heinonen and Rickman [28] and Pankka, Rajala, and Wu [43], this theorem is known to be true for degree $d \geq 3$ branched covers $\partial W \rightarrow \partial W'$ between boundaries of connected,
compact, oriented 3-manifolds $W$ and $W'$, when $\partial W$ has $p \geq 2$ connected components and $W'$ is a PL 3-sphere with the interiors of $p$ disjoint closed 3-balls removed.

With this in place, we need only address the branched cover in the first step of the definition of $f$.

**Example 1.3.9.** With the notation established in Example 1.3.7, we will construct a degree $m$ branched covering map

$$F : T_0 \setminus \text{int} \left( \bigcup_{j=1}^{m} T_j \right) \to B(0,2) \setminus \text{int}(T_0)$$

such that $F|\partial T_j \to \partial T_0 = \varphi_j^{-1}$. Let $\omega : \mathbb{R}^3 \to \mathbb{R}^3$ be the degree $m/2$ winding map

$$\omega : (r, \theta, x_3) \mapsto (r, \theta m/2, x_3).$$

Then $\omega : T_0 \to T_0$ is an unbranched covering map that maps all $T_j$ with odd indices to $\omega(T_1)$ and all $T_j$ with even indices to $\omega(T_2)$. Furthermore, we have that $\omega(T_1)$ and $\omega(T_2)$ are linked twice, such that a rotation about the $x_1$-axis by angle $\pi$ maps $\omega(T_1)$ onto $\omega(T_2)$ and vice versa. Another way of saying this is that $\omega(T_1)$ and $\omega(T_2)$ are symmetric under the involution

$$\iota : (x_1, x_2, x_3) \mapsto (x_1, -x_2, -x_3).$$

Hence $\iota(\omega(T_1)) = \omega(T_2)$ and $\iota(\omega(T_2)) = \omega(T_1)$.

The quotient map $q : T_0 \to T_0/\langle \iota \rangle$ is then a degree 2 sense preserving map having the property that $q(\omega(T_1)) = q(\omega(T_2))$ is a 1-holed torus unknotted in the 3-cell $q(T_0)$. To ensure that $q$ has bounded linear distortion, which ensures that the Berstein and Edmonds extension theorem holds, we consider a PL version of $q$. For more detail on all things piecewise linear, we refer to Munkres’ book [41].
We give $T_0$ a $C^1$-triangulation $g : |U| \to T_0$ by a simplicial complex $U$ in $\mathbb{R}^3$, and we identify $q(T_0)$ with a simplicial complex $V$ in $\mathbb{R}^3$ induced by a homeomorphism $h : |V| \to B(0,2)$ satisfying

- $U$ respects the involution $\iota$,
- $\omega(T_1) \cup \omega(T_2)$ is a subcomplex of $U$,
- $q(\omega(T_1))$ is a subcomplex of $V$,
- the map $q : T_0 \to q(T_0)$ is simplicial,
- after possible refinement, $q \circ \omega \circ \varphi|_{T_0}$ are simplicial and pairwise ambient isotopic.

Note further that $q \circ \omega \circ \varphi_1(T_0)$ and $h^{-1}(T_0)$ are ambient isotopic in $V$. Hence, by an isotopy extension theorem (see [31, p.136]), there exists a homeomorphism $\zeta : \overline{B(0,2)} \to V$ such that $\zeta|_{T_0} = q \circ \omega \circ \varphi_1^{-1}$ and $\zeta = h^{-1}$ on $\partial B(0,2)$, and that $\zeta \circ h : V \to V$ is PL. Finally, $\zeta^{-1} \circ q \circ \omega|_{T_j} = \varphi_j^{-1}$ for all $j \in \{1, \ldots, m\}$. Set $F = \zeta^{-1} \circ q \circ \omega$.

So we see that there is a class of uqr maps to which an extension of Theorem 1.2.26 to wild Cantor sets can apply. This class is however quite small so far. One aim of this work is to construct new examples of uqr maps of polynomial type having a wild Cantor set as their Julia set. This will be accomplished by applying the same methods from [23] to a Cantor set having a higher genus than Antoine’s necklace. To understand the genus of a Cantor set, we must first consider defining sequences.

### 1.3.1 Defining Sequences

The construction of the standard Cantor ternary set $C$ in $\mathbb{R}$ can be accomplished by taking an intersection of nested unions of closed intervals, as described in the previous section. If
we regard $C$ as a subset of the $x_1$-axis in $\mathbb{R}^3$, this construction can be accomplished by way of intersecting closed balls. Similarly, Antoine’s necklace $A$ is usually constructed using intersections of nested chains of linked 1-holed tori.

It is tempting to conclude that the manner of construction of two Cantor sets $C$ and $D$ in $\mathbb{R}^n$ determines whether $C$ and $D$ are equivalently embedded. However, this is not in general the case. Let us consider an alternate way of constructing $C$ in $\mathbb{R}^3$.

**Example 1.3.10.** Consider $C \subset \mathbb{R}^3$ constructed as follows.

- Let $x_0 = (1/2, 0, 0)$, and set $C_0 = \overline{B(x_0, 1/2)}$.
- Define $\varphi_1, \varphi_2 : \mathbb{R}^3 \to \mathbb{R}^3$ by
  \[
  \varphi_1(x) = \frac{1}{3}x \quad \text{and} \quad \varphi_2(x) = \frac{1}{3}x + \left(\frac{2}{3}, 0, 0\right),
  \]
  respectively, for all $x \in \mathbb{R}^3$.
- Inductively define $C_k = \varphi_1(C_{k-1}) \cup \varphi_2(C_{k-1})$ for all $k \geq 1$.
- Set $C = \bigcap_{k=1}^\infty C_k$.

Now let $D_0$ be $C_0$ with a small handle attached, as in Figure 1.3. Inductively define $D_k = \varphi_1(D_{k-1}) \cup \varphi_2(D_{k-1})$ for all $k \geq 1$. Then each $D_k$ is the union of $2^k$ disjoint solid 1-holed tori.
tori. More specifically, each torus in $D_k$ is exactly the corresponding ball from $C_k$, but with a handle attached. These handles do not affect the intersection of $D_k$ with $D_{k+1}$, and so

$$\bigcap_{k=1}^{\infty} D_k = \bigcap_{k=1}^{\infty} C_k = C.$$ 

So it is possible to construct $C$ as an intersection of solid 1-holed tori. This would naively suggest that $C$ and Antoine’s necklace $A$ might be similarly embedded in $\mathbb{R}^3$. However, we know that $C$ is tame, where $A$ is wild.

In fact, we can add as many superfluous handles as we want, and obtain infinitely many alternate constructions of $C$ in $\mathbb{R}^3$.

The takeaway from the previous example is that we cannot necessarily draw conclusions about the way a Cantor set $C$ is embedded in $\mathbb{R}^n$ simply from the particular construction chosen.

**Definition 1.3.11** (see [51], Section 2). A *defining sequence* for a Cantor set $C \subset \mathbb{R}^3$ is a sequence $(M_i)$ of compact 3-manifolds $M_i$ with boundary such that

(i) each $M_i$ consists of disjoint polyhedral cubes with handles,

(ii) $M_{i+1} \subset \text{Int}(M_i)$ for each $i$, and

(iii) $C = \bigcap_i M_i$.

We denote the set of all defining sequences for $C$ by $\mathcal{D}(C)$.

Using different terminology, Armentrout proved in [2] that every Cantor set in $\mathbb{R}^3$ has a defining sequence. However, from Example 1.3.10, it is also clear that every Cantor set $C \subset \mathbb{R}^3$ has infinitely many defining sequences, each able to consist of manifolds with different topological properties. One may consider the usual defining sequences for the Cantor sets $C$
and $\mathcal{A}$ in $\mathbb{R}^3$ to be in some sense ‘canonical,’ but given a Cantor set $C \subset \mathbb{R}^3$ with defining sequence $(M_i)$, there is so far no obvious connection between the manifolds in $(M_i)$ and the properties of $C$.

But there are some observations worth making here. Since $M_{i+1} \subset \text{Int}(M_i)$, and $C = \bigcap_i M_i$, we see that the manifolds that make up $M_i$ separate points in $C$. That is, given $x, y \in C$, there exists $i \in \mathbb{N}$ and a component $M$ of $M_i$ such that $x \in M$ and $y \notin M$, with $\partial M \cap C = \emptyset$. Recall that Antoine’s proof of the wildness of $\mathcal{A}$ was to show that no two points in $\mathcal{A}$ can be separated by a sphere. This implies that there does not exist any defining sequence of $\mathcal{A}$ whose components are all 3-balls. Here we are beginning to see the connection between defining sequences and inequivalently embedded Cantor sets.

One formalization of this connection is through Sher’s definition of equivalent defining sequences (see [51, Section 2]). If $(C_i)$ and $(D_i)$ are two defining sequences for the Cantor set $C \subset \mathbb{R}^3$, we say they are equivalent, and write $(C_i) \sim (D_i)$, if for each $i \in \mathbb{N}$ there exists a homeomorphism $h_i : \mathbb{R}^3 \to \mathbb{R}^3$ such that $h_{i+1}|(\mathbb{R}^3 \setminus C_i) = h_i|(\mathbb{R}^3 \setminus C_i)$ and $h_i(M_i) = N_i$. With this definition, it is not difficult to prove the following theorem.

**Theorem 1.3.12** ([51], Theorem 1). Let $C$ and $D$ be Cantor sets in $\mathbb{R}^3$. Then $C$ and $D$ are ambiently homeomorphic in $\mathbb{R}^3$ if and only if there exist defining sequences $(C_i)$ and $(D_i)$ of $C$ and $D$ respectively with $(C_i) \sim (D_i)$.

While equivalence of defining sequences is certainly a useful concept, it doesn’t make determining whether two Cantor sets are ambiently homeomorphic much easier. This is because any argument trying to rule out ambient homeomorphism would have to be made on the level of every defining sequence for both Cantor sets in question. Fortunately, defining sequences can be used to construct an invariant for Cantor sets: the genus.
1.3.2 The Genus of a Cantor Set

Let $M$ be a cube with handles in $\mathbb{R}^3$. Denote by $g(M)$ the number of handles of $M$. For a disjoint union of cubes with handles $M = \bigcup_{\lambda \in \Lambda} M_\lambda$, we define $g(M) = \sup\{g(M_\lambda) : \lambda \in \Lambda\}$.

**Definition 1.3.13** (see [58], p. 350). Let $(M_i)$ be a defining sequence for a Cantor set $C \subset \mathbb{R}^3$. Define

$$g(C; (M_i)) = \sup\{g(M_i) : i \geq 0\}.$$ 

We then define the genus of the Cantor set $X$ as

$$g(C) = \inf\{g(C; (M_i)) : (M_i) \in \mathcal{D}(C)\}.$$ 

Now let $x \in C$. Denote by $M_i^x$ the union of all the components of $M_i$ containing $x$. Similar to above, define

$$g_x(C; (M_i)) = \sup\{g(M_i^x) : i \geq 0\}.$$ 

Then define the local genus of $C$ at the point $x$ by

$$g_x(C) = \inf\{g_x(C; (M_i)) : (M_i) \in \mathcal{D}(C)\}.$$ 

It follows from Theorem 1.3.12 that, if $C$ and $D$ are ambiently homeomorphic Cantor sets in $\mathbb{R}^3$, we must have that $g(C) = g(D)$. Hence determining the genus of a Cantor set can be a useful shortcut to see that two Cantor sets are not ambiently homeomorphic.

Here we list some properties and observation concerning the genus of a Cantor set.

(i) If $C \subset \mathbb{R}^3$ is a Cantor set, and $x \in C$, then $g_x(C) \leq g(C)$ (see [58, Lemma 1]).

(ii) By a theorem of Bing (see [12]), a Cantor set $C \subset \mathbb{R}^3$ is tame if and only if $g(C) = 0$. 
(iii) By a theorem of Osborne (see [42]), a Cantor set \( C \subset \mathbb{R}^3 \) is tame if and only if \( g_x(C) = 0 \) for all \( x \in C \).

We comment on some of these properties. By (ii), we see that \( g(\mathcal{A}) \geq 1 \), where \( \mathcal{A} \) is any Antoine’s necklace in \( \mathbb{R}^3 \). This follows from the fact that \( \mathcal{A} \) is wild. Note also that the usual defining sequence for \( \mathcal{A} \) proves that \( g(\mathcal{A}) \leq 1 \), and hence \( g(\mathcal{A}) = 1 \). This is worth noting in and of itself, that, if a Cantor set \( C \subset \mathbb{R}^3 \) has a defining sequence where the maximum genus of all manifolds involved equals \( g \in \mathbb{N} \), then we must have that \( g(C) \leq g \). This together with (i) means that there are generally two steps to establishing the genus of a given Cantor set \( C \). An upper bound for \( g(C) \) is given by a particular defining sequence. A lower bound can then be found either by exhibiting a point in \( x \in C \) with the desired local genus, or else by proving that there exist no defining sequences without manifolds of at least the desired genus.

The first example of a Cantor set of genus 2 is due to Babich. Before going over the example, we state the following definition.

**Definition 1.3.14** ([3], Definition 2). Let \( C \subset \mathbb{R}^3 \) be a Cantor set. Suppose that for each point \( x \in C \) and each \( \epsilon > 0 \), there is a \( \delta > 0 \) such that, for each map \( f : S^1 \to \text{Int}(B(x, \delta)) \setminus C \), there is a map \( F : \mathbb{B}^2 \to \text{Int}(B(x, \epsilon)) \) such that \( F|\partial \mathbb{B}^2 = f \) and \( F^{-1}(C) \) is finite. Then \( C \) is scrawny.

**Example 1.3.15** (see [3]). Let \( B_0 \) be a solid 2-holed torus, that is a cube with two handles. Let \( B_{1,1}, \ldots, B_{1,9} \) be solid 2-holed tori contained in \( \text{Int}(B_0) \) and arranged as in Figure 1.4.

Fix homeomorphisms \( \phi_j : B_0 \to B_{1,j} \) for \( j \in \{1, \ldots, 9\} \), and define

\[
B_1 = \bigcup_{j=1}^{9} B_{1,j} = \bigcup_{j=1}^{9} \phi_j(B_0).
\]
Figure 1.4: A chain of 2-holed tori. See also Figure 1 in [3].

Inductively define \( B_k = \bigcup_{j=1}^{n} \phi_j(B_{k-1}) \) for all \( k \geq 2 \), and set

\[
\mathcal{B} = \bigcap_{k=0}^{\infty} B_k.
\]

Then \( \mathcal{B} \) is a Cantor set, which Babich calls Antoine’s eyeglasses. It can be shown using the same methods as for \( \mathcal{A} \) that \( \mathcal{B} \) is wild. Note also that every component of the defining sequence \( (B_k) \) is a solid 2-holed torus, and hence \( g(\mathcal{B}) \leq 2 \).

To prove that \( g(\mathcal{B}) \geq 2 \), note first that \( \mathcal{B} \) is scrawny. This can be seen by studying Figure 1.4. Babich goes on to prove that no scrawny Cantor set has a defining sequence consisting of 1-holed tori. Hence \( g(\mathcal{B}) = 2 \).

**Example 1.3.16** (see [58]). For each \( r \in \mathbb{N} \), there exists a Cantor set \( \mathcal{Z} \subset \mathbb{R}^3 \) with \( g(\mathcal{Z}) = r \). Since tame Cantor sets have genus 0, and \( g(\mathcal{A}) = 1 \), we may assume that \( r \geq 2 \).

Construct a defining sequence \( (\mathcal{Z}_i) \) for \( \mathcal{Z} \) as follows. Fix a point \( x_0 \in \mathbb{R}^3 \) and let \( \mathcal{Z}_0 \) be a cube with \( r \) handles containing \( x_0 \) in its interior. The manifold \( \mathcal{Z}_1 \) will have \( 5r + 1 \) components. One of them, call it \( \mathcal{Z}_1^0 \), is a cube with \( r \) handles having \( x_0 \) in its interior. Each handle of \( \mathcal{Z}_1^0 \) is linked with a chain of five 1-holed tori that runs along the core of some handle in \( \mathcal{Z}_0 \). See Figure 1.5 for details. (Only two handles of \( \mathcal{Z}_0 \) are shown. The remaining
$r - 2$ handles are in the dotted part in the middle.) The union of $Z_1^0$ with these chains is then $Z_1$.

![Diagram of a genus $r$ Cantor set]

Figure 1.5: A defining sequence for a genus $r$ Cantor set. See also Figure 1 in [58].

To construct the manifold $Z_2$, first apply the Antoine’s necklace construction inside each 1-holed torus component of $Z_1$. That is, form a chain of linked 1-holed tori inside each 1-holed torus of $Z_1$. Then construct $5r + 1$ components of $Z_2$ inside $Z_1^0$, embedded the same way $Z_1$ is embedded in $Z_0$. Repeat this process inductively. Then $Z = \bigcap_k Z_k$.

From the defining sequence it is clear that $g(Z) \leq r$. Željko is then able to prove that $g_{\pi_0}(Z) \geq r$, and hence that $g(Z) = r$. See [58, Theorem 5] for more details.

It is unlikely that either of the Cantor sets $B$ or $Z$ are realizable as the Julia set of a uqr map in $\mathbb{R}^3$, regardless of the genus of the latter. In particular, we can see that an analogue of the Fletcher-Wu construction is not possible for either Cantor set. The set $Z$ is not geometrically self similar, and, even though $B$ could be made geometrically self similar by increasing the number (and hence decreasing the size) of the components of $B_1$, it is the branched covering map procedure from Example 1.3.9 that fails here. An involution about the central point of symmetry of $B_1$ would cause the central 2-holed torus to lose a hole.

One aim of this work is to construct a Cantor set $X$ of genus 2 in $\mathbb{R}^3$, along with a uqr map $f$ of polynomial type having $J(f) = X$. In particular, $X$ will be the first explicit construction of a geometrically self-similar Cantor set of genus 2. The construction of $X$
takes place in Chapter 3, and Chapter 4 is concerned with proving that $g(\mathcal{X}) = 2$. Finally, Chapter 5 contains the construction of the map $f$.

We end this section with a cautionary example. Consider the Cantor set $C \subset \mathbb{R}^3$ with a defining sequence having $C_0$ a solid 2-holed torus and $C_1$ a chain of linked solid 2-holed tori arranged as in Figure 1.6. Clearly $g(C) \leq 2$. However, it is possible to construct an alternate defining sequence, showing that $g(C) \leq 1$. Replace $C_0$ with a 1-holed torus $T_0$, as in Figure 1.7. Since every component of $C_1$ contains a chain of 2-holed tori similar to $C_1$ as part of $C_2$, this same replacement can be made with every component of $C_1$. And so on, all the way down the defining sequence. This example serves to reiterate that it is not sufficient for establishing the genus of a Cantor set to simply provide a convenient defining sequence.

![Figure 1.6: A chain of 2-holed tori. The $C_1$ tori in blue are drawn up to homotopy type for simplicity.](image)

![Figure 1.7: The chain $C_1$ with surrounding 1-holed torus $T_0$.](image)
1.3.3 Higher Dimensions

We comment here on Cantor sets and dynamics in dimensions higher than 3. Let $n \geq 4$. There are many known ways of constructing wild Cantor sets in $\mathbb{R}^n$, see for example the constructions of Blankinship [13] and of DeGryse and Osborne [17]. However, given a known Cantor set $C \subset \mathbb{R}^n$, it is in general unlikely that $C$ is realizable as the Julia set of uqr map.

This is due to a number of difficulties. First, it is not possible to simply embed an existing wild Cantor set from $\mathbb{R}^3$ into $\mathbb{R}^n$ with $n \geq 4$ and extend its associated uqr map. Let us be more precise here. Suppose that $C \subset \mathbb{R}^3$ is a wild Cantor set. Now embed $C$ into $\mathbb{R}^4$ via the inclusion $i : \mathbb{R}^3 \to \mathbb{R}^3 \times \{0\} \subset \mathbb{R}^4$. As a consequence of a theorem of Osborne ([42, Theorem 3]), $i(C)$ is a tame Cantor set in $\mathbb{R}^4$.

Additionally, constructing uqr maps with desired properties is no easy task even in $\mathbb{R}^3$, let alone higher dimensions. Similar to the Fletcher-Wu construction shown in Section 1.3, it is of course possible to try to glue together local behavior on a Cantor set with the asymptotic behavior of a uqr power-type map.

This leaves us with yet another difficulty inherent to Cantor sets in higher dimensions. A direct analogue of the Fletcher-Wu construction is not possible with $\mathbb{R}^n$ with $n \geq 4$ because it is not currently known whether any geometrically self-similar wild Cantor sets exist in these dimensions. However, Pankka and Wu recently managed to construct a uqr map on $\mathbb{R}^4$ with a wild Cantor set as Julia set by extending the Berstein and Edmonds theorem (Theorem 1.3.8) and by relaxing the geometric self-similarity to quasi-self-similarity. See [44] for details.

Finally, the proof of the extension of Theorem 1.2.26 to $\mathbb{R}^3$ will rely heavily on defining sequences. At present, there is no established definition for defining sequences in $\mathbb{R}^n$ with $n \geq 4$. And, even if we take the straightforward approach of replacing the term ‘cube with
handles’ in Definition 1.3.11 with a higher-dimensional analogue, it is still not known whether every Cantor set in $\mathbb{R}^n$ even has such a defining sequence.

To avoid these difficulties, this work will remain mostly planted in $\mathbb{R}^3$.

## 1.4 Knot Theory

As much of this work is concerned with the behavior of linked tori in $\mathbb{R}^3$, it is important to have a grasp of some elementary aspects of Knot Theory. For additional detail on this fascinating subject, we refer to the books of Cromwell [16] and Rolfsen [47].

Traditionally, Knot Theory is the study of embeddings of one or more topological circles in Euclidean space. The basic objects of study are knots and links. We provide a preliminary definition of these objects, to be revised later in this section.

**Definition 1.4.1** (Knots and Links, preliminary). A knot $K \subset \mathbb{R}^3$ is a set of points homeomorphic to a circle. A link $L = K_1 \sqcup \cdots \sqcup K_n \subset \mathbb{R}^3$ is a finite disjoint union of knots $K_1, \ldots, K_n$.

The simplest knot in $\mathbb{R}^3$ is a geometric circle. This is called the trivial knot. The simplest link is a disjoint union of trivial knots, all living in the same plane in $\mathbb{R}^3$. This is called the trivial link. To understand what it means for a knot or a link to be non-trivial, we need a robust notion of equivalence of knots and links. We examine these concepts here by way of examples.

First, a comment on intuition. To get acquainted with Knot Theory, it is sometimes helpful to imagine knots and links as bits of string in the real world. In fact, playing with rubber bands, hair ties, carabiners, and actual string can be an effective tool for developing an intuitive notion for knot equivalence. One gets to understand that two knots $K_1$ and $K_2$ should be equivalent if, when realized with string, we can deform $K_1$ into $K_2$ with cutting
or gluing any bits if string. In other words, $K_1$ and $K_2$ should be equivalent if there is some kind of continuous deformation process that begins with $K_1$ and ends with $K_2$.

