The Zorich Transform and Generalizing Koenigs Linearization Theorem to Quasiregular Maps

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This dissertation investigates the role that a new tool called the Zorich transform plays in quasiregular dynamics as a generalization of the logarithmic transform in complex dynamics. In particular we use the Zorich transform to construct analogues of the logarithmic spiral maps and interpolation between radial stretch maps. These constructions are then used to completely classify the orbit space of a quasiregular map.

Also, conditions are given in which a quasiregular map $f : D \to \mathbb{R}^n$, where $D \subset \mathbb{R}^n$ is a domain, that is quasiconformal in a neighborhood of a geometrically attracting fixed point can be conjugated by a quasiconformal map to the asymptotic representation of $f$ in a neighborhood of the fixed point. To find such a quasiconformal map, we construct a sequence of quasiconformal maps that converge to the desired map in a neighborhood of the fixed point.
THE ZORICH TRANSFORM AND GENERALIZING
KOENIGS LINEARIZATION THEOREM
TO QUASIREGULAR MAPS

BY
JACOB PRATSCHER
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DEDICATION

For my uncle and godfather, Richard John Mroz, whose enthusiasm for his work inspired me in my own.
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1.1 Mapping of the Logarithmic Transform

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CHAPTER 1

INTRODUCTION

To begin, we make a few remarks about the notation used throughout the work and about the work itself. The following notation is used: $\mathbb{R}^n$ is the $n$-dimensional real coordinate plane, $\mathbb{C}$ is the complex plane, $\mathbb{B}^n$ is the $n$-dimensional unit ball centered at the origin, $B(x, r)$ is the ball centered at $x \in \mathbb{R}^n$ with radius $r$, $S^{n-1}$ is the unit sphere in $\mathbb{R}^n$, $\| \cdot \|$ represents the operator norm, and $| \cdot |$ is used to represent the standard Euclidean norm. The work from chapter 2, except for Section 2.2.2, and Chapter 3 first appeared in a *Journal of Geometric Analysis* [8]. Also, all figures presented are created by the author.

Informally, the overarching goal of this work is to take concepts from complex dynamics and to find generalizations of such tools and results in quasiregular dynamics. Quasiregular maps are indeed the most natural generalization of analytic functions on the plane to $\mathbb{R}^n$, $n \geq 2$, since the generalized Liouville’s theorem tells us that conformal mappings on $\mathbb{R}^n$, $n \geq 3$, must be Möbius transformations, [20, Theorem I.2.5]. In this chapter, we notice that quasiregular mappings are not differentiable everywhere, unlike holomorphic maps. Gutlyanskii et al [10] introduced the notion of generalized derivatives which is used to help understand the behavior of a quasiregular mapping near such points. A feature of quasiregular mappings, as compared to differentiable mappings, is that there may exist multiple generalized derivatives at a given point. The collection of all generalized derivatives of $f$ at $x_0$ is called the infinitesimal space and denoted by $T(x_0, f)$.

Since there can be more than one generalized derivative, the dynamics of a quasiregular may be quite irregular near a given point. Demonstratively, we show a new example, Example
1.2.25, which describes a situation where there are two nested sequences of annuli, on one the map satisfies $|f(x)| < |x|$ and on the other $|f(x)| > |x|$.

In Section 1.1 we introduce key concepts and tools from complex dynamics that we want to generalize to quasiregular mappings. First we give Koenigs Linearization Theorem and Böttcher’s Theorem for the complex plane. Following this, we identify a key tool used in proving Böttcher’s Theorem called the logarithmic transform. After introducing these concepts, we start with definitions and preliminary results related to quasiregular maps in Section 1.2. In this section, after giving definitions and known theorems for quasiregular maps, we introduce the notion of generalized derivatives. Following this, we define the infinitesimal space and introduce key results related to a simple infinitesimal space. Then we give a description of point evaluation under all generalized derivatives in the infinitesimal space $T(x_0, f)$, or orbit spaces of quasiregular maps. Later in this chapter, we introduce definitions to describe what we mean by a quasiregular mapping to have an attracting fixed point. There is also an introduction to the needed assumptions to be able to generalize Koenigs Linearization Theorem.

1.1 Complex Dynamics

In this section we introduce concepts from classical complex dynamics that are motivations for generalizations to dynamics with quasiregular maps. More background regarding complex dynamics can be found in Milnor [16]. Additionally, background regarding classical complex analysis can be found in Conway [1] and/or in Needham [19].
1.1.1 Koenigs Linearization Theorem and Böttcher’s Theorem

We want to look at two key results in classical complex dynamics. Before we do so, we need to define what it means for a holomorphic function to have an attracting or repelling fixed point.

Suppose that for a domain $D \subset \mathbb{C}$ that $f : D \to \mathbb{C}$ is analytic, with a fixed point we may assume without loss of generality is at $0$. Since $f$ is analytic, then $f$ can be written in the power series form

$$f(z) = \lambda z + a_2z^2 + a_3z^3 + \cdots$$

in a neighborhood of $0$. Then we say that $f$ is superattracting if $\lambda = 0$, geometrically attracting if $|\lambda| < 1$, repelling if $|\lambda| > 1$, and indifferent if $|\lambda| = 1$. Note that $f'(0) = \lambda$ is called the multiplier of the fixed point. We have two theorems that allow us to conjugate a holomorphic map as long as $|\lambda| \neq 1$. In the case of a geometrically attracting (or repelling) fixed point we have Koenigs Linearization Theorem.

**Theorem 1.1.1** (Koenigs Linearization Theorem). *If the multiplier $\lambda$ satisfies $|\lambda| \neq 0, 1$, then there exists a holomorphic function $\phi$ with $\phi(0) = 0$, so that $\phi \circ f(z) = \lambda \phi(z)$ in a neighborhood of the origin. Furthermore, $\phi$ is unique up to multiplication by a nonzero constant.*

Of particular note, $\phi$ is a solution to the functional equation called the Schröder equation. For the superattracting case we have Böttcher’s Theorem.
Theorem 1.1.2 (Böttcher’s Theorem). Suppose that the multiplier $\lambda = 0$ for a holomorphic function $f$ with a fixed point at 0. Let $a_n, n \geq 2$, be the first non-zero term in the power series representation of $f$ in a neighborhood of the origin, that is

$$f(z) = a_n z^n + a_{n+1} z^{n+1} + \cdots.$$ 

Then there exists a holomorphic function $\phi$ such that 

$$\phi \circ f(z) = (\phi(z))^n$$ 

in a neighborhood of the origin. Also, $\phi$ is unique up to multiplication by an $(n-1)$th root of unity.

1.1.2 Logarithmic Transform

The logarithmic transform gives a change of variables that makes calculations more amenable. It was introduced by Eremenko and Lyubich [2] and has found much utility in complex dynamics. A use of the logarithmic transform is in Böttcher’s Theorem, see [16], and its quasiregular generalization in the complex plane [3]. The logarithmic transform was also used in studying the class $\mathcal{B}$ of transcendental entire functions whose singular values are bounded, see [21, Section 5]. Of particular importance, the domain of the logarithmic transform is made of simply connected components which are called tracts. The image of these tracts under the logarithmic transform is simply connected, i.e. the logarithmic transform is a universal covering when the domain is restricted to one of these tracts. Also of importance is that the logarithmic transform is a conformal homeomorphism. However, the map $f$ will
be a universal covering of a corresponding set, but may not be conformal. Let us now define the logarithmic transform.

**Definition 1.1.3** (Logarithmic transform). Let \( f : \mathbb{C} \to \mathbb{C} \) be holomorphic. Let \( V \) be a Jordan domain that contains the singular values of \( f \) and the values 0 and \( f(0) \). Now let \( U := \mathbb{C} \setminus V \) and \( Y = f^{-1}(U) \) and then let \( T := \exp^{-1}(Y) \) and \( W := \exp^{-1}(U) \). Then the **logarithmic transform** of \( f, \tilde{f} : T \to W \) is defined by

\[
\tilde{f}(z) = \log f(e^z)
\]

where the principal branch of the logarithm is taken. Note that each component of \( T \) is called a **tract of** \( \tilde{f} \).
Also, in Figure 1.1.2 we see that the shaded region is a Jordan domain. In this figure, we see that understanding the dynamics of $f$ can be simplified by first understanding the dynamics of $\tilde{f}$. As noted before, the map $\tilde{f}$ typically has stronger properties, like being conformal, that allows us to say more about the dynamics. Another example of simplicity, in Figure 1.1.2, if the restriction of $\tilde{f}$ to $T$ behaves like a scaling, then correspondingly we would know that the restriction of $f$ to $Y$ acts like a power mapping. Also, note that the domain of $\tilde{f}$ is a $2\pi i$ periodic half-plane, which is also a Jordan domain. Then in terms of calculations, we are able to restrict the domain and range to the principal branch of the logarithm.

### 1.2 Quasiregular Maps

In this section we begin by defining quasiregular mappings and then give a brief discussion on tools that have already been developed as analogues to tools from classical complex analysis and dynamics. For a deeper discussion on quasiregular mappings see Rickman [20]. Note that details on linear distortion and distortion bounds can be found in Iwaniec and Martin [12]. Recall that we use $\| \cdot \|$ to represent the operator norm, and $| \cdot |$ to represent the standard Euclidean norm.

**Definition 1.2.1.** Let $n \geq 2$ and $D$ a domain in $\mathbb{R}^n$. Then a continuous mapping $f : D \rightarrow \mathbb{R}^n$ is called *quasiregular* if $f$ is in the Sobolev space $W^{1}_{n,\text{loc}}(D)$ and there exists $K \in [1, \infty)$ so that

$$\|f'(x)\|^n \leq K J_f(x) \text{ a.e.,}$$
where $f'$ is the derivative matrix, and $J_f$ is the Jacobian of $f$. The smallest $K$ here is called the outer dilatation $K_O(f)$ of $f$. If $f$ is quasiregular, then it is also true that

$$J_f(x) \leq K'\ell(f'(x))^n \text{ a.e.}$$

for some $K' \in [1, \infty)$. Here, $\ell(f'(x)) = \inf_{|h|=1} |f'(x)h|$. The smallest $K'$ for which this holds is called the inner dilatation $K_I(f)$ of $f$. The maximal dilatation is then $K(f) = \max\{K_O(f), K_I(f)\}$. We say that $f$ is $K$-quasiregular if $K(f) \leq K$. Note that an injective quasiregular map is called quasiconformal.

We now give an example of a quasiregular map.

**Example 1.2.2.** Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map defined by

$$f(x) = (5x_1, 3x_2, x_3).$$

Then we have

$$f'(x) = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$ 

We can see that $J_f(x) = 15$. Using the matrix norm, we have

$$\|f'(x)\| = \sup_{|h|=1} |f'(x)h| = 5.$$ 

We also have

$$l(f'(x)) = \inf_{|h|=1} |f'(x)h| = 1.$$ 

To satisfy the definition of quasiregular, we see that $K_I = 15$ and $K_O = 25/3$ so that $K = \max\{25/3, 15\} = 15$. Therefore, our linear map $f$ is 15-quasiconformal.
If $D$ is a domain in $\mathbb{R}^n$ with non-empty boundary, then for $x \in D$, we denote by $d(x, \partial D)$ the Euclidean distance from $x$ to $\partial D$. Of utmost importance is that quasiregular mappings have bounded linear distortion. Now we define what linear distortion means.

**Definition 1.2.3.** Let $n \geq 2$, $D \subset \mathbb{R}^n$ a domain, $x \in D$ and $f : D \to \mathbb{R}^n$ be $K$-quasiregular. For $0 < r < d(x, \partial D)$, we define

$$
\ell_f(x, r) = \inf_{|y-x|=r} |f(y) - f(x)|, \quad L_f(x, r) = \sup_{|y-x|=r} |f(y) - f(x)|.
$$

The linear distortion of $f$ at $x$ is

$$
H(x, f) = \limsup_{r \to 0} \frac{L_f(x, r)}{\ell_f(x, r)}.
$$

When $f$ is differentiable at $x$ and the derivative matrix $f'$ is invertible, we have an equivalent definition of linear distortion, [12, Section 6.4], given by

$$
H(x, f) = \|f'\|\|\left( f' \right)^{-1}\|.
$$

We also know that the distortion of a $K$-quasiregular map $f$ is bounded by the linear distortion, that is

$$
K \leq H^{n-1}.
$$

This means that if we know that the linear distortion $H$ of a map $f$ is bounded, then the maximal dilatation $K$ of $f$ must also be bounded. For our applications, we estimate quantities related to the derivative of the maps we construct. However, our mappings are not differentiable everywhere. To conclude quasiconformality, we need to make use of the following theorem which first appeared in [8].
Theorem 1.2.4. Suppose that $f : D \to D'$ is a homeomorphism, and that $E \subset D$ is a set such that $E$ is closed in $D$ and such that $E$ has a $\sigma$-finite $(n-1)$-dimensional Hausdorff measure. Suppose there exist constants $C_1, C_2 > 1$ such that $f$ is differentiable on $D \setminus E$ and

(i) either the matrix $f'$ has an inverse and $\sup_{x \in D \setminus E} H(x, f) \leq C_1$,

(ii) or $\inf_{x \in D \setminus E} J_f(x) \geq \frac{1}{C_2}$ and $\sup_{x \in D \setminus E} \|f'(x)\| \leq C_2$.

Then we can conclude that $f$ is quasiconformal on $D$.

Proof. Given $x \in D \setminus E$, let $U \subset D \setminus E$ be a neighborhood of $x$. First suppose on $U$ that we satisfy condition (ii) so that $J_f$ is bounded below by a positive number, and $\|f'\|$ is bounded on $U$. Note that Iwaniec and Martin, [12, Section 6.4], tells us that

$$K_I \leq K_O^{n-1}.$$  

Since $J_f$ is bounded below on $U$, we have that the matrix $f'$ is invertible. Since we also have $\|f'\|$ being bounded, we can conclude that $K_O$ and hence $K(f)$ is bounded on $U$. That is, $f$ is quasiconformal on $U$. Now suppose on $U$ we satisfy condition (i) so that the matrix $f'$ has an inverse and the linear distortion $H$ from (1.1) is bounded by above on $U$. Then by (1.2) the maximal dilatation

$$K \leq (H')^{n-1}. \tag{1.3}$$

Then $f$ is quasiconformal on $U$. So every point $x \in D \setminus E$ has a neighborhood $U$ with maximal dilatation being bounded by a constant depending on $C_1$ or $C_2$, respectively. Then Väisälä, [23, Theorem 35.1], tells us that $f$ is quasiconformal on $D$. \qed

The local index $i(x, f)$ of a quasiregular mapping $f$ at the point $x$ is

$$i(x, f) = \inf_N \sup_{y \in f(N)} \text{card}(f^{-1}(y) \cap N),$$
where the infimum is taken over all neighborhoods $N$ of $x$. In particular, $f$ is locally injective at $x$ if and only if $i(x, f) = 1$. Note that if $f$ is holomorphic in a neighborhood of $\omega \in \mathbb{C}$, then $f$ has the power series representation

$$f(z) = \sum_{n=d}^{\infty} a_n(z - \omega)^n,$$

where the $d$th term is the first non-zero term in the series. Taylor’s Theorem gives us that there exists an $r > 0$ such that

$$\left| \sum_{n=d+1}^{\infty} a_n(z - \omega)^n \right| < |a_d||z - \omega|^d, \text{ for } 0 < |z - \omega| < r.$$

Then Rouché’s Theorem tells us that $i(\omega, f) = d$.

**Theorem 1.2.5** (Theorem II.4.3, [20]). Let $n \geq 2$, $D \subset \mathbb{R}^n$ be a domain and $f : D \to \mathbb{R}^n$ be a non-constant quasiregular mapping. Then for all $x \in D$,

$$H(x, f) \leq C < \infty,$$

where $C$ is a constant that depends only on $n$ and the product $i(x, f)K_0(f)$.

A useful tool in quasiregular dynamics is the use of normal families. However, we should note that most normal family results are only applicable when there is a uniform bound on the maximal dilatation of the maps in the family. First, let $D \subset \mathbb{R}^n$ be a domain. We let $C(D, \mathbb{R}^n)$ denote the space of continuous functions on $D$. We say that the sequence $f_n \in C(D, \mathbb{R}^n)$ converges **locally uniformly** or **uniformly on compact subsets** if there exists $f \in (D, \mathbb{R}^n)$ such that $f_n$ converges uniformly to $f$ on any given compact $K \subset D$. Also, the sequence $f_n$ **diverges locally uniformly** to infinity if there exists an integer $N$ such that for every compact $K_1 \subset D$ and $K_2 \subset \mathbb{R}^n$ we have that $f_n(K_1) \cap K_2 = \emptyset$ when $n \geq N$. 
Definition 1.2.6 (Normal Families). A set $F \subset C(D, \mathbb{R}^n)$ is called normal if for every sequence of $F$ there is a subsequence which converges locally uniformly to a function $f \in C(D, \mathbb{C})$.

There is a version of Montel’s Theorem for quasiregular mappings due to Miniowitz.

Theorem 1.2.7 ([17]). Let $F$ be a family of $K$-quasiregular mappings defined on a domain $D \subset \mathbb{R}^n$. Then there exists a constant $q = q(n, K)$ so that if $a_1, \ldots, a_q$ are distinct points in $\mathbb{R}^n$ so that $f(D) \cap \{a_1, \ldots, a_q\} = \emptyset$ for all $f \in F$, then $F$ is a normal family.

The constant $q$ arises from Rickman’s version of Picard’s Theorem.

Theorem 1.2.8. [20, Theorem IV.2.1] For every $n \geq 3$ and $K \geq 1$ there exists a positive integer $q_0 = q_0(n, K)$ depending only on $n$ and $K$ such that every $K$-quasiregular mapping $f : \mathbb{R}^n \to \mathbb{R}^n \setminus \{a_1, \ldots, a_q\}$ is constant whenever $q \geq q_0$ and $a_1, \ldots, a_q$ are distinct points in $\mathbb{R}^n$.

The constant $q_0$ is called Rickman’s constant. Note that if we let one of the distinct $q_0$ points be infinity in Theorem 1.2.8 then the $q$ in Theorem 1.2.7 may be $q_0 - 1$.

1.2.1 Generalized Derivatives

One of the advantages to quasiregular mappings is that a quasiregular mapping has more flexibility than holomophic functions. For example, there is no analogue of the Identity Theorem. Recall that the Identity Theorem states for holomorphic functions $f$ and $g$ that $f = g$ on a domain $D$ if $f = g$ on a subset of $D$ that has an accumulation point. As noted in the definition, quasiregular mappings are only required to be differentiable almost everywhere. The overarching goal of this section is to understand how quasiregular mappings behave near points where the derivative does not exist as an analogue to how the derivative describes the local behavior of a differentiable function. For that purpose, Gutlyanskii et
al [10] introduced the notion of generalized derivatives. The normal family machinery mentioned for quasiregular maps are used to conclude that generalized derivatives always exist. Generalized derivatives have been studied in [6, 4, 9, 18].

Let us begin with the definition of a generalized derivative of a quasiregular mapping. Let $f : D \to \mathbb{R}^n$ be quasiregular and let $x_0 \in D$. For $t > 0$, let

$$f_t(x) = \frac{f(x_0 + tx) - f(x_0)}{\rho_f(t)},$$

where $\rho_f(r)$ is the mean radius of the image of a sphere of radius $r$ centered at $x_0$ and given by

$$\rho_f(r) = \left( \frac{\mu[f(B(x_0, r))]}{\mu[B(0, 1)]} \right)^{1/n}.$$  

Here $\mu$ denotes the standard Lebesgue measure. While each $f_t(x)$ is only defined on a ball centered at 0 of radius $d(x_0, \partial D)/t$, when we consider limits as $t \to 0$, we obtain mappings defined on all of $\mathbb{R}^n$. Of course, there is no reason for such a limit to exist. Fix real values $t_0 > 0$ and $r > 0$. Let $t \leq t_0$, then $f(x_0 + tx)$ has a bound on linear distortion, this is true since $f$ is quasiregular. Since $\rho_f$ preserves the volume of the unit ball, then the image of $B(x_0, r)$ under $f_t$ will be contained in a ball of radius $M > 0$ that depends on $t_0$. In particular, we can show that the outer dilatation of $f_t$ is bounded. Theorem 1.2.5 tells us that the linear distortions of $f_t$ for $t \leq t_0$ are all bounded by a constant $C$. Since $f_t(B(X_0, r))$ is contained in a ball of radius $M$, then each $f_t$ omits the needed values in the image for $t \leq t_0$. It follows from Theorem 1.2.7 that for any sequence $t_k \to 0$, there is a subsequence for which we do have local uniform convergence to some non-constant quasiregular mapping.
Definition 1.2.9. Let \( f : D \to \mathbb{R}^n \) be a quasiregular mapping defined on a domain \( D \subset \mathbb{R}^n \) and let \( x_0 \in \mathbb{R}^n \). A generalized derivative \( \varphi \) of \( f \) at \( x_0 \) is defined by

\[
\varphi(x) = \lim_{k \to \infty} f_{t_k}(x),
\]

for some decreasing sequence \((t_k)_{k=1}^{\infty}\), whenever the limit exists. The collection of generalized derivatives of \( f \) at \( x_0 \) is called the infinitesimal space of \( f \) at \( x_0 \) and is denoted by \( T(x_0, f) \).