Take for example the *trefoil knot*, the simplest non-trivial knot in $\mathbb{R}^3$. To build this knot in reality, simply take a length of string, tie an overhand knot, and then connect the two loose ends of the string (see Figure 1.8). Trying to continuously deform the string into a flat circle is not possible. This is not a rigorous proof that the trefoil knot is non-trivial, but it serves as a nice intuitive exercise.

The first mechanism for continuous deformation that may occur to a new student of knot theory is homotopy. We may then be tempted to declare that the knots $K_1$ and $K_2$ should be equivalent if there exists a homotopy $h_t : K_1 \times [0, 1] \to \mathbb{R}^3$ such that $h_0$ is the identity map and $h_1(K_1) = K_2$. However, this definition is useless. This is because homotopy allows curves to pass through themselves. Hence all knots are homotopic to the trivial knot. This breaks our intuition regarding the trefoil knot, so it is inadequate.

The first mechanism for continuous deformation that may occur to a new student of knot theory is homotopy. We may then be tempted to declare that the knots $K_1$ and $K_2$ should be equivalent if there exists a homotopy $h_t : K_1 \times [0, 1] \to \mathbb{R}^3$ such that $h_0$ is the identity map and $h_1(K_1) = K_2$. However, this definition is useless. This is because homotopy allows curves to pass through themselves. Hence all knots are homotopic to the trivial knot. This breaks our intuition regarding the trefoil knot, so it is inadequate.

To avoid curves passing through themselves, the next step may be to stipulate that the homotopy $h_t$ be injective for every value of $t$. Such a homotopy is called an *isotopy*. So we update our definition to say that two knots $K_1$ and $K_2$ should be equivalent if there is an isotopy $h_t : K_1 \times [0, 1] \to \mathbb{R}^3$ such that $h_0$ is the identity map and $h_1(K_1) = K_2$. Initially,
this passes the intuitive test of the trefoil knot, as we cannot seem to undo this knot without self-intersection. However, by a process known as bachelor’s unknotting, we can see that the trefoil knot is actually isotopic to the trivial knot.

In Figure 1.9, we see an illustration of bachelor’s unknotting. An isotopy $h_t$ can shrink

![Figure 1.9: Bachelor’s unknotting.](image)

the knotted portion of the trefoil knot as $t \to 1$, making it vanish when $t = 1$. Hence isotopy is still an inadequate definition. One way to think of the problem that bachelor’s unknotting presents us with is to look at the portion of $\mathbb{R}^3$ that is near the shrinking knot. There is some space ‘inside’ the knot that shrinks as $t \to 1$, and, once $t = 1$, that space seems to disappear. It is squished into the same point as the knot. We need only make one adjustment to our previous definition, and we have the correct notion of knot equivalence.

**Definition 1.4.2.** We say the knots $K_1$ and $K_2$ in $\mathbb{R}^3$ are equivalent if they are ambient isotopic, that is if there exists an isotopy $h_t : \mathbb{R}^3 \times [0, 1] \to \mathbb{R}^3$ such that $h_0$ is the identity map and $h_1(K_1) = K_2$.

Extending the isotopy to apply to the whole of $\mathbb{R}^3$ fixes all the pathological behavior from the previous examples. It even applies to links. Simply replace the word ‘knots’ with ‘links’ in Definition 1.4.2.

Before elaborating on knot and link equivalence, we must revise Definition 1.4.1 to remove some other pathological behavior. For this, we need to consider local flatness.
Definition 1.4.3. Let $K \subset \mathbb{R}^3$ be homeomorphic to a circle. A point $p \in K$ is called \textit{locally flat} if there is some neighborhood $U \ni p$ such that the pair $(U, U \cap K)$ is homeomorphic to the unit ball $B(0, 1)$ plus a diameter. If $p$ is locally flat for all $p \in K$, we say $K$ is \textit{locally flat}.

Example 1.4.4. This example is due to Fox and Artin [25]. Let $K$ be a geometric circle in $\mathbb{R}^3$. Construct the set $K_1$ by deforming part of $K$ as seen in Figure 1.10. Inductively construct $K_i$ by repeating the highlighted section of $K_1$ $i$ times. Note that the knot $K_i$ can clearly be ‘undone’ by pulling from the right, when we regard $K_i$ as a loop of string.

Now consider the set $K_\infty$ constructed similarly to the $K_i$, except that the highlighted section is scaled down in size by a constant factor every time it is repeated. See Figure 1.11. It can be shown that this curve is continuous at the limit point of the shrinking sections, and

![Figure 1.10: The knot $K_1$. The section marked by a blue dotted square is repeated $i$ times to form $K_i$.](image1)

![Figure 1.11: Construction of $K_\infty$.](image2)
is hence still homeomorphic to a circle. However, the set \( K_\infty \) is not ambiently isotopic to the trivial knot. What’s happening here is the \( K_\infty \) fails to be locally flat at the limit point. A knot which contains points that are not locally flat is called wild.

To avoid dealing with wild knots, we revise our previous definition of knots and links to exclude this behavior.

**Definition 1.4.5.** A knot \( K \subset \mathbb{R}^3 \) is a locally flat set of points homeomorphic to a circle. A link \( L = K_1 \sqcup \cdots \sqcup K_n \subset \mathbb{R}^3 \) is a finite disjoint union of knots \( K_1, \ldots, K_n \).

Let us return briefly to the notion of knot and link equivalence. Suppose that \( h_t : \mathbb{R}^3 \times [0,1] \to \mathbb{R}^3 \) is an isotopy such that \( h_0 \) is the identity, and \( h_1(K_1) = K_2 \) for the knots \( K_1 \) and \( K_2 \). Note that \( h_1 \) is then an orientation-preserving homeomorphism of \( \mathbb{R}^3 \). We may ask whether equivalence of knots may be established by orientation preserving ambient homeomorphisms instead of ambient isotopy. To see that this possible, consider the following theorem.

**Theorem 1.4.6** (see [24], Theorem 15). Let \( h : \mathbb{R}^3 \to \mathbb{R}^3 \) be an orientation-preserving homeomorphism. Then \( h \) is isotopic to the identity mapping on \( \mathbb{R}^3 \).

Hence it is possible to use whichever definition of knot equivalence suits the framework of the problem at hand.

The simplest non-trivial link in \( \mathbb{R}^3 \) is the **Hopf link**. This link is exemplified by the union of the two circles \( C_1 : x_1^2 + x_2^2 = 1, x_3 = 0 \) and \( C_2 : x_1 = 0, (x_2 - 1)^2 + x_3^2 = 1 \). For intuition, it can be thought of as the result of connecting two climbing carabiners. We will be concerned with linking behavior between not just circles, but rather with handlebodies. As such, we can regard two connected carabiners as a Hopf link between two solid 1-holed tori in \( \mathbb{R}^3 \). Henceforth, the word ‘link’ will refer to finite disjoint unions of handlebodies, rather than simply circles.
**Definition 1.4.7.** A link $L$ is *split* if there is a 2-sphere $S$ embedded in the link complement $\mathbb{R}^3 \setminus L$ so that there are some components of $L$ on each side of $S$.

**Remark 1.4.8.** Let $L$ be a link consisting of three solid 1-holed tori, $T_1, T_2,$ and $T_3$ such that $T_1$ and $T_2$ form a Hopf link, and both sublinks $T_1 \cup T_3$ and $T_2 \cup T_3$ are split links. Then $L$ itself is a split link, even though $T_1 \cup T_2$ is not a split link. In this sense, we can divide $L$ into the two non-split pieces $T_1 \cup T_2$ and $T_3$, which we may think of as 'link components' of $L$. 
CHAPTER 2

SPIDER’S WEBS

The main aim of this chapter is to generalize Theorem 1.2.26 to cover uniformly quasiregular mappings of polynomial type on $\mathbb{R}^3$ whose Julia sets are wild Cantor sets. Let us formally state this theorem here.

**Theorem 2.0.1.** Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be a non-injective uqr map of polynomial type with $J(f)$ a Cantor set, and let $x_0 \in J(f)$ be a repelling fixed point. If $\varphi \in Df(x_0)$ and $L$ is a corresponding linearizer satisfying $f \circ L = L \circ \varphi$, then $A(L)$ is a spider’s web. Moreover, if $(M_i)_{i=1}^\infty$ is a defining sequence for $J(f)$, and $M_i^{x_0}$ is the component of $M_i$ containing $x_0$, then for all sufficiently large $i$, $A(L)$ contains continua homeomorphic to $\partial M_i^{x_0}$.

Note: It is not currently known whether any uqr maps of transcendental type exist on $\mathbb{R}^3$, and it is possible that none exist. If none exist, then the condition “of polynomial type” can be dropped from the statement of Theorem 2.0.1, as it will be implied.

Along the way to proving Theorem 2.0.1, we will prove some results about generalized derivatives, and an important result concerning repelling periodic points.

### 2.1 Lemmas on Generalized Derivatives

We require the following lemma, which allows us to rescale a generalized derivative.

**Lemma 2.1.1.** Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be uqr, $f(0) = 0$, with $f$ locally injective at 0 and suppose that $\varphi \in Df(0)$ arises as the local uniform limit of $f(s_k x)/s_k$ as $s_k \to \infty$. If either $r_j \to \infty$
is an increasing sequence, or \( r_j \to 0 \) is a decreasing sequence, then any locally uniform limit \( \varphi_0 \) of a convergent subsequence of \( \varphi(r_jx)/r_j \) is also in \( \mathcal{D}f(0) \).

**Proof.** First, we assume that \( r_j \to \infty \) is an increasing sequence. Fix \( R > 0 \). Given \( \epsilon > 0 \), there exists \( J \in \mathbb{N} \) so that if \( j \geq J \) and \( |x| \leq R \), then

\[
\left| \varphi_0(x) - \frac{\varphi(r_jx)}{r_j} \right| < \frac{\epsilon}{2}.
\]  

(2.1)

We are assuming here that the \( j \) are indices giving the subsequence converging to \( \varphi_0 \).

Now fix \( j \geq J \). Since \( f(s_kx)/s_k \to \varphi \) locally uniformly on \( \mathbb{R}^n \), there exists \( K(j) \in \mathbb{N} \) depending on \( j \) so that \( r_j s_{K(j)} \to 0 \) as \( j \to \infty \) and if \( |x| \leq R \), that is \( |r_jx| \leq r_jR \), then

\[
\left| \frac{\varphi(r_jx)}{r_j} - \frac{f(s_kr_jx)}{s_kr_j} \right| < \frac{\epsilon}{2},
\]

(2.2)

for \( k \geq K(j) \). From (2.1) and (2.2) it then follows that, if \( j \geq J \) and \( k \geq K(j) \), we have

\[
\left| \varphi_0(x) - \frac{f(s_kr_jx)}{s_kr_j} \right| < \epsilon,
\]

for all \( |x| \leq R \). So \( \varphi_0 \) is a local uniform limit of \( f(s_kr_jx)/(s_kr_j) \), where \( s_kr_j \to 0 \) as \( j \to \infty \). Hence \( \varphi_0 \in \mathcal{D}f(0) \).

Next, we assume that \( r_j \to 0 \) is a decreasing sequence. We again suppose that \( R > 0 \) is fixed and that (2.1) holds for \( |x| \leq R \) and appropriate indices \( j \geq J \). Now fix such a \( j \geq J \). Since \( f(r_js_kx)/(r_js_k) \) converges locally uniformly to \( \varphi(r_jx)/r_j \) as \( k \to \infty \), find \( K(j) \in \mathbb{N} \) large enough so that for \( k \geq K(j) \),

\[
\left| \frac{\varphi(r_jx)}{r_j} - \frac{f(r_js_kx)}{r_js_k} \right| < \frac{\epsilon}{2},
\]

for \( |x| \leq R \). By the same reasoning as above, we conclude that \( \varphi_0 \in \mathcal{D}f(x_0) \). \( \square \)
The rescaling in the preceding lemma then allows us to make precise statements about the growth rate of a loxodromic repelling generalized derivative and its iterates.

**Lemma 2.1.2.** Suppose that $x_0$ is a repelling fixed point of a uqr map $f$ and that $\varphi \in Df(x_0)$.

Then:

(a) there exist $C_3 > 1$ and $N \in \mathbb{N}$ so that $2|x| \leq |\varphi^N(x)| \leq C_3|x|$ for all $x$.

(b) Given $0 < s < t$, there exists $C_4 > 1$ so that for all $m \in \mathbb{N}$ we have

$$\frac{M(t, \varphi^m)}{m(s, \varphi^m)} \leq C_4.$$

Note: The reason for the labeling of $C_3$ and $C_4$ is to be notationally consistent with the constants $C_1$ and $C_2$ from Lemma 1.2.10. Also, before this lemma, it is worth pointing out that the $N$ in part (a) is necessary, as the following example shows.

**Example 2.1.3.** Define the quasiconformal maps $g_1, g_2$ on $\mathbb{C}$ as follows:

$$g_1(x + iy) = \begin{cases} x + iy, & x \geq 0, \\ \frac{x}{10} + iy, & x < 0, \end{cases} \quad g_2(x + iy) = \begin{cases} -\frac{x}{10} - iy, & x \geq 0, \\ -x - iy, & x < 0, \end{cases}$$

and then let

$$g(z) = \begin{cases} g_1(z), & |z| \leq 1, \\ g_{int}(z), & 1 < |z| < 2, \\ g_2(z), & |z| \geq 2, \end{cases}$$

where $g_{int}$ is a quasiconformal interpolation of $g_1$ and $g_2$ guaranteed by Sullivan's Annulus Theorem. We refer to the paper of Tukia and Väisälä [56] for a detailed discussion of the Annulus Theorem in the quasiconformal category. Now define $f(z) = g(2g^{-1}(z))$. Then $f$ is a loxodromic repelling uqc map with $f(1) = g(2) = -1/5$, and hence $m(1, f) \leq 1/5$. Thus $m(r, f) \geq r$ does not hold for this mapping.
Proof of Lemma 2.1.2. Since $x_0$ is repelling, $\varphi$ is a loxodromic repelling uqc map. We must then have that $M(r, \varphi) > r$ for all $r > 0$. We first claim that there is $C > 1$ such that

$$M(r, \varphi) \geq Cr \quad (2.3)$$

for all $r$. If this is not the case, then there is a non-negative monotonic sequence $r_k$ with either $r_k \to 0$ or $r_k \to \infty$ such that

$$\frac{M(r_k, \varphi)}{r_k} \to 1.$$ 

Consider the sequence $\varphi(r_kx)/r_k$. By standard normal family results, this has a convergent subsequence with limit $\varphi_0$. By Lemma 2.1.1, $\varphi_0 \in Df(0)$. However, $M(1, \varphi_0) = 1$, meaning that $\varphi_0$ is not loxodromic repelling. This contradicts Theorem 1.2.20, namely that if one element of $Df(0)$ is loxodromic repelling, then they all must be.

Next, we apply Theorem 1.2.8 to $\varphi^m$. For any $m \in \mathbb{N}$, if $|x| = |y| = r$ with $|\varphi^m(x)| = M(r, \varphi^m)$ and $|\varphi^m(y)| = m(r, \varphi^m)$, then

$$\frac{M(r, \varphi^m)}{m(r, \varphi^m)} \leq \eta(1). \quad (2.4)$$

Now, by induction on $m$, (2.3) implies that

$$M(r, \varphi^m) \geq C^mr$$

for all $r > 0$. Then (2.4) implies that

$$m(r, \varphi^m) \geq \frac{C^mr}{\eta(1)}.$$
for all $r > 0$. Hence we can choose $N$ large enough so that $C^N/\eta(1) \geq 2$ to obtain the left hand inequality in (a).

For the right hand inequality, recall from Section 1.2.2 that $f$ satisfies a local Lipschitz condition (see [30, Lemma 4.1]). That is, in a neighborhood of 0, $f$ satisfies $|f(x)| \leq L|x|$ for some $L > 1$. Hence, in a possibly smaller neighborhood of 0, $f^N$ satisfies $|f^N(x)| \leq L^N|x|$. The process of taking the limit of $f^N(s_k x)/s_k$ as $s_k \to 0$ to find $\varphi^N$ implies that $|\varphi^N(x)| \leq L^N|x|$ for all $x \in \mathbb{R}^n$. Setting $C_3 = L^N$ gives the right hand inequality of (a).

To prove (b), choose $x$ with $|x| = t$ and $|\varphi^m(x)| = M(t, \varphi^m)$, and choose $y$ with $|y| = s$ and $|\varphi^m(y)| = m(s, \varphi^m)$. Then, again by Theorem 1.2.8, we have

$$\frac{M(t, \varphi^m)}{m(s, \varphi^m)} = \frac{|\varphi^m(x) - \varphi^m(0)|}{|\varphi^m(y) - \varphi^m(0)|} \leq \eta \left( \frac{|x - 0|}{|y - 0|} \right) = \eta \left( \frac{t}{s} \right).$$

Setting $C_4 = \eta(t/s)$ proves (b).

### 2.2 Density of Repelling Periodic Points

Before proving Theorem 2.0.1, there is an important detail that needs establishing. In the given statement, we say ‘let $x_0 \in J(f)$ be a repelling fixed point.’ A priori, there is no reason to expect $f$ to have a repelling periodic point. We may wish to appeal to Theorem 1.2.28 to say that the set of repelling periodic points is dense in $J(f)$, but this does not work. Since $J(f)$ is a Cantor set, Theorem 1.3.6 implies that $F(f)$ has only one connected component, and so Theorem 1.2.28 does not apply. However, in this section we will prove that the set of repelling periodic points is nonetheless dense in $J(f)$. Note that the following theorem is in fact slightly stronger than that, more than guaranteeing the existence of the required repelling periodic point for Theorem 2.0.1.
Theorem 2.2.1. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a uqr map of polynomial type, and suppose that $J(f)$ is a Cantor set. Then every periodic point of $f$ is in $J(f)$ and, moreover, is repelling. The repelling periodic points are thus dense in $J(f)$.

To prove this theorem, we need the following results about point separation in Cantor sets. We believe the former to be well-known, for example [57, Lemma 3.1] is the two-dimensional version.

Lemma 2.2.2. Let $X$ be a Cantor set in $\mathbb{R}^n$, let $x \in X$, and let $\delta > 0$. Then there exists a neighborhood $U$ of $x$ such that $\text{diam}(U) \leq \delta$ and $\partial U \cap X = \emptyset$.

Proof. Let $B = \overline{B(x, \delta/2)}$. Since $X \cap B$ is totally disconnected, there exists $Y \subset X \cap B$ that is clopen in $X \cap B$ such that $x \in Y$. Since $Y$ is in particular closed in $X \cap B$, there exists $C \subset B$ closed in $\mathbb{R}^n$ such that $C \cap (X \cap B) = Y$. Now since $Y$ is open in $X$, $X \setminus Y$ is closed in $X$, and so there exists a closed set $D \subset \mathbb{R}^n$ with $X \cap D = X \setminus Y$. Since $\mathbb{R}^n$ is $T_4$, we can choose open sets $U \supset C$ and $V \supset D$ in $\mathbb{R}^n$ such that $U \cap V = \emptyset$. Since $B$ is also $T_4$, we can choose $U$ so that $U \subset B$, ensuring that $\text{diam}(U) \leq \delta$. Since $U$ and $V$ are disjoint, and $U$ and $V$ are both open, we have that $\overline{U} \cap V = \emptyset$. Since $Y \subset U$ and $X \setminus Y \subset V$, we see that $\partial U \cap X = \emptyset$.\[\square\]

We refine this lemma to ensure that the boundary of the neighborhood is in a controlled annular region.

Lemma 2.2.3. Let $X$ be a Cantor set in $\mathbb{R}^n$, let $x \in X$, and let $0 < r < s$. Then there exists a neighborhood $U$ of $x$ so that $\partial U \subset A(x, r, s)$ and $\partial U \cap X = \emptyset$.

Proof. If $A(x, r, s) \cap X = \emptyset$, then let $U$ be any neighborhood of $x$ with $\partial U \subset A(x, r, s)$, and we are done.
So suppose that $A(x, r, s) \cap X \neq \emptyset$. Let $B = \{y : |x - y| < (r + s)/2\}$. If $\partial B \cap X = \emptyset$ we are again done. Otherwise, for each $y \in \partial B \cap X$, apply Lemma 2.2.2 to $y$ and $\delta = (s - r)/100$ to find a neighborhood $U_y$ of $y$ with $\partial U_y \cap X = \emptyset$. We may then take

$$U = B \cup \left( \bigcup_{y \in \partial B \cap X} U_y \right),$$

since this is a neighborhood of $x$ whose boundary is necessarily contained in $A(x, r, s)$, and whose boundary is disjoint from $X$.

**Proof of Theorem 2.2.1.** Suppose that $f : \mathbb{R}^n \to \mathbb{R}^n$ is a $K$-uqr map of polynomial type and that $J(f)$ is a Cantor set. Recall that $\mathbb{R}^n \setminus J(f)$ is connected (Theorem 1.3.6). This together with the fact that $J(f)$ is compact in $\mathbb{R}^n$ give that $F(f)$ is connected. Since $I(f)$ is an open connected neighborhood of infinity (see [21]), we must have that $F(f) = I(f)$.

By a result from Siebert’s thesis [52, Satz 4.3.4], $J(f)$ is contained in the closure of the set of periodic points of $f$. Let $x_0$ be such a periodic point, say of period $p$. Since $F(f) = I(f)$, $x_0$ cannot be in $F(f)$, and so we have that $x_0 \in J(f)$. Henceforth, write $g = f^p$ so that $x_0$ is a fixed point of $g$.

Now let $R > 0$ be such that $g$ is injective on $B(x_0, R)$. That such an $R$ exists follows from the fact that otherwise $x_0$ would be super-attracting by 1.2.21, and would hence be in $F(g)$.

Let $\delta > 0$. Since $J(g) = J(f)$ and $F(g) = F(f)$, we have by Lemma 2.2.3, applied to $r = \delta/2$ and $s = \delta$, that there exists a neighborhood $U_\delta$ of $x_0$ such that $\gamma_\delta = \partial U_\delta$ is contained in $I(g) \cap A(x_0, \delta/2, \delta)$. This follows since $J(g) = \partial I(g)$, and again since $I(g) = F(g)$.

We define $L(x_0, r, g) = \max\{|g(x) - x_0| : |x - x_0| = r\}$ and $l(x_0, r, g) = \min\{|g(x) - x_0| : |x - x_0| = r\}$. Furthermore, we denote by $L_m$ and $l_m$ the maximum and minimum of
\{ |g^m(x) - x_0| : x \in \gamma_\delta \}, respectively. Note first that as long as \( g^{m-1}(U_\delta) \subset B(x_0, R) \), we have

\[ L_m \leq L(x_0, \delta, g^m), \quad \text{and} \quad l_m \geq l(x_0, \delta/2, g^m). \]

Moreover, with the same assumption, we have that \( g^m \) is \( K \)-quasiconformal on \( U_\delta \). It then follows from [22, Theorem 1.1] that there exists \( C^* > 1 \) depending only on \( m, n \) and \( K_O(g^m) \) (which is uniformly bounded over \( m \) since \( f \) is uqr) such that

\[ \frac{L(x_0, \delta, g^m)}{l(x_0, \delta/2, g^m)} \leq C^*. \quad (2.5) \]

Hence \( L_m/l_m \leq C^* \).

Now find \( C > 1 \) sufficiently large so that the forward orbit of any \( x \in B(x_0, R/C^*) \cap I(g) \) must first pass through \( A(x_0, R/(CC^*), R/C^*) \) before leaving \( B(x_0, R/C^*) \). Let \( \delta < R/(C(C^*)^2) \), and find \( M \) maximal so that \( g^M(\gamma_\delta) \cap A(x_0, R/(CC^*), R/C^*) \) is non-empty. By (2.5), we then have that

\[ g^M(\gamma_\delta) \subset A(x_0, R/(C(C^*)^2), R), \]

and hence that \( U_\delta \subset g^M(U_\delta) \). By the topological classification of repelling fixed points (Definition 1.2.18), \( x_0 \) is a repelling fixed point of \( g^M \). Hence, by [30, Proposition 4.6], \( x_0 \) is a repelling periodic point of \( f \).

\[ \square \]

### 2.3 Positive Lower Order of Growth

We consider one last thing before proving Theorem 2.0.1. Recall from Sections 1.2.1 and 1.2.2 that one reason we are interested in the fast escaping set of a transcendental type qr map \( L \) is that, under certain conditions, we may use \( A(L) \) to draw conclusions about \( J(L) \) since \( J(L) = \partial A(L) \). In particular, this equality holds if \( L \) has positive lower order. In this
section, we not only show that arbitrary Poincaré linearizers have positive lower order, but give explicit estimates for both the order and lower order.

Recall that the order of growth of a quasiregular mapping $L : \mathbb{R}^n \to \mathbb{R}^n$ is defined by

$$
\rho_L = \limsup_{r \to \infty} (n-1) \frac{\log \log M(r, L)}{\log 2},
$$

and the lower order is

$$
\lambda_L = \liminf_{r \to \infty} (n-1) \frac{\log \log M(r, L)}{\log 2}.
$$

**Theorem 2.3.1.** Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a $K$-uqr mapping of polynomial type of degree $d > K$ with repelling fixed point $x_0$ and let $L$ be a linearizer of $f$ at $x_0$ conjugating $f$ to $\varphi \in \mathcal{D}f(x_0)$. Recall $C_3$ and $N$ from Lemma 2.1.2 applied to $L$. Then

$$
\frac{N \log d - \log K}{\log C_3} \leq \rho_L \leq \frac{N \log d + \log K}{\log 2},
$$

and the same is true for the lower order $\lambda_L$. In particular, since $C_3 \geq 2$, $\lambda_L$ is positive.

**Proof.** With $N$ from Lemma 2.1.2, write $F = f^N$ and $\Phi = \varphi^N$. Since $f$ is $K$-uqr, so is $F$. Also note that the degree of $F$ is $d^N$. Let $R_0$ be as in Lemma 1.2.10 applied to $F$, and suppose that $M(r, L) \geq R_0$ for $r \geq r_1$. Fix $r_0 \geq r_1$, and let $r \in [r_0, 2r_0]$. Then, by Lemma 1.2.10 and the fact that $M(r, L)$ is increasing in $r$, we have in particular

$$
M(r, F^j \circ L) \leq C_2^{\frac{q_j((d^N K)^{1/(n-1)}))}{(d^N K)^{1/(n-1)}}} M(2r_0, L)^{(d^N K)^{j/(n-1)}},
$$

where $q_j(y) = (y^j - 1)/(y - 1)$. Since $F^j \circ L = L \circ \Phi^j$, we have by Lemma 2.1.2 that

$$
M(2^j r, L) \leq M(r, L \circ \Phi^j).
$$
Since $\log q_j(y) = j \log y + O(1)$, we can combine (2.6) and (2.7) to obtain that

$$\log \log M(2^j r, L) \leq j \log(d^K)^{1/(n-1)} + O(1)$$

uniformly for $r \in [r_0, 2r_0]$ as $j \to \infty$. Hence

$$\log \log M(2^j r, L) \leq \frac{\log(d^K)^{1/(n-1)} \log 2^jr}{\log 2} + O(1),$$

which gives that

$$\frac{\log \log M(2^j r, L)}{\log 2^j r} \leq \frac{\log(d^K)^{1/(n-1)}}{\log 2} + o(1)$$

uniformly for $[r_0, 2r_0]$ as $j \to \infty$. Taking the lim sup and lim inf gives the upper bound for both $\rho_L$ and $\lambda_L$.

Similarly, now let $r \in [r_0, C_3 r_0]$. By Lemma 1.2.10, we then have

$$C_{jj}((d/K)^{1/(n-1)}) M(r_0, L)^{(d/K)^{1/(n-1)}} \leq M(r, F^j \circ L). \quad (2.8)$$

Again similarly to above, Lemma 2.1.2 gives that

$$M(r, L \circ \Phi^j) \leq M(C_3^j r, L). \quad (2.9)$$

Combining (2.8) and (2.9) yields

$$j \log \left(\frac{d^K}{K}\right)^{1/(n-1)} - O(1) \leq \log \log M(C_3^j r, L),$$

uniformly for $r \in [r_0, C_3 r_0]$ as $j \to \infty$. Hence

$$\frac{\log(d^K)^{1/(n-1)} \log C_3^j r}{\log C_3} - O(1) \leq \log \log M(C_3^j r, L),$$
giving
\[
\frac{\log(d^N/K)^{1/(n-1)}}{\log C_3} - o(1) \leq \frac{\log \log M(C_3^j r, L)}{\log C_3^j r}
\]
uniformly for \( r \in [r_0, C_3 r_0] \) as \( j \to \infty \). Again, taking the lim sup and lim inf gives the lower bound for \( \rho_L \) and \( \lambda_L \).

\[\square\]

### 2.4 The Proof of Theorem 2.0.1

The method of proving Theorem 2.0.1 is based on that of Mihaljević-Brandt and Peter [37], but more technical difficulties need to be overcome. This is due to linearizing to a uniformly quasiconformal map and not necessarily a dilation. We can replace \( f \) from Theorem 2.0.1 by \( g := f^N \) so that the conclusions of Lemma 2.1.2 hold with \( \Phi := \varphi^N \) and, importantly, the degree \( d \) of \( g \) is larger than \( K \). Then \( g : \mathbb{R}^3 \to \mathbb{R}^3 \) is a \( K \)-uqr map and we have \( g \circ L = L \circ \Phi \).

We begin with some lemmas on the growth of \( L \).

**Lemma 2.4.1.** Recall \( C_1 \) from Lemma 1.2.10 and \( C_3 \) from Lemma 2.1.2 applied to \( \Phi \). There exists \( R_1 > 0 \) such that if \( r > R_1 \), then
\[
\log M(C_3^m r, L) \geq \log M(r, L) \prod_{i=1}^{m-1} \left( \left( \frac{d}{K} \right)^{1/2} + \frac{\log C_1}{\log M(C_3^i r, L)} \right)
\]
for all \( m \in \mathbb{N} \).

**Proof.** Let \( r \) be large, precisely how large will be determined shortly. Let \( y \in S_r \) such that \( |L(y)| \geq |L(x)| \) for all \( x \in S_r \). Let \( w = |L(y)| \) so that \( |w| = M(r, L) \). Suppose that \( r \) is
large enough so that $|w| > R_0$, where $R_0$ is from Lemma 1.2.10 applied to $g$. Then by the functional equation $g \circ L = L \circ \Phi$, Lemma 1.2.10, and Lemma 2.1.2, we have

$$\log M(C_3r, L) \geq \log M(r, L \circ \Phi)$$

$$= \log M(r, g \circ L)$$

$$= \log M(L(S_r), g)$$

$$\geq \log |g(w)|$$

$$\geq \log C_1 + (d/K)^{1/2} \log |w|$$

$$= \log C_1 + (d/K)^{1/2} \log M(r, L).$$

To see that the result is true for all $m \in \mathbb{N}$, we iterate the previous argument with $C_3^i r$ playing the role of $r$ where $i$ increments from $m - 1$ down to 1 as follows:

$$\log M(C_3^m r, L) \geq \log C_1 + (d/K)^{1/2} \log M(C_3^{m-1} r, L)$$

$$= \log M(C_3^{m-1} r, L) \left( \frac{\log C_1}{\log M(C_3^{m-1} r, L)} + (d/K)^{1/2} \right)$$

$$\geq (\log C_1 + (d/K)^{1/2} \log M(C_3^{m-2} r, L)) \left( \frac{\log C_1}{\log M(C_3^{m-1} r, L)} + (d/K)^{1/2} \right)$$

$$= \log M(C_3^{m-2} r, L) \prod_{i=m-2}^{m-1} \left( \frac{\log C_1}{\log M(C_3^i r, L)} + (d/K)^{1/2} \right)$$

$$\geq \cdots \geq \log M(r, L) \prod_{i=1}^{m-1} \left( \frac{\log C_1}{\log M(C_3^i r, L)} + (d/K)^{1/2} \right).$$
Lemma 2.4.2. Let $\mu > 1$ and recall $C_3$ from Lemma 2.1.2 applied to $\Phi$. There exist $R_2 > 0$ and $M \in \mathbb{N}$ such that for any $R > R_2$, the sequence defined by

$$r_m = C_3^m M^m(R,L)$$

satisfies

$$M(r_m, L) > r_{m+1}^\mu,$$

for $m \geq M$.

Proof. Assume that $R$ is large. Set $\beta = (d/K)^{1/2}$. With the sequence $r_m$ defined by (2.10), applying Lemma 2.4.1 with $r = M^m(R, L)$ yields

$$\log M(r_m, L) = \log M(C_3^m M^m(R,L), L) \geq \prod_{i=1}^{m} \left( \beta + \frac{\log C_1}{\log M(C_3^m M^m(R,L), L)} \right) \cdot \log M(M^m(R,L), L) \geq \left( \beta - \frac{\log C_1}{\log R} \right)^m \log M^{m+1}(R, L).$$

Now,

$$\log r_{m+1}^\mu = \mu \log (C_3^{m+1} M^{m+1}(R,L)) = \mu(m + 1) \log C_3 + \mu \log M^{m+1}(R,L).$$

Hence the result is true if

$$\log M^{m+1}(R,L) \left( \left( \beta - \frac{\log C_1}{\log R} \right)^m - \mu \right) > \mu(m + 1) \log C_3.$$
Since \( \mu(m+1) \log C_3 \) is much smaller than \( \log M^{m+1}(R, L) \) for large \( m \), this is so if we choose \( R \) large enough and \( m \) large enough so that \( \beta^m > \mu \). \( \square \)

**Lemma 2.4.3.** Let

\[
\mu > \left( \frac{\log C_3}{\log 2} \right) \left( \frac{\log d + \log K}{\log d - \log K} \right). \tag{2.11}
\]

There exists \( R_3 > 0 \) such that for \( r > R_3 \) there is a continuum \( \Gamma^r \) separating \( S_r \) and \( S_{r^m} \) such that

\[
m(\Gamma^r, L) > M(r, L).
\]

**Proof.** There exists a neighborhood \( U \) of 0 such that \( L|_U \) is injective, and we may assume that \( U \subset \mathbb{B}^3 \). Let \( \delta > 0 \) be small enough that \( B(x_0, \delta) \subset L(U) \). Since \( J(f) \) is a Cantor set with defining sequence, there exists a continuum \( \gamma_\delta \subset B(x_0, \delta) \cap F(f) \) so that \( \gamma_\delta \) separates \( x_0 \) from infinity. Let \( \Gamma_\delta = L^{-1}(\gamma_\delta) \cap U \). Then \( \Gamma_\delta \) is a continuum which separates 0 from infinity. Find \( 0 < s < t < 1 \) so that \( \Gamma_\delta \subset A(s, t) \).

Let \( r \) be large. Exactly how large will be specified later. Since \( \Phi^m \) is \( K \)-quasiconformal for all \( m \), the modulus of the ring domain \( \Phi^m(A(s, t)) \) is uniformly bounded. By taking \( r \) as large as we like, we can make the modulus of \( A(r, r^\mu) \) arbitrarily large. Hence by Lemma 2.1.2 we can guarantee that for large \( r \) there exists some \( m \in \mathbb{N} \) so that \( \Phi^m(A(s, t)) \subset A(r, r^\mu) \). We wish to estimate how many iterates of \( \Phi \) we need to apply to ensure the image of \( A(s, t) \) is contained in \( A(r, r^\mu) \).

Find \( p_1 \) minimal so that \( \Phi^{p_1}(A(s, t)) \subset A(r, r^\mu) \). By Lemma 2.1.2 (a), we can guarantee that \( \Phi^{p_1}(S_s) \) meets \( A(r, C_3r) \), otherwise \( p_1 \) would not be minimal. Hence we actually have \( \Phi^{p_1} \subset A(r, C_3C_4r) \) by Lemma 2.1.2 (b). Since this construction requires that \( M(t, \Phi^{p_1}) \leq C_3C_4r \), Lemma 2.1.2 (a) again implies that

\[
2^{p_1} t \leq C_3C_4r. \tag{2.12}
\]
Similarly, find $p_2$ maximal so that $\Phi^{p_2}(A(s, t)) \subset A(r, r^\mu)$. As above, the fact that $p_2$ is chosen maximal means that $\Phi^{p_2}(A(s, t)) \subset A(r^\mu/(C_3C_4), r^\mu)$. Since this requires $m(s, \Phi^{p_2}) \geq r^\mu/(C_3C_4)$, Lemma 2.1.2 (a) implies that

$$C_3^{p_2} s \geq \frac{r^\mu}{C_3C_4}. \tag{2.13}$$

From (2.12) and (2.13), we conclude by taking logarithms that

$$p_2 \geq p_1 \left( \frac{\mu \log 2}{\log C_3} \right) + O(1). \tag{2.14}$$

Here $p_1$ and $p_2$ depend on $r$, and we write $O(1)$ here and below for constants independent of $r$.

We clearly need $r$ large enough for these constructions to make sense, and in particular we require $A(r, C_3C_4r)$ and $A(r^\mu/(C_3C_4), r^\mu)$ to be disjoint. This is certainly the case as long as

$$C_3C_4r < \frac{r^\mu}{C_3C_4},$$

that is if

$$r > (C_3C_4)^{2/(\mu-1)}.$$

Next, find $R_0 > 0$ large enough so that Lemma 1.2.10 can be applied to iterates of $g$. Find $j \in \mathbb{N}$ minimal so that $g^j(\gamma_\delta) \subset \{ x : |x| > R_0 \}$. Define $\Gamma^r := \Phi^{p_2}(\Gamma_\delta)$. It follows immediately from the fact that $\Phi^{p_2}(A(s, t)) \subset A(r, r^\mu)$ that $\Gamma^r$ separates $S_r$ and $S_{r^\mu}$.
We first estimate the minimum modulus on $\Gamma^r$:

$$
\log m(\Gamma^r, L) = \log m(\Gamma^r, g^{p_2} \circ L \circ \Phi^{-p_2})
\leq \log m(\Gamma_\delta, g^{p_2})
\geq \log m(R_0, g^{p_2-j})
\geq q_{p_2-j}((d/K)^{1/2}) \log C_1 + (d/K)^{(p_2-j)/2} \log R_0.
$$

Next,

$$
\log M(r, L) = \log M(r, g^{p_1} \circ L \circ \Phi^{-p_1})
\leq \log M(\Phi^{-p_1}(S_r), g^{p_1} \circ L)
\leq \log M(1, g^{p_1} \circ L)
\leq \log M(R_0, g^{p_1})
\leq q_{p_1}((dK)^{1/2}) \log C_2 + (dK)^{(p_1)/2} \log R_0.
$$

We may assume that $N$ is chosen large enough so that $(dK)^{1/2} \geq 2$. Using $y_j^{1-1} \leq q_j(y) \leq y_j$, for $y_j \geq 2$, (2.14) and writing $\mu_1 = \mu \log 2/\log C_3$, these two chains of inequalities imply that we require

$$
\left( \frac{d}{K} \right)^{(p_1\mu_1)/2+O(1)} \log C_1 + \left( \frac{d}{K} \right)^{(p_1\mu_1)/2+O(1)} \log R_0 \geq (dK)^{p_1/2} \log(C_2R_0).
$$

Taking logarithms, this reduces to

$$
p_1 \left( \frac{\mu_1 \log(d/K)}{2} - \frac{\log(dK)}{2} \right) \geq O(1).
$$
By (2.11), the second factor on the left hand side is strictly positive, which means that for large enough $r$, and hence large enough $p_1$, this inequality is satisfied.  

Recall Theorem 1.2.27 on a sufficient condition for $A(L)$ to be a spider’s web. This is the condition we will use to establish Theorem 2.0.1.

**Proof of Theorem 2.0.1.** Recall Lemmas 1.2.10, 2.4.1, 2.4.2, and 2.4.3. Then we let $R > \max\{R_0, R_1, R_2, R_3\}$ and for $m \in \mathbb{N}$, let $r_m = C_3^m M^m(R, L)$ and let $\mu$ satisfy (2.11). From all of this, it follows that there exists a continuum $\Gamma^r_m$ separating $S_{r_m}$ and $S_{r_m}^\mu$ such that

$$m(\Gamma^r_m, L) > M(r_m, L).$$

We define $G_m$ to be the interior of $\Gamma^r_m$. Then by construction, every $G_m$ is a bounded topologically convex domain with

$$G_m \supset \{x \in \mathbb{R}^3 : |x| < r_m\} \supset \{x \in \mathbb{R}^3 : |x| < M^m(R, L)\} = B(0, M^m(R, L)).$$

It follows further from Lemma 2.4.3 that

$$m(\partial G_m, L) = m(\Gamma^r_m, L) > M(r_m, L) > r_m^\mu > \max_{x \in \partial G_{m+1}} |x|,$$

and hence $G_{m+1}$ is contained in a bounded component of $\mathbb{R}^3 \setminus L(\partial G_m)$. Thus we have fulfilled the condition of Theorem 1.2.27 for $A(L)$ to be a spider’s web.

The final statement of the theorem follows immediately by choosing the $\gamma_\delta$ in the proof of Lemma 2.4.3 appropriately.  

**Remark 2.4.4.** Recall from Section 1.3.3 that there is currently no definition of defining sequences for Cantor sets in dimensions 4 and above. Recall also that, even with a definition in place, it is not known whether every higher-dimensional Cantor set actually has a defining
sequence. However, if a definition was formalized, it would likely involve the intersections of disjoint unions of compact $n$-manifolds with boundary. Consider then the statement of Theorem 2.0.1 with $\mathbb{R}^3$ replaced by $\mathbb{R}^n$. The only additional assumption required for the proof to carry over into higher dimensions is that the Cantor set $J(f)$ has a defining sequence. So, if progress is made concerning higher-dimensional defining sequences, it will immediately increase the generality of Theorem 2.0.1.

Recall from Section 1.3 that there exists a class of uqr maps of polynomial type having a wild Cantor set as their Julia set. If $f$ is a map from this class, then $J(f)$ is an Antoine’s necklace. We then conclude from Theorem 2.0.1 that, if $L$ is a linearizer for $f$ at a repelling fixed point, then $A(L)$ is a spider’s web containing arbitrarily large surface 1-holed tori. We wish to have a larger class of maps to which Theorem 2.0.1 can apply. Specifically, we aim to construct a class of maps having as their Julia set a Cantor set of genus 2. This will be accomplished by a similar method as in [23]. Recall that this method requires having a geometrically self-similar Cantor set. So far, no explicit construction of a geometrically self-similar Cantor set of genus 2 has ever been given. In the following chapters, we will construct such a Cantor set, prove that it has genus 2, and then construct the desired uqr map.
CHAPTER 3
A GEOMETRICALLY SELF-SIMILAR CANTOR SET

To eventually construct a new class of uqr maps to which Theorem 2.0.1 applies, we must first construct a geometrically self-similar Cantor set of genus 2. No such Cantor set has ever been constructed, so it is in particular not known how many 2-holed tori are minimally required to be in each chain to allow for geometric self-similarity. Recall that Antoine’s necklace requires at least 20 tori in each chain to be geometrically self-similar, and that this bound is sharp.

In this chapter, we construct a geometrically self-similar wild Cantor set $\mathcal{X}$, and give an explicit upper bound for the minimum number of 2-holed tori required in each chain. Many decisions made in the construction of $\mathcal{X}$ are not intended to be optimal for minimizing this bound, and are made simply for computational convenience. So it is possible that geometrically self-similar Cantor sets of genus 2 exist requiring fewer 2-holed tori than used in this chapter. We prove that $g(\mathcal{X}) = 2$ in the next chapter.

3.1 Square Tori

In constructing a geometrically self-similar Cantor set, we will make several geometrically convenient choices. There is no topological reason for any of the choices throughout this chapter, though they will make many computations much simpler. The first is that we will focus on tori with piecewise linear core curves.
Let us begin by constructing a solid 1-holed torus in $\mathbb{R}^3$ out of a square. Start with a circle of radius $R > 0$ in the $x_3 = 0$ plane. Circumscribe a square around the circle such that the sides of the square are parallel to the $x_1$- and $x_2$-axes, respectively. Now let $Y$ be the result of thickening the square by some value $0 < r < R$ with respect to the $\infty$-metric in the $x_3 = 0$ plane. We then obtain a 1-holed torus $T := Y \times [-r, r]$ (see Figure 3.1). The core square has sides of length $2R$, and the resulting torus is made of beams with square cross-sections with sides of length $2r$.

By allowing two such squares to touch at a corner, we obtain a core curve which looks like an angular figure-eight. Thickening this curve similarly to above gives us a solid 2-holed torus. A cross-section showing the holes can be seen in Figure 3.2. This resulting torus has the following geometric measurements:

- The core curve has total length $16R$. 

![Figure 3.1: A square 1-holed torus.](image)

![Figure 3.2: A square 2-holed torus.](image)
• The length (distance in the $x_3 = 0$ plane from end to end) of the torus is $2\sqrt{2}(2R + r)$.

• The width (distance across the hole perpendicular to the length) is $2\sqrt{2}(R + r)$.

• The $x_3$-height is $2r$.

Call this torus $X_0$ with core curve $\gamma$.

Our aim is now to construct a chain of linked 2-holed tori inside of $X_0$ to form the defining sequence for the inductive definition of a Cantor set. Each such 2-holed torus is further to be geometrically similar to $X_0$. In other words, for a suitably-chosen integer $m$, we wish to find 2-holed tori $X_{1,1}, \ldots, X_{1,m}$ contained in the interior of $X_0$ which link to form a chain, and such that each $X_{1,j}$ is a scaled down version of $X_0$, with common scaling factor $k$. Moreover, the structure of the chain is essential to creating a Cantor set whose genus is 2, so we will additionally demand the $X_{1,j}$ are arranged along $\gamma$ in a very specific way. So, given a desirable arrangement for the 2-holed tori, we need a scaling factor $k$ and an integer $m$ such that $m$ 2-holed tori having size $k$ times that of $X_0$ can be placed in the given arrangement. Naturally, $k$ and $m$ will determine each other. Note then that the pair $(k, m)$ depends on the choice for $r$. Before we determine a formula for finding pairs $(k, m)$, let us discuss the desired arrangement of 2-holed tori along $\gamma$.