To exhibit the behavior of generalized derivatives, we consider some simple examples.

Example 1.2.10. Let \( w \in \mathbb{C} \setminus \{0\} \) and define \( f(z) = wz \). we will calculate the generalized derivative of \( f \) at 0. Note that in \( \mathbb{C} \), we have that \( \mu(B(0,1)) = \pi \) and \( \mu(f(B(0,t))) = \pi |w|^2 t^2 \), so that the mean radius, for \( t > 0 \), is

\[
\rho_f(t) = \left( \frac{\pi |w|^2 t^2}{\pi} \right)^{1/2} = |w| t.
\]

Then for \( t > 0 \) we have

\[
f_t(z) = \frac{f(tz)}{\rho_f(t)} = \frac{wtz}{|w| t} = e^{i \arg w} z.
\]

Consequently, \( T(0, f) \) consists only of the map \( \varphi(z) = e^{i \arg w} z \).

Example 1.2.11. Let \( d \in \mathbb{N} \) and define \( f(z) = z^d \). First note that \( \mu(f(B(0,t))) = \pi t^d \).

Then at \( z_0 = 0 \), we have

\[
\rho_f(t) = \left( \frac{\mu(f(B(0,t)))}{\mu(B(0,1))} \right)^{1/2} = \left( \frac{\pi (t^d)^2}{\pi} \right)^{1/2} = t^d.
\]

The generalized derivative of \( f \) at 0 is

\[
\lim_{t \to 0} f_t(z) = \lim_{t \to 0} \frac{(tz)^d}{t^d} = z^d.
\]
Therefore $T(0, f)$ consists only of the map $\varphi(z) = z^d$.

Note that in Examples 1.2.10 and 1.2.11 we see that the generalized derivative $\varphi$ preserves the measure of the unit ball. These examples illustrate the informal property that generalized derivatives maintain the shape of $f$ near $x_0$, but they lose information on the scale of $f$.

In general, if a quasiregular map $f$ is in fact differentiable at $x_0 \in \mathbb{R}^n$ and is non-degenerate, i.e. $f'(x_0) \neq 0$, then $T(x_0, f)$ consists only of a scaled multiple of the derivative of $f$ at $x_0$. The reason for the scaling is the use of $\rho_f(r)$ in the definition of $f$. We may in fact replace $\rho_f(r)$ by $L_f(x_0, r), l_f(x_0, r)$ or any other quantity comparable to $\rho_f(r)$. In the special case of uniformly quasiregular mappings, that is, quasiregular mappings with a uniform bound on the distortion of the iterates, it was proved in [11] that at fixed points with $i(x_0, f) = 1$, they are bi-Lipschitz at $x_0$. That is,

**Proposition 1.2.12.** [11, Lemma 4.1] Suppose that $f : D \rightarrow \mathbb{R}^n$ is a $K$-uniformly quasiregular map, that $0$ is a fixed point of $f$, and that $i(0, f) = 1$. Then there exist $L \geq 1$ and neighborhood $U$ of the origin such that

$$
\frac{1}{L} |x| \leq |f(x)| \leq L|x| \text{ for every } x \in U.
$$

In this special case one may replace $\rho_f(r)$ with $r$ itself. In general, quasiregular mappings are only locally Hölder continuous and so it does not suffice to use $r$ instead of $\rho_f(r)$.

### 1.2.2 Infinitesimal Space

We now note some results from [10] that gives some properties of the infinitesimal space.
Theorem 1.2.13. [10, Theorem 2.7] Let \( f : D \to \mathbb{R}^n, n \geq 2, \) be a nonconstant \( K \)-quasiregular mapping and let \( x_0 \in D \). Then the infinitesimal space \( T(x_0, f) \) is not empty and each \( g \in T(x_0, f) \) is a nonconstant \( K \)-quasiregular mapping of \( \mathbb{R}^n \) onto \( \mathbb{R}^n \) such that

1. \( g(0) = 0 \),
2. \( \mu(g(B(0, 1))) = \mu(B(0, 1)) \),
3. \( i(0, g) = i(x_0, f) \),
4. \( \deg(g) = i(x_0, f) \), and
5. \( g(z) \to \infty \) as \( z \to \infty \).

Notice that the above theorem, for example condition (4), epitomizes that the global behavior of \( g \) is a reflection of the local behavior of \( f \) at \( x_0 \).

Corollary 1.2.14. [10, Corollary 2.8] Let \( f : D \to \mathbb{R}^n, n \geq 2, \) be a quasiregular mapping. The map \( f \) is locally injective at \( x_0 \in D \) if and only if each \( g \in T(x_0, f) \) is a quasiconformal mapping of \( \mathbb{R}^n \to \mathbb{R}^n \).

Let us note the following definition:

Definition 1.2.15. Let \( f : D \to \mathbb{R}^n \) be quasiregular on a domain \( D \) and let \( x_0 \in D \). If the infinitesimal space \( T(x_0, f) \) consists of only one element, then \( T(x_0, f) \) is called simple.

We get further useful results that allow us to approximate a quasiregular mapping by a map involving the generalized derivative when the infinitesimal space is simple. As in [10], we now introduce the equivalence relation \( \sim \). Let \( v, w : D \to \mathbb{R}^n \) and \( 0 \in D \), then

\[ v(x) \sim w(x) \text{ as } x \to 0 \]
means
\[ |v(x) - u(x)| = o(|v(x)| + |w(x)|). \]

For functions \( f, g : \mathbb{R} \to \mathbb{R}, f(x) = o(g(x)) \) as \( x \to 0 \) if and only if given \( \epsilon > 0 \), there exists \( \delta > 0 \) so that
\[ \frac{|f(x)|}{|g(x)|} < \epsilon \text{ for } |x| < \delta. \]

Moreover, the equivalence relation \( v(x) \sim w(x) \) is equivalent to either

\[ v(x) - w(x) = o(v(x)), \quad (1.6) \]

or

\[ v(x) - w(x) = o(w(x)). \quad (1.7) \]

We want to be able to relate a quasiregular mapping to the map’s corresponding generalized derivative. To do so, we use the following theorem.

**Theorem 1.2.16.** [10, Proposition 4.7] Let \( f : D \to \mathbb{R}^n, n \geq 2, \) be a nonconstant quasiregular mapping and \( x_0 \in D \). If \( T(x_0, f) = \{g\} \) is simple with \( f(0) = 0 \in D \), then \( f \) has the representation
\[ f(x) \sim \rho_f(|x|)g\left(\frac{x}{|x|}\right) =: D(x). \quad (1.8) \]

Also, we have asymptotic homogeneity of \( \rho_f \), that is there exists \( d > 0 \) such that for all \( s > 0 \)
\[ \rho_f(st) \sim s^d \rho_f(t) \quad (1.9) \]
as \( t \to 0 \), and \( t > 0 \). We call \( D(x) \) the asymptotic representation of \( f \).
We can see that if $f$ is locally injective, then $g \in T(x_0, f)$ is quasiconformal so that $D(x)$ is injective as well. In particular, we have the following versions of the chain rule and the inverse function theorem.

**Theorem 1.2.17.** [5, Theorem 1.2 and 1.3] Suppose $f,h : \mathbb{R}^n \rightarrow \mathbb{R}^n$ are quasiregular mappings which fix 0 and $T(0, f)$ and $T(0, h)$ are simple, with $T(0, f) = \{g_f\}$ and $T(0, h) = \{g_h\}$. Then there is a constant $C > 0$ so that $T(0, f \circ h) = \{g_{f \circ h}\} = \{C(g_f \circ h)\}$. Additionally, if $f$ is quasiconformal in a neighborhood $U$ of 0 then there exists a constant $c > 0$ such that $T(0, f^{-1}) = \{cg^{-1}\}$.

Gutlyanskii et al, [10], describes $D$ in terms of maps with bounded length distortion.

**Definition 1.2.18.** [14, Bounded Length Distortion] Let $L \geq 1$ and let $D$ be a domain in $\mathbb{R}^n$, $n \geq 2$. A continuous mapping $h : D \rightarrow \mathbb{R}^n$ is said to be of $L$-bounded length distortion, also $L$-BLD, if $h$ is discrete, open, sense-preserving and

$$\frac{l(\alpha)}{L} \leq l(h(\alpha)) \leq Ll(\alpha),$$

for every path $\alpha$ in $D$. Here $l(\alpha)$ denotes the length of the path $\alpha$.

We say that $h : D \rightarrow \mathbb{R}^n$ is of bounded length distortion, and write $h \in BLD$ if $h$ is $L$-BLD for some $L \geq 1$.

Note that $h \in BLD$ if and only if $h$ is quasiregular and $|h'(x)|$ is essentially bounded away from 0 and $\infty$. For an in depth discussion of BLD maps, see [14].

**Theorem 1.2.19.** [10, Proposition 4.18] Let $f : D \rightarrow \mathbb{R}^n$, $n \geq 2$, be a nonconstant quasiregular mapping and $x_0 \in D$. If $T(x_0, f) = \{g\}$, then $g$ has the representation:

$$g(z) = h(s(z))$$  \hspace{1cm} (1.10)
where the mapping \( s : \mathbb{R}^n \to \mathbb{R}^n \) defined by

\[
s(z) = z|z|^{d-1}, \quad d > 0,
\]

(1.11)

is a radial stretching of \( \mathbb{R}^n \) and the mapping \( h : \mathbb{R}^n \to \mathbb{R}^n, h(0) = 0, \)

\[
h(w) = \begin{cases} 
|w| \cdot g(w/|w|), & w \neq 0 \\
0, & w = 0 
\end{cases}
\]

(1.12)

Moreover, \( h \) is homogeneous of degree 1, that is,

\[
h(tw) = th(w), \quad t > 0,
\]

and of the class \( BLD(\mathbb{R}^n) \). The mapping \( g \) is of the class \( BLD(\mathbb{R}^n) \) if and only if \( d = 1 \).

1.2.3 Orbit Spaces

It is possible for quasiregular mapping to have more than one generalized derivative at a given point. It was observed in Fletcher and Wallis [9, Corollary 2.11] that the infinitesimal space can contain uncountably many mappings. Here we want to explore the possibilities for the shape of all maps from an infinitesimal space evaluated at a point.

If \( x \in D \) and \( \mathcal{F} \subset C(D, \mathbb{R}^n) \), denote by \( E_x : \mathcal{F} \to \mathbb{R}^n \) the point evaluation map, that is, if \( f \in \mathcal{F} \) then \( E_x(f) = f(x) \).
Definition 1.2.20. Let \( f : D \to \mathbb{R}^n \) be a quasiregular mapping defined on a domain \( D \subset \mathbb{R}^n \) and let \( x_0 \in D \). Then the orbit of a point \( x \in \mathbb{R}^n \) under the infinitesimal space \( T(x_0, f) \) is defined by

\[
\mathcal{O}(x) = E_x(T(x_0, f)) = \{ \varphi(x) : \varphi \in T(x_0, f) \}.
\]

Fletcher and Wallis show that the orbit space is the accumulation set of a curve.

Theorem 1.2.21. \([9, \text{Theorem 2.10}]\) Let \( f : D \to \mathbb{R}^n \) be a quasiregular mapping defined on a domain \( D \subset \mathbb{R}^n \) and let \( x_0 \in D \). Then for any \( x \in \mathbb{R}^n \) the orbit space \( \mathcal{O}(x) \) is the accumulation set of the curve \( t \mapsto f_t(x) \), where \( f_t(x) \) is defined by (1.4).

Moreover, \([7, \text{Theorem 1.5}]\) shows that for any \( x \in \mathbb{R}^n \) we have that \( \mathcal{O}(x) \) lies in a ring \( \{ y \in \mathbb{R}^n : 1/C' \leq |y| \leq C'' \} \) for some constant \( C' \geq 1 \) depending only on \( |x|, n, K_O(f) \) and \( i(x_0, f) \). Then Fletcher and Wallis were able to show that if the orbit of a point under all elements in the infinitesimal space contains more than one point, then it contains a continuum that must necessarily be contained in \( \mathbb{R}^n \setminus \{0\} \).

Corollary 1.2.22. \([9, \text{Corollary 2.11}]\) Let \( f : D \to \mathbb{R}^n \) be a quasiregular mapping defined on a domain \( D \subset \mathbb{R}^n \) and let \( x_0 \in D \). Then the infinitesimal space \( T(x_0, f) \) either consists of one element or uncountably many.

In two dimensions, Fletcher and Wallis are able to give the converse statement of conclusions made from Theorem 1.2.21. In particular, they show that any compact connected set away from the origin can be realized as the orbit space of a quasiconformal map.

Theorem 1.2.23. \([9, \text{Theoreom 2.12}]\) Let \( X \subset \mathbb{R}^2 \setminus \{0\} \) be a non-empty, compact and connected set. Then there exists a quasiregular mapping \( f : \mathbb{R}^2 \to \mathbb{R}^2 \) for which \( X \) is the image of the point evaluation map \( E_{e_1} : T(0, f) \to \mathbb{R}^2 \) for \( e_1 = (1, 0) \).

What we see is that any compact connected set in \( \mathbb{R}^2 \setminus \{0\} \) arises as an orbit space of some quasiconformal map. To prove this theorem, Fletcher and Wallis relied on computa-
tions involving the complex dilatation of a quasiconformal map, which can only be used in the plane. We complete the realization of the orbit space by showing that every compact connected set in $\mathbb{R}^n \setminus \{0\}$, for $n \geq 3$, arises as an orbit space of a quasiconformal map.

**Theorem 1.2.24** (Complete Realization of the Orbit Space). Let $X \subset \mathbb{R}^n \setminus \{0\}$ be a non-empty, compact and connected set. Then there exists a quasiconformal mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ for which $X$ is the image of the point evaluation map $E_{e_1} : T(0, f) \to \mathbb{R}^n$ for $e_1 = (1, 0, ..., 0)$.

Make note that the point $e_1$ is a choice made as a starting point, and that any point really could have been chosen. For the ease of calculations, $e_1$ is convenient. We prove this result in Chapter 3.

### 1.2.4 Attracting Fixed Points of Quasiregular Maps

For a quasiregular map $f : \mathbb{R}^n \to \mathbb{R}^n$, we want to understand the behavior near an attracting fixed point. However, since $f$ is not necessarily differentiable everywhere we can not rely on a power series representation to define what it means to classify a fixed point as (super)attracting or repelling. We also saw it is possible for a quasiregular map to have infinitely many generalized derivatives at a fixed point. When there are infinitely many generalized derivatives, the behavior of the map may vary drastically on different scales. In other words, it may be possible for a quasiregular mapping to exhibit attracting and repelling behavior as we approach the origin. To illustrate, we are motivated by [4, Proposition 1.7] to construct the following example.
Example 1.2.25. Our goal is to construct a quasiconformal map where on some spherical shells of large modulus of the unit ball attracts to the origin and on other spherical shells of large modulus of the unit ball repels from the origin. First let

\[ r_{2m} = -m^2 - 5m, \]
\[ r_{2m+1} = -m^2 - 6m - 1, \]
\[ s_{2m} = -m^2 - 6m - 4, \]
\[ s_{2m+1} = -m^2 - 7m - 5, \]
\[ C_m = -\frac{2}{3}m^2 - 2m - \frac{5}{3}, \text{ and} \]
\[ D_m = 2m^2 + 14m + 11. \]

Then for \( 0 \leq m \in \mathbb{Z} \) we want to define \( f : (0, 1) \to (0, 1) \) by

\[
f(x) = \begin{cases} 
    x/e & \text{if } x \in (e^{r_{2m+1}}, e^{r_{2m}}] \\
    e^{C_m}x^{1/3} & \text{if } x \in (e^{s_{2m}}, e^{r_{2m+1}}] \\
    ex & \text{if } x \in (e^{s_{2m+1}}, e^{s_{2m}}] \\
    e^{D_m}x^3 & \text{if } x \in (e^{r_{2m+2}}, e^{s_{2m+1}}] 
\end{cases} \tag{1.13}
\]

Note that the choice of \( r_k, s_k, C_m, \) and \( D_m \) are so that \( f \) is continuous.
Now let $F : \mathbb{B}^n \to \mathbb{R}^n$, where $\mathbb{B}^n$ is the unit ball in $\mathbb{R}^n$, be defined by

$$F(x) = f(|x|) \frac{x}{|x|} = \begin{cases} 
  x/e & \text{if } |x| \in (e^{r_{2m}+1}, e^{r_{2m}}] \\
  e^{C_m}x|\frac{x}{|x|}|^{-2/3} & \text{if } |x| \in (e^{s_{2m}}, e^{r_{2m}+1}] \\
  ex & \text{if } |x| \in (e^{s_{2m}+1}, e^{s_{2m}}] \\
  e^{D_m}x|x|^2 & \text{if } |x| \in (e^{r_{2m}+2}, e^{s_{2m}+1}] \\
  0 & \text{if } x = 0 
\end{cases} \quad (1.14)$$

In each spherical shell $F$ is of the form $Cx|x|^\alpha - 1$, where $C$ is the respective constant and $\alpha \in \{1/3, 1, 3\}$, which we know is quasiconformal when $\alpha > 0$. In this case we know that $F$ has infinitely many generalized derivatives at the origin. But $F$ has two nested sequences of annuli, on one the map satisfies $|f(x)| < |x|$ and on the other $|f(x)| > |x|$. We see that finding generalized derivatives in this case does not reveal much about the behavior of $F$ near the origin.

We see from the above example that it may not be dynamically interesting to compare quasiregular maps to their generalized derivatives in cases when there are infinitely many generalized derivatives. Therefore we restrict our attention to fixed point of quasiregular maps with a simple infinitesimal space. Also, dynamically speaking we want to avoid maps where we are attracting in one direction but repelling in another. For example the smooth linear map $f(x, y) = (3x, y/2)$ has 0 as a fixed point where the map repels away from 0 in the $x$ direction, but is attracting in the $y$ direction. Let us now define what we mean when we say a fixed point of a quasiregular map is geometrically attracting or superattracting.

**Definition 1.2.26 (Geometrically Attracting Fixed Point).** Suppose $f : \mathbb{R}^n \to \mathbb{R}^n$ is quasiconformal in a neighborhood of 0 with $f(0) = 0$. Then 0 is called a *geometrically attracting*
fixed point of \( f \) if there exist constants \( r > 0 \) and \( 0 < \lambda < 1 \) such that
\[ |f(x)| < \lambda |x| \]
whenever \( |x| < r \).

**Definition 1.2.27** (Geometrically Superattracting Fixed Point). Suppose \( f : \mathbb{R}^n \rightarrow \mathbb{R}^n \) is quasiregular with \( f(0) = 0 \) and \( i(f, 0) \geq 2 \). We say 0 is a *geometrically superattracting* fixed point if there exist constants \( r > 0 \) and \( 0 < \lambda < 1 \) such that
\[ |f(x)| < \lambda |x| \]
whenever \( |x| < r \).

Clearly \( f^m(x) \rightarrow 0 \) as \( m \rightarrow \infty \) for \( |x| < r \) in the attracting and superattracting cases.