The linking between neighbors in the chain should be as follows. Let $\gamma_j$ be the core curve of $X_{1,j}$. Then $\gamma_j$ consists of two square segments. We want one square segment of $\gamma_j$ to form a Hopf link with one square segment of $\gamma_{j-1}$, and the other square segment of $\gamma_j$ to form a Hopf link with one square segment of $\gamma_{j+1}$ (if $j = 1$, replace $j - 1$ with $m$, and if $j = m$, replace $j + 1$ with 1). In other words, a chain is formed by linking each 2-holed torus with two other 2-holed tori, one in each hole. See Figure 3.3 for an illustration. The angle between two linked 2-holed tori here is assumed to be $\pi/2$ for simplicity. To make sure that neighboring 2-holed tori are disjoint, we must replace the condition $r < R$ with a more
Figure 3.3: A link between three 2-holed tori.

restrictive one. However, this new condition will later be overridden anyway, so we ignore it for now.

Since the curve $\gamma$ is self-intersecting, it is not sufficient for every $X_{1,j}$ to be linked with exactly two other 2-holed tori. Rather, we require that there be a four-way linking of 2-holed tori at the intersection point of the figure-eight. This is going to be the key in ensuring that the genus of the resulting Cantor set is actually 2. See Figure 3.4 for an illustration of a four-way linking between the core curves of some 2-holed tori. Given a specific choice of

Figure 3.4: A four-way linking between figure-eight core curves. Thickening by a small value $r > 0$ yields a link between 2-holed tori.
positioning and orientation for the four 2-holed tori in question, we will need an additional bound on the thickness \( r \). If the 2-holed tori are too thick, they cannot be mutually disjoint in the four-way linking.

### 3.2 The Four-Way Linking

It will suffice to consider a four-way linking of 1-holed tori, since our 2-holed tori can then be obtained by adding a second hole to each torus.

For computational simplicity, place the self-intersection of \( \gamma \) at the origin, and orient \( \gamma \) so that the segments emanating from the intersection point follow the \( x_1 \)- and \( x_2 \)-axes, respectively. Choose four square tori, call them \( T_1, \ldots, T_4 \) with core curves \( \gamma_1, \ldots, \gamma_4 \), respectively. Orient the tori so that each one forms a diamond with respect to the \( x_1x_2 \)-plane, i.e. such that the intersections of the core curves and the \( x_1x_2 \)-plane happen along the diagonal of the squares. Such a diamond can be seen in Figure 3.5. Then position the intersection with

![Figure 3.5: A square 1-holed torus oriented as a diamond.](image)

the \( x_1x_2 \)-plane as in Figure 3.6. The coordinates of the intersection points between the \( \gamma_j \) and the \( x_1x_2 \)-plane are chosen to be:

- \( T_1 : (3\sqrt{2}R/4, 0, 0), (-5\sqrt{2}R/4, 0, 0) \)
Figure 3.6: The four-way linking, before rotation.

- $T_2: (5\sqrt{2}R/4, 0, 0), (-3\sqrt{2}R/4, 0, 0)$

- $T_3: (0, 1\sqrt{2}R/4, 0), (0, -7\sqrt{2}R/4, 0)$

- $T_4: (0, 7\sqrt{2}R/4, 0), (0, -1\sqrt{2}R/4, 0)$

There is nothing special about these choices, other than that they will allow room for some positive value of $r$ to be sufficiently small so that the four 1-holed tori are in fact disjoint.

Once the 1-holed tori are rotated appropriately, the core curves of the tori no longer intersect each other, and the core curves will pairwise form Hopf links. As labeled here, $T_1$ and $T_2$ will form a chain going in the $x_1$-direction, and $T_3$ and $T_4$ will form a chain going in the $x_2$-direction, with the four-way linking in the middle (see Figure 3.6). More precisely, we will rotate the tori as follows:

- $T_1$ is rotated about the $x_1$-axis by an angle of $-3\pi/8$  

- $T_2$ is rotated about the $x_1$-axis by an angle of $3\pi/8$  

- $T_3$ is rotated about the $x_2$-axis by an angle of $\pi/8$  

- $T_4$ is rotated about the $x_2$-axis by an angle of $-\pi/8$
With this arrangement, note that $T_1$ and $T_2$ are symmetric via a rotation by an angle $\pi/2$ about the $x_3$-axis, and that the same is true for $T_3$ and $T_4$.

To determine a bound on the thickness coefficient $r$ of the tori, we must bound the minimal distances between the tori once they are rotated. To do this, we will inscribe the square beams that the tori consist of inside right circular cylinders, which will have radius $\sqrt{2}r$. Clearly the distance between any two square beams will then be at least as large as the distance between the corresponding cylinders. See Figure 3.7 for an illustration. We can

Figure 3.7: The distance between cylinders.

then compute the distances between the line segments that make up the $\gamma_j$, and subtract $2\sqrt{2}r$ (twice the radius of a single cylinder) to have a valid and relatively close lower bound for the distances between the tori.

Thanks to the symmetry of the position and orientation of the tori, we can concentrate on the four line segments $L'_1, \ldots, L'_4$ of the $\gamma_j$ marked in Figure 3.8, each having $x_3 \geq 0$. Before rotation, the given segments have the following vector equations:
Figure 3.8: Lines used in distance estimations.

\[ L_1' : \begin{pmatrix} 3\sqrt{2}/4 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \quad L_2' : \begin{pmatrix} 5\sqrt{2}/4 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \]

\[ L_3' : \begin{pmatrix} 0 \\ \sqrt{2}/4 \\ 0 \end{pmatrix} + t \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \quad L_4' : \begin{pmatrix} 0 \\ -\sqrt{2}/4 \\ 0 \end{pmatrix} + t \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \]

each for \( 0 \leq t \leq \sqrt{2}R \).

After the rotations described in (3.1) - (3.4), the equations become the following:

\[ L_1 : \begin{pmatrix} 3\sqrt{2}/4 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ \sqrt{2 + \sqrt{2}/2} \\ \sqrt{2 - \sqrt{2}/2} \end{pmatrix} \quad L_2 : \begin{pmatrix} 5\sqrt{2}/4 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ \sqrt{2 + \sqrt{2}/2} \\ \sqrt{2 - \sqrt{2}/2} \end{pmatrix} \]

\[ L_3 : \begin{pmatrix} 0 \\ \sqrt{2}/4 \\ 0 \end{pmatrix} + t \begin{pmatrix} -\sqrt{2 - \sqrt{2}/2} \\ -1 \\ \sqrt{2 + \sqrt{2}/2} \end{pmatrix} \quad L_4 : \begin{pmatrix} 0 \\ -\sqrt{2}/4 \\ 0 \end{pmatrix} + t \begin{pmatrix} \sqrt{2 - \sqrt{2}/2} \\ 1 \\ \sqrt{2 + \sqrt{2}/2} \end{pmatrix}, \]
again for $0 \leq t \leq \sqrt{2}R$. Let $\tau_j$ denote the cylinder around $L_j$ with radius $\sqrt{2}r$. By the distance argument above, we have that

$$d(T_i, T_j) > d(\tau_i, \tau_j) = d(L_i, L_j) - 2\sqrt{2}r,$$

where the equality is true because the thickening of the $L_j$ into cylinders happens perpendicular to the $L_j$, and the minimal distance between two lines runs perpendicular to both lines as in Figure 3.7. So we need to find $r$ so that $d(L_i, L_j) - 2\sqrt{2}r > 0$ for each pair $(i, j)$.

### 3.2.1 A Bound on $r$

We are now ready to bound $r$ in terms of $R$ to ensure that the tori of the four-way linking are in fact disjoint. Recall that the distance between the lines $p + tu$ and $q + sv$ is computed with the formula

$$d = \frac{|(p - q) \cdot (u \times v)|}{|u \times v|}.$$
$L_1$ to $L_2$:

$$d(L_1, L_2) = \left| \begin{pmatrix} -\sqrt{2}R/2 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} \begin{pmatrix} -1 \\ \sqrt{2 + \sqrt{2}/2} \end{pmatrix} \times \begin{pmatrix} -1 \\ \sqrt{2 - \sqrt{2}/2} \end{pmatrix} \end{pmatrix} \right|$$

$$= \left| \begin{pmatrix} -1 \\ \sqrt{2 + \sqrt{2}/2} \\ \sqrt{2 - \sqrt{2}/2} \end{pmatrix} \times \begin{pmatrix} -1 \\ -\sqrt{2 + \sqrt{2}/2} \\ \sqrt{2 - \sqrt{2}/2} \end{pmatrix} \right|$$

$$= \left| \begin{pmatrix} -\sqrt{2}/2 \\ 0 \\ \sqrt{2 + \sqrt{2}/2} \end{pmatrix} \right| = \frac{|-R/2|}{\sqrt{1/2 + 2 + \sqrt{2}}}$$

$$= \frac{R}{2\sqrt{\frac{5}{2} + \sqrt{2}}}.$$

So we want $r$ such that $\frac{R}{2\sqrt{\frac{5}{2} + \sqrt{2}}} - 2\sqrt{2}r > 0$, giving

$$r < \frac{R}{4\sqrt{5 + 2\sqrt{2}}} \approx 0.08935 \cdot R. \quad (3.5)$$
\[d(L_1, L_3) = \frac{\left| \begin{array}{ccc}
3\sqrt{2}R/4 \\
-\sqrt{2}R/4 \\
0
\end{array} \right| \cdot \left| \begin{array}{ccc}
\sqrt{2} + \sqrt{2}/2 \\
\sqrt{2} - \sqrt{2}/2
\end{array} \right| \times \left| \begin{array}{ccc}
-\sqrt{2} - \sqrt{2}/2 \\
\sqrt{2} + \sqrt{2}/2
\end{array} \right|}{\left| \begin{array}{ccc}
\sqrt{2} + \sqrt{2}/2 \\
\sqrt{2} - \sqrt{2}/2
\end{array} \right| \cdot \left| \begin{array}{ccc}
-\sqrt{2} - \sqrt{2}/2 \\
\sqrt{2} + \sqrt{2}/2
\end{array} \right|}
\]

\[= \frac{\left| \begin{array}{ccc}
3\sqrt{2}R/4 \\
-\sqrt{2}R/4 \\
0
\end{array} \right| \cdot \left| \begin{array}{ccc}
(\sqrt{2} + \sqrt{2}/2) \\
(\sqrt{2} - 2)/4 + \sqrt{2} + \sqrt{2}/2
\end{array} \right|}{\left| \begin{array}{ccc}
(\sqrt{2} + \sqrt{2}/2) \\
(\sqrt{2} - 2)/4 + \sqrt{2} + \sqrt{2}/2
\end{array} \right|}
\]

\[= \frac{\left| \begin{array}{ccc}
3\sqrt{2}R/4 \\
-\sqrt{2}R/4 \\
0
\end{array} \right| \cdot \left| \begin{array}{ccc}
(\sqrt{2} + \sqrt{2} + 2\sqrt{2} - \sqrt{2})/4 \\
(\sqrt{2} - 2 + 2\sqrt{2} + \sqrt{2})/4
\end{array} \right|}{\left| \begin{array}{ccc}
(\sqrt{2} + \sqrt{2} + 2\sqrt{2} - \sqrt{2})/4 \\
(\sqrt{2} - 2 + 2\sqrt{2} + \sqrt{2})/4
\end{array} \right|}
\]

\[= \frac{(3\sqrt{2}(2 + \sqrt{2} + 2\sqrt{2} - \sqrt{2})/16) \cdot R - (\sqrt{2}(\sqrt{2} - 2 + 2\sqrt{2} + \sqrt{2})/16) \cdot R}{\sqrt{23/8 + \sqrt{2}/2 + \sqrt{4 + 2\sqrt{2}/4 + \sqrt{4 - 2\sqrt{2}/4 + \sqrt{2 - \sqrt{2}/2 - \sqrt{2 + \sqrt{2}/2}}}}}
\]

\[= \frac{2 + 4\sqrt{2} + 3\sqrt{4 - 2\sqrt{2} - \sqrt{4 + 2\sqrt{2}}}}{8\sqrt{23/8 + \sqrt{2}/2 + \sqrt{4 + 2\sqrt{2}/4 + \sqrt{4 - 2\sqrt{2}/4 + \sqrt{2 - \sqrt{2}/2 - \sqrt{2 + \sqrt{2}/2}}}}}} R.
\]

Let \( c_1 \) be the coefficient of \( R \). Then we want \( r \) such that \( c_1 R - 2\sqrt{2}r > 0 \), giving

\[ r < \frac{c_1 R}{2\sqrt{2}} \approx 0.1840 \cdot R. \quad (3.6) \]
Let $c_2$ be the coefficient of $R$. Then we want $r$ such that $c_2 R - 2\sqrt{2} r > 0$, giving

$$r < \frac{c_2 R}{2\sqrt{2}} \approx 0.1735 \cdot R.$$  \hfill (3.7)
Let $c_3$ be the coefficient of $R$. Then we want $r$ such that $c_3R - 2\sqrt{2}r > 0$, giving

$$r < \frac{c_3R}{2\sqrt{2}} \approx 0.3510 \cdot R.$$ (3.8)
$L_2$ to $L_4$:

$$d(L_2, L_4) = \left| \begin{pmatrix} \frac{5\sqrt{2}R}{4} \\ \frac{\sqrt{2}R}{4} \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -\frac{1}{\sqrt{2} + \sqrt{2}/2} \\ \sqrt{2} - \sqrt{2}/2 \end{pmatrix} \times \begin{pmatrix} \frac{\sqrt{2} - \sqrt{2}/2}{\sqrt{2} + \sqrt{2}/2} \\ \frac{1}{\sqrt{2} + \sqrt{2}/2} \end{pmatrix} \right|$$

$$= \left| \begin{pmatrix} \frac{5\sqrt{2}R}{4} \\ \frac{\sqrt{2}R}{4} \\ 0 \end{pmatrix} \cdot \begin{pmatrix} \frac{\sqrt{2} - \sqrt{2}/2}{\sqrt{2} + \sqrt{2}/2} \\ \frac{1}{\sqrt{2} + \sqrt{2}/2} \end{pmatrix} \right|$$

Let $c_4$ be the coefficient of $R$. Then we want $r$ such that $c_4 R - 2\sqrt{2}r > 0$, giving

$$r < \frac{c_4 R}{2\sqrt{2}} \approx 0.3633 \cdot R. \quad (3.9)$$
\[ L_3 \text{ to } L_4:\]

\[
d(L_3, L_4) = \left| \begin{pmatrix} 0 & \sqrt{2}R/2 \\ \sqrt{2}R/2 & 0 \end{pmatrix} \cdot \left( \begin{pmatrix} -\sqrt{2} - \sqrt{2}/2 \\ -1 \\ \sqrt{2} + \sqrt{2}/2 \end{pmatrix} \times \begin{pmatrix} \sqrt{2} - \sqrt{2}/2 \\ 1 \\ \sqrt{2} + \sqrt{2}/2 \end{pmatrix} \right) \right| \\
= \left| \begin{pmatrix} 0 & \sqrt{2} + \sqrt{2}/2 \\ \sqrt{2} + \sqrt{2}/2 & 0 \end{pmatrix} \cdot \begin{pmatrix} -\sqrt{2} + \sqrt{2}/2 \\ \sqrt{2}/2 \\ 0 \end{pmatrix} \right| = \frac{|R/2|}{\sqrt{2} + \sqrt{2} + \frac{1}{2}} \\
= \frac{R}{2\sqrt{\frac{3}{2}} + \sqrt{2}}.
\]

Finding \( r \) such that \( \frac{R}{2\sqrt{\frac{3}{2}} + \sqrt{2}} - 2\sqrt{2}r > 0 \) gives

\[
r < \frac{R}{4\sqrt{5} + 2\sqrt{2}}. \tag{3.10}
\]

Due to symmetry, this is the same as the result in (3.5).

Since the smallest of the bounds from (3.5) - (3.10) is given by (3.5), we now have that this equation is a sufficient condition on \( r \) for the four-way linking to be possible. We summarize this in the following lemma.

**Lemma 3.2.1.** Let \( \gamma_1, \ldots, \gamma_4 \) be squares with side length \( 2R \) oriented as in Section 3.2. Suppose \( r > 0 \) satisfies

\[
r < \frac{R}{4\sqrt{5} + 2\sqrt{2}}.
\]
and let $T_1, \ldots, T_4$ be the square tori obtained by thickening $\gamma_1, \ldots, \gamma_4$ by $r$ as in Section 3.1. Then $T_1, \ldots, T_4$ are disjoint and any pair of the $\gamma_1, \ldots, \gamma_4$ form a Hopf link with each other.

### 3.3 Bounding $m$ and Constructing the Chain

We are now ready to go through the details of constructing the chain of linked 2-holed tori that forms the first inductive level in the construction of our Cantor set. As in Section 3.1, let $X_0$ be a 2-holed torus with size coefficient $R$ and thickness coefficient $r$ subject to the constraint of Lemma 3.2.1. Recall from the same section that, in order to define the first chain $X_1$, we need to place bounds on the pair $(k, m)$, where $k$ is the scaling coefficient and $m$ is the number of 2-holed tori in the chain. To achieve this, we will first place a preliminary bound on $k$, then use this bound in finding a minimal number $m$ subject to the $k$-bound required to construct the chain.

Contrary to the Antoine’s Necklace bound in [59], the value we find for $m$ is almost certainly not sharp. That is, it should be possible to construct a geometrically self-similar Cantor set of genus 2 with fewer than $m$ 2-holed tori in the $X_1$-chain. So we can think of $m$ as an upper bound on the minimal number of 2-holed tori it takes to build such a Cantor set.

We now wish to construct solid 2-holed tori $X_{1,1}, \ldots, X_{1,m}$ along the curve $\gamma$ with a four-way linking at the central point of $\gamma$. Denote by $\gamma_j$ the core curve of $X_{1,j}$ similar to $\gamma$ for $X_0$. The $X_{1,j}$ will have size coefficient $kR$ and thickness coefficient $kr$. To bound $k$ initially, we stipulate that the length of each $X_{1,j}$ fit inside half the thickness of $X_0$. In other words,

$$2\sqrt{2}k(2R + r) < r.$$
This ensures that, no matter the orientation of $X_{1,j}$, as long as $\gamma_j$ touches $\gamma$, we have that $X_{1,j}$ is entirely contained in the interior of $X_0$ (see Figure 3.9 for an illustration). Solving for $k$ in the above inequality yields

$$k < \frac{r}{2\sqrt{2}(2R + r)}. \quad (3.11)$$

Henceforth assume that $k$ satisfies (3.11).

To find a minimal value for $m$, we must now discuss the precise construction of $X_1$. For simplicity, position $X_0$ in $\mathbb{R}^3$ such that the self-intersection point of $\gamma$ is at the origin, the length of $X_0$ runs along the $x_1$-axis, and the width of $X_0$ runs along the $x_2$-axis (see Figure 3.10). We can then regard $\gamma$ as split into eight line segments, two in each quadrant of the $x_1x_2$-plane. Let $\gamma'$ be the segment in the first quadrant starting at the origin. If we arrange some small 2-holed tori carefully along $\gamma'$, we can generate the remaining 2-holed tori via rotational symmetries. If done properly, the four-way linking at the origin will happen, and be consistent with the calculations in Section 3.2. Note that in this construction, $m$ will always be a multiple of 8.
Recall that the length of \( \gamma' \) is \( 2R \). Choose a point \( p_1 \) on \( \gamma' \) to be \( \sqrt{2kR}/4 \) Euclidean units from the origin. Now let \( n \) be a sufficiently large even number, and position points \( p_2, \ldots, p_n \) sequentially along \( \gamma' \) such that \( d(p_i, p_{i+1}) = 3\sqrt{2}kR \) for \( i = 1, \ldots, n - 1 \), and such that \( p_n \) is \( 3\sqrt{2}kR - \sqrt{2}kR/4 \) Euclidean units short of the terminal point of \( \gamma' \). Note that \( n \) and \( k \) determine each other, so when we say that \( n \) is ‘sufficiently large,’ we equivalently mean that \( k \) is small enough to satisfy (3.11). Finally, let \( p_{n+1} \) be the point \( \sqrt{2}kR/4 \) units past the terminal point of \( \gamma' \), still in line with \( p_1, \ldots, p_n \).

We will use \( p_1, \ldots, p_{n+1} \) as the anchoring points for the first \( n \) 2-holed tori \( X_{1,1}, \ldots X_{1,n} \). To make this discussion easier, we define orientation vectors for the 2-holed tori. For \( j \in \{1, \ldots, m\} \), let \( v_{j,1} \) and \( v_{j,2} \) we the unit vectors in the direction of the length and width of \( X_{1,j} \), respectively, as in Figure 3.11.

Now the points \( p_1, \ldots, p_{n+1} \) will determine the 2-holed tori \( X_{1,1}, \ldots X_{1,n} \) as follows. Note first that the distance between \( p_i \) and \( p_{i+1} \) is the same as the distance from a lengthwise corner of \( \gamma_i \) to the midpoint of the opposing hole of \( X_{1,i} \). So, for each \( i \), position \( X_{1,i} \) such that \( p_i \) lies in the center of one hole of \( X_{1,i} \), and \( p_{i+1} \) lies at the opposite terminal corner of \( \gamma_i \) (see Figure 3.12). This means that \( v_{i,1} = \frac{\sqrt{2}}{2} (1, 1, 0) \) for each \( i \). To ensure the linking of
consecutive tori, if $i$ is an odd integer between 1 and $n$, let

$$\mathbf{v}_{i,2} = \left\langle \frac{\sqrt{4 + 2\sqrt{2}}}{4}, \frac{-\sqrt{4 + 2\sqrt{2}}}{4}, \frac{\sqrt{2} - \sqrt{2}}{2} \right\rangle,$$

which has an angle of $3\pi/8$ with respect to the $x_3$-axis, and let

$$\mathbf{v}_{i+1,2} = \left\langle \frac{-\sqrt{4 - 2\sqrt{2}}}{4}, \frac{\sqrt{4 - 2\sqrt{2}}}{4}, \frac{\sqrt{2} + \sqrt{2}}{2} \right\rangle,$$

which has an angle of $\pi/8$ with respect to the $x_3$-axis. This way, the widths of the 2-holed tori $X_{1,i}$ and $X_{1,i+1}$ are perpendicular to each other for each $i \in \{1, \ldots, n\}$, guaranteeing linking with the given spacing.

Note the following:
• \(X_{1,1}\) will be part of the four-way linking, and is consistent in angle and position with the 1-holed torus \(T_2\) from the four-way linking computations in Section 3.2.

• The number of tori \(n\) is even.

• The core curve of \(X_{1,n}\) extends past the terminal point of \(\gamma'\) by \(\sqrt{2kR}/4\) units in the \(x_1x_2\)-plane.

Let \(\rho_1\) denote a clockwise (with respect to the \(x_1x_2\)-plane) rotation by the angle \(\pi/2\) around the vertical line through the point \((2\sqrt{2}R, 0, 0)\) (the center of the right hole of \(X_0\)). Then for \(i = 1, \ldots, n\), define \(X_{1,i+n} = \rho_1(X_{1,i})\). Then for \(i = n+1, \ldots, 2n\), we have that

\[
\mathbf{v}_{i,1} = \frac{\sqrt{2}}{2} (1, -1, 0)
\]

and

\[
\mathbf{v}_{i,2} = \begin{cases} 
  \left\langle \frac{-\sqrt{4+2\sqrt{2}}}{4}, \frac{-\sqrt{4+2\sqrt{2}}}{4}, \frac{\sqrt{2}-\sqrt{2}}{2} \right\rangle & \text{if } i \text{ is odd}, \\
  \left\langle \frac{\sqrt{4-2\sqrt{2}}}{4}, \frac{\sqrt{4-2\sqrt{2}}}{4}, \frac{\sqrt{2}+\sqrt{2}}{2} \right\rangle & \text{if } i \text{ is even}.
\end{cases}
\]

We can then see that \(X_{1,n}\) and \(X_{1,n+1}\) are linked in the same manner as the 1-tori \(T_3\) and \(T_2\) in Section 3.2. That is, the relative positions and angles of \(\gamma_n\) and \(\gamma_{n+1}\) are the same as the relative positions and angles between the core squares of the 1-holed tori \(T_3\) and \(T_4\) when we first described the process of four-way linking. So far, we have a chain of \(2n\) linked 2-holed tori along the portion of \(\gamma\) in the first quadrant.