### 1.2.4.1 Strategy for Generalizing Koenigs Linearization Theorem

We give a brief sketch of the approach we take, while labeling and designating needed assumptions to reach the goal of generalizing Koenigs Linearization Theorem to quasiregular maps. Recall from our discussion above, that we require the infinitesimal space of a quasiregular map at a fixed point to be simple. Let \( f : D \rightarrow \mathbb{R}^n \) be quasiregular, where \( T(0, f) = \{g\} \) is simple. Our goal is to relate a locally injective map which has 0 as an attracting fixed point to the generalized derivative of \( f \) at 0. Naturally, our first assumption is

**a)** 0 is a geometrically attracting fixed point of \( f \), with constant \( r > 0 \) and \( 0 < \lambda < 1 \).

Since \( T(0, f) \) is simple, (1.8) from Theorem 1.2.16 suggests we can conjugate \( f \) to \( D \) in a neighborhood of 0. Our goal then is to construct a quasiconformal map \( \psi \) that performs such a conjugation. Also, since we have (1.8), that is
\[ f(x) \sim \rho_f(|x|)g \left( \frac{x}{|x|} \right) =: D(x) \]
we can write
\[ f(x) = D(x) + E(x), \]
where \( E(x) \) is the error term defined by \( E(x) = f(x) - D(x) \).
We construct a sequence \((\psi_k)\) of functions that converge to the quasiconformal map \(\psi\) that we are looking for. We use the asymptotic representation of \(f, D\) in (1.8), to construct our sequence of functions \((\psi_k)\) in the following manner:

\[
\psi_1 = D^{-1} \circ f, \quad \text{and} \\
\psi_{k+1} = D^{-1} \circ \psi_k \circ f, \quad 1 \leq k \in \mathbb{Z}.
\]

Since we know that the composition of two quasiconformal mappings is quasiconformal and then once we know that both \(D\) and \(f\) are both quasiconformal in a neighborhood of the origin, then for each \(k\) we have that \(\psi_k\) is quasiconformal in the same neighborhood of the origin as well. To be sure that our sequence is well defined, we need to restrict the domain of \(f\) so that the range of \(f\) is a subset of the range of \(D\). To understand the behavior of \(f, D,\) and \(\psi_k, k \geq 1,\) we will be using a tool called the Zorich transform. The definition of the Zorich transform is postponed until Chapter 2. Note that some of the assumptions involve the Zorich transform version of our mappings, where the Zorich transform of a map \(f\) is denoted by \(\tilde{f}\). In fact, the Zorich transform of each \(\psi_k\) is quasiconformal in a half-beam which corresponds to the neighborhood of the origin where each \(\psi_k\) is quasiconformal. In Section 2.2.2 we show that \(D\) is quasiconformal is a neighborhood of the origin by analyzing \(\tilde{D}\). For \(g\tilde{D}\) to be quasiconformal we need \(\tilde{f} \in \text{BLD}\), the purpose of this condition is discussed in Section 2.2.2.

Let us define \(E_k(x) = \psi_k(x) - x\) for each \(k\). Since \(\psi_k \sim x\) as \(x \to 0\), we expect each \(E_k\) to be a small error term. Using (1.8), we show that \(\psi_1(x) = x + E_1(x)\) where \(|E_1(x)| = o(|x|)\). We correspondingly show that each \(|E_k(x)|\) is bounded. To analyze the sequence \((\psi_k)\) we will be taking the Zorich transform of each \(\psi_k\), denoted by \(\tilde{\psi}_k\), and show that \(\tilde{\psi}_k = X + \tilde{E}_k(X)\). We should note that \(\tilde{E}_k\) is not the Zorich transform of \(E_k\) but is the corresponding error term defined by \(\tilde{E}_k(X) = \tilde{\psi}_k(X) - X\). In fact, we see that \(\tilde{E}_k\) and \(E_k\) are indirectly related.
Observe that each $|\tilde{E}_k(X)|$ is bounded by a function involving $|\tilde{E}_1(X)|$ and $L$, where $L$ is the bi-Lipschitz constant from the following theorem:

**Theorem 1.2.28.** Given a mapping $f : D \rightarrow \mathbb{R}^n$ that is quasiconformal in a neighborhood of the origin with $T(0,f) = \{g\}$ and $\tilde{f} \in \text{BLD}$, then the Zorich Transform $\tilde{D}$ is $L$-bi-Lipschitz, where $D$ is the asymptotic representation of $f$ as defined in (1.8).

This theorem is proved in Section 2.2.2. To have a uniform bound on $|\tilde{E}_k(X)|$, we need a gap between consecutive terms of $\tilde{E}_1(\tilde{f}^k(X))$. To such an end, we introduce assumption

**b)** $|E_1(x)| < c|x|^\alpha$, where $c$ is a positive constant and $\alpha > 1$ such that $L^2 \lambda^{\alpha-1} < 1$, where $\lambda$ is from assumption (a).

To show that $\psi$ is quasiconformal in a neighborhood of 0, we use Theorem 1.2.7. To do so, we show that the maximal dilatation of $\psi_k$ is uniformly bounded. For this end, we need assumption

**c)** the derivative $[\tilde{D}^{-1}]'(X)$ has the condition

$$\|[[\tilde{D}^{-1}]'(X) - [\tilde{D}^{-1}]'(Y)]\| \leq L_3 |X - Y| \text{ a.e.}$$

(1.15)

where $L_3 \geq 1$, and that

$$\|\tilde{E}'(X)\| \leq c_1 e^{\beta x_n} \text{ a.e.},$$

(1.16)

where $\tilde{E} = \tilde{f} - \tilde{D}$, $c_1$ is a constant, and $\beta$ is a positive constant such that $L^2 \lambda^\beta < 1$, with $\lambda$ from assumption (a) and $L$ from Theorem 1.2.28.
In particular, the condition (1.15) allows us to write \( \| \tilde{E}'_k(X) \| \) in terms of \( \| \tilde{E}'(X) \| \) almost everywhere. The assumption of (1.16) gives us a gap when we pre-compose \( \tilde{E}'(X) \) with iterates of \( \tilde{f} \), so that we may have a uniform bound on \( \| \tilde{E}'_k(X) \| \) a.e.

We now state our generalization of Koenigs Linearization Theorem for quasiregular maps:

**Theorem 1.2.29.** Let \( f \) be quasiconformal in a neighborhood of \( 0 \) where \( 0 \) is a geometrically attracting fixed point of \( f \), \( T(0, f) = \{ g \} \) is simple and \( \tilde{f} \in BLD \), where \( D(x) = \rho_f(|x|)g(x/|x|) \), \( |E_1(x)| < c|x|^{\alpha} \), where \( c \) is a positive constant and \( \alpha > 1 \) such that \( L^2\lambda^{\alpha-1} < 1 \), and the derivative \( [\tilde{D}^{-1}]'(X) \) has the condition

\[
\|[\tilde{D}^{-1}]'(X) - [\tilde{D}^{-1}]'(Y)\| \leq L_3|X - Y| \text{ a.e.}
\]

where \( L_3 \geq 1 \), and that

\[
\|\tilde{E}'(X)\| \leq c_1e^{\beta X} \text{ a.e.,}
\]

where \( c_1 \) and \( \beta \) are positive constants such that \( L^2\lambda^\beta < 1 \). Then there exists a quasiconformal map \( \psi \) where

\[
\circ f = D \circ \psi,
\]

in a neighborhood of \( 0 \).

**1.2.4.2 Confirming Theorem 1.2.29 is a Generalization**

Let \( f \) be a holomorphic function such that \( 0 \) is fixed with multiplier \( 0 < |\lambda| < 1 \). Since this means that

\[
f(z) = \lambda z + \sum_{n=2}^{\infty} a_n z^n,
\]
we know that \( f \) is bijective in a neighborhood of the origin. Hence, there is an \( r > 0 \) so that \(|f(z)| < c_2 \lambda |z|\) where \( 0 < c_2 < \frac{1}{\lambda} \) for \(|z| < r\), satisfying condition (a). Note that we denote \( a_1 = \lambda \) for simplicity in some calculations. Now we need to determine what the generalized derivative of \( f \) is. To do so, we determine the mean radius function from (1.8), that is

\[
\rho_f(s) = \left( \frac{\mu[f(B(0, s))]^2}{\mu[B(0, 1)]} \right)^{1/2},
\]

where \( \mu \) is the standard Lebesgue measure. In the plane we know that \( \mu[B(0, 1)] = \pi \).

Now we need to determine the area of the image of \( B(0, s) \) under the holomorphic function \( f \), where \( s \) is a nonnegative real number. Using polar coordinates, that \( f \) is injective in a neighborhood of 0, and the power series representation of \( f \) we have

\[
\mu[f(B(0, s))] = \int_{B(0, s)} J_f(z) \, dx \, dy = \int_{B(0, s)} |f'(z)| \, dx \, dy = \int_{B(0, s)} f'(z) \overline{f}'(z) \, dx \, dy = \int_0^s \int_0^{2\pi} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_n \overline{a_m} n \lambda^{n+m-1} e^{i(n-m)\theta} \, d\theta \, dr = \pi \sum_{n=1}^{\infty} n s^{2n} |a_n|^2.
\]

Note that if \( n \neq m \) we have

\[
\int_0^{2\pi} e^{i(n-m)\theta} \, d\theta = \frac{1}{n-m} e^{2\pi i(n-m)\theta} - \frac{1}{n-m} = 0.
\]
This gives us the following calculations for the mean radius:

\[
\rho_f(s) = \left( \frac{\pi \sum_{n=1}^{\infty} n s^{2n} |a_n|^2}{\pi} \right)^{1/2} \\
= \left( s^2 |\lambda|^2 + \sum_{n=2}^{\infty} n s^{2n-2} |a_n|^2 \right)^{1/2} \\
= s|\lambda| \left( 1 + \sum_{n=2}^{\infty} \frac{n s^{2n-2} |a_n|^2}{|\lambda|^2} \right)^{1/2}.
\]

In this case, we can notice that \( \rho_f \) is a real analytic function. From (1.2.9) we have that the generalized derivative of \( f \) at 0 is

\[
g(z) = \lim_{s \to 0} \frac{f(sz) - f(0)}{\rho_f(s)} \\
= \lim_{s \to 0} \frac{f(sz) - f(0)}{s|\lambda| \left( 1 + \sum_{n=2}^{\infty} \frac{n s^{2n-2} |a_n|^2}{|\lambda|^2} \right)^{1/2}} \\
= \frac{f'(0) z}{|\lambda|} = \frac{\lambda}{|\lambda|} z.
\]

We can see that the generalized derivative is a scaled version of the map we would conjugate to in the classical Koenigs Linearization Theorem. Using (1.8) we have that the asymptotic representation of our holomorphic function is

\[
D(z) = \rho_f(|z|) g \left( \frac{z}{|z|} \right) \\
= |z||\lambda| \left( 1 + \sum_{n=2}^{\infty} \frac{n |z|^{2n-2} |a_n|^2}{|\lambda|^2} \right)^{1/2} \cdot \frac{\lambda z}{|\lambda||z|} \\
= \lambda z \left( 1 + \sum_{n=2}^{\infty} \frac{n |z|^{2n-2} |a_n|^2}{|\lambda|^2} \right)^{1/2}.
\]
Even though $\rho_f$ is real analytic, $\mathcal{D}$ is not holomorphic, unless $a_n = 0$ for $n \geq 2$. However, $\mathcal{D}$ is asymptotically conformal, that is for $B_s = \{ z : |z| < s \}$ we have that maximal dilatation of $\mathcal{D}$ restricted to $B_s$, $K(\mathcal{D}|_{B_s})$, goes to 1 as $s$ goes to 0, i.e. $K(\mathcal{D}|_{B_s}) \to 1$ as $s \to 0$. As we restrict ourselves to smaller and smaller domains about the origin, $\mathcal{D}$ becomes closer and closer to a holomorphic mapping. In this case, that resembles the fact that $\mathcal{D}(z) \sim \lambda z$ in a small enough neighborhood. From Theorem 1.2.17 we have that $\mathcal{D}^{-1}(z) \sim \frac{1}{\lambda} z$. Then the following calculations are made,

$$\psi_1(z) = \mathcal{D}^{-1} \circ f(z)$$

$$\sim \frac{1}{\lambda} \left( \lambda z + \sum_{n=2}^{\infty} a_n z^n \right)$$

$$= z + \sum_{n=2}^{\infty} \frac{a_n}{\lambda} z^n.$$  

By Taylor’s Theorem, there exists an $r_1 > 0$ and a positive constant $C$ so that

$$|E_1(z)| = |\psi_1(z) - z|$$

$$\sim \left| z + \sum_{n=2}^{\infty} \frac{a_n}{\lambda} z^n - z \right|$$

$$\leq C|z|^2 \text{ for } |z| < r_1.$$  

We see that condition (b) is satisfied. In the holomorphic case we use the logarithmic transform and take the principal branch of the logarithm for the range. Also, let $Z = X + iY$
be a point in the range of the principal branch of the logarithm where $X$ and $Y$ are real numbers, we have that the logarithmic transform of $D$ is

$$\tilde{D}(Z) = \log(D(e^Z))$$

$$= \log \left( \lambda e^Z \left( 1 + \sum_{n=2}^{\infty} \frac{n|a_n|^2 e^{X(2n-2)}}{|\lambda|^2} \right)^{1/2} \right)$$

$$= Z + \log(\lambda) + \frac{1}{2} \log \left( 1 + \sum_{n=2}^{\infty} \frac{n|a_n|^2 e^{X(2n-2)}}{|\lambda|^2} \right).$$

There exists an $M_1 < 0$ so that when $X < M_1$ we have that

$$\sum_{n=2}^{\infty} \frac{n|a_n|^2 e^{X(2n-2)}}{|\lambda|^2} < 1.$$  

Using the power series representation for the logarithm we have

$$\frac{1}{2} \log \left( 1 + \sum_{n=2}^{\infty} \frac{n|a_n|^2 e^{X(2n-2)}}{|\lambda|^2} \right) = \frac{1}{2} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \left( \sum_{n=2}^{\infty} \frac{n|a_n|^2 e^{X(2n-2)}}{|\lambda|^2} \right)^k$$

$$= \sum_{n=1}^{\infty} b_n e^{2nX},$$

where the $b_n$'s are the appropriate real constants. This gives us that

$$\tilde{D}(Z) = Z + \log(\lambda) + \sum_{n=1}^{\infty} b_n e^{2nX},$$

so that as long as we take $M_1$ negative enough, we have

$$\tilde{D}^{-1}(Z) = Z - \log(\lambda) + \sum_{n=0}^{\infty} c_n e^{nX},$$
where the \(c_j\)'s are the appropriate constants. Note that

\[
[D^{-1}]'(Z) = \begin{pmatrix} 1 + \sum_{n=1}^{\infty} nc_ne^{nx} & 0 \\ 0 & 1 \end{pmatrix}.
\]

Let \(W = A + iB\) in the range of the principal branch of the logarithm. Then using the power series for the exponential function, Taylor’s Theorem and that \(|X - A| \leq |Z - W|\), there is an \(L_3\) so that

\[
||[D^{-1}]'(Z) - [D^{-1}]'(W)|| = |\sum_{n=1}^{\infty} nc_ne^{nx} - \sum_{n=1}^{\infty} nc_ne^{nA}| 
\leq L_3|X - A| \leq L_3|Z - W|.
\]

Hence we have that (1.15) is satisfied. We know that

\[
\tilde{D}(Z) = Z + \log(\lambda) + \sum_{n=1}^{\infty} b_ne^{2nx}
\]

and

\[
\tilde{f}(Z) = Z + \log(\lambda) + \log \left(1 + \sum_{n=2}^{\infty} \frac{a_ne^{Z(n-1)}}{\lambda}\right).
\]

Also note that we define \(\tilde{E}\) by \(\tilde{E}(Z) = \tilde{f}(Z) - \tilde{D}(Z)\). There exists an \(M_2 < 0\) so that when \(X < M_2\) we can use the Taylor series form of the logarithm, which with \(e^{x+iy} = e^x \cos(Y) + ie^x \sin(Y)\) gives us

\[
\tilde{f}(Z) = Z + \log(\lambda) + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \left(\sum_{n=2}^{\infty} \frac{a_ne^{Z(n-1)}}{\lambda}\right)^k
\]

\[
= Z + \log(\lambda) + \sum_{n=1}^{\infty} d_ne^{nx}.
\]
where $d'_n s$ are appropriate constants. Let $M = \min\{M_1, M_2\}$, so that when $X < M$ we have

$$
\tilde{E}(Z) = \sum_{n=1}^{\infty} d_ne^{nz} - \sum_{n=1}^{\infty} b_ne^{2nX}.
$$

Using the triangle inequality with respect to norms, and Taylor’s theorem there is constant $c_1$ so that

$$
\|\tilde{E}'(Z)\| = \|d/dZ \left( \sum_{n=1}^{\infty} d_ne^{nz} - \sum_{n=1}^{\infty} b_ne^{2nX} \right)\|
\leq \left\| \sum_{n=1}^{\infty} d_ne^{nz} \right\| + \left\| \sum_{n=1}^{\infty} b_ne^{2nX} \right\|
\leq \sum_{n=1}^{\infty} n|d_n|e^{nx} + \sum_{n=1}^{\infty} 2n|b_n|e^{2nx}
\leq C_1 e^X.
$$

Therefore, for $X < M$ we satisfy (1.16) and hence have completed confirming condition (c).

Note that having $X < M$ is acceptable since we ultimately want our conditions to be true in a neighborhood of the origin. We see that our generalization applies for holomorphic maps.

Note that $D(z) \neq \lambda z$, however $D$ retains more information about the behavior of $f$ near the origin and could be argued to be a better candidate than conjugating to $\lambda z$. 
CHAPTER 2
THE ZORICH TRANSFORM

The purpose of this chapter is to define and discuss the properties of the Zorich transform. This new tool is an analogue to the logarithmic transform. Section 2.1 defines the class of Zorich maps which is used to define the Zorich transform in Section 2.2. Also, Section 2.1 proves that the Zorich maps are quasiregular, note that a Zorich map is an analogue to the exponential map on the plane. Zorich maps were first introduced by Zorich in [24].

Of particular note is Section 2.2.2 which gives a special property of the Zorich Transform of the asymptotic representation of a quasiregular mapping $f$. That is $\tilde{D}$, where $D$ is defined by (1.8), is bi-Lipschitz in a lower half beam. As a consequence, we have a deeper understanding of the mean radius function $\rho_f$. To be able to prove these key results, we show that the asymptotic representation $\mathcal{D}$ of a quasiregular map $f$, where $f$ is injective in a neighborhood of the origin and the infinitesimal space $T(0, f)$ is simple, is indeed quasiconformal in a neighborhood of the origin.

2.1 The Zorich Map

In this section we define and discuss some properties of the Zorich Transform, but to do so we must recall the definition of a Zorich map. We start with the following definition:
Definition 2.1.1. A function \( g : D \to \mathbb{R}^n \), where \( D \subset \mathbb{R}^n \), is infinitesimally bi-Lipschitz if there is a constant \( L \geq 1 \) such that

\[
\frac{1}{L} \leq \liminf_{x \to a} \frac{|g(x) - g(a)|}{|x - a|} \leq \limsup_{x \to a} \frac{|g(x) - g(a)|}{|x - a|} \leq L,
\]

for all \( a \in D \).

Note that if we let \( a = x \in D \) and \( x = x + \epsilon \), with \( \epsilon = (\epsilon_1, ..., \epsilon_n) \) where \( |\epsilon| < r \) for some small \( r > 0 \), then it is sufficient to show that

\[
\frac{1}{L}|\epsilon| = \frac{1}{L}|x - (x + \epsilon)| \leq |g(x) - g(x + \epsilon)| \leq L|x - (x + \epsilon)| = L|\epsilon|
\]

to satisfy the definition of infinitesimally bi-Lipschitz.

Here we define the class of Zorich maps. Let \( g : D \to \mathbb{R}^n \), where \( D \subset \mathbb{R}^{n-1} \times \{0\} \) such that \( \bar{D} \) is a \((n-1)\)-polytope which creates a discrete group when continuously reflected in the \((n-2)\)-faces. The group \( G \) that is acting on \( \bar{D} \) is isomorphic to \( \mathbb{Z}^{n-1} \times P \), where \( P \) is a point group of rotations, see [22]. Also, we must have \( g(D) \) is the upper unit sphere (upper in terms of \( g(x_1, ..., x_{n-1}, 0) = (y_1, ..., y_n) \) is on the unit sphere where \( y_n \geq 0 \)) and \( g \) is infinitesimally bi-Lipschitz. We can extend the domain \( D \) of \( g \) to the domain \( \mathbb{R}^{n-1} \times \{0\} \) where a reflection in a \((n-2)\)-face of \( \bar{D} \) in the pre-image corresponds to reflection of the half unit sphere across the \( y_1, ..., y_{n-1} \)-plane so that \( y_n \leq 0 \). Then as we keep reflecting in the \((n-2)\)-faces of the corresponding cells in the pre-image, we appropriately reflect the half unit sphere. Let us denote this extension of \( g \) as the function \( h : \mathbb{R}^{n-1} \times \{0\} \to \mathbb{R}^n \).

We define a Zorich Map \( \mathcal{Z} : \mathbb{R}^n \to \mathbb{R}^n \setminus \{0\} \) where \( n \geq 3 \) to be

\[
\mathcal{Z}(x_1, ..., x_n) = e^{x_n} h(x_1, ..., x_{n-1}, 0).
\]
Note that these maps are infinite to one. Also note that $\mathcal{Z}$ is strongly automorphic with respect to $G$, see [12].

**Theorem 2.1.2.** If $g : D \to \mathbb{R}^n$, where $D \subset \mathbb{R}^{n-1} \times \{0\}$ such that $D$ is a $(n-1)$-polytope as defined above with $g(D)$ being the upper unit sphere and $g$ is infinitesimally bi-Lipschitz, then

$$
\mathcal{Z}(x) = e^{x_n} h(x_1, \ldots, x_{n-1}, 0)
$$

is quasiregular in $\mathbb{R}^n$, where $h : \mathbb{R}^{n-1} \times \{0\} \to \mathbb{R}^n$ is the extension of $g$ by reflections as defined earlier.

**Proof.** Since $h$ is extended by reflections in $(n-2)$-faces of $D$, we can restrict our attention to $h|_D = g$. Note that we can see that since $g$ is infinitesimally bi-Lipschitz that

$$
\mathcal{Z}|_D(x) = e^{x_n} g(x_1, \ldots, x_{n-1}, 0)
$$

is absolutely continuous on lines. Also, since we are multiplying each coordinate in the image of $g$ by $e^{x_n}$ we can see that $\mathcal{Z}|_D$ must also be locally $L^n$-integrable. All that is left to show is that $\mathcal{Z}_g$ has bounded distortion

Since $g$ is infinitesimally bi-Lipschitz, there is a $L \geq 1$ such that

$$
\frac{1}{L} \leq \liminf_{\epsilon \to 0} \frac{|g(x + \epsilon) - g(x)|}{|\epsilon|} \leq \limsup_{\epsilon \to 0} \frac{|g(x + \epsilon) - g(x)|}{|\epsilon|} \leq L,
$$

for all $x \in D$, $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$. The linear distortion is

$$
H(x, \mathcal{Z}) = \limsup_{r \to 0} \frac{\max_{|\epsilon| = r} |\mathcal{Z}(x + \epsilon) - \mathcal{Z}(x)|}{\min_{|\epsilon| = r} |\mathcal{Z}(x + \epsilon) - \mathcal{Z}(x)|}
= \limsup_{r \to 0} \frac{\max_{|\epsilon| = r} |e^{x_n} (e^{\epsilon_n} g(x_1 + \epsilon_1, \ldots, x_{n-1} + \epsilon_{n-1}, 0) - g(x_1, \ldots, x_{n-1}, 0))|}{\min_{|\epsilon| = r} |e^{x_n} (e^{\epsilon_n} g(x_1 + \epsilon_1, \ldots, x_{n-1} + \epsilon_{n-1}, 0) - g(x_1, \ldots, x_{n-1}, 0))|}.
$$
Note that
\[ \lim_{x \to 0} \frac{e^x - 1}{x} = 1, \]
so there is a > 0, a ∈ R, such that |e^{\epsilon_n} - 1| = a|\epsilon_n|, where a → 1 as \( \epsilon_n \to 0 \). For notation, let \( \bar{x} = (x_1, \ldots, x_{n-1}, 0) \). Also note that \( r^2 = |\epsilon|^2 = |\bar{\epsilon}|^2 + |\epsilon_n|^2 \), so that \( |\epsilon_n|^2 = r^2 - |\bar{\epsilon}|^2 \). This leads to
\[ |\mathcal{Z}(x + \epsilon) - \mathcal{Z}(x)| = e^{\epsilon_n} |e^{\epsilon_n} g(\bar{x} + \bar{\epsilon}) - g(\bar{x})| \]
\[ = e^{\epsilon_n} |e^{\epsilon_n} (g(\bar{x} + \bar{\epsilon}) - g(\bar{x})) + g(\bar{x}) (e^{\epsilon_n} - 1) |. \]

Notice that \( g(\bar{x} + \bar{\epsilon}) - g(\bar{x}) \) describes how the first \( n - 1 \) coordinates map onto the unit sphere. In particular for a point \( A \) on the unit sphere, \( g(\bar{x} + \bar{\epsilon}) - g(\bar{x}) \) moves point \( A \) to point \( B \), still on the unit sphere, by a distance of \( c|\bar{\epsilon}| \) where \( \frac{1}{L} \leq c \leq L \), since \( h \) is bi-Lipschitz. Then \( e^{\epsilon_n} - 1 \) moves point \( B \) orthogonally from the unit sphere to a point \( C \) by a distance of \( |e^{\epsilon_n} - 1| = a|\epsilon_n| \). Let \( L' \) be the distance from point \( A \) to point \( C \), in particular
\[ L' = |e^{\epsilon_n} (g(\bar{x} + \bar{\epsilon}) - g(\bar{x})) + g(\bar{x}) (e^{\epsilon_n} - 1) |. \]

One can also notice that \( \angle ABC = \pi/2 + \delta \) with \( \delta > 0 \) where \( \delta \to 0 \) as \( r \to 0 \). The linear distance \( L' \) is then
\[ L'^2 = (c|\bar{\epsilon}|)^2 + a^2|\epsilon_n|^2 - 2ac|\bar{\epsilon}||\epsilon_n| \cos \left( \frac{\pi}{2} + \delta \right) \]
\[ = c^2|\bar{\epsilon}|^2 + a^2r^2 - a^2|\bar{\epsilon}|^2 + 2ac|\bar{\epsilon}||\epsilon_n| \delta. \]

Since \( c \leq L \), we have that
\[ L'^2 \leq L^2|\bar{\epsilon}|^2 + a^2r^2 + 2L|\bar{\epsilon}||\epsilon_n| \delta \leq r^2(L^2 + a^2 + 2La\delta). \]
For $\epsilon$ sufficiently small, we can have $a$ close enough to 1 and $\delta$ small enough so that

$$L' \leq r\sqrt{L^2 + a^2 + 2La\delta} \leq 2r\sqrt{L^2 + 1}.$$  

We also have that

$$L^2 \geq \frac{1}{L^2} |\bar{\epsilon}|^2 + a^2r^2 - a^2|\bar{\epsilon}|^2 + \frac{2}{L}a|\bar{\epsilon}||\epsilon_n|\delta$$

$$\geq \frac{r^2}{L^2} - \frac{|\epsilon_n|^2}{L^2} + a^2|\epsilon_n|^2$$

$$= \frac{r^2}{L^2} + \frac{(a^2L^2 - 1)|\epsilon_n|^2}{L^2}.$$ 

Since $L^2 \geq 1$ we have $a^2L^2 \geq a^2$ which gives us $a^2L^2 - 1 \geq a^2 - 1$. We have that

$$L^2 \geq \frac{r^2}{L^2} + \frac{(a^2 - 1)|\epsilon_n|^2}{L^2}.$$ 

Since $r^2 = |\bar{\epsilon}|^2 + |\epsilon_n|^2$, we know that $|\epsilon_n| \in [0, r]$. If $a^2 - 1 \geq 0$, then

$$L^2 \geq \frac{r^2}{L^2}$$

which means

$$L' \geq \frac{r}{L} \geq \frac{r}{2\sqrt{L^2 + 1}}.$$ 

If $a^2 - 1 < 0$ we have

$$L^2 \geq \frac{r^2}{L^2} + \frac{(a^2 - 1)r^2}{L^2} = \frac{a^2r^2}{L^2}.$$ 

Since $a \to 1$ as $r \to 0$, we can find $r$ small enough so that $a > \frac{1}{2}$. Then we have

$$L^2 \geq \frac{(1/2)^2r^2}{L^2} = \frac{r^2}{4L^2}.$$
Again, we get

\[ L' \geq \frac{r}{2L} > \frac{r}{2\sqrt{L^2 + 1}}. \]

Then we have that our linear distortion

\[
H(x, Z) = \limsup_{r \to 0} \frac{\max_{|\epsilon| = r} |e^{\epsilon_n} g(x_1 + \epsilon, \ldots, x_{n-1} + \epsilon_{n-1}, 0) - g(x_1, \ldots, x_{n-1}, 0)|}{\min_{|\epsilon| = r} |e^{\epsilon_n} g(x_1 + \epsilon, \ldots, x_{n-1} + \epsilon_{n-1}, 0) - g(x_1, \ldots, x_{n-1}, 0)|}
\leq \frac{2r\sqrt{L^2 + 1}}{2\sqrt{L^2 + 1}} = 4(L^2 + 1) \leq 8L^2.
\]

From (1.1) and (1.2) we have that the maximal dilatation \( K \) of \( Z \) is bounded by

\[
(H(x, Z))^{n-1} = (8L^2)^{n-1} \text{ for all } x \in D.
\]

Therefore \( Z \) is quasiregular.

For the constructions of our maps we deal with a particular Zorich map. We define

\[
g(x_1, \ldots, x_{n-1}, 0) = \left( \frac{x_1 \sin M(x_1, \ldots, x_{n-1})}{\sqrt{x_1^2 + \cdots + x_{n-1}^2}}, \ldots, \frac{x_{n-1} \sin M(x_1, \ldots, x_{n-1})}{\sqrt{x_1^2 + \cdots + x_{n-1}^2}}, \cos M(x_1, \ldots, x_{n-1}) \right),
\]

where \( M(x_1, \ldots, x_{n-1}) = \max \{|x_1|, \ldots, |x_{n-1}|\} \), which maps the \([-\pi/2, \pi/2]^{n-1}\) cube to the half unit sphere in \( \mathbb{R}^n \) where \( y_n \geq 0 \) in the image. By considering the limit of \( g \) as \((x_1, \ldots, x_{n-1}, 0)\) goes to the origin, we can extend \( g \) by continuity so that \( g(0, \ldots, 0, 0) = (0, \ldots, 0, 1) \).

**Proposition 2.1.3.** The function \( g \) defined in (2.1) is infinitesimally bi-Lipschitz.

**Proof.** Notice that \( g \) maps the \([-\pi/2, \pi/2]^{n-1}\) cube to the half unit sphere in \( \mathbb{R}^n \) where \( y_n \geq 0 \) in the image. The calculations for \( n > 3 \) are very similar, but even more tedious than
the calculations for \(n = 3\). We show that for \(n = 3\) that for \(g : [-\pi/2, \pi/2]^2 \to \mathbb{R}^3\) defined by

\[
g(x, y, 0) = \left(\frac{x \sin M(x, y)}{\sqrt{x^2 + y^2}}, \frac{y \sin M(x, y)}{\sqrt{x^2 + y^2}}, \cos M(x, y)\right)
\]

where \(M(x, y) = \max\{|x|, |y|\}\), is infinitesimally bi-Lipschitz, and then by a similar argument we can conclude that all other \(g\) functions for \(n > 3\) are also infinitesimally bi-Lipschitz.

Without loss of generality, since \(g\) is symmetric in the square, we restrict ourselves to

\[
A := \{(x, y, z) \in [-\pi/2, \pi/2]^2 \times \{0\} : x \geq |y|\},
\]

so that \(M(x, y) = x\) for \((x, y, z) \in A\). Note that when we take \((x, y, z) \in A\) we can omit the origin, as a single point has Lebesgue measure zero, and so our map still has bounded distortion and is quasiregular. First we note some useful Taylor series expansions:

\[
\cos \epsilon = 1 - \frac{\epsilon^2}{2} + o(\epsilon^2),
\]

\[
\sin \epsilon = \epsilon + o(\epsilon^2),
\]

and

\[
((x + \epsilon)^2 + (y + \delta)^2)^{-1/2} = (x^2 + y^2)^{-1/2} \left(1 - \frac{\epsilon x + \delta y}{x^2 + y^2} + o(|(\epsilon, \delta)|^2)\right).
\]

Here we take \(\epsilon\) and \(\delta\) to be small enough so that \((x + \epsilon, y + \delta) \in A\) for our calculations. One can ask about how we handle the distortion about the boundary of \(A\). The following calculations are similar with same final estimates when we consider the other triangle quadrants, which gives our infinitesimally bi-Lipschitz result for \(h\). We have

\[
|g(x, y) - g(x + \epsilon, y + \delta)|^2 = |(u, v, w)|^2,
\]
where

\[
u = \frac{x \sin x}{\sqrt{x^2 + y^2}} - \frac{(x + \epsilon) \sin(x + \epsilon)}{\sqrt{(x + \epsilon)^2 + (y + \delta)^2}}
\]

\[
v = \frac{y \sin x}{\sqrt{x^2 + y^2}} - \frac{(y + \delta) \sin(x + \epsilon)}{\sqrt{(x + \epsilon)^2 + (y + \delta)^2}}
\]

and

\[
w = \cos x - \cos(x + \epsilon).
\]

Using the Taylor series above, we have the following calculations,

\[
u^2 = \left(\frac{x \sin x}{\sqrt{x^2 + y^2}} - \frac{(x + \epsilon) \sin(x + \epsilon)}{\sqrt{(x + \epsilon)^2 + (y + \delta)^2}}\right) \left(1 - \frac{\epsilon x + \delta y}{x^2 + y^2}\right)^2 + o((\epsilon, \delta)^2)
\]

\[
= \frac{1}{x^2 + y^2} \left(\frac{-\epsilon x^2 \sin x}{x^2 + y^2} + \frac{\delta y x \sin x}{x^2 + y^2} + \epsilon x \cos x + \epsilon \sin x\right)^2 + o((\epsilon, \delta)^2)
\]

\[
= \frac{1}{x^2 + y^2} \left(\frac{\epsilon^2 x^4 \sin^2 x}{(x^2 + y^2)^2} + \frac{2 \epsilon \delta x^3 y \sin^2 x}{(x^2 + y^2)^2} - \frac{2 \epsilon^2 x^2 \sin^2 x}{x^2 + y^2} - \frac{2 \epsilon^2 x^2 \sin^2 x}{x^2 + y^2} + \frac{\delta^2 x^2 y^2 \sin^2 x}{(x^2 + y^2)^2}\right)
\]

\[
+ \frac{1}{x^2 + y^2} \left(\frac{-2 \epsilon \delta x^2 y \sin x \cos x}{x^2 + y^2} - \frac{2 \epsilon \delta x y \sin^2 x}{x^2 + y^2} + \epsilon^2 x^2 \cos^2 x + 2 \epsilon x \sin x \cos x + \epsilon^2 \sin^2 x\right)
\]

\[
+ o((\epsilon, \delta)^2),
\]

\[
v^2 = \left(\frac{y \sin x}{\sqrt{x^2 + y^2}} - \frac{(y + \delta) \sin(x + \epsilon)}{\sqrt{(x + \epsilon)^2 + (y + \delta)^2}}\right) \left(1 - \frac{\epsilon x + \delta y}{x^2 + y^2}\right)^2 + o((\epsilon, \delta)^2)
\]

\[
= \frac{1}{x^2 + y^2} \left(\frac{-\epsilon y \sin x}{x^2 + y^2} + \frac{\delta y^2 \sin x}{x^2 + y^2} + \epsilon y \cos x + \delta \sin x\right)^2 + o((\epsilon, \delta)^2)
\]

\[
= \frac{1}{x^2 + y^2} \left(\frac{\epsilon^2 y^2 \sin^2 x}{(x^2 + y^2)^2} + \frac{2 \epsilon \delta y^3 \sin^2 x}{(x^2 + y^2)^2} - \frac{2 \epsilon^2 y^2 \sin x \cos x}{x^2 + y^2} - \frac{2 \epsilon \delta y \sin^2 x}{x^2 + y^2} + \frac{\delta^2 y^4 \sin^2 x}{(x^2 + y^2)^2}\right)
\]

\[
+ \frac{1}{x^2 + y^2} \left(\frac{-2 \epsilon \delta y^3 \sin x \cos x}{x^2 + y^2} - \frac{2 \delta^2 y^2 \sin^2 x}{x^2 + y^2} + \epsilon^2 y^2 \cos^2 x + 2 \epsilon \delta y \sin x \cos x + \delta^2 \sin^2 x\right)
\]

\[
+ o((\epsilon, \delta)^2),
\]
and

\[ w^2 = (\cos x - (\cos x \cos \epsilon - \sin \epsilon \sin x))^2 + o(|(\epsilon, \delta)|^2) \]

\[ = \left( \cos x - \cos x + \frac{\epsilon^2}{2} \cos x + \epsilon \sin x \right)^2 + o(|(\epsilon, \delta)|^2) \]

\[ = \epsilon^2 \sin^2 x + o(|(\epsilon, \delta)|^2). \]

Then separating into \( \epsilon^2, \delta^2, \) and \( \epsilon \delta \) terms we have

\[ |(u, v, w)|^2 = u^2 + v^2 + w^2 \]

\[ = \frac{\epsilon^2 \sin^2 x}{(x^2 + y^2)^3} (x^4 - 2x^4 - 2x^2 y^2 + 2x^2 y^2 + y^4 + x^2 y^2) \]

\[ + \frac{\delta^2 \sin^2 x}{(x^2 + y^2)^3} (x^2 y^2 + y^4 - 2x^2 y^2 - 2y^4 + x^4 + 2x^2 y^2 + y^4) \]

\[ + \frac{\epsilon \delta \sin^2 x}{(x^2 + y^2)^3} (2x^3 y - 2x^3 y - 2xy^3 + 2xy^3 - 2x^3 y - 2xy^3) + o(|(\epsilon, \delta)|^2) \]

\[ = \epsilon^2 \left( 1 + \frac{y^2 \sin^2 x}{(x^2 + y^2)^2} \right) + \delta^2 \left( \frac{x^2 \sin^2 x}{(x^2 + y^2)^2} \right) \]

\[ - 2\epsilon \delta \left( \frac{xy \sin^2 x}{(x^2 + y^2)^2} \right) + o(|(\epsilon, \delta)|^2). \]

Here we have that

\[ |g(x, y) - g(x+\epsilon, y+\delta)|^2 = \epsilon^2 \left( 1 + \frac{y^2 \sin^2 x}{(x^2 + y^2)^2} \right) + \delta^2 \left( \frac{x^2 \sin^2 x}{(x^2 + y^2)^2} \right) - 2\epsilon \delta \left( \frac{xy \sin^2 x}{(x^2 + y^2)^2} \right) + o(|(\epsilon, \delta)|^2). \]

We can notice that the term

\[ \epsilon^2 \left( 1 + \frac{y^2 \sin^2 x}{(x^2 + y^2)^2} \right) + \delta^2 \left( \frac{x^2 \sin^2 x}{(x^2 + y^2)^2} \right) - 2\epsilon \delta \left( \frac{xy \sin^2 x}{(x^2 + y^2)^2} \right) \]
is a quadratic form in \((\epsilon, \delta)\) with corresponding matrix

\[
B = \begin{pmatrix}
1 + \frac{y^2 \sin^2 x}{(x^2 + y^2)^2} & \frac{-xy \sin^2 x}{(x^2 + y^2)^2} \\
\frac{-xy \sin^2 x}{(x^2 + y^2)^2} & \frac{x^2 \sin^2 x}{(x^2 + y^2)^2}
\end{pmatrix}
\]

Since we are in quadratic form the eigen-values and -vectors tell us how much and in what direction we have distortion. If the eigen-values are bounded above and below by positive constants, then we have that our map \(h\) is infinitesimally bi-Lipschitz. That is, if the eigen-values \(\lambda\) have the bounding \(\frac{1}{L} \leq \lambda \leq L\) for some \(L \geq 1\), then we have

\[
\frac{1}{L} (\epsilon^2 + \delta^2) \leq \epsilon^2 \left(1 + \frac{y^2 \sin^2 x}{(x^2 + y^2)^2}\right) + 2\epsilon \delta \left(\frac{-xy \sin^2 x}{(x^2 + y^2)^2}\right) + \delta^2 \left(\frac{x^2 \sin^2 x}{(x^2 + y^2)^2}\right) \leq L (\epsilon^2 + \delta^2),
\]

which when we consider the small error term gives us an \(\tilde{L} \geq 1\) such that

\[
\frac{1}{L} (\epsilon^2 + \delta^2) \leq |g(x, y) - g(x + \epsilon, y + \delta)|^2 \leq \tilde{L} (\epsilon^2 + \delta^2).
\]

To find our eigen-values, we have

\[
det(\lambda I - B) = \lambda^2 - \lambda \left(1 + \frac{y^2 \sin^2 x}{(x^2 + y^2)^2}\right) + \frac{x^2 y^2 \sin^4 x}{(x^2 + y^2)^4}
\]

\[
+ \left(1 + \frac{y^2 \sin^2 x}{(x^2 + y^2)^2}\right) \left(\frac{x^2 \sin^2 x}{(x^2 + y^2)^2}\right)
\]

\[
= \lambda^2 - \lambda \left(1 + \frac{\sin^2 x}{x^2 + y^2}\right) + \frac{x^2 \sin^2 x}{(x^2 + y^2)^2},
\]

so that \(det(\lambda I - B) = 0\) when

\[
\lambda = \frac{1}{2} \left(1 + \frac{\sin^2 x}{x^2 + y^2} \pm \sqrt{\frac{(x^2 - \sin^2 x)^2 + 2x^2y^2 + y^4 + 2y^2 \sin^2 x}{(x^2 + y^2)^2}}\right).
\]
For the rest of the calculations, we use the following facts about $\sin x/x$.