Now apply \((\rho_1)^2\), that is, for \(i = 1, \ldots, 2n\) define \(X_{1,i+2n} = (\rho_1)^2(X_{1,i})\). Similarly to the previous rotation, \(X_{1,2n}\) and \(X_{1,2n+1}\) are linked, again in the same manner as \(T_3\) and \(T_2\) from Section 3.2. Additionally, \(X_{1,1}\) and \(X_{1,4n}\) are now linked, again in a way corresponding to \(T_2\) and \(T_3\) (in that order).

Finally, let \(\rho_2\) denote a rotation by the angle \(\pi\) around the \(x_3\)-axis. For \(i = 1, \ldots, 4n\), define \(X_{1,i+4n} = \rho_2(X_{1,i})\). We then have a chain \(X_1 = \bigcup_{j=1}^{m} X_{1,j}\) of \(m = 8n\) linked 2-holed tori along \(\gamma\). By the rotational symmetry of the four-way linking pointed out in Section 3.2,
X\(_1\) has the desired four-way linking at the origin, consisting of the tori \(X_{1,1}, X_{1,4n}, X_{1,4n+1},\) and \(X_{1,8n}\), corresponding to the 1-holed tori \(T_2, T_3, T_4,\) and \(T_1\), respectively.

Note that, since \(n\) is even, we have that \(m = 8n\) is a multiple of 16. To find a minimal \(m\) for which this construction is possible, recall that each pair of consecutive points from \(p_1, \ldots, p_n\) determines a line segment of length \(3\sqrt{2}kR\). So the total length of the path along which the 2-holed tori \(X_{1,1}, \ldots, X_{1,n}\) are arranged is \(3\sqrt{2}kRn\). But this is equal to the length of \(\gamma'\), which is \(2R\). This gives us the equation

\[
3\sqrt{2}kRn = 2R.
\]

Replacing \(n\) with \(m/8\) and rearranging yields

\[
m = \frac{16}{3\sqrt{2}k}. \tag{3.12}
\]

So, if we choose \(k\) to satisfy (3.11) and such that \(m\) in (3.12) is a multiple of 16, then we can construct the chain \(X_1 = \bigcup_{j=1}^{m} X_{1,j}\). Note that \(k\) must be of the form \(\frac{1}{3\sqrt{2}l}\) for some positive integer \(l\). Alternately, we can substitute the bound for \(k\) in (3.11) into (3.12) and not have to consider \(k\) anymore. However, it is conceptually helpful to keep the scaling coefficient in mind for the final construction.

**Example 3.3.1.** Let \(R = 1\). Then \(r = 0.08\) satisfies Lemma 3.2.1, and any \(k \leq 0.013\) satisfies (3.11). Under these conditions, the maximum value of \(k\) for which \(m\) in (3.12) is a multiple of 16 is \(k = \frac{1}{6\sqrt{2}}\), yielding a chain of \(m = 32\) tori.

It bears repeating that this process of constructing \(X_1\) was not optimal. Relaxing the constraint on \(k\) and arranging the \(X_{1,j}\) differently can almost certainly yield a geometrically self-similar Cantor set using fewer than 32 2-holed tori in \(X_1\). However, the value \(m = 32\) is likely the smallest number of 2-holed tori possible using this particular construction.
method. Furthermore, considering that the minimal number of 1-tori required to construct a geometrically self-similar Antoine’s necklace is 20 (see [59]), having a construction using 32 2-holed tori is likely close to optimal.

3.4 The Cantor Set

Let $X_0$ be a genus 2 solid torus as in Section 3.1, with size coefficient $R$ and thickness coefficient $r$ satisfying Lemma 3.2.1. Let $k$ satisfy (3.11), and accordingly let $m$ be a multiple of 16 satisfying (3.12). Let $X_{1,1}, \ldots, X_{1,m}$ be genus 2 solid tori arranged as in Section 3.3, and fix sense-preserving similarities $\phi_j : X_0 \to X_{1,j}$ for $j = 1, \ldots, m$. Define $X_1 = \bigcup_{j=1}^m X_{1,j}$.

Now for $n \geq 2$, define

$$X_n = \bigcup_{j=1}^m \phi_j(X_{n-1}).$$

Then the set

$$\mathcal{X} = \bigcap_{n=0}^{\infty} X_n$$

is a Cantor set with genus at most 2.

We leave the proof that $\mathcal{X}$ has genus 2 for the next chapter. For now, we will prove that $\mathcal{X}$ is wild. To accomplish this, we will adapt a proof of the wildness of Antoine’s necklace, as found in Chapter 18 of Moise’s book [40]. The goal is to show that $\mathbb{R}^3 \setminus \mathcal{X}$ is not simply connected.

To begin this proof, we first introduce additional geometric structure. Consider the 2-holed tori $X_{1,1}, \ldots, X_{1,m}$, the constituent tori of the chain $X_1$. For each $X_{1,j}$, define two square 2-cells $D_j$ and $D'_j$ such that $\partial D_j$ and $\partial D'_j$ are “longitudinal squares” in $\partial X_{1,j}$; that is $D_j$ and $D'_j$ fill in the holes of $X_{1,j}$. To be more precise, consider the orientation along the core curve $\gamma$ of $X_0$ as shown in Figure 3.13. As in Section 3.3, the indices of the $X_{1,j}$
increase along this orientation, and the tori $X_{1,1}, X_{1,m/2}, X_{1,m/2+1},$ and $X_{1,m}$ are involved in the four-way linking. For $X_{1,j}$, we designate $D_j$ as the disk that appears first as we traverse $\gamma$, and $D'_j$ as the disk that appears second. Hence $D_j$ is punctured by $X_{1,j-1}$ and $D'_j$ is punctured by $X_{1,j+1}$, modulo $m$. See Figure 3.14. The four disks that are involved in the four-way linking are then $D_1, D'_{m/2}, D_{m/2+1},$ and $D'_m$, and they are each punctured by the other three 2-holed tori.
Lemma 3.4.1. Let the $X_{1,j}$ and the $D_j$ and $D'_j$ be defined as above, where $j \in \{1, \ldots, m\}$. Then $\partial X_0$ is a retract of the set

$$X_0 \setminus \left[ \bigcup_{j=1}^{m} X_{1,j} \cup \bigcup_{j=1}^{m} (D_j \cup D'_j) \right].$$

Proof. Note that the set

$$\bigcup_{j=1}^{m} X_{1,j} \cup \bigcup_{j=1}^{m} (D_j \cup D'_j)$$

contains the core curve $\gamma$ of $X_0$. Clearly $\partial X_0$ is a retract of $X_0 \setminus \gamma$, so let $r : X_0 \setminus \gamma \to \partial X_0$ be such a retraction. Restricting $r$ to the set

$$X_0 \setminus \left[ \bigcup_{j=1}^{m} X_{1,j} \cup \bigcup_{j=1}^{m} (D_j \cup D'_j) \right]$$

then gives the desired retraction. \hfill \Box

Now set

$$A_j = D_j \setminus X_{1,j-1} \text{ for } j \neq 1, m/2 + 1,$$

and

$$A'_j = D'_j \setminus X_{1,j+1} \text{ for } j \neq m/2, m.$$

Each $A_j$ and $A'_j$ is then a 1-annulus. At the four-way linking, set

- $A_1 = D_1 \setminus (X_{1,m/2+1} \cup X_{1,m/2} \cup X_{1,m}),$
- $A'_{m/2+1} = D'_{m/2+1} \setminus (X_{1,1} \cup X_{1,m/2} \cup X_{1,m}),$
- $A_{m/2} = D_{m/2} \setminus (X_{1,1} \cup X_{1,m/2+1} \cup X_{1,m}),$ and
- $A'_m = D'_m \setminus (X_{1,1} \cup X_{1,m/2+1} \cup X_{1,m/2}).$

These are then all 3-annuli.
For the upcoming result, we need variations on the following theorem.

**Theorem 3.4.2** ([40], Theorem 16.5). Let \( J_1, J_2, \) and \( J_3 \) be plane polygons, forming sequential Hopf links, let \( D \) be the plane 2-cell whose boundary is \( J_2 \), and suppose that \( D \) is simply punctured by \( J_1 \) and \( J_3 \). Let \( p \) be a closed path in

\[
U = D \setminus (J_1 \cup J_2 \cup J_3).
\]

If \( p \cong e \) in \( \mathbb{R}^3 \setminus (J_1 \cup J_3) \), then \( p \cong e \) in \( U \).

Note first that this theorem also applies without the presence of \( J_3 \). We require this version of Theorem 3.4.2 to handle the case when \( A_j \) (or \( A_j' \)) is a 1-annulus. To address the case when \( A_j \) (or \( A_j' \)) is a 3-annulus, we modify Theorem 3.4.2 in the following lemma.

**Lemma 3.4.3.** Let \( J_1, J_2, J_3, \) and \( J_4 \) be plane polygons, forming pairwise Hopf links as in the four-way linking from Section 3.2. Let \( D \) be the plane 2-cell whose boundary is, without loss of generality, \( J_1 \), and suppose that \( D \) is simply punctured by \( J_2, J_3, \) and \( J_4 \). Let \( p \) be a closed path in

\[
U = D \setminus (J_1 \cup J_2 \cup J_3 \cup J_4).
\]

If \( p \cong e \) in \( \mathbb{R}^3 \setminus (J_2 \cup J_3 \cup J_4) \), then \( p \cong e \) in \( U \).

**Proof.** We illustrate the linking of \( J_1, \ldots, J_4 \) in Figure 3.15. This is analogous with the four-way linking illustrated in Figure 3.4.

Let \( P_0 \) be a base point in \( U \), as in Figure 3.16. Since \( U \) is a 3-annulus, \( \pi_1(U) \) is freely generated by the paths \( g_1, g_2, \) and \( g_3 \), as illustrated. Note that these paths are all non-trivial in \( \mathbb{R}^3 \setminus (J_2 \cup J_3 \cup J_4) \). Furthermore, as elements of \( \pi_1(\mathbb{R}^3 \setminus (J_2 \cup J_3 \cup J_4)) \), the paths \( g_1, g_2, \) and \( g_3 \) have no relations. So \( \{g_1, g_2, g_3\} \) freely generates a subgroup of \( \pi_1(\mathbb{R}^3 \setminus (J_2 \cup J_3 \cup J_4)) \). Hence, if \( p \) is a path in \( U \) with base point \( P_0 \) such that \( p \cong e \) in \( \mathbb{R}^3 \setminus (J_2 \cup J_3 \cup J_4) \), then in particular \( p \cong e \) in \( U \).  
\( \square \)
We now return to our 2-holed tori.

**Lemma 3.4.4.** Let $p$ be a closed path in $\mathbb{R}^3 \setminus X_1$. If $p \simeq e$ in $\mathbb{R}^3 \setminus X_1$, then $p \simeq e$ in $\mathbb{R}^3 \setminus X_0$.

**Proof.** From now, and for the rest of this chapter, if $f : A \to B$ is a map, then $|f|$ denotes the image $f(A)$.

Recall the $A_j$ and $A'_j$, $j \in \{1, \ldots, m\}$, defined above. Unless a distinction is required, we will use the symbol $A_j$ to refer to an arbitrary element of the set of all $A_j$ and $A'_j$. Suppose without loss of generality that $p$ is a PL mapping, and let

$$\Phi : [0, 1]^2 \to \mathbb{R}^3 \setminus X_1$$

be a PL contraction of $p$ to $e$. We can choose $|p|$ and $|\Phi|$ in general position relative to $A_j$ in the following sense: there exists a triangulation $K$ of $[0, 1]^2$ such that, if $\sigma^2$ is a 2-cell of

---

**Figure 3.15:** A four-way linking of plane polygons. Approximate locations of intersections with $D$ marked in red.
Figure 3.16: The 3-annulus $U$ with generators of $\pi_1(U)$.

$K$, and $\Phi(\sigma^2)$ intersects $A_j$, then $\Phi|A_j$ is a simplicial homeomorphism, and $A_j$ contains no vertex of $\Phi(\sigma^2)$.

Let

$$J = \Phi^{-1}(A_j \cap |\Phi|).$$

Note that the set $\Phi(J) = A_j \cap |\Phi|$ could be an arbitrary 1-dimensional polygon without isolated points. However, note also that, since $J$ contains no vertex of $K$, we must have that $J \cap \sigma^2$ is a linear interval joining two points of $\partial \sigma^2$ and containing no vertex of $\sigma^2$. Furthermore, if $\sigma^2_1 \cap \sigma^2_2 = \sigma^1$, and $P \in \sigma^1 \cap J$, then $P$ is the common endpoint of the linear intervals $\sigma^2_1 \cap J$ and $\sigma^2_2 \cap J$. Hence $J$ is locally Euclidean, and we can write $J$ as a finite union of disjoint polygons

$$J = \bigcup_{i=1}^{n} J_i.$$

Let $J_i$ be a component of $J$ that is the boundary of some 2-cell $d_i$ in $[0,1]^2$, such that $d_i$ contains no other component of $J$. Consider the mapping

$$p_i = \Phi|J_i : J_i \to A_j.$$ 

Such a $p_i$ can be regarded as a closed path in $A_j$, with a certain base point $P_0$. 
Now we need to be careful with subscripts and primes. Suppose briefly that $A_j$ is not one of the 3-annuli at the four-way linking of $X_1$. Then, since

$$\Phi(d_i) \subset \mathbb{R}^3 \setminus X_{1,i-1} \text{ (or } \mathbb{R}^3 \setminus X_{1,i+1} \text{ depending)},$$

we have that

$$p_i \cong e \text{ in } \mathbb{R}^3 \setminus X_{1,i-1} \text{ (or } \mathbb{R}^3 \setminus X_{1,i+1}).$$

Without loss of generality, suppose $A_j = A_1$. Then similarly we conclude that

$$p_i \cong e \text{ in } \mathbb{R}^3 \setminus (X_{1,m/2} \cup X_{1,m/2+1} \cup X_{1,m}).$$

In either case, if $A_j$ is a 1-annulus, we apply Theorem 3.4.2, and if $A_j$ is a 3-annulus, we apply Lemma 3.4.3, and we get that

$$p_i \cong e \text{ in } \text{Int}(A_j).$$

Therefore $p_i$ can be extended to give us a PL mapping $\Phi_i : d_i \to A_j$. We can then define a new contraction

$$\Phi' : [0,1]^2 \to \mathbb{R}^3 \setminus X_1$$

by defining $\Phi'|d_i = \Phi_i$ and $\Phi' = \Phi$ elsewhere. Let $N$ be a small connected neighborhood of $d_i$ in $[0,1]^2$. Since $N \setminus d_i$ is connected, so is its image under $\Phi'$. Hence $\Phi'(N)$ approaches $A_j$ from only one side. Therefore we can pull $\Phi(N)$ off of $A_j$. This gives us another new contraction $\Phi''$. By passing from $\Phi$ to $\Phi''$, if we repeat the given setup, we have reduced the number of components of $J$ by at least one. Thus, in a finite number of steps, we get a contraction

$$\Psi : [0,1]^2 \to \mathbb{R}^3 \setminus X_1$$
such that

$$|\Psi| \cap A_j = \emptyset.$$  

We can then reset our notation by defining $\Phi := \Psi$.

We intend to perform this operation for every element in the set of $A_j$ and $A'_j$; $j \in \{1, \ldots, m\}$. We must first verify that, if $|\Phi|$ is already disjoint from, say $A'_{j-1}$, this procedure applied to $A_j$ does not suddenly cause $|\Phi| \cap A'_{j-1}$ to be nonempty. Toward this, note that each $A_j$ (or $A'_j$) intersects its neighbor(s) either in a single linear interval as in Figure 3.17, or else, if we have one of the 3-annuli at the four-way linking, intersects its three neighbors

![Figure 3.17](image)

in three linear intervals in one of two possible configurations as in Figure 3.18. As in the preceding discussion, if $p_i : J_i \to A_j$ is a closed path in $A_j$, then $p_i$ is contractible in $A_j \setminus A'_{j-1}$ (or whichever combination of subscripts and primes applies). Hence it is possible to pull $|\Phi|$ off the $A_j$ and $A'_j$ one at a time in such a way that $|\Phi|$ stays disjoint from previous annuli.

![Figure 3.18](image)
Hence after, say, \(k\) steps we have a contraction

\[
\Psi_k : [0, 1]^2 \to \mathbb{R}^3 \setminus X_1
\]

such that \(|\Psi_k| \cap A_j = \emptyset\) and \(|\Psi_k| \cap A'_j = \emptyset\) for each \(j\). It then follows that

\[
|\Psi_k| \cap \left[ \bigcup_{j=1}^m X_{1,j} \cup \bigcup_{j=1}^m (D_j \cup D'_j) \right] = \emptyset.
\]

Let \(r\) be a retraction

\[
r : X_0 \setminus \left[ \bigcup_{j=1}^m X_{1,j} \cup \bigcup_{j=1}^m (D_j \cup D'_j) \right] \to \partial X_0
\]

guaranteed by Lemma 3.4.1. Define \(r|((\mathbb{R}^3 \setminus X_0))\) to be the identity, and finally let

\[
\rho = r \circ \Psi_k : [0, 1]^2 \to \mathbb{R}^3 \setminus \text{Int}(X_0).
\]

To get a retraction of \(p\) in \(\mathbb{R}^3 \setminus X_0\), we need only pull the image of \(\rho\) slightly off of \(\partial X_0\) into \(\mathbb{R}^3 \setminus X_0\). \(\square\)

**Lemma 3.4.5.** Let \(p\) be a closed path in \(\mathbb{R}^3 \setminus X_0\), and suppose that \(p \simeq e\) in \(\mathbb{R}^3 \setminus X\). Then \(p \simeq e\) in \(\mathbb{R}^3 \setminus X_0\).

**Proof.** Suppose without loss of generality that \(p\) is PL and that there is a PL contraction

\[
\Phi : [0, 1]^2 \to \mathbb{R}^3 \setminus X
\]
of \( p \). If \( |\Phi| \cap X_n \) is nonempty for all \( n \), then \( |\Phi| \cap X \) is nonempty, which is false. Hence \( |\Phi| \cap X_n = \emptyset \) for some \( n \). Let \( C \) be a component of \( X_{n-1} \), and let

\[
X_0' = C \text{ and } X_1' = C \cap X_n.
\]

Then \( X_0' \) and \( X_1' \) are related in the same way as \( X_0 \) and \( X_1 \) via the defining similarities. By the previous lemma, there is a contraction \( |\Phi'| \) of \( p \) to \( e \) in \( \mathbb{R}^3 \setminus X_0' \) that does not intersect any component of \( X_{n-1} \). Hence, after a finite number of steps, we obtain a contraction of \( p \) in \( X_{n-1} \). Iterating this argument on \( n \), we conclude that \( p \) is contractible in \( \mathbb{R}^3 \setminus X_0 \).

**Theorem 3.4.6.** The Cantor set \( \mathcal{X} \) is wild.

**Proof.** We have that \( \pi_1(\mathbb{R}^3 \setminus X_0) \) is nontrivial. Hence there exists a closed path \( p \subset \mathbb{R}^3 \setminus X_0 \) that is not contractible in \( \mathbb{R}^3 \setminus X_0 \). By contraposition of the previous lemma, \( p \) is also not contractible in \( \mathbb{R}^3 \setminus \mathcal{X} \). So \( \pi_1(\mathbb{R}^3 \setminus \mathcal{X}) \) is nontrivial, and hence \( \mathcal{X} \) is wild.

It remains to be shown that \( g(\mathcal{X}) = 2 \). It is tempting to consider the genus 2 Cantor sets \( \mathcal{B} \) and \( \mathcal{Z} \) of Babich and Željko (Examples 1.3.15 and 1.3.16, respectively) to try to find a suitable method to prove this claim. This is however not possible. Babich shows in [3] that \( \mathcal{B} \) is a scrawny Cantor set (see Definition 1.3.14), and then shows that scrawny Cantor sets all have genus at least 2. The Cantor set \( \mathcal{X} \) appears not to be scrawny. In fact, it seems likely that any meridional disk \( D \) of \( X_0 \) actually intersects \( \mathcal{X} \) in a planar Cantor set, let alone in infinitely many points. The proof that \( \mathcal{Z} \) has its desired genus as in [58] relies heavily on \( \mathcal{Z} \) containing a central point of rotational symmetry. The Cantor set \( \mathcal{X} \) has no such point. As such, we must devise our own method of proving that \( g(\mathcal{X}) = 2 \). That is the content of the following chapter.
CHAPTER 4
THE GENUS OF THE CANTOR SET

In the previous chapter, we constructed a geometrically self-similar wild Cantor set \( \mathcal{X} \) with a defining sequence consisting of solid 2-holed tori. With the given defining sequence, it is clear that the genus of \( \mathcal{X} \) in the sense of Definition 1.3.13 is at most 2. In this chapter, we prove that the genus of \( \mathcal{X} \) actually equals 2, and in the following chapter we construct a uqr map \( f \) of polynomial type having \( \mathcal{X} \) as its Julia set. The map \( f \) then satisfies the hypotheses of Theorem 2.0.1.

The proof that \( g(\mathcal{X}) \geq 2 \) will use contradiction. We will suppose that \( \mathcal{X} \) has a defining sequence consisting of compact manifolds with boundary having genus at most 1. Picking out one such solid 1-holed torus, call it \( T \), it will be possible to show that \( T \) does not interact nicely with the defining sequence for \( \mathcal{X} \) from the previous chapter. In particular, to avoid violating the decreasing nature of defining sequences (condition (ii) of Definition 1.3.11), \( \partial T \) will have to separate points of \( \mathcal{X} \) in a way that is impossible for a genus 1 surface.

4.1 Alternate Defining Sequences

For convenience, the defining sequence for \( \mathcal{X} \) given in Chapter 3 will henceforth be called the standard defining sequence of \( \mathcal{X} \). Suppose that \( g(\mathcal{X}) = 1 \), that is suppose that there exists a defining sequence \( (M_i) \) for \( \mathcal{X} \) such that \( g(\mathcal{X}; (M_i)) = 1 \). Let us unpack what the previous statement implies.
First note that, although \( g(\mathcal{X}; (M_i)) \) is defined using a supremum, since \( g(M) \) is a non-negative integer for any cube with handles \( M \), we can replace that sup with a max. Hence \((M_i)\) must contain a solid 1-holed torus as one of its components.