**Lemma 2.1.4.** If $f(x) = \frac{\sin x}{x}$, then $f$ is decreasing on $(0, \pi/2)$ and $f : [0, \pi/2] \to [2/\pi, 1]$.

Assuming that $x \neq 0$, we now show that $\lambda > 0$. Also note that

$$
\lambda = \frac{1}{2a} (-b \pm \sqrt{b^2 + 4ac}) = \frac{1}{2} \left( -b \pm \sqrt{b^2 - 4c} \right)
$$

where

$$a = 1,$$

$$b = -\left( 1 + \frac{\sin^2 x}{x^2 + y^2} \right), \text{ and}$$

$$c = \frac{x^2 \sin^2 x}{(x^2 + y^2)^2}.$$

Notice that $-b > 0$ and $c > 0$. We also have

$$b^2 - 4c = \frac{(x^2 - \sin^2 x)^2 + 2x^2 y^2 + y^4 + 2y^2 \sin^2 x}{(x^2 + y^2)^2} > 0,$$

then $|b| > \sqrt{b^2 - 4c}$. This gives us

$$\lambda = -b \pm \sqrt{b^2 - 4c} > 0.$$

Since we have $x \geq |y|$, with $x \neq 0$ since we are not at the origin, then

$$\lambda \leq \frac{1}{2} \left( 1 + \frac{\sin^2 x}{x^2} + \sqrt{\frac{(x^2)^2}{x^4} + \frac{2x^4}{x^4} + \frac{x^4}{x^4} + \frac{2x^2 \sin^2 x}{x^4}} \right)
$$

$$\leq \frac{1}{2} \left( 1 + 1 + \sqrt{1 + 2 + 1 + 2} \right) = 1 + \frac{\sqrt{6}}{2}.$$
Let $p = -b$ and $q = \sqrt{b^2 - 4c}$, then $\lambda = p \pm q$. We showed that

$$\lambda \leq p + q < 1 + \frac{\sqrt{6}}{2}.$$  

Also, note

$$p^2 - q^2 = (-b)^2 - \left(\sqrt{b^2 - 4c}\right)^2 = b^2 - b^2 + 4c = 4c$$

$$= 4 \frac{x^2 \sin^2 x}{(x^2 + y^2)^2} \geq 4 \frac{x^2 \sin^2 x}{(x^2 + x^2)^2} = 4 \frac{x^2 \sin^2 x}{4x^4}$$

$$= \frac{\sin^2 x}{x^2} \geq \frac{4}{\pi^2},$$

since we have $(x, y) \in A$. Here we want to show that $\lambda \geq p - q$ is bounded from below. We know that

$$p + q \leq 1 + \frac{\sqrt{6}}{2}, \text{ and }$$

$$p - q \geq \frac{4}{\pi^2}.$$  

This leads to the following calculation,

$$\lambda \geq p - q = \frac{p^2 - q^2}{p + q} \geq \frac{4}{\pi^2} \frac{p^2}{p + q}$$

$$\geq \frac{4}{\pi^2} \frac{1}{1 + \frac{\sqrt{6}}{2}} = \frac{8}{\pi^2 (2 + \sqrt{6})}.$$  

Since $\frac{\pi^2 (2 + \sqrt{6})}{8} > 1 + \frac{\sqrt{6}}{2}$, then we can let $L = \frac{\pi^2 (2 + \sqrt{6})}{8}$, so that $g(x, y)$ is infinitesimally bi-Lipschitz.  

$\square$
2.2 Defining the Zorich Transform

In this section we discuss how to define the Zorich transform and the corresponding properties. For a given Zorich map, we define a Zorich transform \( \tilde{f} \) for a continuous function \( f : \mathbb{R}^n \to \mathbb{R}^n \) to be

\[
Z \circ \tilde{f}(x) = f \circ Z(x).
\]

We discuss the domain of \( \tilde{f} \) in a little bit. The difficulty here is how the Zorich map is defined, every \((n - 2)\) face of \( \bar{D} \times \mathbb{R} \) is in the branch set, whereas with the exponential map we have no branch set. Under a Zorich transform, it may be possible for a neighborhood of a point to move partially through one these \((n - 2)\) faces causing the neighborhood to split apart due to the Zorich map being defined using reflections. In other words, it is possible for a sequence to converge to a single point in the domain, whereas in the range of the Zorich transformation, the image of the sequence has subsequences that converge to two or more distinct points.

However, choose \( C \) to be \( \bar{D} \cup \hat{D} \), where \( \hat{D} \) is one of the corresponding adjacent reflections of \( D \) in one of the \((n - 2)\)-faces of \( \bar{D} \). In \( \mathbb{R}^{n - 1} \), \( C \) is a fundamental set under the group action of \( G \). In particular, we have an equivalence relation on \( \mathbb{R}^{n - 1} \) defined by the group action of \( G \) on \( C \). In a natural way, we can extend this equivalence relation to an equivalence relation \( \sim \) on \( \mathbb{R}^n \) by letting the group action \( G \) act appropriately on \( B = C \times \mathbb{R} \). Note that \( B \) is a fundamental set under the appropriate group action of \( G \) on \( \mathbb{R}^n \). We can define \( Z : \mathbb{R}^n / \sim \to \mathbb{R}^n \setminus \{0\} \), where under the equivalence class, for a fixed \( z \in \mathbb{R} \), we identify points on the boundary \( \hat{D} \times \{z\} \) to points on the boundary of \( \bar{D} \times \{z\} \). As a consequence, open neighborhoods on the boundary of \( B \) may seem disconnected when viewed as a set of \( \mathbb{R}^n \), but is really connected under the quotient space. In particular, \( Z \) is a homeomorphism from a fundamental set \( B \), as an equivalence class, to \( \mathbb{R}^n \setminus \{0\} \). Correspondingly, we can
define \( Z^{-1} : \mathbb{R}^n \setminus \{0\} \to B \). From here on out, when we discuss a fundamental set \( B \), we are treating it as our base for an equivalence class of our quotient space. In particular, \( \tilde{f} \) maps from the equivalence class of \( B \) to the equivalence class of \( B \), for simplicity we define \( \tilde{f} : B \to B \).

As a result of “restricting” our attention to \( B \) by considering the quotient space, we can see that \( Z \circ Z^{-1} \) is the identity map on \( B \). Furthermore, when we have quasiregular maps \( f \) and \( g \) we get that

\[
\tilde{f} \circ \tilde{g} = Z^{-1} \circ f \circ Z \circ Z^{-1} \circ g \circ Z = Z^{-1} \circ f \circ g \circ Z = \tilde{f} \circ \tilde{g}.
\]

For our particular Zorich map \( Z \) defined by \( g \) from (2.1) we choose our fundamental set to be

\[
B := \left( \left( \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right] \times \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right] \right)^{n-2} \right) \cup \left( \left( \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right] \times \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right] \right)^{n-2} \right) \times \mathbb{R}
\]

Then our Zorich transform is continuous from \( B \) to \( B \). From here on out, when we reference the Zorich map \( Z \) we mean \( Z : B \to \mathbb{R}^n \setminus \{0\} \) with respect to the Zorich map defined by the corresponding \( g \).

### 2.2.1 Properties of the Zorich Transform

Now that we have the Zorich Transform defined, we make note of some of the properties of \( \tilde{f} \) for a quasiregular mapping \( f \). Since the corresponding function \( g \) with the Zorich map \( Z \) is infinitesimally bi-Lipschitz, the distortion of \( f \) under conjugation with the Zorich map is still uniformly bounded from above and vice versa. That is, we have
Proposition 2.2.1. A map $f : D \to \mathbb{R}^n$ is quasiregular if and only if the Zorich transform $\tilde{f} = Z^{-1} \circ f \circ Z$ is quasiregular, with $\tilde{f} : U \to B$, with $U \subseteq B$. Also, if $f$ is quasiconformal from $\mathbb{R}^n$ to $\mathbb{R}^n$, then $\tilde{f} : B \to B$ is also quasiconformal.

In certain cases we can define a Zorich transform globally. For example, if we are in $n \geq 3$ dimensions Mayer [15] gives us an example where $\tilde{f}$ is multiplication by an integer (note that we are starting with the Zorich Transform first) and then solves the Schröder equation $f \circ h = h \circ \tilde{f}$ by letting $h$ be the Zorich map which results in giving us a power type map $f$.

Let us look at a couple of more examples when we are in three dimensions, and the Zorich map is defined by $g$ in (2.1).

Example 2.2.2. If $A_{\theta,l}$ is a rotation by $\theta$ about the line $l$ which passes through the origin, then we want to find $\tilde{f}$ such that $\tilde{f} \circ Z = Z \circ A_{\theta,l}$. Now, for trivial rotation $A_{\theta,l}$, where $\theta = 2k\pi$, $k \in \mathbb{Z}$, we can define $\tilde{f}$ globally. In this case $\tilde{f}$ is the identity.

Now let us consider a non-trivial rotation about $l$. This case forces $\tilde{f}$ to be defined from $B$ to $B$. In this consideration, we are looking at a fixed height $r$ where $Z$ maps onto the sphere of radius $e^r$. For simplicity of our conversation, we can let $r = 0$ so that we are looking at the unit sphere. Suppose $l$ is the $z$-axis, then the points $(0, 0, 1)$ and $(0, 0, -1)$ are fixed under $A_{\theta,l}$. Also, the unit circle on the $xy$-plane maps onto itself. The pre-image of $(0, 0, 1)$ and $(0, 0, -1)$ under $Z$ are the center points in the corresponding squares in $B$. The pre-image of the unit circle on the $xy$-plane under $Z$ is the boundary of the first square. For any circle of radius $s < 1$ centered at $(0, 0, 1)$ or $(0, 0, -1)$ on the unit sphere is a corresponding square centered about the corresponding pre-image of a fixed point. For $A_{\theta,l}$ we can define $\tilde{f}$ to be a “rotation” about these squares, so that when we apply $Z$ we get the image of $A_{\theta,l}$. That is, we get Figure 2.2.2:
Figure 2.1: On Left Zorich Transform of Rotation about z-axis illustrated on Right

We can also see, that if we were to try to extend $\tilde{f}$ by reflections so that $\tilde{f} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$, then at the slice level our map would no longer be continuous. For example, we have Figure 2.2.2.

Figure 2.2: Non-restricted Domain for a Rotation

We see that if we were to place a neighborhood around a corner of four squares, the neighborhood would split into two different directions. We saw earlier that in special cases we can have a Zorich transform defined from $\mathbb{R}^n$ to $\mathbb{R}^n$, but this previous example shows that to guarantee continuity of our Zorich transforms we need to restrict the domain and codomain to the fundamental set $B$. 
Now, suppose $A_{\theta,l}$ is a rotation about a line that is not the $z$-axis. As in the above example, $A_{\theta,l}$ has two fixed points, and a great circle in the unit sphere that maps onto itself. Around the fixed points we can find the pre-images of the circles centered about the fixed points on the circle under $Z$. Then we can define $\tilde{f}$ as rotations about the pre-images of those circles centered about the fixed points. For example, if we rotated the sphere about a line that goes through two branch points, then the great circle must go through the other two branch points of $Z$. Then $\tilde{f}$ has the type of flow map found in Figure 2.2.2.

![Figure 2.3: Zorich Transform of a Rotation that passes through branch points of Zorich Map](image)

In particular, if we have a different rotation we have a different flow map. Restricting ourselves to the unit sphere, to determine what a flow map looks like, we first need to find the points fixed on the unit circle under the rotation. From here we need to determine the pre-image of the great circle that maps unto itself under the rotation. Then from here determine the pre-image of circles on the unit sphere centered at any of the fixed points.

Now that we have a bit of an understanding of how Zorich transforms behave, let us construct the maps we need to prove Theorem 1.2.24. We use Zorich transforms to show that these maps are indeed quasiregular, actually they are injective, so we can even say they are quasiconformal.
2.2.2 The Zorich Transform of the Asymptotic Representation of a Quasiregular Mapping

Before we can prove anything about the Zorich transform of the asymptotic representation $D$, we first show that $D$ is quasiconformal in a neighborhood of the origin given that $\tilde{f} \in \text{BLD}$. Since (1.8) tells us that $D(x) = \rho_f(|x|)g\left(\frac{x}{|x|}\right)$, in the proof we will see that quasiconformality of $D$ will depend on $\rho_f(|x|)$. To examine the behavior of $\rho_f(|x|)$ we will be examining $\tilde{\rho}_f$, the logarithmic transform of $\rho_f$. It may be possible for a curve $\gamma \in B$ of finite length, where $B$ is the fundamental set corresponding to the Zorich map, to be mapped to a non-rectifiable curve. This may cause $\tilde{\rho}_f'$ to be unbounded from above which would mean that $\tilde{D}$ is not quasiconformal. To account for this, we assume that $\tilde{f} \in \text{BLD}$.

**Theorem 2.2.3.** Given a mapping $f : D \rightarrow \mathbb{R}^n$ that is quasiconformal in a neighborhood of the origin with $T(0, f) = \{g\}$ and $\tilde{f} \in \text{BLD}$, then the asymptotic representation of $f$, $D$ as defined in (1.8) is quasiconformal in a neighborhood of the origin.

**Proof.** Corollary 1.2.14 tells us that $g$ is quasiconformal on all of $\mathbb{R}^n$. From (1.8) we know that $D(x) = \rho_f(|x|)g\left(\frac{x}{|x|}\right)$. From Fletcher and Wallis [9, Lemma 3.1] we know that $\rho_f$ is a continuous increasing function. Since $\rho_f$ is a continuous increasing function, then we know that $\rho_f$ is differentiable almost everywhere. Because $g$ is quasiconformal, we know that $D$ must also be differentiable almost everywhere. Let $R > 0$ so that $f$ is quasiconformal when $|x| < R$.

Using the Zorich transform we show that $D(x) = \rho_f(|x|)g(x/|x|)$ is quasiconformal on $B(0, R) \setminus \{0\}$, and hence by [23, Theorem 35.1] we conclude that $D$ is quasiconformal on $B(0, R)$. Let $(X_1, ..., X_n) = X \in B$ where $B$ is the fundamental set corresponding to the
Zorich map $Z : B \to \mathbb{R}^n$. By the definition of the Zorich map with infinitesimal bi-Lipschitz map $G$ and $X_n < \ln(R)$, we have that

$$
\tilde{D}(X) = Z^{-1}(\rho_f(e^{X_n})g(Z(X)))
= (0, \ldots, 0, \ln(\rho_f(e^{X_n})|g(Z(X))|)) + G^{-1}\left(\frac{g(Z(X))}{|g(Z(X))|}\right)
= (0, \ldots, 0, \ln(\rho_f(e^{X_n}))) + Z^{-1}(g(Z(X)))
$$

Proposition 2.2.1 tells us that $\tilde{g}$ is quasiconformal since $g$ is quasiconformal. Hence the linear distortion of $X \in B$ in the $X_1, \ldots, X_{n-1}$ directions or the “side-to-side” direction in the fundamental set $B$ must be bounded away from 0 and $\infty$. The quasiconformality of $\tilde{D}$ depends on the behavior of the linear distortion of $D$ in the $X_n$th direction. To this end, we examine the behavior of $\tilde{\rho}_f(X_n) = \ln(\rho_f(e^{X_n}))$ for $X_n < \ln(R)$. Once we show that the linear distortion in the $X_n$th direction of $\tilde{D}$ is uniformly bounded away from 0 and $\infty$, then the linear distortion of $\tilde{D}$ is bounded and $\tilde{D}$ is quasiconformal for $X_n < \ln(R)$ by (1.1) and (1.2).

We now show that the linear distortion of $D$ is bounded in the $X_n$th direction. Without loss of generality, let $t, s < \ln(R)$ with $t < s$ and consider

$$
\tilde{\rho}_f(s) - \tilde{\rho}_f(t).
$$

For $r < R$, we know that $\rho_f$ gives the average radius of the image of a circle with radius $r$ under $f$. This means that $\tilde{\rho}_f$ is the average “height” of the image of a line perpendicular to the $X_n$ axis in the fundamental set $B$ under $\tilde{f}$.

Then (2.2) is the difference of the average height of the image of a line with “height” $t$ under $\tilde{f}$ and the average height of the image of a line of “height” $s$ under $\tilde{f}$ where both lines
are in the fundamental set $B$. Since $\tilde{f} \in \text{BLD}$ there is a constant $L_1 \geq 1$ such that $\tilde{f}$ is of $L_1$-BLD.

Let

$$\tilde{H}(t) := \{X \in B : X_n = t\},$$

$$\tilde{B}(t) := \{X \in B : X_n < t\},$$

$$\tilde{S}(t,s) := \{X \in B : t \leq X_n \leq s\},$$

and let $A_t$ be a line that is parallel to the $X_n$ axis in $B$. Now let $X, Y \in \tilde{S}(t,s) \cap A_t$ such that $|X - Y| = s - t$. Let $\sigma$ be a path joining $X$ and $Y$ and $\gamma$ be a path joining $\tilde{f}(X)$ and $\tilde{f}(Y)$. Note that $\sigma$ and $\gamma$ of a particular length may not be unique in the quotient space. Also note that the length of a path $\sigma$ can be written as

$$l(\sigma) = \int_{\sigma} |dT|,$$

and the length of $\gamma$ as

$$l(\gamma) = \int_{\gamma} |d\tau|.$$

Then we write

$$|X - Y| = \inf_{\sigma} \int_{\sigma} |dT|$$

and

$$|\tilde{f}(X) - \tilde{f}(Y)| = \inf_{\gamma} \int_{\gamma} |d\tau|,$$

where the infimums are taken over all paths joining $X$ to $Y$ and $\tilde{f}(X)$ to $\tilde{f}(Y)$, respectively. Since $\tilde{f}|_B$ is injective and of $L_1$-BLD we have

$$\frac{|X - Y|}{L_1} = \inf_{\sigma} \frac{\int_{\sigma} |dT|}{L_1} \leq \inf_{\gamma=\tilde{f}(\sigma)} \int_{\gamma} |d\tau| \leq L_1 \inf_{\sigma} \int_{\sigma} |dT| = L_1 |X - Y|. $$
Since $|X - Y| = s - t$ we have

$$\frac{s - t}{L_1} \leq |\tilde{f}(X) - \tilde{f}(Y)| \leq L_1(s - t). \quad (2.3)$$

We see that the image of the points $X \in \tilde{H}(s) \cap A_l$ and $y \in \tilde{H}(t) \cap A_l$ can only differ by at least $(s - t)/L_1$ and at most $L_1(s - t)$. Then the difference of the average “height” of $\tilde{f}(H(s))$ and the average “height” of $\tilde{f}(H(t))$ can be no smaller than $(s - t)/L_1$ and no larger than $L_1(s - t)$, that is

$$\frac{s - t}{L_1} \leq \tilde{\rho}_f(r + t) - \tilde{\rho}_f(r) \leq L_1(s - t). \quad (2.4)$$

Hence $\tilde{\rho}_f$ is $L_1$-bi-Lipschitz for $X \in B(\ln(R))$ which means that the linear distortion in the $X_n$th direction in $\tilde{B}(\ln(R))$ is uniformly bounded away from 0 and $\infty$. \hfill \Box

Using the definition of Zorich transform and Theorem 1.2.19 we can get the following result about the Zorich transform of the asymptotic representation of a locally injective quasiregular map.

**Theorem 1.2.8.** Given a mapping $f : D \to \mathbb{R}^n$ that is quasiconformal in a neighborhood of the origin with $T(0, f) = \{g\}$ and $\tilde{f} \in \text{BLD}$, then there exists and $L \geq 1$ and $M_1 < 0$ such that the Zorich Transform $\tilde{D}$ is $L$-bi-Lipschitz when $X_n < M_1$, where $D$ is the asymptotic representation of $f$ as defined in (1.8).

**Proof.** Theorem 2.2.3 gives us that $D$ is quasiconformal. Also, Theorem 1.2.19 tells us that we can write $g(x) = h(s(x))$, where $h \in \text{BLD}$, and $s(x) = x|x|^{d-1}$ for some $d > 0$. Notice that $s(x/|x|) = x/|x|$, so that $g|_{S^{n-1}} = h|_{S^{n-1}}$ is BLD. Theorem 4.13 by Gutlyanskii et. al.

states that a map is BLD if and only if the map is locally bi-Lipschitz. In particular, the unit sphere $S^{n-1}$ is compact, so that $h|_{S^{n-1}}$ is $L_2$-bi-Lipschitz. Let us define $\bar{X}$ to be

$$\bar{X} = (X_1, \ldots, X_{n-1}, 0).$$

Since

$$D(Z(X)) = \rho_f(|Z(X)|)h\left(\frac{Z(X)}{|Z(X)|}\right) = \rho_f(e^{X_n})h(G(\bar{X})), $$

we have that

$$\tilde{D}(X) = G^{-1}\left(\frac{h(G(\bar{X}))}{|h(G(\bar{X}))|}\right) + (0, \ldots, 0, \ln \left(|h(G(\bar{X}))|\rho_f(e^{X_n})\right),$$

(2.5)

where we can note that

$$\ln \left(|h(G(\bar{X}))|\rho_f(e^{X_n})\right) = \ln |h(G(\bar{X}))| + \ln \rho_f(e^{X_n}).$$

To determine $\tilde{D}$ is bi-Lipschitz in a lower half-beam, we look at the “up-down” and “side-to-side” directions separately under $\tilde{D}$. It is clear that $\tilde{D}$ restricted to the ‘side-to-side’ or “horizontal” direction is bi-Lipschitz, since $G$ is $L_1$-bi-Lipschitz and $h|_{S^{n-1}}$ is $L_2$-bi-Lipschitz. That is, $G^{-1}(h(G(\bar{X}))/|h(G(\bar{X}))|)$ is $C_1$-bi-Lipschitz where the constant depends on $L_1$ and $L_2$.

Next we use the quasiconformality of $\tilde{D}$ to show $\tilde{D}$ itself is locally bi-Lipschitz. Recall that as we are working in the quotient space where $B$ is the corresponding fundamental domain, we modify the definitions of bi-Lipschitz and BLD according to the induced metric on the quotient space. Since $D$ is quasiconformal, we know that $\tilde{D}$ is also quasiconformal.
This implies that the linear distortion is bounded, in other words, there exists an $H \geq 1$ such that for all $X \in B$,

$$\limsup_{r \to 0} \frac{L_\tilde{D}(X,r)}{l_\tilde{D}(X,r)} \leq H.$$ 

In particular, for a fixed $X \in B$, we can find an $r_0$ so that

$$L(X,r) \leq 2Hl(X,r),$$

for $r \leq r_0$. Let $\bar{L}_\tilde{D}(X,r)$ and $\bar{l}_\tilde{D}(X,r)$ be $L_\tilde{D}(X,r)$ and $l_\tilde{D}(X,r)$ where $X$ is only evaluated over the “horizontal” direction, i.e. we let $F(\bar{X}) = G^{-1}\left(\frac{h(\bar{G}(\bar{X}))}{h(G(\bar{X}))}\right)$. Then for a fixed $X_n$ we let $\bar{L}_\tilde{D}(X,r) = L_F(\bar{X},r)$ and $\bar{l}_\tilde{D}(X,r) = l_F(\bar{X},r)$. From above we know that in the “side-to-side” direction that $\tilde{D}$ is $C_1$-bi-Lipschitz. Hence, for $r \leq r_0$ and $X_n$ fixed we have for $|X - Y| = |ar{X} - \bar{Y}| = r$ that

$$\frac{r}{C_1} = \frac{|\bar{X} - \bar{Y}|}{C_1} \leq |F(X) - F(Y)| \leq C_1|X - Y| = C_1 r,$$

which gives us

$$\frac{r}{C_1} \leq \bar{l}_\tilde{D}(X,r), \text{ and } \bar{L}_\tilde{D}(X,r) \leq C_1 r$$

for $r \leq r_0$. Clearly we have $\bar{L}_\tilde{D}(X,r) \leq L_\tilde{D}(X,r)$ and $l_\tilde{D}(X,r) \leq \bar{L}_\tilde{D}(X,r)$. Then for $r \leq r_0$ we know that

$$\frac{r}{C_1} \leq \bar{l}_\tilde{D}(X,r) \leq L_\tilde{D}(X,r) \leq L_\tilde{D}(X,r) \leq 2Hl_\tilde{D}(X,r),$$

and

$$C_1 r \geq \bar{L}_\tilde{D}(X,r) \geq \bar{l}_\tilde{D}(X,r) \geq l_\tilde{D}(X,r) \geq \frac{\bar{L}_\tilde{D}(X,r)}{2H}.$$
so that \( L_{\tilde{D}}(X,r) \leq 2HC_1 r \) and \( l_{\tilde{D}}(X,r) \geq \frac{r}{2HC_1} \). This implies that \( \tilde{D} \) is locally \( 2HC_1 \)-bi-Lipschitz. By [10, Theorem 4.13] we have that \( \tilde{D} \) is \( 2HC_1 \)-BLD. Note that the length of a path \( \sigma \) from \( X \) to \( Y \) in \( B \) can be written as

\[
l(\sigma) = \int_{\sigma} |dT|.
\]

Let \( \gamma \) be a path from \( \tilde{D}(X) \) to \( \tilde{D}(Y) \). Then we can write

\[
|X - Y| = \inf_{\sigma} \int_{\sigma} |dT|
\]

and

\[
|\tilde{D}(X) - \tilde{D}(Y)| = \inf_{\gamma} \int_{\gamma} |d\tau|,
\]

where the infimums are taken over all paths joining \( X \) to \( Y \) and \( \tilde{D}(X) \) to \( \tilde{D}(Y) \), respectively. Note that these paths may not be unique since we are working over the quotient space. Since \( \tilde{D} \) is injective and \( 2HC_1 \)-BLD we have

\[
\frac{|X - Y|}{2HC_1} = \inf_{\sigma} \int_{\sigma} \frac{|dT|}{2HC_1} \leq \inf_{\gamma=\tilde{D}(\sigma)} \int_{\gamma} |d\tau| \leq 2HC_1 \inf_{\sigma} \int_{\sigma} |dT| = 2HC_1 |X - Y|.
\]

Therefore, we have that \( \frac{|X - Y|}{2HC_1} \leq |\tilde{D}(X) - \tilde{D}(Y)| \leq 2HC_1 |X - Y| \), in other words \( \tilde{D} \) is \( L \)-bi-Lipschitz with \( L = 2HC_1 \).

\[\square\]

Theorem 1.2.16 tells us that \( \rho_f \) can be represented by the asymptotic homogeneity of (1.9). In the proof above, we saw that we know from [9, Lemma 3.1] that the mean radius \( \rho_f \) is an increasing continuous function. As a consequence of the above theorem, we show that \( \rho_f \) is locally bi-Lipschitz.
Corollary 2.2.4. Given a mapping \( f : D \to \mathbb{R}^n \) that is quasiconformal in a neighborhood of the origin with \( T(0, f) = \{g\} \) and \( \tilde{f} \in BLD \), then \( \ln \circ \rho_f \circ \exp \) is \( L \)-bi-Lipschitz and \( \rho_f \) is locally bi-Lipschitz with constant depending on \( \rho_f(y)/y \) where \( y \) is fixed, that is

\[
\frac{\rho_f(y)}{2Ly} |x - y| \leq |\rho_f(x) - \rho_f(y)| \leq \frac{2L\rho_f(y)}{y} |x - y|.
\]

Proof. We showed that \( \tilde{D} \) is bi-Lipschitz in the Theorem 1.2.28. Also, we have

\[
\tilde{D}(X) = G^{-1}\left(\frac{h(G(\bar{X})))}{|h(G(X))|}\right) + (0, ..., 0, \ln \left(|h(G(\bar{X}))|\rho_f(e^X)\right)),
\]

where we can note that

\[
\ln \left(|h(G(\bar{X}))|\rho_f(e^X)\right) = \ln |h(G(\bar{X}))| + \ln \rho_f(e^X).
\]

Let \( \tilde{\rho}_f = \ln \circ \rho_f \circ \exp \), then \( \tilde{\rho}_f \) is \( L \)-bi-Lipschitz since

\[
|\tilde{\rho}_f(X_n + h) - \tilde{\rho}_f(X_n)| = |\tilde{D}(X + (0, ..., 0, h)) - \tilde{D}(X)|.
\]

Without loss of generality, let \( x = e^t \) and \( y = e^s \) with \( t > s \). Then

\[
\frac{1}{L'} |t - s| \leq |\ln(\rho_f(e^t)) - \ln(\rho_f(e^s))| \leq L'|t - s|
\]

implies that

\[
\frac{1}{L'} |\ln(x) - \ln(y)| \leq |\ln(\rho_f(x)) - \ln(\rho_f(y))| \leq L'|\ln(x) - \ln(y)|.
\]
From [9, Lemma 3.1] we know that $\rho_f$ is a continuous increasing function, so that $\ln(\rho_f(e^t)) \geq \ln(\rho_f(e^s))$. Dropping absolute value signs we have

$$\ln \left( \frac{x}{y} \right)^{1/L} \leq \ln \left( \frac{\rho_f(x)}{\rho_f(y)} \right) \leq \ln \left( \frac{x}{y} \right)^L.$$

Using algebraic manipulation and the fact that natural logarithm is an increasing function we have

$$\left( \frac{x - y + 1}{y} \right)^{1/L} \leq \frac{\rho_f(x) - \rho_f(y)}{\rho_f(y)} + 1 \leq \left( \frac{x - y + 1}{y} \right)^L$$

so that

$$\left( \frac{x - y + 1}{y} \right)^{1/L} - 1 \leq \frac{\rho_f(x) - \rho_f(y)}{\rho_f(y)} \leq \left( \frac{x - y + 1}{y} \right)^L - 1.$$

Using binomial series representation for $\left( \frac{x - y + 1}{y} \right)^{1/L}$ and $\left( \frac{x - y + 1}{y} \right)^L$ we have

$$\left( \frac{x - y + 1}{y} \right)^{1/L} = 1 + \frac{x - y}{L y} + \frac{1/L(1/L - 1)(x - y)^2}{2!y^2} + \frac{1/L(1/L - 1)(1/L - 2)(x - y)^3}{3!y^3} + \cdots$$

and

$$\left( \frac{x - y + 1}{y} \right)^L = 1 + \frac{L(x - y)}{y} + \frac{L(L - 1)(x - y)^2}{2!y^2} + \frac{L(L - 1)(L - 2)(x - y)^3}{3!y^3} + \cdots,$$

where both converge for $x < 2y$, since both $x$ and $y$ are positive. Since $L \geq 1$ we know that $0 \leq 1/L \leq 1$, which gives us

$$1 + \frac{x - y}{L y} + \frac{(1/L)(1/L - 1)(x - y)^2}{2!y^2} + \frac{(1/L)(1/L - 1)(1/L - 2)(x - y)^3}{3!y^3} + \cdots \geq 1 + \frac{x - y}{L y} \geq 1 + \frac{x - y}{2L y},$$
so that
\[
\left( \frac{x - y}{y} + 1 \right)^{1/L} \geq 1 + \frac{x - y}{2Ly}.
\]
Since
\[
\left( \frac{x - y}{y} + 1 \right)^L = 1 + \frac{L(x - y)}{y} + \frac{L(L - 1)(x - y)^2}{2!y^2} + \frac{(L)(L - 1)(L - 2)(x - y)^3}{3!y^3} + \cdots
\]
is convergent for $x - y < y$ we can find an $r_1 < y$ so that when $(x - y) < r_1$ we have
\[
\frac{L(L - 1)(x - y)^2}{2!y^2} + \frac{(L)(L - 1)(L - 2)(x - y)^3}{3!y^3} + \cdots < \frac{L(x - y)}{y}.
\]
Hence for $x - y < r_1$, we have
\[
1 + \frac{x - y}{2Ly} - 1 \leq \frac{\rho_f(x) - \rho_f(y)}{\rho_f(y)} \leq 1 + \frac{2L(x - y)}{y} - 1
\]
Therefore, for $|x - y| < r_1$ we have
\[
\frac{\rho_f(y)}{2L} |x - y| \leq |\rho_f(x) - \rho_f(y)| \leq \frac{2L\rho_f(y)}{y} |x - y|.
\]
CHAPTER 3
THE COMPLETE REALIZATION OF ORBIT SPACES

The goal of this chapter is to prove Theorem 1.2.24, that is

**Theorem 1.2.24** (The Complete Realization of the Orbit Space). Let \( X \subset \mathbb{R}^n \setminus \{0\} \) be a non-empty, compact and connected set. Then there exists a quasiconformal mapping \( f : \mathbb{R}^n \to \mathbb{R}^n \) for which \( X \) is the image of the point evaluation map \( E_{e_1} : T(0, f) \to \mathbb{R}^n \) for \( e_1 = (1, 0, \ldots, 0) \).

In Section 3.1 we construct the needed maps to prove Theorem 1.2.24. Of particular importance, we use the new tool of the Zorich Transform to show that the needed constructions are indeed quasiconformal. Section 3.1.1 introduces a maps that stretch radially which are the basis for constructing an interpolation between the radial stretch maps in Section 3.1.2 and maps that stretch radially and spiral in Section 3.1.4. Norm calculations to show the maps in Section 3.1.1 and 3.1.2 are similar and hence are preformed in Section 3.1.3. The norm calculations for the maps that stretch radially and spiral need a deeper consideration and are included in Section 3.1.4. In Section 3.2 we prove Theorem 1.2.24 and complete the realization of orbit spaces.

For this chapter we define the Zorich map using the infinitesimally bi-Lipschitz map \( g \) from (2.1) in Section 2.1, that is

\[
g(x_1, \ldots, x_{n-1}, 0) = \left( \frac{x_1 \sin M(x_1, \ldots, x_{n-1})}{\sqrt{x_1^2 + \cdots + x_{n-1}^2}}, \ldots, \frac{x_{n-1} \sin M(x_1, \ldots, x_{n-1})}{\sqrt{x_1^2 + \cdots + x_{n-1}^2}}, \cos M(x_1, \ldots, x_{n-1}) \right).
\]
Also, for the Zorich transform we choose our fundamental set to be

\[ B := \left( \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right] \times \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right]^{n-2} \right) \cup \left( \left( \frac{\pi}{2}, \frac{3\pi}{2} \right) \times \left( -\frac{\pi}{2}, \frac{\pi}{2} \right)^{n-2} \right) \times \mathbb{R}. \]

In this case, \( G \) is the group isomorphic to \( G = \mathbb{Z}^{n-1} \times P \), where \( P \) is the appropriate point group of rotations, that acts on \( \bar{D} = [-\pi/2, \pi/2]^{n-1} \).

### 3.1 Constructions of Maps Using the Zorich Transform

#### 3.1.1 A Radial Stretch Map

Note that we are using coordinates \((x_1, \ldots, x_n)\) for points in the fundamental set \( B \) and \((y_1, \ldots, y_n)\) in the image of \( Z \). Also, we use the convention that if

\[ g(x_1, \ldots, x_n) = (f(x_1, \ldots, x_n)x_1, \ldots, f(x_1, \ldots, x_n)x_n), \]

then we write

\[ g(x_1, \ldots, x_n) = f(x_1, \ldots, x_n)(x_1, \ldots, x_n). \]

As a starting point for the maps we are using to prove our result, we define a map that stretches a sphere centered at the origin radially onto an ellipsoid centered at the origin by a factor of \( K \geq 1 \) in the \( y_n \) direction. We can do this by the function \( R : \mathbb{R}^n \to \mathbb{R}^n \) defined by

\[ (y_1, \ldots, y_n) \mapsto \frac{K}{\sqrt{K^2 + (1 - K^2) \cos^2 \varphi}} (y_1, \ldots, y_n), \]
where
\[
\varphi = \cos^{-1} \left( \frac{y_n}{\sqrt{y_1^2 + \cdots + y_n^2}} \right).
\]
Consider a fixed \(x_n\), so that we have a slice of \(B\) at height \(x_n\). The image of the slice under \(Z\) is a sphere of radius \(y_n = e^{x_n}\). By looking at the image of \(Z\), we can determine
\[
\varphi = \cos^{-1} \left( \frac{e^{x_n} \cos M(x_1, \ldots, x_{n-1})}{\sqrt{e^{2x_n}x_1^2 \sin^2 M(x_1, \ldots, x_{n-1}) + \cdots + e^{2x_n}x_{n-1}^2 \sin^2 M(x_1, \ldots, x_{n-1}) + e^{2x_n} \cos^2 M(x_1, \ldots, x_{n-1})}} \right),
\]
which gives us
\[
\varphi = \cos^{-1} (\cos M(x_1, \ldots, x_{n-1})) = M(x_1, \ldots, x_{n-1}).
\]
Then we can define our Zorich transformation \(\tilde{R} : B \to B\) by
\[
\tilde{R}(x_1, \ldots, x_n) = \left( x_1, \ldots, x_{n-1}, x_n \ln \left( \frac{K}{\sqrt{K^2 + (1 - K^2) \cos^2 M(x_1, \ldots, x_{n-1})}} \right) \right).
\]
Notice that under the Zorich transform the first \(n - 1\) coordinates are unchanged, but the \(n\)th coordinate rises according to the first \(n - 1\) coordinates, and as we approach the center of the \(n - 1\) coordinates we get closer to achieving maximum stretch.

To define a Zorich transformation for a stretch by \(K \geq 1\) in any direction, we can conjugate \(R\) by a rotation and \(\tilde{R}\) by the corresponding Zorich transformation of the rotation. As long as \(\tilde{R}\) is quasiconformal, then all the other corresponding maps from conjugation are quasiconformal, we are reduced to the case of looking at \(R\). To show that \(R\) is quasiconformal, we just need to show that the corresponding Zorich transform \(\tilde{R}\) is quasiconformal. Let
\[
V = \ln \left( \frac{K}{\sqrt{K^2 + (1 - K^2) \cos^2 M(x_1, \ldots, x_{n-1})}} \right).
\]
For now, we look at

\[ A_1 := \left\{ (x_1, \ldots, x_n) \in \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right]^{n-1} \times \{x_n\} : x_1 > |x_i| \text{ for } 2 \leq i \leq n-1 \right\}, \]

so that \( M(x_1, \ldots, x_{n-1}) = x_1 \). We can compute

\[ V_{x_i} = \begin{cases} \frac{(1-K^2) \cos x_1 \sin x_i}{K^2+(1-K^2) \cos^2 x_1} & \text{if } i = 1 \\ 0 & \text{if } 2 \leq i \leq n \end{cases}. \]

Note that \( K^2 + (1-K^2) \cos^2 x_1 = 1 + (K^2 - 1) \sin^2 x_1 \in [1, K^2] \) so that

\[ |V_{x_1}| = \left| \frac{(1-K^2) \cos x_1 \sin x_1}{K^2+(1-K^2) \cos^2 x_1} \right| \leq K^2 - 1. \tag{3.1} \]

We are restricting ourselves to the set \( A_1 \) because when we analyze other pyramid sections of the cube \( [-\pi/2, \pi/2]^{n-1} \) the only change would be that \( V_{x_1} \) would now be zero and the \( i \)th location of the derivative is as above with \( x_1 \) replaced with \( x_i \), which does not change norm calculations of the derivative matrix \( \tilde{R}' \) or \( (\tilde{R}')^{-1} \) for the regions \( A_i \). From here it is a relatively simple calculation to show \( \tilde{R}' \) and \( (\tilde{R}')^{-1} \) have bounded norm in \( A_1 \) and all other \( A_i \)'s. We perform the norm calculations for the radial stretch and radial stretch interpolation map together, which can be found in Section 3.1.3. Since we have finitely many \( A_i \)'s and the boundaries of each pyramid section forms a closed set which is a \( \sigma \)-finite \((n-1)\)-dimensional Hausdorff measurable set, Theorem 1.2.4 tells us that \( \tilde{R} \) is quasiconformal on \( B \). Using this radial stretch map, we define the radial stretch interpolation maps.
3.1.2 Radial Stretch Interpolation Maps