Furthermore, the defining sequence \((M_i)\) must actually contain arbitrarily small solid 1-holed tori. To see this, suppose to the contrary that there exists \( N \in \mathbb{N} \) such that, for all \( n > N \), the manifold \( M_n \) is a disjoint union of closed balls. Since \( M_{i+1} \subset \text{Int}(M_i) \) for each \( i \) (see Definition 1.3.11), we can see that
\[
\mathcal{X} = \bigcap_{i \geq 1} M_i = \bigcap_{i > N} M_i,
\]
or in other words, the first finitely many terms of the defining sequence are irrelevant. Then \((M_i)_{i > N}\) is a defining sequence of \( \mathcal{X} \) consisting entirely of closed balls. We would then conclude that \( g(\mathcal{X}) = 0 \), which is a contradiction.

So, given \( \epsilon > 0 \) arbitrarily small, we can find a solid 1-holed torus \( T_\epsilon \) which is a component of \((M_i)\) satisfying \( \text{diam}(T_\epsilon) \leq \epsilon \). Given such a \( T_\epsilon \), we can now bring back the standard defining sequence and ask how its components interact with \( T_\epsilon \). In particular, we can consider how \( T_\epsilon \) intersects with the 2-holed tori of the standard defining sequence.

Let \( X_{i,j} \) be such a 2-holed torus, and assume that \( T_\epsilon \cap X_{i,j} \) is non-empty. For all arguments in this chapter, we would like to have the property that \( \partial T_\epsilon \cap \partial X_{i,j} = \emptyset \). Just from the definition of a defining sequence, there is no immediate reason that this should be case for an arbitrarily chosen \( T_\epsilon \). However, we can still make this assumption, after observing the following. First, since \( T_\epsilon \) can be chosen to be arbitrarily small, we can assume that \( T_\epsilon \subset \text{Int}(X_{i,j}) \) for some \( X_{i,j} \) with \( i \) relatively small. Now, it is possible that \( X_{i,j} \) contains a component of the standard defining sequence, call it \( X_{i',j'} \), such that \( \partial T_\epsilon \cap \partial X_{i',j'} \neq \emptyset \). But notice that \( \mathcal{X} \cap T_\epsilon \) and \( \partial T_\epsilon \) are disjoint compact sets in a \( T_4 \) space. This means that \( d(\mathcal{X} \cap T_\epsilon, \partial T_\epsilon) = \delta > 0 \). It is then possible to find components of the standard defining sequence...
sequence all having diameter less than $\delta$ that cover $\mathcal{X} \cap T_\epsilon$, and so, by going finitely many steps down from $X_{i,j}$ along the inductive construction of the Cantor set from Chapter 3, we get a collection of 2-holed tori whose boundary does not intersect $\partial T_\epsilon$. Using the same $T_4$ reasoning on $\mathcal{X} \cap (\overline{X_{i,j} \setminus T_\epsilon})$, we can now guarantee that, at some level, say $i + N$, $\partial T_\epsilon \cap \partial X_{i+N} = \emptyset$.

There is an important consequence to note here. Since $T_\epsilon$ is a component of a defining sequence for $\mathcal{X}$, we have that $T_\epsilon \cap \mathcal{X}$ is nonempty. Also, by the previous paragraph, the boundary of $T_\epsilon$ does not intersect the boundary of any 2-holed torus at the $i + N$ level. In fact, this is also true at the $i + n$ level for every $n \geq N$. So, we now have that, from the $i + N$ level on, $\text{Int}(T_\epsilon)$ contains some components of the standard defining sequence, and that $\partial T_\epsilon$ separates those components from the remaining 2-holed tori at the same level. In other words, from some level on, $T_\epsilon$ contains only whole 2-holed tori and separates them from all the other whole 2-holed tori. Since we only ever need to go finitely many steps from a 2-holed torus containing $T_\epsilon$ to a point where $T_\epsilon$ has this separation property, we will just assume that the boundary of any 1-holed torus from $(M_i)$ does not intersect the boundary of any 2-holed torus from the standard defining sequence.

4.2 Linking Lemmas

The heavy lifting in most of the proofs of this chapter will be accomplished by the two lemmas in this section. These lemmas establish a kind of transitive property of linking. We believe these facts to be well-known, but we include their proofs as we were unable to find a reference.

First, we introduce some terms. Recall Definition 1.4.7, that of a split link, and that this definition applies for disjoint unions of handlebodies as well. Conversely, if $L$ is a link of
handlebodies for which there does not exist a 2-sphere that separates some components of \( L \), we will call \( L \) a *chain* of handlebodies. With this in mind, consider the following definition.

**Definition 4.2.1.** Let \( K \) and \( L \) be disjoint chains of handlebodies in \( \mathbb{R}^3 \). We will say \( K \) and \( L \) are *unlinked* if there exists a 2-sphere \( S \) embedded in \( \mathbb{R}^3 \setminus (K \cup L) \) such that \( K \) and \( L \) are on opposite sides of \( S \). If no such sphere exists, we will say \( K \) and \( L \) are *linked*.

This definition may seem unnecessary, as it simply replaces the terms split and non-split. We introduce it nonetheless, for two reasons. First, we are particularly interested in non-split links, and it seems more appropriate to refer to the pertinent concept in positive rather than negative terms. Second, referring to two specific unions of handlebodies, here \( K \) and \( L \), and addressing the linking relationship between them is easier with Definition 4.2.1. That is, we regard \( K \) and \( L \) as entities in their own right, each possessing the property of being non-split, and then turn our attention to the relationship between \( K \) and \( L \). This will help us maintain the correct mental image of the relationships between all the different chains later in the chapter.

With this in mind, we turn to the linking lemmas.

**Lemma 4.2.2.** Let \( T \subset \mathbb{R}^3 \) be a solid 1-holed torus, and let \( H \subset T \) be a chain of handlebodies that is linked with \( \mathbb{R}^3 \setminus T \). If \( C \) is a chain of handlebodies in \( \mathbb{R}^3 \setminus T \) that is linked with \( T \), then \( C \) is also linked with \( H \).

**Proof.** Suppose for the sake of contradiction that \( C \) and \( H \) are unlinked. Then there exists a 2-sphere \( S \) embedded in \( \mathbb{R}^3 \setminus (C \cup H) \) that separates \( C \) and \( H \) from each other. Let \( U^+ \) and \( U^- \) be the components of \( \mathbb{R}^3 \setminus (C \cup H) \) containing \( H \) and \( C \), respectively. Note that \( \partial U^+ = \partial U^- = S \).

Now, since \( T \) and \( C \) are linked, we must have that some subset of \( T \) lies in \( U^- \). With that in mind, consider the set \( \partial(U^+ \cup T) \). This set consists of points of \( S \), together with the
points of $\partial T$ that live in $U^-$. More precisely, it is $S$ together with all the points of $\partial T$ living in $U^-$, with all the interiors of the components of the 2-dimensional intersection $\text{Int}(T) \cap S$ removed. It is possible that some subsets of $T$ protrude into $U^-$ and then have a piece that intersects $\overline{U^+}$ in a curve or point, see Figure 4.1 for an illustration of such a case. In fact,

![Figure 4.1: An undesirable intersection of $T$ and $S$.](image)

such an intersection may not even be a well-defined manifold. Such subsets of $T$ contribute structure to $\partial(U^+ \cup T)$ that is undesirable for a later part of the argument, so we will avoid them in the following way.

Define the set

$$A = \{ x \in \partial(U^+ \cup T) \mid \text{no neighborhood of } x \text{ is homeomorphic to a disk} \},$$

that is, $A$ is the collection of points where $\partial(U^+ \cup T)$ fails to be a 2-manifold. Then, since $A$ and $C$ are disjoint closed sets, there is a minimum distance between them, call it $\delta_1 > 0$. To ensure that later on we do not accidentally pick up additional points from $T$, consider the closed set $\overline{T \cap U^+}$. This set is also disjoint from $A$, since points of $A$ necessarily arise on the boundary of parts of $\text{Int}(T)$ living in $U^-$. So there is another minimum distance $\delta_2 > 0$ between $\overline{T \cap U^+}$ and $A$. Let $\delta = \min\{\delta_1, \delta_2\}$.
Now consider the open cover 

\[ \{ B(x, \delta/2) \mid x \in A \} \]

of \( A \). Since \( A \) is compact, there exist \( x_1, \ldots, x_n \) for some \( n \in \mathbb{N} \) such that

\[ \{ B(x_i, \delta/2) \mid 1 \leq i \leq n \} \]

is an open cover of \( A \). Note that the open set \( V = \bigcup_{i=1}^{n} B(x_i, \delta/2) \) has the property that \( \partial V \) separates \( A \) from both \( C \) and \( \overline{T \cap U^+} \). Note also that \( \partial V \) is a compact 2-manifold without boundary consisting of finitely many connected components. Now consider the set \( M = (S \setminus V) \cup (\partial V \cap U^+) \). Then \( M \) is the result of attaching the half of \( \partial V \) living in \( U^+ \) to \( S \) and removing the 2-dimensional interior of \( S \cap V \). So \( M \) is itself a 2-sphere, \( M \) still separates \( H \) from \( C \), and \( M \) has no undesirable intersections with \( T \). For notational convenience, redefine \( S \) to be \( M \), and let \( U^+ \) and \( U^- \) be the connected components of \( \mathbb{R}^3 \setminus S \) containing \( H \) and \( C \), respectively.

Now that \( \partial(U^+ \cup T) \) no longer has points where it fails to be a 2-manifold, we conclude that it is a 2-manifold. In fact, \( \partial(U^+ \cup T) \) is connected, compact, and by the discussion in [34], orientable. Hence by the Classification of Closed Surfaces (see for example [33, Theorem 4.14]), \( \partial(U^+ \cup Y) \) is determined entirely by its genus \( g \).

Suppose that \( g = 0 \). Then \( \partial(U^+ \cup T) \) is a 2-sphere separating \( \text{Int}(T) \) from \( C \). In particular, we now have that the core curve \( t \) of \( T \) is separated from \( C \) by a 2-sphere. But this contradicts the fact that \( T \) and \( C \) are linked.

Now suppose that \( g > 0 \). Then \( \partial(U^+ \cup T) \) is a surface handlebody with, say, \( n \) handles for some \( n \in \mathbb{N} \). This means that there exist \( n \) properly embedded topological circles \( c_1, \ldots, c_n \subset \partial(U^+ \cup T) \) such that, removing \( c_1, \ldots, c_n \) from \( \partial(U^+ \cup T) \), making a small
amount of space along the gaps resulting from the removal, and attaching $2n$ disks to the removal sites, the resulting surface is a 2-sphere (see Figure 4.2). Due to the fact that all the handles of $\partial(U^+ \cup T)$ arise from $T \cap U^-$, the circles $c_1, \ldots, c_n$ can all be chosen to be subsets of $T \cap U^-$. We can then fill in each circle $c_i$ to form a disk $d_i \subset T \cap U^-$. We claim that at least one of these disks must intersect the core curve $t$ of $T$, and is hence a meridional disk of $T$.

Suppose that this is not the case, that is, suppose that there exists a choice of circles $c_1, \ldots, c_n \subset \partial T \cap U^-$ such that each $c_i$ can be filled in to form a disk $d_i \subset T \cap U^-$. We then remove the $c_i$ from $\partial(U^+ \cup T)$, make some space as in Figure 4.2, and fill in one side of each removal site with the corresponding disk $d_i$. Since each $d_i$ is a closed set disjoint from the closed set $t$, there is an $\epsilon_i$-neighborhood of $d_i$ that is still disjoint from $t$. This means we can attach another disk $d_i'$ to the other side of the removal side that is also disjoint from $t$. The resulting surface is a 2-sphere that separates $t$ from $C$. This is a contradiction since the linking of $T$ and $C$ implies that $t$ and $C$ are linked.

Hence we can find a meridional disk $D$ of $T$ lying in $U^-$, where $\partial D$ is one of the circles $c_1, \ldots, c_n$. If we can show that $H \cap D \neq \emptyset$, then we obtain a contradiction since $D \subset U^-$ and $H \cap U^- = \emptyset$.

To that end, suppose that $H \cap D = \emptyset$. Since $H$ and $D$ are both compact, there is a minimum distance $d > 0$ between them. It follows that $H$ is contained in a subset $B$ of $T$ that is homeomorphic to $\mathbb{B}^3$. But then $\partial B$ is a 2-sphere separating $H$ from $\mathbb{R}^3 \setminus T$. This is contradiction, since $H$ and $\mathbb{R}^3 \setminus T$ are linked. \qed
Lemma 4.2.3. Let $T \subset \mathbb{R}^3$ be a solid 1-holed torus, and let $H \subset T$ be a chain of handlebodies. If $C$ is a chain of handlebodies in $\mathbb{R}^3 \setminus T$ that is linked with $H$, then $C$ is also linked with $T$. Furthermore, this implies that $H$ is linked with $\mathbb{R}^3 \setminus T$.

Proof. Suppose for the sake of contradiction that $C$ and $T$ are unlinked. Then there is a 2-sphere $S$ embedded in $\mathbb{R}^3 \setminus (C \cup T)$ separating $C$ and $T$. But, since $H \subset T$, we have that $C$ and $H$ are separated by $S$, and are hence unlinked. This is a contradiction.

For the second claim, notice that $C \subset \mathbb{R}^3 \setminus T$ cannot be separated from $H$ be a 2-sphere. So $H$ is also linked with $\mathbb{R}^3 \setminus T$. \hfill \Box

4.3 Getting a Handle on Separation

Armed with the linking lemmas, and the assumption from Section 4.1 that a 1-holed torus can be chosen to only contain whole 2-holed tori, we are ready to begin proving that $g(\mathcal{X}) > 1$. To accomplish this, we will use induction to prove that a 1-holed torus from $(M_i)$ cannot separate any subcollection of 2-holed tori at a given level of the standard defining sequence from the rest of the 2-holed tori at the same level. This will prove that almost all of the 2-holed tori of the standard defining sequence are in some sense essential to the structure of $\mathcal{X}$.

First, a quick note to explain why the previous sentence said ‘almost all’ instead of simply ‘all’. Recalling the notation of Chapter 3, consider $X_0$, the largest 2-holed torus of the standard defining sequence. In the construction of the Cantor set $\mathcal{X}$, we take a countable intersection, and so the first finitely many terms don’t impact the structure of $\mathcal{X}$. In particular, it is perfectly possible to replace $X_0$ with a solid 1-holed torus, or even a ball, as long as it contains all the 2-holed tori of $X_1$. For many of the arguments of this chapter,
this easy method of replacing $X_0$ is undesirable. So we begin here by introducing notation for the remainder of the chapter, which unfortunately conflicts with that of Chapter 3.

Henceforth, let $X_0$ be a 2-holed torus from the standard defining sequence at the $N$th level for some $N \geq 1$, and let $T$ be a solid 1-holed torus such that $T \subset \text{Int}(X_0)$. Let $X_{1,1}, \ldots, X_{1,m}$ be the $(N + 1)$-level 2-holed tori contained in $X_0$, and define $X_1 = \bigcup_{i=1}^{m} X_{1,i}$. We will sometimes call $X_1$ the *full chain*, and we will call the tori $X_{1,1}, \ldots, X_{1,m}$ the *1-elements* of $X_1$. The reasoning for the numerical prefix will become clear in a later section.

To discuss the structure of $X_1$, we will use the following terms.

**Definition 4.3.1.** Let $Y$ be the union of some non-empty subset of the set of 1-elements of $X_1$. We will call $Y$ a *subchain* of the full chain $X_1$. By the *complement* of a subchain $Y$, we mean $X_1 \setminus Y$, the complementary subchain. If $Y$ is not a split link, then call $Y$ an *arc* of 1-elements.

Let us unpack the concept of an arc. First, it is worth noting that, given an arc $Y$, it is not necessarily the case that $X_1 \setminus Y$ is an arc (see Figure 4.3 for an example). However, if $Y$ is an arbitrary subchain of $X_1$, we can regard $Y$ as consisting of a finite collection of disjoint arcs which are as large as possible. To be more precise, suppose $Y_1 \subset Y$ is an arc. If there does not exist any arc $Y'$ such that $Y_1 \subsetneq Y' \subseteq Y$, then we will say $Y_1$ is a *maximal arc* of $Y$ (see Figure 4.4 for an example of a maximal vs non-maximal arc). Thus $Y$ can be decomposed uniquely into disjoint maximal arcs.
To see one reason why maximal arcs are useful, suppose that $Y_1$ is a maximal arc of a subchain $Y$. We then have that $Y_1$ is linked with $X_1 \setminus Y$, since otherwise $Y_1$ would not be maximal. This means that the maximal arcs of a subchain are in some sense the largest link-connected components of $Y$ that link directly with $X_1 \setminus Y$.

It is the decomposition of a subchain $Y$ into maximal arcs that illustrates why we focus on arcs. We will show in this section that the boundary of the torus $T$ cannot separate $Y$ from its complement $X_1 \setminus Y$. Now, if $\partial T$ did separate $Y$ from $X_1 \setminus Y$, then in particular, $\partial T$ would separate any maximal arc $Y_1$ of $Y$ from $X_1 \setminus Y_1$. Hence we can restrict our attention to investigating separation of arcs from their complements, rather than arbitrary subchains.

So let $Y$ be an arc of $X_1$, and without loss of generality suppose that $Y \subset \text{Int}(T)$. To discuss how $\partial T$ might try to separate $Y$ from $X_1 \setminus Y$, we need to get a handle on how $Y$ is linked with $X_1 \setminus Y$. In particular, we would like to know when $Y$ is linked with its complement in multiple locations. Let us first consider an example where this is not the case. Suppose that $Y$ is one of the two loops of $X_1$. In other words, here $Y$ consists of half of the 1-elements of $X_1$, and $Y$ forms a loop of 1-elements around one of the holes of $X_0$. In this case, $Y$ is linked with $X_1 \setminus Y$ at the four-way linking point, so in some sense $Y$ is linked with its complement in only one location. It is the case that two different 1-elements of $Y$ participate in the linking with $X_1 \setminus Y$, but those two 1-elements are also linked with each other.
The above situation is not typical. In most cases, $Y$ links with its complement via two 1-elements of $Y$ that are not themselves linked. We will address this scenario separately from the one above where $Y$ is a loop of 1-elements.

**Lemma 4.3.2.** Let $Y$ be an arc of $X_1$ that is not a loop of 1-elements, and suppose that $Y \subset \text{Int}(T)$. Then $\partial T$ cannot separate $Y$ from $X_1 \setminus Y$.

*Proof.* Let $X_{1,i}$ and $X_{1,j}$ be elements of $X_1 \setminus Y$ that are linked with $Y$. Note that if there is only one 1-element in $X_1 \setminus Y$, then $X_1 \setminus Y$ is itself an arc linked with two complementary 2-holed tori. So without loss of generality, assume $i \neq j$.

For the sake of discussing the structure of the 2-holed tori involved, regard a 2-holed torus as a solid cube with handles, where by *handle* we mean a subset of the 2-holed torus containing exactly one generator of its first homology group. With this in mind, since $Y$ is not a loop of 1-elements, we can choose $X_{1,i}$ and $X_{1,j}$ so that not both of these complementary tori are involved in the four-way linking of $X_1$. This ensures that $X_{1,i}$ and $X_{1,j}$ are not themselves linked.

Now let $Y_1$ be the 1-element of $Y$ that is linked with $X_{1,i}$, and denote by $y_1$ the handle of $Y_1$ that is linked with the handle $x_i$ of $X_{1,i}$. Similarly, let $Y_2$ be the 1-element of $Y$ that is linked via the handle $y_2$ with the handle $x_j$ of $X_{1,j}$. Again, since $Y$ is not a loop of 1-elements, and since at most one of $X_{1,i}$ and $X_{1,j}$ is involved in the four-way linking, we have that $Y_1 \neq Y_2$, and that neither $y_1$ nor $y_2$ is linked with both of $x_i$ and $x_j$.

Furthermore, both $y_1$ and $y_2$ are contained in $T$, and both $x_i$ and $x_j$ are contained in $\mathbb{R}^3 \setminus T$. It then follows from Lemma 4.2.3 that $x_i$ and $x_j$ are both linked with $T$. The same lemma additionally implies that in particular $y_1$ is linked with $\mathbb{R}^3 \setminus T$. But then by Lemma 4.2.2, $y_1$ is also linked with $x_j$. So $Y$ is linked with both $x_i$ and $x_j$. This is a contradiction. Hence $\partial T$ cannot separate $Y$ from $X_1 \setminus Y$. \qed
Lemma 4.3.3. Let $Y$ be a loop of $X_1$, and suppose that $Y \subset \text{Int}(T)$. Then $\partial T$ cannot separate $Y$ from $X_1 \setminus Y$.

Proof. Suppose for the sake of contradiction that $\partial T$ separates $Y$ from $X_1 \setminus Y$. Let $X'_0$ be the $N$-level 2-holed torus that is linked with $X_0$ through the handle of $X_0$ containing $Y$ (see Figure 4.5 for illustration). Since $T \subset \text{Int}(X_0)$, we have that $X'_0 \subset \mathbb{R}^3 \setminus T$. Note that $Y$ is linked with $X'_0$, and so Lemma 4.2.3 implies that $T$ is linked with $X'_0$. Additionally, since $T$ separates $Y$ from $X_1 \setminus Y$ at the four-way linking of $X_1$, Lemma 4.2.3 also gives us that $T$ is linked with a 1-element of $X_1 \setminus Y$ that is involved in the four-way linking, call it $X_{1,j}$.

Let $Y_1$ be a 1-element of $Y$ that is linked with $X_{1,j}$. Then, again by Lemma 4.2.3, $Y_1$ is linked with $\mathbb{R}^3 \setminus T$. So, by Lemma 4.2.2, $Y_1$ is linked with $X_0$. But this is impossible, since $Y_1$ is an $(N + 1)$-level 2-holed torus, and in the construction of the standard defining sequence, 2-holed tori from different levels are not linked with each other. Hence $\partial T$ cannot separate $Y$ from $X_1 \setminus Y$. \hfill $\square$

We expand on the previous two lemmas in the following theorem. This completes our examination of $X_1$.

Theorem 4.3.4. Let $X_0$ be a 2-holed torus from the standard defining sequence that is not the first one, and let $X_1 = \bigcup_{j=1}^m \varphi_j(X_0)$. If $T \subset \text{Int}(X_0)$ is a solid 1-holed torus, then $\partial T$ can neither separate any proper subchain of $X_1$ from its complement, nor $X_1$ itself from $\partial X_0$. 

Figure 4.5: The linking of $Y$ and $X'_0$. 

linked with $X'_0$, and so Lemma 4.2.3 implies that $T$ is linked with $X'_0$. Additionally, since $T$ separates $Y$ from $X_1 \setminus Y$ at the four-way linking of $X_1$, Lemma 4.2.3 also gives us that $T$ is linked with a 1-element of $X_1 \setminus Y$ that is involved in the four-way linking, call it $X_{1,j}$.

Let $Y_1$ be a 1-element of $Y$ that is linked with $X_{1,j}$. Then, again by Lemma 4.2.3, $Y_1$ is linked with $\mathbb{R}^3 \setminus T$. So, by Lemma 4.2.2, $Y_1$ is linked with $X_0$. But this is impossible, since $Y_1$ is an $(N + 1)$-level 2-holed torus, and in the construction of the standard defining sequence, 2-holed tori from different levels are not linked with each other. Hence $\partial T$ cannot separate $Y$ from $X_1 \setminus Y$. \hfill $\square$

We expand on the previous two lemmas in the following theorem. This completes our examination of $X_1$.

Theorem 4.3.4. Let $X_0$ be a 2-holed torus from the standard defining sequence that is not the first one, and let $X_1 = \bigcup_{j=1}^m \varphi_j(X_0)$. If $T \subset \text{Int}(X_0)$ is a solid 1-holed torus, then $\partial T$ can neither separate any proper subchain of $X_1$ from its complement, nor $X_1$ itself from $\partial X_0$. 


Proof. Let $Y$ be a proper subchain of $X_1$. Then $Y$ can be subdivided into maximal arcs. Let $Y_1$ be such an arc. Lemmas 4.3.2 and 4.3.3 show that, regardless of whether $Y_1$ is a loop of 1-elements or not, $\partial T$ cannot separate $Y_1$ from $X_1 \setminus Y_1$, and hence $\partial T$ cannot separate $Y$ from its complement.

Now suppose that $X_1 \subset \text{Int}(T)$. Let $X_0'$ and $X_0''$ be the two $N$-level 2-holed tori that are linked with $X_0$, one through each handle of $X_0$. Since $T \subset \text{Int}(X_0)$, we have that $X_0'$ and $X_0''$ are both contained in $\mathbb{R}^3 \setminus T$. Since $X_0'$ and $X_0''$ are both linked with $X_1$, albeit through different loops of $X_1$, by Lemma 4.2.3 $T$ is also linked with both $X_0'$ and $X_0''$. Additionally, both loops of $X_1$ are linked with $\mathbb{R}^3 \setminus T$. But then by Lemma 4.2.2, in particular $X_0'$ is linked with both loops of $X_1$, which contradicts the construction of the standard defining sequence. Hence $\partial T$ cannot separate $X_1$ from $\partial X_0$.

\[ \square \]

4.4 Separating $X_2$

Before moving on to the inductive step, it is instructive to examine the case of separation at the next level down in the standard defining sequence. The structure of subchains gets much more complicated, and it is not necessarily obvious that the eventual inductive argument is correct without first becoming oriented at this new level.

To start, let $X_2$ be the union of all $(N + 2)$-level 2-holed tori contained in the $N$-level 2-holed torus $X_0$. We will be breaking down the possible structures of a subchain of $X_2$ into several cases. Rather than front-loading all of the terminology, definitions for the types of structures will be given as needed. In each case, we will prove that $\partial T$ cannot separate subchains of the given type from their complements. The cases will be addressed in increasing order of structural complexity.
Adapting terminology from the previous section, we call the individual 2-holed tori whose union is $X_2$ the 2-elements of $X_2$. The function of the numerical prefix will become clear shortly.

**Definition 4.4.1.** Let $Y$ be the union of some non-empty subset of the set of 2-elements of $X_2$. We will call $Y$ a subchain of the full chain $X_2$.

**Definition 4.4.2.** If a subchain $Y$ of $X_2$ consists entirely of all the 2-elements contained inside a single $X_1$-level 2-holed torus, i.e. if $Y = \bigcup_{j=1}^{m} \varphi_i(\varphi_j(X_0))$ for some $i \in \{1, \ldots, m\}$, then $Y$ is called a 1-element of $X_2$.

It should be noted that a 1-element in the $X_2$ setting is different from a 1-element in the $X_1$ setting. There is however a nice relationship between the two, specifically, a 1-element in the $X_2$ setting is simply the intersection of $X_2$ with a 1-element in the $X_1$ setting.

To address the simplest kind of structure for a subchain $Y$ of $X_2$, we reintroduce the concept of arcs from the previous section, though there is a notational change. Again, the reason for the change will become clear later on.

**Definition 4.4.3.** We say $Y$ is an arc of 2-elements of $X_2$ if $Y$ consists of tori from the same 1-element of $X_2$, and $Y$ is not a split link. If in particular $Y$ is exactly one of the loops of a given 1-element, we will call $Y$ a loop of 2-elements.

This definition gives us the simplest structural case for a subchain $Y$ of $X_2$. As in the previous section, if $Y$ is an arbitrary subchain of a single 1-element of $X_2$, then $Y$ decomposes into maximal arcs of 2-elements, so focusing on arcs is justified. Also, we will assume without loss of generality that $Y \subset \text{Int}(T)$, as in the previous section.

**Lemma 4.4.4.** Let $Y$ be an arc of 2-elements from a 1-element $M$ of $X_2$. If $Y \subset \text{Int}(T)$, then $\partial T$ cannot separate $Y$ from $X_2 \setminus Y$. 
The proof of this lemma will proceed by using the exact argument from the proof of Lemma 4.3.2 and a slight modification of the argument from the proof of Lemma 4.3.3. We include the adapted set-up for the $X_2$ setting, and refer to the $X_1$ proofs where the argument becomes identical.

**Proof of Lemma 4.4.4.** Assume for the sake of contradiction that $\partial T$ separates $Y$ from $X_2 \setminus Y$. Suppose then that $Y$ is not a loop of 2-elements. We then have, without loss of generality, that $Y$ is linked with two distinct 2-elements of $M \setminus Y$, say $M_1$ and $M_2$, and that this linking takes place via two different handles of 2-elements of $Y$, call the handles $y_1$ and $y_2$. Additionally, we may again assume that not both of $M_1$ and $M_2$ are involved in the four-way linking in $M$. Applying Lemmas 4.2.2 and 4.2.3 as in the proof of Lemma 4.3.2, we obtain that in particular $M_1$ is linked with both handles $y_1$ and $y_2$, which contradicts the construction of the standard defining sequence.

Now suppose that $Y$ is a loop of 2-elements. Then we need only make a single adjustment to the argument in the proof of Lemma 4.3.3 to be finished. Specifically, replace $X'_0$ in the given proof with the $(N+1)$-level 2-holed torus that is linked with $Y$.

For the second structural case, assume $Y$ is a union of arcs of 2-elements, where not all the maximal arcs come from the same 1-element of $X_2$. This is the first case that is meaningfully different from the $X_1$ setting. However, we can quickly reduce this case down to only one situation. Suppose that one of the maximal arcs of $Y$, call it $Y_1$, is not a loop of 2-elements. Then the argument from the proof of Lemma 4.3.2 applies exactly as it did for Lemma 4.4.4. So, if $Y$ contains any maximal arcs that are not loops, $\partial T$ cannot separate $Y$ from $X_2 \setminus Y$.

Furthermore, suppose that $Y$ contains a maximal arc $Y_1$ that is a loop of 2-elements. If $Y$ does not contain the loop of 2-elements that is linked with $Y_1$, then the argument in the proof of Lemma 4.3.3 as used in Lemma 4.4.4 applies, and $\partial T$ cannot separate $Y$ from
$X_2 \setminus Y$. The only situation remaining is if $Y$ does contain that loop. So, denote by $Y_2$ the loop of 2-elements that is linked with $Y_1$, and suppose that $Y$ contains both of these loops as maximal arcs. See Figure 4.6 for an illustration. In this case, the argument of Lemma 4.3.3 does not apply to either $Y_1$ or $Y_2$, since $T$ contains part of the linking 2-holed torus for both $Y_1$ and $Y_2$. So we handle this situation separately.

**Lemma 4.4.5.** Suppose that $Y$ consists of two loops of 2-elements, say the loops $Y_1$ and $Y_2$ contained in the 1-elements $M_1$ and $M_2$, respectively. Suppose also that $Y_1$ and $Y_2$ are linked. If $Y \subset \text{Int}(T)$, then $\partial T$ cannot separate $Y$ from $X_2 \setminus Y$.

**Proof.** Suppose for the sake of contradiction that $\partial T$ separates $Y$ from $X_2 \setminus Y$. Note that a 2-element of $Y_1$, say $y_1$, is linked with a 2-element $y_1'$ of $M_1 \setminus Y_1$. Here $y_1$ and $y_1'$ are involved in the four-way linking at the center of $M_1$. Similarly, a 2-element $y_2$ of $Y_2$ is linked with a 2-element $y_2'$ of $M_2 \setminus Y_2$ at the four-way linking in $M_2$. Since $y_1'$ and $y_2'$ are both contained in $\mathbb{R}^3 \setminus T$, we have by Lemma 4.2.3 that $y_1'$ and $y_2'$ are both linked with $T$. Moreover, Lemma 4.2.3 also implies that in particular $y_1$ is linked with $\mathbb{R}^3 \setminus T$. It then follows from Lemma 4.2.2 that $y_1$ is also linked with $y_2'$. But this contradicts the construction of the standard defining sequence, since $y_1$ and $y_2'$ are 2-elements of disjoint 1-elements. Hence $\partial T$ cannot separate $Y$ from $X_2 \setminus Y$. \qed
Combining Lemmas 4.4.4 and 4.4.5, we have so far shown that \( \partial T \) cannot separate \( Y \) from \( X_2 \setminus Y \) if \( Y \) consists entirely of arcs of 2-elements. To move on to the next structural case for a subchain of \( X_2 \), we introduce a different kind of arc.

**Definition 4.4.6.** We say \( Y \) is an *arc of 1-elements* of \( X_2 \) if \( Y \) is a union of 1-elements, and \( Y \) is a non-split link. If the \( X_1 \)-level tori containing \( Y \) form a loop in the \( X_1 \) setting, we call \( Y \) a *loop of 1-elements*.

We are now able to differentiate between different types of arcs in the \( X_2 \) setting, one consisting of 2-elements from the same 1-element, and one consisting directly of 1-elements. When \( Y \) is an arc of 1-elements, we can show that \( \partial T \) doesn’t separate nicely by considering two cases, depending on whether \( Y \) is a loop or not. As previously in this section, the arguments are practically identical to those of Lemmas 4.3.2 and 4.3.3.

**Lemma 4.4.7.** Let \( Y \) be an arc of 1-elements of \( X_2 \) that is not a loop of 1-elements. If \( Y \subset \text{Int}(T) \), then \( \partial T \) cannot separate \( Y \) from \( X_2 \setminus Y \).

*Proof.* Suppose for the sake of contradiction that \( \partial T \) separates \( Y \) from \( X_2 \setminus Y \). Let \( M_1 \) and \( M_2 \) be 1-elements of \( X_2 \setminus Y \) that are linked with \( Y \). Note that, in contrast to the case involving arcs of 2-elements, \( M_1 \) and \( M_2 \) are linked with \( Y \) via loops of 2-elements. That is, there exist loops of 2-elements, say \( m_1 \subset M_1 \) and \( m_2 \subset M_2 \), as well as loops \( y_1 \) and \( y_2 \) contained in \( Y \), such that \( m_1 \) is linked with \( y_1 \) and \( m_2 \) is linked with \( y_2 \). Similar to the proof of Lemma 4.3.2, we can assume that \( m_1 \neq m_2, y_1 \neq y_2 \), and that not both of \( M_1 \) and \( M_2 \) are involved in the four-way linking in \( X_1 \). This ensures that, in particular, \( y_1 \) is not linked with \( m_2 \).

Summarizing the relevant points, we have that

(i) \( m_1 \) is linked with \( y_1 \),

(ii) \( m_2 \) is linked with \( y_2 \).
(iii) $y_1$ and $y_2$ are both contained in $\text{Int}(T)$,

(iv) $m_1$ and $m_2$ are both contained in $\mathbb{R}^3 \setminus T$.

It follows from Lemma 4.2.3 that both $m_1$ and $m_2$ are linked with $T$, and that in particular $y_1$ is linked with $\mathbb{R}^3 \setminus T$. But then by Lemma 4.2.2, $y_1$ is also linked with $m_2$, which is a contradiction. Hence $\partial T$ cannot separate $Y$ from $X_2 \setminus Y$.

**Lemma 4.4.8.** Let $Y$ be a loop of 1-elements of $X_2$. Then $\partial T$ cannot separate $Y$ from $X_2 \setminus Y$.

**Proof.** Suppose for the sake of contradiction that $\partial T$ separates $Y$ from $X_2 \setminus Y$. Let $X'_0$ be the $N$-level 2-holed torus that is linked with $X_0$ through the handle containing $Y$. Since $T$ does not contain $X'_0$, Lemma 4.2.3 implies that $X_0$ is linked with $T$, and that $Y$ is linked with $\mathbb{R}^3 \setminus T$. So, by Lemma 4.2.2, a 1-element of $Y$ is linked with $X'_0$. This violates the construction of the standard defining sequence. Hence $\partial T$ cannot separate $Y$ from $X_2 \setminus Y$.

There now remains only one structural case for $Y$, namely when $Y$ consists of a mixture of arcs of 2-elements and arcs of 1-elements. Before proving anything about this case, we should clarify the notion of maximality. We consider an arc of 1-elements $Y_1 \subset Y$ to be maximal if there is no arc of 1-elements $Y'_1$ such that $Y_1 \subsetneq Y'_1 \subseteq Y$, and we consider an arc of 2-elements $Y_2 \subset Y$ to be maximal if there is no arc of 2-elements $Y'_2$ such that $Y_2 \subsetneq Y'_2 \subseteq Y$.

To illustrate a subtlety, consider a subchain $Y$ of $X_2$ that is the union of two subchains $Y_1$ and $Y_2$, as shown in Figure 4.7. Here $Y_1$ is a 1-element (and hence an arc of 1-elements), and $Y_2$ is an arc of 2-elements that is linked with $Y_1$. Note that $Y$ is a non-split link, and yet $Y_1$ and $Y_2$ are disjoint maximal arcs of $Y$. This is because we do not consider a link of arcs of different element types to itself be an arc.

All of this means that, when we take an arbitrary subchain $Y$ of $X_2$, we can break it down into disjoint maximal arcs. Then each arc is either an arc of 1-elements or of 2-elements, and
two maximal arcs can be linked with each other, provided they are different types of arcs. So we can see that it is again justified to restrict our attention to arcs. With that in mind, we tackle the situation of mixed-level arcs.

**Lemma 4.4.9.** Let $T \subset X_0$ be a solid 1-holed torus. If $Y$ is a subchain of $X_2$ consisting of some arcs of 2-elements and some arcs of 1-elements of $X_2$, and if $Y \subset \text{Int}(T)$, then $\partial T$ cannot separate $Y$ from $X_2 \setminus Y$.

**Proof.** Suppose for the sake of contradiction that $\partial T$ separates $Y$ from $X_2 \setminus Y$. Let $Y_1$ be a maximal arc of 2-elements of $Y$. If $Y_1$ is an arc of 2-elements that is not a loop, then the same reasoning as in the first paragraph of the proof of Lemma 4.4.4 applies, and we achieve a contradiction. So suppose $Y_1$ is a loop of 2-elements. Let $Y'_1$ be the loop of 2-elements that is linked with $Y_1$. If $Y'_1 \subset X_2 \setminus Y$, then the same reasoning as in the second paragraph of the proof of Lemma 4.4.4 applies. So suppose that $Y'_1$ is contained in $Y$. If $Y'_1$ is itself a maximal arc of 2-elements in $Y$, then Lemma 4.4.5 applies and we are done.

The possibility that has not been previously covered is to suppose that $Y'_1$ is part of a maximal arc of 1-elements in $Y$. Denote by $Y_2$ the maximal arc of 1-elements containing $Y'_1$. There are now two cases, either $Y_2$ is a loop of 1-elements, or it is not a loop.

**Case 1:** Suppose that $Y_2$ is a loop of 1-elements. Let $X'_0$ be the $N$-level 2-holed torus that is linked with $X_0$ through the handle containing $Y_2$. Applying the linking lemmas as in
the proof of Lemma 4.3.3, we conclude that a 2-element $y_1$ of $Y_1$ is linked with $X'_0$, which contradicts the construction of the standard defining sequence.

**Case 2:** Suppose that $Y_2$ is not a loop of 1-elements. Then $Y_2$ is linked with $X_2 \setminus Y_2$ in at least two locations, one of which being the linking between $Y_2$ and $Y_1$. At a second location, $Y_2$ is either linked directly with $X_2 \setminus Y$, or else $Y_2$ is linked with another loop of 2-elements in $Y$. In the former case, we apply the linking lemmas to show that a 2-element $y_1$ of $Y_1$ is linked with a 1-element of $X_2 \setminus Y$, a contradiction.

In the latter case, let $M_2$ be a 1-element of $X_2 \setminus Y_2$ that is linked with $Y_2$, where $Y_1$ is not contained in $M_2$. In other words, $M_2$ represents the second location where $Y_2$ links with its complement. Since $Y_2$ is a maximal arc of 1-elements, we must have that $M_2$ contains a maximal arc of 2-elements $M'_2$ of $Y$. But then $M'_2$ is linked with $X_2 \setminus Y$, and so we can apply the linking lemmas to show that a 2-element $y_1$ of $Y_1$ is linked with a 2-element contained in $M_2$. But this also contradicts the construction of the standard defining sequence, since 2-elements contained in different 1-elements are not linked. Hence $\partial T$ cannot separate $Y$ from $X_2 \setminus Y$.

Combining Lemmas 4.4.4, 4.4.5, 4.4.7, 4.4.8, and 4.4.9, we are ready to prove the separation theorem for $X_2$.

**Theorem 4.4.10.** Let $T \subset X_0$ be a solid 1-holed torus. Then $\partial T$ can neither separate any proper subchain of $X_2$ from its complement, nor can $\partial T$ separate $X_2$ from $\partial X_0$.

**Proof.** Let $Y$ be a proper subchain of $X_2$. Then $Y$ can be subdivided into finitely many maximal non-split subchains, which must be arcs of 2-elements, arcs of 1-elements, or non-split links of mixed-level arcs. Lemmas 4.4.4, 4.4.5, 4.4.7, 4.4.8, and 4.4.9 show that, regardless of the precise structure of $Y$, we cannot have $\partial T$ separating $Y$ from $X_2 \setminus Y$.

Now suppose for the sake of contradiction that $X_2 \subset \text{Int}(T)$. Apply the same argument as the second paragraph of the proof of Theorem 4.3.4 to obtain a contradiction. \hfill \Box
At this stage it is worth taking note of something. We handled the $X_2$ setting by breaking down the possible structure of a subchain $Y$ into several cases. However, every case was handled by using the linking lemmas (Lemmas 4.2.2 and Lemmas 4.2.3) in practically the same manner. In fact, in terms of how the linking lemmas are applied, all the cases for $Y$ boil down to two qualitatively different approaches. If $Y$ was linked with its complement $X_2 \setminus Y$ in at least two different locations, we applied the linking lemmas similarly to the proof of Lemma 4.3.2. That is, we showed that a part of $Y$ had to be linked simultaneously with two different parts of $X_2$. This was the case in every application but one.

The only times we had to reach outside of $X_2$ to contradict the construction of the standard defining sequence was if $Y$ contained a loop of 1-elements as a maximal arc. An example of this was Lemma 4.4.8, where we had the unique situation where $Y$ was linked with $X_2 \setminus Y$ in only one location, namely at the site of the $X_1$-level four-way linking. Here we applied the linking lemmas as in the proof of Lemma 4.3.3, exploiting the fact that separation via $\partial T$ now forced a 1-element of $Y$ to be linked with the $N$-level 2-holed torus $X'_0$. The fact that this situation was unique follows from the existence of the four-way linking at the $X_1$-level. Even when $Y$ contained a loop of 2-elements, being linked with $X_2 \setminus Y$ in an $X_2$-level four-way linking, there was always at least one other location where $Y$ was linked with its complement. This uniqueness (up to rotational symmetry) of a subchain that links with its complement in exactly one location will play a pivotal role in the following section, and will be the focus of an inductive step.

### 4.5 Inductive Proof of Genus

Let $X_n$ be the union of all $(N + n)$-level 2-holed tori contained in the $N$-level torus $X_0$. We begin this section by generalizing notation and definitions from the previous sections.
Definition 4.5.1. For \( i \in \{1, \ldots, n\} \), define an \( i \)-element of \( X_n \) as the union of all \((N+n)\)-level 2-holed tori contained in a single \((N+i)\)-level 2-holed torus.

Definition 4.5.2. Let \( Y \) be a proper subchain of \( X_n \). We say \( Y \) is an arc of \( i \)-elements of \( X_n \) if \( Y \) consists entirely of \( i \)-elements, and the \( i \)-level 2-holed tori containing the \( i \)-elements of \( Y \) form a non-split link. In particular, if the given \( i \)-level 2-holed tori form a loop, then we call \( Y \) a loop of \( i \)-elements.

Definition 4.5.3. Let \( Y \) be a proper subchain of \( X_n \). An arc \( Y_1 \) of \( i \)-elements in \( Y \) is considered maximal if there exists no arc of \( i \)-elements \( Y' \) such that \( Y_1 \subsetneq Y' \subseteq Y \).

It bears emphasizing again that, given an arbitrary proper subchain \( Y \) of \( X_n \), we can decompose \( Y \) into a finite union of disjoint maximal arcs. Thus restricting our attention to separation of arcs from their complements is appropriate.

Recall from the end of the previous section that we intend to use the linking lemmas (Lemmas 4.2.2 and 4.2.3) to establish that a \( \partial T \) cannot separate a subchain \( Y \) from its complement in \( X_n \), and that we aim to do this by splitting the type of structure of \( Y \) into two cases: one where \( Y \) is linked with its complement in exactly one location, and one where \( Y \) is linked with its complement in at least two locations. First, we will use induction to show that the former case only occurs when \( Y \) is a loop of 1-elements.

Lemma 4.5.4. There is a unique (up to rotational symmetry) subchain \( Y \) of \( X_n \) that is linked with its complement in exactly one location. More precisely, such a \( Y \) must be a loop of 1-elements in \( X_n \).

Proof. We will proceed by induction on \( n \). The case when \( n = 1 \) is covered in Section 4.3.

Suppose that there is a unique (up to rotational symmetry) subchain \( Y_{n-1} \) of \( X_{n-1} \) that is linked with its complement \( X_{n-1} \setminus Y_{n-1} \) in exactly one location, and that in this case \( Y_{n-1} \) is a loop of 1-elements.
Now let $Y$ be a proper subchain of $X_n$. We will break down the structure of $Y$ into two cases.

Case 1: Suppose that $Y$ contains no 1-elements. Then $Y$ is either contained in a single 1-element of $X_n$, or else $Y$ is a union of subchains contained in distinct 1-elements of $X_n$. Recall that $X_n$ is a union of $(N + n)$-level 2-holed tori from the standard defining sequence, all contained inside $X_0$, a certain $N$-level 2-holed torus. Let $Y_1$ be the portion of $Y$ living in a single $(N + 1)$-level 2-holed torus, call it $M_1$. In other words, let $Y_1 = Y \cap M_1$. Since $Y$ contains no 1-elements, $Y_1$ is not all of $M_1 \cap X_n$, and hence $Y_1$ is linked with $X_n \setminus Y$ in $M_1$. Certainly, if $Y_1$ is linked with $X_n \setminus Y$ in more than one location, then so is $Y$ itself. So assume that $Y_1$ is linked with $X_n \setminus Y$ in exactly one location.

Without loss of notational generality, suppose that $\varphi_1$ is the similarity from Chapter 3 that maps $X_0$ to $M_1$. Then $\varphi_1^{-1}(Y_1)$ is a subchain of $X_{n-1}$ that is linked with its complement in exactly one location. By the inductive hypothesis, $\varphi_1^{-1}(Y_1)$ is a loop of 1-elements in $X_{n-1}$. So $Y_1$ is a loop of 2-elements of $X_n$. Denote by $Y_2$ the loop of 2-elements of $X_n$ that is linked with $Y_1$, and let $M_2$ be the $(N + 1)$-level 2-holed torus containing $Y_2$.