We want to take a spherical shell and stretch the outer shell by a factor of $K \geq 1$ and the inner shell by a factor of $L \geq 1$ in the same direction, when the inner and outer parts of the shell never cross from the different stretching. We define a map by stretching in the $y_n$ direction, but we can get any direction by conjugation our function by rotations as before mentioned with radial stretch map. Let $s, t \in \mathbb{R}$ with $s < t$ such that we are stretching by a factor of $L$ when $y_n = e^s$ and a factor of $K$ when $y_n = e^t$, with

$$|\ln(K/L)| < \frac{t-s}{2}.$$ 

Letting $\nu = \frac{x_n-s}{t-s} = \frac{\ln|y|-s}{t-s}$, we can define the radial interpolation map to be

$$R_1(y_1, ..., y_n) = \left( \frac{K}{\sqrt{K^2 + (1-K^2)\cos^2 \varphi}} \right)^\nu \left( \frac{L}{\sqrt{L^2 + (1-L^2)\cos^2 \varphi}} \right)^{1-\nu} (y_1, ..., y_n),$$

(3.2)

with domain

$$\{y \in \mathbb{R}^n : e^s \leq |y| \leq e^t\}.$$ 

We have the corresponding Zorich transform defined on

$$\{x \in B : s \leq x_n \leq t\}$$

and is defined by

$$\tilde{R}_1(x_1, ..., x_n) = (x_1, ..., x_{n-1}, x_n + V_1),$$
where

\[
V_I = \ln \left( \frac{K}{\sqrt{K^2 + (1 - K^2) \cos^2 M'}} \right) \frac{x_n - s}{t - s} + \ln \left( \frac{L}{\sqrt{L^2 + (1 - L^2) \cos^2 M'}} \right) \frac{x_n - t}{s - t},
\]

with

\[M' = M(x_1, ..., x_{n-1}).\]

As before, we look at

\[
\left\{ (x_1, ..., x_n) \in \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right]^{n-1} \times \{x_n\} : x_1 > |x_i| \text{ for } 2 \leq i \leq n - 1 \right\},
\]

so that \(M(x_1, ..., x_{n-1}) = x_1\). This gives us

\[
(V_I)_{x_i} = \begin{cases} 
\frac{(1-K^2) \cos x_1 \sin x_1}{K^2+(1-K^2) \cos^2 x_1} \frac{x_n - s}{t - s} + \frac{(1-L^2) \cos x_1 \sin x_1}{L^2+(1-L^2) \cos^2 x_1} \frac{x_n - t}{s - t} & \text{if } i = 1 \\
0 & \text{if } 2 \leq i \leq n - 1 \\
\frac{1}{t - s} \ln \left( \frac{K \sqrt{L^2 + (1 - L^2) \cos^2 x_1}}{L \sqrt{K^2 + (1 - K^2) \cos^2 x_1}} \right) & \text{if } i = n
\end{cases}
\]

Note again that \(K^2 + (1 - K^2) \cos^2 x_1 = 1 + (K^2 - 1) \sin^2 x_1\) so that by similar calculations as before we have

\[
|(V_I)_{x_1}| \leq K^2 + L^2 - 2,
\]

as before we have

\[
|(V_I)_{x_n}| = \left| \frac{1}{t - s} \ln \left( \frac{K \sqrt{L^2 + (1 - L^2) \cos^2 x_1}}{L \sqrt{K^2 + (1 - K^2) \cos^2 x_1}} \right) \right|
\]

\[
\leq \frac{1}{t - s} \left| \ln \left( \frac{K}{L} \right) \right|
\]

\[
< \frac{1}{t - s} \frac{t - s}{2} = \frac{1}{2}.
\]
so that
\[ |(V_I)_{xn}| < \frac{1}{2}. \quad (3.4) \]

We can use these calculations to show that the norms of the matrices \( \tilde{R}' \) and \( (\tilde{R}_I')^{-1} \) are bounded in each pyramid section \( A_i \), which may be found in the next section. Using a similar argument to conclude that \( \tilde{R} \) is quasiconformal, we can now conclude that \( \tilde{R}_I \) is quasiconformal so that \( R_I \) is quasiconformal.

### 3.1.3 Norm Calculations for \( \tilde{R} \) and \( \tilde{R}_I \)

Our goal in this section is to show that the maximal dilatation is bounded. To do so, we show that \( \|R'\|\|R')^{-1}\| \) and \( \|\tilde{R}_I'\|\|\tilde{R}_I'^{-1}\| \) are bounded above by a real number greater than one.

Let \( A \) be either \( \tilde{R}' \) or \( \tilde{R}_I' \), this means that

\[
A = \begin{pmatrix}
1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
C & 0 & \cdots & 1 + \epsilon
\end{pmatrix},
\]

where

\[
C = \begin{cases}
V_{x_1} & \text{if } A = \tilde{R}' \\
(V_I)_{x_1} & \text{if } A = \tilde{R}_I'
\end{cases}
\]

and

\[
\epsilon = \begin{cases}
0 & \text{if } A = \tilde{R}' \\
(V_I)_{xn} & \text{if } A = \tilde{R}_I'
\end{cases}
\]
Using (3.1) and (3.3) we have

$$|C| \leq K^2 + L^2 - 2 < K^2 + L^2. \quad (3.5)$$

From the derivative of $\tilde{R}$ and from (3.4) we have

$$|\epsilon| < \frac{1}{2}. \quad (3.6)$$

Also note that

$$A^{-1} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ -\frac{C}{1+\epsilon} & 0 & \cdots & \frac{1}{1+\epsilon} \end{pmatrix}.$$ 

Using (3.5) and (3.6) we have that

$$\|A\| = \sup_{|y|=1} |Ay| = \sup_{|y|=1} \sqrt{y_1^2 + \cdots + y_{n-1}^2 + (Cy_1 + (1+\epsilon)y_n)^2}$$
$$= \sup_{|y|=1} \sqrt{y_1^2 + \cdots + y_{n-1}^2 + y_n^2 + C^2y_1^2 + 2Cy_1y_n + \epsilon^2y_n^2}$$
$$\leq \sqrt{1 + (K^2 + L^2)^2 + 2(K^2 + L^2)^2 + \frac{1}{4}} = \sqrt{\frac{5}{4} + (K^2 + L^2)^2 + 3(K^2 + L^2)} = H_1.$$
We also have by Equations (3.5) and (3.6) that

\[ \| A^{-1} \| = \sup_{\| y \|=1} | A^{-1} y | = \sup_{\| y \|=1} \sqrt{y_1^2 + \cdots + y_{n-1}^2 + \left( \frac{-C}{1+\epsilon} y_1 + \frac{1}{1+\epsilon} y_n \right)^2} \]

\[ = \sup_{\| y \|=1} \sqrt{y_1^2 + \cdots + y_{n-1}^2 + \frac{C^2}{(1+\epsilon)^2} y_1^2 - \frac{2C}{(1+\epsilon)^2} y_1 y_n + \frac{1}{(1+\epsilon)^2} y_n^2} \]

\[ \leq \sup_{\| y \|=1} \sqrt{y_1^2 + \cdots + y_{n-1}^2 + (K^2 + L^2) y_1^2 + 2(K^2 + L^2)|y_1||y_n| + y_n^2} \]

\[ \leq \sqrt{1 + (K^2 + L^2)^2 + 2(K^2 + L^2)} = \sqrt{(1 + (K^2 + L^2))^2} \]

\[ = 1 + K^2 + L^2 = H_2. \]

Then we have \( H = \| A \| \| A^{-1} \| \leq H_1 H_2 = H'. \) Therefore, \( \tilde{R} \) and \( \tilde{R}_I \) have bounded maximal dilatation in each pyramid section \( A_i. \)

### 3.1.4 Radial Stretch with Spiraling Map

As previously done, we first show that a specific radial stretch map with spiraling is quasiconformal, and then by conjugation with rotations, or Zorich transforms of rotations, we get that all the other radial stretch with spiraling maps are quasiconformal. We show that the derivative matrix of the Zorich transform of the radial stretch map with spiraling exists and is bounded from above in particular regions. Also, we show that the Jacobian of the Zorich transform of the radial stretch map with spiraling is positive, i.e. is sense-preserving, in those same regions.

First, we define the radial stretch map with spiraling \( R_s : \mathbb{R}^n \to \mathbb{R}^n \) to be

\[ R_s(y_1, \ldots, y_n) = \frac{K}{\sqrt{K^2 + (1 - K^2) \sum_{i=1}^{n} y_i^2}} (u, v, y_3, \ldots, y_n) \quad (3.7) \]
with

\[ u = y_1 \cos \left( \alpha \ln \sqrt{y_1^2 + \cdots + y_n^2} \right) - y_2 \sin \left( \alpha \ln \sqrt{y_1^2 + \cdots + y_n^2} \right) \]

\[ v = y_1 \sin \left( \alpha \ln \sqrt{y_1^2 + \cdots + y_n^2} \right) + y_2 \cos \left( \alpha \ln \sqrt{y_1^2 + \cdots + y_n^2} \right), \]

where \( \alpha \) is a fixed real number. This map dilates by a factor of \( K \geq 1 \) in the \( y_1 \) direction while simultaneously rotating in the \( y_1, y_2 \)-plane, creating a spiral. Here on out, we are looking at the Zorich transform of the above map, \( \tilde{R}_s : B \to B \) where \( B \) is the before mentioned fundamental domain of the Zorich map, to show that \( \tilde{R}_s \) is quasiconformal and hence \( R_s \) is also quasiconformal. Let

\[ M = M(x_1, ..., x_{n-1}) = \max\{|x_1|, ..., |x_{n-1}|\} \text{ and,} \]

\[ m = m(x_1, ..., x_{n-1}) = \min \left\{ \frac{1}{|x_1 \cos(\alpha x_n) - x_2 \sin(\alpha x_n)|}, \frac{1}{|x_1 \sin(\alpha x_n) + x_2 \cos(\alpha x_n)|}, \frac{1}{|x_3|}, ..., \frac{1}{|x_{n-1}|} \right\}, \]

where \( \alpha \) is the same as in the definition of \( R_s \). The Zorich transform of \( R_s \) is defined by

\[ \tilde{R}_s(x_1, ..., x_n) = (u_1, ..., u_n) \]

with

\[ u_i = \begin{cases} 
Mm(x_1 \cos(\alpha x_n) - x_2 \sin(\alpha x_n)) & \text{for } i = 1 \\
Mm(x_1 \sin(\alpha x_n) + x_2 \cos(\alpha x_n)) & \text{for } i = 2 \\
Mm x_i & \text{for } 3 \leq i \leq n - 1 \\
x_n + \ln K - \frac{1}{2} \ln \left( K^2 + (1 - K^2) \frac{x_1^2 \sin^2 M}{x_1^2 + \cdots + x_{n-1}^2} \right) & \text{for } i = n 
\end{cases}. \]
We need to discuss bounding on $\|\tilde{R}_s\|$ where the derivative exist. For $x_1, \ldots, x_{n-1}$ not all zero,

$$M(u_1, \ldots, u_{n-1}) = M(x_1, \ldots, x_{n-1}),$$

since $m$ cancels with one of the following, $(x_1 \cos(\alpha x_n) - x_2 \sin(\alpha x_n))$, $(x_1 \sin(\alpha x_n) + x_2 \cos(\alpha x_n))$, $x_3, \ldots, x_{n-1}$, leaving one of the $u_i$ as $\pm M$. By definition of $m$ we have that $|u_j| \leq |u_i|$ for $1 \leq j \leq n - 1$. We indeed have that $Z \circ \tilde{R}_s = R_s \circ Z$.

To keep $\tilde{R}_s$ injective and sense preserving we can choose $\alpha$ to be sufficiently small so that

$$J_{\tilde{R}_s} > 2^{-(n+1)/2}.$$

Before further discussion, recall that $B = C \times \mathbb{R}$. We want $\alpha$ to be small enough so that when we apply $\tilde{R}_s$ the images of $C \times \{z_1\}$ and $C \times \{z_2\}$, where $z_1, z_2 \in \mathbb{R}$ and $z_1$ and $z_2$ are close together, do not intersect. This corresponds to the images of two spheres, which are close together, spiral slow enough under $R_s$ so that the images do not intersect. In particular, our map remains injective, and hence is a homeomorphism. Also, below, we split the function into the regions where the mapping is differentiable, we also note that the regions where we are not differentiable form a closed set that is also a $\sigma$- finite $(n-1)$-dimensional Hausdorff measurable set. In each region where we are differentiable, we show that $\|\tilde{R}_s\|$ is bounded from above. Using this upper bound along with the lower bound for the Jacobian allows us to use Theorem 1.2.4 to conclude that $\tilde{R}_s$ is quasiconformal, and hence $R_s$ is also quasiconformal. We now introduce calculations which help in the understanding of the lower bound for the Jacobian and the bounds for the the norms of the derivative matrix in all the regions where the derivative exists.

First note that the Zorich transform of $R_s$ is defined by

$$\tilde{R}_s(x_1, \ldots, x_n) = (u_1, \ldots, u_n)$$
with

\[ u_i = \begin{cases} 
Mm(x_1 \cos(\alpha x_n) - x_2 \sin(\alpha x_n)) & \text{for } i = 1 \\
Mm(x_1 \sin(\alpha x_n) + x_2 \cos(\alpha x_n)) & \text{for } i = 2 \\
Mmx_i & \text{for } 3 \leq i \leq n - 1 \\
x_n + \ln K - \frac{1}{2} \ln \left(K^2 + (1 - K^2) \frac{x_1^2 \sin^2 M}{x_1^2 + \cdots + x_{n-1}^2}\right) & \text{for } i = n
\end{cases} \]

where

\[ M = M(x_1, \ldots, x_{n-1}) = \max\{|x_1|, \ldots, |x_{n-1}|\} \quad \text{and,} \]

\[ m = m(x_1, \ldots, x_{n-1}) = \min \left\{ \left| \frac{1}{x_1 \cos(\alpha x_n) - x_2 \sin(\alpha x_n)} \right|, \left| \frac{1}{x_1 \sin(\alpha x_n) + x_2 \cos(\alpha x_n)} \right|, \left| \frac{1}{x_3} \right|, \ldots, \left| \frac{1}{x_{n-1}} \right| \right\}. \]

We discuss bounding \( \tilde{R}_s \) and \( J_{\tilde{R}_s} \) where the derivatives exist. Also, for \( x_1, \ldots, x_{n-1} \) not all zero,

\[ M(u_1, \ldots, u_{n-1}) = M(x_1, \ldots, x_{n-1}) \]

since \( m \) cancels with one of the following, \( (x_1 \cos(\alpha x_n) - x_2 \sin(\alpha x_n)) \), \( (x_1 \sin(\alpha x_n) + x_2 \cos(\alpha x_n)) \), \( x_3, \ldots, x_{n-1} \), leaving one of the \( u_i \) as \( \pm M \). By definition of \( m \) we have that \( |u_j| \leq |u_i| \) for \( 1 \leq j \leq n - 1 \). We indeed have that \( Z \circ \tilde{R}_s = R_s \circ Z \). A useful calculation is that if either \( x_1 \) or \( x_2 \) are not zero, then

\[ \frac{\sqrt{x_1^2 + x_2^2}}{\sqrt{2}} \leq \max\{|x_1 \cos(\alpha x_n) - x_2 \sin(\alpha x_n)|, |x_1 \sin(\alpha x_n) + x_2 \cos(\alpha x_n)|\} \leq \sqrt{x_1^2 + x_2^2}, \]

which gives us

\[ \frac{1}{\sqrt{x_1^2 + x_2^2}} \leq \min \left\{ \left| \frac{1}{x_1 \cos(\alpha x_n) - x_2 \sin(\alpha x_n)} \right|, \left| \frac{1}{x_1 \sin(\alpha x_n) + x_2 \cos(\alpha x_n)} \right| \right\} \leq \frac{\sqrt{2}}{\sqrt{x_1^2 + x_2^2}}, \tag{3.8} \]
For $R_s$ to be quasiregular we want $R_s'$ and $(R_s')^{-1}$ to be bounded. For $1 \leq i \leq n - 1$, let

$$A_i := \left\{ (x_1, \ldots, x_{n-1}, x_n) \in \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right]^{n-1} \times \mathbb{R} : x_i > |x_j| \text{ for } j \neq i, 1 \leq j \leq n - 1 \right\}.$$ 

We break our calculations into three cases, when $(x_1, \ldots, x_n) \in A_1$, $(x_1, \ldots, x_n) \in A_2$, and $(x_1, \ldots, x_n) \in A_j$ for $3 \leq j \leq n - 1$.

**Case I:** Suppose that $(x_1, \ldots, x_n) \in A_1$, so that $M = x_1$. First note that the solution sets of the equations $x_j = x_1 \cos(\alpha x_n) - x_2 \sin(\alpha x_n)$, $x_j = x_1 \sin(\alpha x_n) + x_2 \cos(\alpha x_n)$, and $x_1 \cos(\alpha x_n) - x_2 \sin(\alpha x_n) = x_1 \sin(\alpha x_n) + x_2 \cos(\alpha x_n)$ are closed, $\sigma$-finite $(n-1)$-dimensional Hausdorff measurable sets. Note that our function is not differentiable on these sets as well. The following three sub-cases address the different regions we can be in $A_1$ which are bounded by the solution sets described, or the boundary of $A_1$.

**Sub-case a:** Suppose that

$$m = \frac{1}{x_1 \cos(\alpha x_n) - x_2 \sin(\alpha x_n)},$$

if we had $-m$ the derivative calculations have opposite signs and the bounding would work the same. Also if $M = -x_1$, the following calculations would also just be of opposite sign and does not significantly change.
For this case we have

\[ u_1 = x_1, \]

\[ u_2 = (x_1^2 \sin(\alpha x_n) + x_1 x_2 \cos(\alpha x_n))(x_1 \cos(\alpha x_n) - x_2 \sin(\alpha x_n))^{-1}, \]

\[ u_i = x_1 x_i (x_1 \cos(\alpha x_n) - x_2 \sin(\alpha x_n))^{-1} \text{ for } 3 \leq i \leq n - 1 \text{ and,} \]

\[ u_n = x_n + \ln(K) - \frac{1}{2} \ln \left( K^2 + \left(1 - K^2 \right) \frac{x_1^2 \sin^2 x_1}{x_1^2 + \cdots + x_{n-1}^2} \right). \]

We now have the derivative matrix

\[
\tilde{R}_s' = \begin{pmatrix}
1 & 0 & 0 & 0 & \cdots & 0 & 0 \\
(u_2)_{x_1} & (u_2)_{x_2} & 0 & 0 & \cdots & 0 & (u_2)_{x_n} \\
(u_3)_{x_1} & (u_3)_{x_2} & (u_3)_{x_3} & 0 & \cdots & 0 & (u_3)_{x_n} \\
\vdots & \ddots & \cdots & \vdots & \ddots & \cdots & \vdots \\
(u_{n-1})_{x_1} & (u_{n-1})_{x_2} & 0 & 0 & \cdots & (u_{n-1})_{x_{n-1}} & (u_{n-1})_{x_n} \\
(u_n)_{x_1} & (u_n)_{x_2} & (u_n)_{x_3} & (u_n)_{x_4} & \cdots & (u_n)_{x_{n-1}} & 1
\end{pmatrix},
\]

where

\[
(u_2)_{x_i} = \begin{cases}
\frac{(x_1^2 - x_2^2) \sin(\alpha x_n) \cos(\alpha x_n) - 2 x_1 x_2 \sin^2(\alpha x_n)}{(x_1 \cos(\alpha x_n) - x_2 \sin(\alpha x_n))^2} & i = 1 \\
x_1^2 / (x_1 \cos(\alpha x_n) - x_2 \sin(\alpha x_n))^2 & i = 2 \\
0 & 3 \leq i \leq n - 1 \\
(\alpha x_1^3 + \alpha x_1 x_2^2) / (x_1 \cos(\alpha x_n) - x_2 \sin(\alpha x_n))^2 & i = n
\end{cases}
\]
for $3 \leq j \leq n - 1$ we have

$$(u_j)_{x_i} = \begin{cases} 
-x_2x_j \sin(\alpha x_n) / (x_1 \cos(\alpha x_n) - x_2 \sin(\alpha x_n))^2 & i = 1 \\
x_1x_j \sin(\alpha x_n) / (x_1 \cos(\alpha x_n) - x_2 \sin(\alpha x_n))^2 & i = 2 \\
0 & i \neq 1, 2, j, n, \\
(x_1^2 \cos(\alpha x_n) - x_1x_2 \sin(\alpha x_n)) / (x_1 \cos(\alpha x_n) - x_2 \sin(\alpha x_n))^2 & i = j \\
(\alpha x_1^2 x_j \sin(\alpha x_n) + \alpha x_1x_2 x_j \cos(\alpha x_n)) / (x_1 \cos(\alpha x_n) - x_2 \sin(\alpha x_n))^2 & i = n
\end{cases}$$

and

$$(u_n)_{x_1} = \frac{(K^2 - 1) \left[ (x_1 \sin^2 x_1 + x_1^2 \sin x_1 \cos x_1) \left( x_1^2 + \cdots + x_{n-1}^2 \right)^{-1} - \frac{x_1^2 \sin^2 x_1}{(x_1^2 + \cdots + x_{n-1}^2)} \right]}{K^2 + (1 - K^2) \frac{x_1^2 \sin^2 x_1}{x_1^2 + \cdots + x_{n-1}^2}},$$

for $2 \leq i \leq n - 1$ we have

$$(u_n)_{x_i} = \frac{(1 - K^2)x_i x_1^2 \sin^2 x_i / \left( x_1^2 + \cdots + x_{n-1}^2 \right)^2}{K^2 + (1 - K^2) \frac{x_1^2 \sin^2 x_1}{x_1^2 + \cdots + x_{n-1}^2}},$$

and

$$(u_n)_{x_n} = 1.$$
Here we give bounding for the partial derivatives. Using the fact that \((x_1, ..., x_n) \in A_1\) and (3.8), and the fact that we are in sub-case a), we have

\[
|(u_2)_x| = \left| \frac{(x_1^2 - x_2^2) \sin(\alpha x_n) \cos(\alpha x_n) - 2x_1 x_2 \sin^2(\alpha x_n)}{(x_1 \cos(\alpha x_n) - x_2 \sin(\alpha x_n))^2} \right| \\
\leq \frac{2(x_1^2 + x_2^2 + 2x_1^2)}{x_1^2 + x_2^2} \\
\leq 2 (1 + 2) = 6.
\]