There are now two sub-cases. If $Y_2 \subset X_n \setminus Y$, then $Y_1$ is linked with $X_n \setminus Y$ in two locations: with a 2-element at the four-way linking of $M_1$, and with $Y_2$. So suppose that $Y_2$ is not contained in $X_n \setminus Y$. Since $Y$ contains no 1-elements, $Y \cap M_2$ is a proper subchain of $X_n \cap M_2$. But then $Y \cap M_2$ must be linked with $X_n \setminus Y$ in $M_2$, and so $Y$ is linked with $X_n \setminus Y$ in at least two locations.

Case 2: Suppose that $Y$ contains some 1-elements. Re-purposing previous notation, let $Y_1$ be a maximal arc of 1-elements. Suppose that $Y_1$ is not a loop of 1-elements. Then $Y_1$ is linked with $X_n \setminus Y_1$ in at least two locations, say with the 1-elements $E_1$ and $E_2$. Let $E'_1$ and $E'_2$ be the loops of 2-elements contained in $E_1$ and $E_2$, respectively, that are linked with $Y_1$. If both $E'_1$ and $E'_2$ are contained in $X_n \setminus Y$, then $Y_1$ is linked with $X_n \setminus Y$ in two locations, and hence so is $Y$. So suppose without loss of generality that $E'_1$ is not contained in $X_n \setminus Y$.
Consider the subchain \( Y \cap E_1 \) of \( Y \). Note that \( Y \cap E_1 \) is not all of \( E_1 \), since otherwise \( Y \) would not be maximal, and note that \( Y \cap E_1 \) is nonempty, since otherwise \( E'_1 \subset X_n \setminus Y \). Applying the same inverse similarity idea as in Case 1, we conclude that the only way \( Y \cap E_1 \) can contribute only one linking point of \( Y \) with \( X_n \setminus Y \) is if \( Y \cap E_1 = E'_1 \), the loop of 2-elements linked with \( Y_1 \). So \( Y \) is linked with \( X_n \setminus Y \) at the four-way linking of 2-elements in \( E_1 \). But by the exact same reasoning, the four-way linking of 2-elements in \( E_2 \) must contribute another linking point of \( Y \) with \( X_n \setminus Y \). Hence \( Y \) is linked with \( X_n \setminus Y \) in at least two locations.

Finally, suppose that \( Y_1 \) is a loop of 1-elements, linked with \( X_n \setminus Y_1 \) at the four-way linking of 1-elements. Again, let \( E_1 \) and \( E_2 \) be the two 1-elements of \( X_n \setminus Y_1 \) that are linked with \( Y_1 \). Applying the same reasoning with the inverse similarity and the inductive hypothesis as above, if either of \( E_1 \) or \( E_2 \) contain more of \( Y \), then \( Y \) is linked with \( X_n \setminus Y \) in at least two locations. So we are left with the case when \( Y = Y_1 \), and indeed, \( Y \) is now linked with \( X_n \setminus Y \) in exactly one location, at the four-way linking of 1-elements.

Having established the uniqueness of subchains linked with their complement in a single location, we are now equipped to use the linking lemmas. For reference, the proofs of Lemmas 4.5.5 and 4.5.6 are conceptually modeled after the proofs of Lemmas 4.3.3 and 4.3.2, respectively.

**Lemma 4.5.5.** Let \( Y \) be a proper subchain of \( X_n \) that is linked with \( X_n \setminus Y \) in exactly one location. If \( Y \subset \text{Int}(T) \), then \( \partial T \) cannot separate \( Y \) from \( X_n \setminus Y \).

**Proof.** First, by Lemma 4.5.4, we have that \( Y \) is a loop of 1-elements. Suppose for the sake of contradiction that \( \partial T \) separates \( Y \) from \( X_n \setminus Y \). Let \( X'_0 \) be the \( N \)-level 2-holed torus that is linked with \( X_0 \) through the handle containing \( Y \). Let \( M_1 \) be one of the 1-elements of \( X_n \setminus Y \) that is linked with \( Y \) at the four-way linking in \( X_0 \). Finally, let \( Y_1 \) be one of the 1-elements of \( Y \) linked with \( M_1 \).
Since $T$ contains $Y$, and $Y$ is linked with $X_0$, Lemma 4.2.3 shows that $X_0$ is linked with $T$. Similarly by Lemma 4.2.3, $M_1$ is linked with $T$, and $Y_1$ is linked with $\mathbb{R}^3 \setminus T$. But then by Lemma 4.2.2, $Y_1$ is linked with $X_0$. This contradicts the construction of the standard defining sequence, since no 1-elements contained in $X_0$ are linked with $X_0'$. Hence $\partial T$ cannot separate $Y$ from $X_n \setminus Y$.

\begin{proof}
Suppose for the sake of contradiction that $\partial T$ separates $Y$ from $X_n \setminus Y$. There are two cases for each location. The subchain $Y$ will be linked with its complement either via an $i$-element $Y_i$ of $Y$ linked with an $i$-element $M_i$ of $X_n \setminus Y$, or via a loop of $i$-elements $Y'_i$ linked with a loop of $i$-elements $M'_i$ of $X_n \setminus Y$. For the second location, either a $j$-element $Y_j$ of $Y$ is linked with a $j$-element $M_j$ of $X_n \setminus Y$, or a loop of $j$-elements $Y'_j$ of $Y$ is linked with a loop of $j$-elements $M'_j$ of $X_n \setminus Y$. Applying the linking lemmas (Lemmas 4.2.2 and 4.2.3) in each case gives us the following situations:

(i) The $i$-element $Y_i$ is linked with the $j$-element $M_j$ through the same loop of $(j + 1)$-elements in $Y_i$ that is linked with $M_i$. If $i \neq j$, this is contradiction since elements of different levels are not linked. Even if $i = j$, this is still a contradiction, since $Y_i$ and $Y_j$ can be assumed not to be linked, and then $M_j$ cannot be linked with both $Y_i$ and $Y_j$ through the same loop of $(j + 1)$-elements.

(ii) The $i$-element $Y_i$ is linked with the loop of $j$-elements $M'_j$. If $i \neq j + 1$, this is an immediate contradiction, since an $i$-element can only be linked with a loop if the loop consists of $(j + 1)$-elements. Even if $i = j + 1$, having $M'_j$ be linked with $Y_i$ would imply that $Y'_j \subset Y_i$, which it is not.

(iii) The loop $Y'_i$ is linked with the $j$-element $M_j$. The same problems arise as in (ii).
\end{proof}
(iv) The loop $Y'_i$ is linked with the loop $M'_j$. This is essentially the same situation as (i), just shifted by one level.

Having exhausted all cases, we conclude that $\partial T$ cannot separate $Y$ from $X_n \setminus Y$.  

We are now equipped to prove that the genus of $\mathcal{X}$ is 2. Before doing so, it is worth recalling some facts about defining sequences discussed in Section 4.1.

**Remark 4.5.7.**

(i) For all $x \in \mathcal{X}$, $g_x(\mathcal{X}) \leq g(\mathcal{X})$.

(ii) If $g_x(\mathcal{X}) = 1$ for some $x \in \mathcal{X}$, then there exists a defining sequence $(M_i)$ for $\mathcal{X}$ having arbitrarily small toroidal neighborhoods of $x$.

(iii) Given a toroidal component $T$ of $(M_i)$, there exists $N \in \mathbb{N}$ such that, for all $n \geq N$, $\partial T$ does not intersect the boundary of any 2-holed torus from the $n$-th level of the standard defining sequence.

**Theorem 4.5.8.** *The genus of the Cantor set $\mathcal{X}$ is 2.*

**Proof.** Since the standard defining sequence is a defining sequence consisting of 2-holed tori, we know that the genus $g(\mathcal{X}) \leq 2$. Furthermore, by Theorem 3.4.6, $\mathcal{X}$ is a wild Cantor set, and hence $g(\mathcal{X}) > 0$. It remains to be shown that $g(\mathcal{X}) \neq 1$.

Let $x \in \mathcal{X}$. We will show that $g_x(\mathcal{X}) > 1$. Since $g_x(\mathcal{X}) \leq g(\mathcal{X})$, the result will then follow.

Suppose for the sake of contradiction that $g_x(\mathcal{X}) = 1$, that is there exists a defining sequence $(M_i)$ for $\mathcal{X}$ containing arbitrarily small solid 1-holed tori accumulating to $x$. Let $y$ be another point in $\mathcal{X}$. Then there exists a 1-holed torus $T$ from $(M_i)$ such that $\text{diam}(T) < d(x, y)/2$, $x \in T$, and $y \notin T$. Note that $\partial T \cap \mathcal{X} = \emptyset$.

By the decreasing nature of defining sequences, there exists a positive integer $n$ such that the $n$-level 2-holed tori $Y_n$ and $Y'_n$ from the standard defining sequence with $x \in Y_n$ and
$y \in Y_n'$ are so small that $Y_n \subset \text{Int}(T)$ and $Y_n' \subset \mathbb{R}^3 \setminus T$. In other words, $\partial T$ separates $Y_n$ from $Y_n'$. In fact, we can assume that $n$ is so large that $\partial T$ does not intersect any $n$-level 2-holed tori.

Furthermore, since $\text{diam}(T) < d(x, y)/2$, there exists a maximum integer $i \in \{0, \ldots, n\}$ such that there exists an $i$-level 2-holed torus $Y_i$ containing both $Y_n$ and $Y_n'$ as $(n - i)$-elements, and with $T \subset \text{Int}(Y_i)$. Let $X_n$ be the union of all $n$-level 2-holed tori contained in $Y_i$. Since $\partial T$ separates $Y_n$ from $Y_n'$, and since $\partial T$ does not intersect any $n$-level 2-holed tori, we must have that $\partial T$ separates some subchain of $X_n$ containing $Y_n$ from its complement, containing $Y_n'$. But, by Lemmas 4.5.5 and 4.5.6, $\partial T$ cannot accomplish this separation. This is a contradiction.

Hence $g_x(\mathcal{X}) > 1$, showing that $g(\mathcal{X}) > 1$. We conclude that $g(\mathcal{X}) = 2$. \hfill \square

An attentive reading of the previous proof reveals that we have actually established a stronger result, stated here as a corollary.

**Corollary 4.5.9.** For all $x \in \mathcal{X}$, the local genus $g_x(\mathcal{X}) = 2$. 
Having established that the Cantor set $\mathcal{X}$ constructed in Chapter 3 has genus 2, we are finally ready to construct a uqr map $f : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ to which Theorem 2.0.1 can apply. We formalize this statement in the following results.

**Theorem 5.0.1.** Let $m \in \mathbb{N}$ be a sufficiently large square that is also a multiple of 16. Then there exists a Cantor set $\mathcal{X} \subset \mathbb{R}^3$ of genus 2 and a uqr map $f : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ of degree $m$ whose Julia set $J(f)$ is $\mathcal{X}$.

**Corollary 5.0.2.** Let $f$ be as in Theorem 5.0.1, and suppose $x_0 \in J(f)$ is a repelling fixed point of $f$. If $L$ is a Poincaré linearizer of $f$ at $x_0$, then $\partial A(L) = J(L)$, and $A(L)$ is a spider’s web. Furthermore, $J(f)$ is the closure of the set of repelling periodic points of $f$.

**Proof.** That $\partial A(L) = J(L)$ follows from $L$ having positive lower order (Theorem 2.3.1), and that $A(L)$ is a spider’s web follows from Theorem 2.0.1. The last statement follows from Theorem 2.2.1. \qed

### 5.1 A Basic Covering Map

Proving Theorem 5.0.1 amounts to constructing the map $f$. We adapt the method of Fletcher and Wu detailed in Chapter 1. Recall that this method involved defining a map to be a branched covering map inside of a solid 1-holed torus, a uqr power map outside of...
a ball, and stitching together those parts via the Berstein and Edmonds extension theorem, Theorem 1.3.8. The only step of this process which needs to be adapted is the branched covering map.

To that end, recall the coefficients $R$ and $r$ from Chapter 3, describing the size and thickness, respectively, of the solid 2-holed torus $X_0$. Set $R = 1$ and assume $r$ satisfies Lemma 3.2.1. Assume also that $X_0$ is positioned in $\mathbb{R}^3$ as in Figure 3.10, with central point of rotational symmetry at the origin, with length along the $x_1$-axis, and width along the $x_2$-axis. Note that $X_0$ is contained in $\text{Int}(B(0,4))$. We now construct a BLD degree $m$ branched covering map

$$F : X_0 \setminus \text{Int}\left(\bigcup_{j=1}^{m} X_{1,j}\right) \to B(0,4) \setminus \text{Int}(X_0)$$

satisfying $F|\partial X_{1,j} : \partial X_{1,j} \to \partial X_0 = \phi_j^{-1}$ for the tori $X_{1,1}, \ldots, X_{1,m}$ fixed in Section 3.3.

Let $\iota_1$ be the involution

$$\iota_1 : (x_1, x_2, x_3) \mapsto (-x_1, x_2, -x_3).$$

By construction, the 2-holed tori $X_{1,1}, \ldots, X_{1,m}$ are symmetric with respect to $\iota_1$. The quotient $q_1 : X_0 \to X_0/\langle \iota_1 \rangle$ is then a degree 2 sense-preserving map satisfying

- $q_1(X_0)$ is a 1-holed torus unknotted in $\mathbb{R}^3$,
- $q_1(X_{1,j}) = q_1(X_{1,m-j+1})$ is a 2-holed torus unknotted in $q_1(X_0)$,
- $\bigcup_{j=1}^{m} q_1(X_{1,j})$ is a chain of $m/2$ linked 2-holed tori following a core curve of the 1-holed torus $q_1(X_0)$. 
For the sake of convenient geometry, we modify $q_1(X_0)$ in a few ways. First, translate $q_1(X_0)$ so that the center of the hole of the 1-holed torus is at the origin. Then apply a map that is radial with respect to the $x_3$-axis, making $q_1(X_0)$ round in two senses:

- the core curve traced by the chain $\bigcup_{j=1}^m q_1(X_{1,j})$ is a circle in the $x_1x_2$-plane centered at the origin;
- every cross section of $q_1(X_0)$ taken perpendicular to the above core curve is a geometric disk.

Additionally, we deform $\text{Int}(q_1(X_0))$ so that all the 2-holed tori $q_1(X_{1,1}), \ldots, q_1(X_{1,m})$ satisfy $\rho(q_1(X_{1,j})) = \rho(q_1(X_{1,j+2}))$ for $j \in \{1, \ldots, m - 2\}$, $\rho(q_1(X_{1,m-1})) = q_1(X_{1,1})$, and $\rho(q_1(X_{1,m})) = q_1(X_{1,2})$, where $\rho$ is the rotation about the $x_3$-axis by an angle $8\pi/m$,

$$\rho(r, \theta, x_3) = (r, \theta + 8\pi/m, x_3).$$

This deformation is made to preserve the fact that all the $q_1(X_{1,j})$ remain geometrically similar to each other. Finally, if necessary, rotate $q_1(X_0)$ around the $x_3$-axis to ensure that $q_1(X_{1,1}) = q_1(X_{1,m/2+1})$ and $q_1(X_{1,m/2}) = q_1(X_{1,m})$ are linked with the $x_1$-axis such that they are symmetric with respect to a rotation about the $x_1$-axis by an angle $\pi$. For the sake of notational simplicity, assume that the map $q_1$ already incorporates all of these modifications.

Let $\omega : \mathbb{R}^3 \to \mathbb{R}^3$ be the degree $m/4$ winding map

$$\omega(r, \theta, x_3) = (r, \theta m/4, x_3).$$

Then $\omega : q_1(X_0) \to q_1(X_0)$ is an unbranched cover that maps all $q_1(X_{1,j})$ with odd indices to $\omega(q_1(X_{1,1}))$ and all $q_1(X_{1,j})$ with even indices to $\omega(q_1(X_{1,2}))$. By construction, $\omega(q_1(X_{1,1}))$ and $\omega(q_1(X_{1,2}))$ are linked inside $q_1(X_0)$ as in Figure 5.1, and are symmetric to each other.
via a rotation about the $x_1$-axis by an angle $\pi$. Let $\iota_2$ be the involution for this rotation, that is

$$\iota_2 : (x_1, x_2, x_3) \mapsto (x_1, -x_2, -x_3).$$

The quotient $q_2 : q_1(X_0) \to q_1(X_0)/\langle \iota_2 \rangle$ is then a degree 2 sense preserving map under which $q_2(\omega(q_1(X_{1,1}))) = q_2(\omega(q_1(X_{1,2})))$ is a 2-holed torus unknotted in the 3-cell $q_2(q_1(X_0))$. For more details on such constructions, see [47, p. 294]. Assuming $q_2$ incorporates some more translation and deformation, the map $q_2 \circ \omega \circ q_1$ is a degree $m$ branched cover from $X_0$ onto $\overline{B(0,4)}$ mapping each $X_{1,j}$ onto $X_0$. To obtain a BLD, and hence $qr$, cover, we consider a PL version of this map.

Give $X_0$ a $C^1$-triangulation $g : |U| \to X_0$ by a simplicial complex $U$ that respects the involutions $\iota_1$ and $\iota_2$, and has both $g^{-1}(q_1^{-1}(\bigcup X_{1,j}))$ and $g^{-1}(\omega(q_1(X_{1,1})) \cup \omega(q_1(X_{1,2})))$ as subcomplexes. We then identify $q_1(X_0)$ with a simplicial complex $V$ via $h : |V| \to q_1(X_0)$ such that $q_1 \circ g : U \to q_1(X_0)$ is simplicial. It then follows that

- $h^{-1}(\bigcup q_1(X_{1,j}))$ is a subcomplex of $V$,
- $h^{-1}(\omega(q_1(X_{1,1})) \cup \omega(q_1(X_{1,2})))$ is a subcomplex of $V$, and
\bullet h \text{ respects } i_2.

Finally, identify \( q_2(q_1(X_0)) \) with a simplicial complex \( W \) via \( i : |W| \to q_2(q_1(X_0)) \) such that \( q_1 \circ q_1 \circ g \) is simplicial. Then \( i^{-1}(q_2(\omega(q_1(X_{1,1})))) \) is a subcomplex of \( W \).

Refine \(|U|\) and \(|W|\) if necessary to ensure that \( q_2 \circ \omega \circ q_1 \circ \phi_j | X_0 \) are simplicial and ambient isotopic. This is possible since \( q_2 \circ \omega \circ q_1 \circ \phi_j \) embeds \( X_0 \) unknottedly into \(|W|\). Then there exists a PL map \( \eta : |W| \to |W| \) which is identity on \( \partial |W| \) so that \( \eta \circ i^{-1} | X_0 = q_2 \circ \omega \circ q_1 \circ \phi_j | X_0 \). Set \( \zeta = \eta \circ i^{-1} \), and then \( F := \zeta^{-1} \circ q_2 \circ \omega \circ q_1 \) is a BLD degree \( m \) branched covering satisfying \( F|\partial X_{1,j} = \phi_j^{-1} \).

5.2 A Genus 2 Julia Set

To construct the map \( f \) from Theorem 5.0.1, let \( m = d^2 \) be a sufficiently large square that is a multiple of 16 and let \( \mathcal{X} \) be the Cantor set from the previous section. Write \( B_0 = B(0, 4) \), \( B_{-1} = B(0, 4^d) \), and write \( \mathbb{R}^3 \) as two disjoint unions, one for the domain and one for the codomain of \( f \), as follows:

\[
\mathbb{R}^3 = X_1 \cup (X_0 \setminus X_1) \cup (B_0 \setminus X_0) \cup (\mathbb{R}^3 \setminus B_0)
\]

and

\[
\mathbb{R}^3 = X_0 \setminus (B_0 \setminus X_0) \cup (B_{-1} \setminus B_0) \cup (\mathbb{R}^3 \setminus B_{-1}).
\]

The uqr map is then defined in the following parts.

(i) Set \( f : \overline{X_0 \setminus X_1} \to \overline{B_0 \setminus X_0} \) to be the degree \( m \) branched covering map \( F \) from the previous section.

(ii) Extend \( f \) into \( X_1 \) by setting \( f|X_1 \) to be \( \phi_j^{-1} : X_{1,j} \to X_0 \) for each \( j \in \{1, \ldots, m\} \).
(iii) Define $f : \mathbb{R}^3 \setminus \text{Int}(B_0) \to \mathbb{R}^3 \setminus \text{Int}(B_{-1})$ to be the restriction of the uqr map $g$ of degree $m$ from Theorem 1.2.24. By definition of the map $g$, it maps $S(0,4)$ onto $S(0,4^d)$. We further remark that $g$ is orientation-preserving.

(iv) Since $f|\partial B_0$ is a BLD degree $m$ branched cover onto $\partial B_{-1}$ and $f|\partial X_0$ is also a BLD degree $m$ branched cover onto $\partial B_0$, we can extend the boundary map to a BLD degree $m$ branched cover $f : B_0 \setminus \text{Int}(X_0) \to B_{-1} \setminus \text{Int}(B_0)$ by the Berstein and Edmonds extension theorem, Theorem 1.3.8.

Then $f : \mathbb{R}^3 \to \mathbb{R}^3$ is indeed a quasiregular map. Finally, we prove Theorem 5.0.1 with the following two lemmas.

**Lemma 5.2.1.** The map $f$ is a uniformly quasiregular mapping of polynomial type.

*Proof.* Let $x \in \mathbb{R}^3$. If the orbit of $x$ under $f$ always remains in $X_1$, then, since $f|X_1$ is a conformal similarity, the dilatation of $f^n$ at $x$ will always equal 1. Suppose then that the orbit of $x$ leaves $X_1$. Then, after iterating through at most finitely many conformal maps and at most two quasiregular maps, $f^{n_0}(x_0) \in \mathbb{R}^3 \setminus \overline{B(0,4)}$ for some $n_0 \in \mathbb{N}$. From this point on, $f$ agrees with the uqr power map $g$ of degree $m$, and hence the dilatation will remain bounded. In summary, the orbit of $x$ consists of finitely many conformal maps, at most two quasiregular maps, and then a uqr map. So the dilatation of $f^n$ remains uniformly bounded on all of $\mathbb{R}^3$ as $n \to \infty$. Hence $f$ is uqr.

Since $f$ has finite degree $m$, $f$ is of polynomial type. \[\Box\]

**Lemma 5.2.2.** The Julia set of $f$ is equal to $X$.

*Proof.* Let $x \in \mathbb{R}^3$. If the orbit of $x$ under $f$ at any point leaves $X_0$, then $x \in I(f)$, as it is pushed to infinity by the uqr power map. By construction, if $x$ does not leave $X_0$, then $x \in X$. 

If \( x \in \mathcal{X} \), then any sufficiently small neighborhood of \( x \) will intersect the boundary of \( X_n \) for some \( n \). Since \( \partial X_n \subset I(f) \) by the preceding argument, we conclude that \( \mathcal{X} = \partial I(f) \).

Since \( f \) is uqr, we have by discussion in Chapter 1 that \( \partial I(f) = J(f) \). Hence \( J(f) = \mathcal{X} \).  \( \square \)
REFERENCES