Using similar methods we have the following bounds:

\[
|(u_2)_x| \leq \begin{cases} 
  6 & i = 1 \\
  2 & i = 2 \\
  0 & 3 \leq i \leq n - 1 \\
  8|\alpha| & i = n
\end{cases},
\]

and

\[
|(u_j)_x| \leq \begin{cases} 
  2 & i = 1 \\
  2 & i = 2 \\
  0 & 3 \leq i \leq n - 1, i \neq j \\
  4 & i = j \\
  8|\alpha| & i = n
\end{cases}.
\]

We need to use slightly different tactics to calculate a bound for the partial derivative \((u_n)_x\). We know that

\[
0 \leq \frac{x_1^2 \sin^2 x_1}{x_1^2 + \cdots + x_{n-1}^2} \leq 1,
\]
so that
\[ 1 \leq K^2 + (1 - K^2) \frac{x_1^2 \sin^2 x_1}{x_1^2 + \cdots + x_{n-1}^2} \leq K^2. \]

We also use the fact that \( x_1 \geq \sin(x_1) \) for \( x_1 \geq 0 \). From here, we have

\[
|(u_n)_{x_1}| = \left| \frac{(K^2 - 1)}{K^2 + (1 - K^2) \frac{x_1^2 \sin^2 x_1}{x_1^2 + \cdots + x_{n-1}^2}} \right| \times \left| \frac{(x_1 \sin^2 x_1 + x_1^2 \sin x_1 \cos x_1) (x_1^2 + \cdots + x_{n-1}^2)^{-1} - x_1^3 \sin^2 x_1 (x_1^2 + \cdots + x_{n-1}^2)^{-2}}{K^2 + (1 - K^2) \frac{x_1^2 \sin^2 x_1}{x_1^2 + \cdots + x_{n-1}^2}} \right|
\]

\[
\leq |(K^2 - 1)| \times \left| (x_1 \sin^2 x_1 + x_1^2 \sin x_1 \cos x_1) (x_1^2 + \cdots + x_{n-1}^2)^{-1} - x_1^3 \sin^2 x_1 (x_1^2 + \cdots + x_{n-1}^2)^{-2} \right|
\]

\[
\leq |K^2 - 1| \left( 1 + 1 + |x_1|^5(x_1^2 + \cdots + x_{n-1}^2)^{-2} \right)
\]

\[
\leq |K^2 - 1|(2 + \pi/2) \leq 4(K^2 - 1).
\]

We also have for \( 2 \leq i \leq n - 1 \) that

\[
|(u_n)_{x_i}| = \left| \frac{(1 - K^2)x_1^2x_i \sin^2 x_1}{(x_1^2 + \cdots + x_{n-1}^2)^2 \left( K^2 + (1 - K^2) \frac{x_1^2 \sin^2 x_1}{x_1^2 + \cdots + x_{n-1}^2} \right)} \right|
\]

\[
\leq (K^2 - 1) \frac{|x_1|^3}{(x_1^2 + \cdots + x_{n-1}^2)^2}
\]

\[
\leq (K^2 - 1).
\]

In conclusion, we have the bounds

\[
|(u_n)_{x_i}| \leq \begin{cases} 
4(K^2 - 1) & i = 1 \\
(K^2 - 1) & 2 \leq i \leq n - 1 \\
1 & i = n
\end{cases}
\]
The above bounds are not sharp, but for our result of $\tilde{R}_s$ to be quasiregular, all we need to know is that these partial derivatives are bounded above by some constant value, so that $\|\tilde{R}_s\|$ is bounded. We also want $J_{\tilde{R}_s}$ to be bounded from below, so that we can use Theorem 1.2.4. To the end of bounding $J_{\tilde{R}_s}$ from below, we can notice that the only terms that appear without an $\alpha$ multiplying them occur when we multiply the diagonal of $\tilde{R}_s'$ together. That is, the non-alpha term of the Jacobian is

$$Q := \frac{x_1^2(x_1^2 \cos(\alpha x_n) - x_1 x_2 \sin(\alpha x_n))^{n-3}}{(x_1 \cos(\alpha x_n) - x_2 \sin(\alpha x_n))^{2(n-2)}}.$$ 

Notice that using (3.8), we have that

$$Q \geq \frac{x_1^{n-1}}{(x_1 \cos(\alpha x_n) - x_2 \sin(\alpha x_n))^{n-1}} \geq \frac{x_1^{n-1}}{\sqrt{x_1^2 + x_2^2}^{n-1}} \geq \frac{x_1^{n-1}}{\sqrt{2x_1^2}^{n-1}} = 2^{-(n-1)/2}.$$ 

For here, we can choose $\alpha$ so that $|\alpha| > 0$ is sufficiently small so that the alpha terms have absolute value less than $\frac{1}{2}Q$. That is,

$$J_{\tilde{R}_s} > \frac{1}{2}Q \geq 2^{-(n+1)/2}.$$ 

**Sub-case b:** For the case when

$$m = \frac{1}{x_1 \sin(\alpha x_n) + x_2 \cos(\alpha x_n)}$$

we have
\[ u_1 = (x_1^2 \cos(\alpha x_n) - x_1 x_2 \sin(\alpha x_n))(x_1 \sin(\alpha x_n) + x_2 \cos(\alpha x_n))^{-1}, \]

\[ u_2 = x_1, \]

\[ u_i = x_1 x_i (x_1 \sin(\alpha x_n) + x_2 \cos(\alpha x_n))^{-1} \] for \( 3 \leq i \leq n - 1 \) and,

\[ u_n = x_n + \ln(K) - \frac{1}{2} \ln \left( K^2 + \left( 1 - K^2 \right) \frac{x_1^2 \sin^2 x_1}{x_1^2 + \cdots + x_{n-1}^2} \right). \]

This gives us the derivative matrix

\[ \tilde{R}_s' = \begin{pmatrix}
(u_1)_{x_1} & (u_1)_{x_2} & \cdots & (u_1)_{x_n} \\
1 & 0 & \cdots & 0 \\
(u_3)_{x_1} & (u_3)_{x_2} & \cdots & (u_3)_{x_n} \\
\vdots & \ddots & \cdots & \vdots \\
(u_n)_{x_1} & \cdots & (u_n)_{x_{n-1}} & 1 
\end{pmatrix}, \]

where

\[ (u_1)_{x_i} = \begin{cases} 
\frac{(x_1^2 - x_2^2) \sin(\alpha x_n) \cos(\alpha x_n) + 2 x_1 x_2 \cos^2(\alpha x_n)}{(x_1 \sin(\alpha x_n) + x_2 \cos(\alpha x_n))^2} & \text{if } i = 1 \\
- \frac{x_1^2}{(x_1 \sin(\alpha x_n) + x_2 \cos(\alpha x_n))^2} & \text{if } i = 2 \\
0 & \text{if } 3 \leq i \leq n - 1 \\
- \frac{\alpha x_1^3 - \alpha x_1 x_2^2}{(x_1 \sin(\alpha x_n) + x_2 \cos(\alpha x_n))^2} & \text{if } i = n
\end{cases} \]
and for $3 \leq j \leq n - 1$ we have

\[
(u_j)_{x_i} = \begin{cases}
  x_2 x_j \cos(\alpha x_n)/(x_1 \sin(\alpha x_n) + x_2 \cos(\alpha x_n))^2 & i = 1 \\
  -x_1 x_j \cos(\alpha x_n)/(x_1 \sin(\alpha x_n) + x_2 \cos(\alpha x_n))^2 & i = 2 \\
  0 & i \neq 1, 2, j, n \\
  (x_1^2 \sin(\alpha x_n) + x_1 x_2 \cos(\alpha x_n))/(x_1 \sin(\alpha x_n) + x_2 \cos(\alpha x_n))^2 & i = j \\
  (-\alpha x_1^2 x_j \cos(\alpha x_n) + \alpha x_1 x_2 x_j \sin(\alpha x_n))/(x_1 \sin(\alpha x_n) + x_2 \cos(\alpha x_n))^2 & i = n
\end{cases}
\]

The partial derivatives of $u_n$ are the same as sub-case a. Also, by looking at the similarities we can see that all of these derivatives are bounded from above, and that we can choose $\alpha$ small enough so that $J_{\tilde{R}_s} > 2^{-(n+1)/2}$.

**Sub-case c:** Let $m = x_j^{-1}$ for $3 \leq j \leq n - 1$, we have

\[
u_1 = (x_1^2 \cos(\alpha x_n) - x_1 x_2 \sin(\alpha x_n)) x_j^{-1},
\]
\[
u_2 = (x_1^2 \sin(\alpha x_n) + x_1 x_2 \cos(\alpha x_n)) x_j^{-1},
\]
\[
u_i = x_1 x_i x_j^{-1} \text{ for } 3 \leq i \leq n - 1, i \neq j,
\]
\[
u_j = x_1 \text{ and,}
\]
\[
u_n = x_n + \ln(K) - \frac{1}{2} \ln \left(K^2 + (1 - K^2) \frac{x_1^2 \sin^2 x_1}{x_1^2 + \cdots + x_{n-1}^2}\right).
\]

We have the derivative matrix

\[
\tilde{R}_s' = \begin{pmatrix}
  (u_1)_{x_1} & (u_1)_{x_2} & \cdots & (u_1)_{x_n} \\
  \vdots & \ddots & \cdots & \vdots \\
  (u_n)_{x_1} & \cdots & (u_n)_{x_{n-1}} & 1
\end{pmatrix}
\]
where

\[
(u_1)_{x_i} = \begin{cases}
(2x_1 \cos(\alpha x_n) - x_2 \sin(\alpha x_n)) x_j^{-1} & i = 1 \\
-x_1 \sin(\alpha x_n)x_j^{-1} & i = 2 \\
0 & 3 \leq i \leq n - 1, i \neq j, \\
(-x_1^2 \cos(\alpha x_n) + x_1 x_2 \sin(\alpha x_n)) x_j^{-2} & i = j, \\
(-\alpha x_1^2 \sin(\alpha x_n) - \alpha x_1 x_2 \cos(\alpha x_n)) x_j^{-1} & i = n
\end{cases}
\]

\[
(u_2)_{x_i} = \begin{cases}
(2x_1 \sin(\alpha x_n) + x_2 \cos(\alpha x_n)) x_j^{-1} & i = 1 \\
x_1 \cos(\alpha x_n)x_j^{-1} & i = 2 \\
0 & 3 \leq i \leq n - 1, i \neq j, \\
(-x_1^2 \sin(\alpha x_n) - x_1 x_2 \cos(\alpha x_n)) x_j^{-2} & i = j, \\
(\alpha x_1^2 \cos(\alpha x_n) - \alpha x_1 x_2 \sin(\alpha x_n)) x_j^{-1} & i = n
\end{cases}
\]

\[
(u_j)_{x_i} = \begin{cases}
1 & i = 1 \\
0 & i \neq 1
\end{cases},
\]

for 3 \leq k \leq n - 1, k \neq j, we have

\[
(u_k)_{x_i} = \begin{cases}
x_k x_j^{-1} & i = 1 \\
0 & i \neq 1, k, j \\
x_1 x_j^{-1} & i = k \\
-x_1 x_k x_j^{-2} & i = j
\end{cases}.
\]
Note that the partial derivatives of $u_n$ are the same as in the previous two cases and are bounded. Since we are assuming that $(x_1, ..., x_n) \in A_1$ where the point at the origin is not included, and that $M = x_1 \neq 0$, this means that $x_1 > |x_i|$ for all $2 \leq i \leq n - 1$. For $m = 1/|x_j|$ for some $j$, the definition of $m$ and (3.8) give

$$\frac{1}{|x_j|} \leq \min\left\{ \frac{1}{|x_1 \cos(\alpha x_n) - x_2 \sin(\alpha x_n)|}, \frac{1}{|x_1 \sin(\alpha x_n) + x_2 \cos(\alpha x_n)|} \right\} \leq \frac{\sqrt{2}}{\sqrt{x_1^2 + x_2^2}}. \quad (3.9)$$

Using the fact that $(x_1, ..., x_n) \in A_1$ and (3.9), for $l = 1, 2$ we have

$$| (u_l)_{x_i} | \leq \begin{cases} 6 & i = 1 \\ 2 & i = 2 \\ 0 & 3 \leq i \leq n - 1, i \neq j \\ 4 & i = j \\ 8|\alpha| & i = n \end{cases}$$

and for $k \neq j, 3 \leq k \leq n - 1$, we have

$$| (u_k)_{x_i} | \leq \begin{cases} 2 & i = 1, j, k \\ 0 & i \neq 1, j, k \end{cases}.$$
In this case, we can calculate the Jacobian by first taking the determinate across $j$th row, so that

$$J_{\tilde{R}_s} = (-1)^{j+1} \det \begin{pmatrix} (u_1)_{x_2} & \cdots & (u_1)_{x_n} \\ \vdots & \ddots & \vdots \\ (u_{j-1})_{x_2} & \cdots & (u_{j-1})_{x_n} \\ (u_j)_{x_2} & \cdots & (u_j)_{x_n} \\ \vdots & \ddots & \vdots \\ (u_n)_{x_2} & \cdots & (u_n)_{x_n} \end{pmatrix}.$$ 

Now take the determinate down the column where we take the partial derivative with respect to $x_n$, then the Jacobian is

$$J_{\tilde{R}_s} = (u_1)_{x_2}(u_2)_{x_j} \left( \prod_{3 \leq i \leq n-1, i \neq j} (u_i)_{x_i} \right) - (u_1)_{x_j}(u_2)_{x_2} \left( \prod_{3 \leq i \leq n-1, i \neq j} (u_i)_{x_i} \right)$$

$$+ (-1)^{n+1}(u_1)_{x_n} \det \begin{pmatrix} (u_2)_{x_1} & \cdots & (u_2)_{x_{n-1}} \\ \vdots & \ddots & \vdots \\ (u_n)_{x_1} & \cdots & (u_n)_{x_{n-1}} \end{pmatrix} + (-1)^{n}(u_2)_{x_n} \begin{pmatrix} (u_1)_{x_1} & \cdots & (u_1)_{x_{n-1}} \\ (u_3)_{x_1} & \cdots & (u_3)_{x_{n-1}} \\ \vdots & \ddots & \vdots \\ (u_n)_{x_1} & \cdots & (u_n)_{x_{n-1}} \end{pmatrix},$$

so that the term without being multiplied by $\alpha$ is

$$\frac{x_1^2 \cos(\alpha x_n) - x_1 x_2 \sin(\alpha x_n))x_1 \cos(\alpha x_n)x_1^{n-4} + (-x_1^2 - x_1 x_1 \cos(\alpha x_n))(-x_1 \sin(\alpha x_n))x_1^{n-4}}{x_j^{n-1}}$$

$$= \frac{x_1^{n-1}(x_1^3 \sin^2(\alpha x_n) + x_1^2 \cos^2(\alpha x_n))}{x_j^{n-1}} = \frac{x_1^{n-1}}{x_j^{n-1}} > 1.$$
We need $\alpha$ to be sufficiently small where we have

$$J_{\tilde{R}_s} > \frac{1}{2} \left( \frac{x_1^2 \cos(\alpha x_n) - x_1 x_2 \sin(\alpha x_n)}{x_j^{n-1}} x_1^{n-4} \right) x_1 \cos(\alpha x_n) x_1^{n-4} \right) + \frac{(-x_1^2 + x_1 x_1 \cos(\alpha x_n))(x_2 \sin(\alpha x_n)) x_1^{n-4} \right)}{x_j^{n-1}} x_1^{n-4} \right) > 2^{-(n+1)/2}.$$

The last inequality shows that all we need do is to choose $\alpha$ in finitely many cases, so that the Jacobian is bounded from below by $2^{-(n+1)/2}$. In other words, we can let $\alpha$ be the minimal in size from sub-cases a, b, and c, then we obtain $\|\tilde{R}_s\|$ is bounded in each region.

**Case II:** We have the case where $M = x_2$, i.e. $(x_1, \ldots, x_n) \in A_2$, which is similar to the case when $M = x_1$. Running through similar calculations as in case I we can show that $\tilde{R}_s$ has bounded derivative matrix, where the derivative matrix is invertible. Moreover, we show that the Jacobian is bounded from below giving us that the inverse derivative matrix is bounded as well in the corresponding regions.

**Case III:** Let $(x_1, \ldots, x_n) \in A_j$ for some $3 \leq j \leq n - 1$, so that $M = x_j$, with $x_j > |x_i|$ for $1 \leq i \leq n - 1$, $i \neq j$, and that $x_j \neq 0$. Here we also break this case into three sub-cases for the same reasoning as in case I.

**Sub-case a:** Suppose that

$$m = \frac{1}{x_1 \cos(\alpha x_n) - x_2 \sin(\alpha x_n)},$$

which means, by definition of $m$ that

$$\frac{1}{x_1 \cos(\alpha x_n) - x_2 \sin(\alpha x_n)} \leq \frac{1}{x_j}.$$ (3.10)
This means that

\[ x_1 \cos(\alpha x_n) - x_2 \sin(\alpha x_n) \geq x_j > 0, \]

which also implies that either \( x_1 \neq 0 \) or \( x_2 \neq 0 \). Since either \( x_1 \) or \( x_2 \) are not zero we have that (3.8) holds. We also have the inequality

\[ \sqrt{x_1^2 + x_2^2} \geq x_1 \cos(\alpha x_n) - x_2 \cos(\alpha x_n) \geq x_j. \]  

(3.11)

For this case we have

\[ u_1 = x_j, \]

\[ u_2 = (x_1 x_j \sin(\alpha x_n) + x_2 x_j \cos(\alpha x_n)) (x_1 \cos(\alpha x_n) - x_2 \sin(\alpha x_n))^{-1}, \]

\[ u_i = x_i x_j (x_1 \cos(\alpha x_n) - x_2 \sin(\alpha x_n))^{-1} \text{ for } 3 \leq i \leq n - 1, i \neq j \]

\[ u_j = x_j^2 (x_1 \cos(\alpha x_n) - x_2 \sin(\alpha x_n))^{-1} \text{ and,} \]

\[ u_n = x_n + \ln(K) - \frac{1}{2} \ln \left( K^2 + (1 - K^2) \frac{x_j^2 \sin^2 x_j}{x_1^2 + \cdots + x_{n-1}^2} \right). \]

Define \( \Omega := \{1, 2, j, n\} \). We have the corresponding derivative matrix

\[ \tilde{R}_s' = \begin{pmatrix}
(u_1)_{x_1} & (u_1)_{x_2} & \cdots & (u_1)_{x_n} \\
\vdots & \ddots & \ddots & \vdots \\
(u_n)_{x_1} & \cdots & (u_n)_{x_{n-1}} & 1
\end{pmatrix}, \]

where

\[ (u_1)_{x_i} = \begin{cases}
0 & i \neq j \\
1 & i = j
\end{cases}. \]
\((u_2)_{x_i} = \begin{cases} 
-x_2 x_j / (x_1 \cos(\alpha x_n) - x_2 \sin(\alpha x_n))^2 & i = 1 \\
x_1 x_j / (x_1 \cos(\alpha x_n) - x_2 \sin(\alpha x_n))^2 & i = 2 \\
0 & i \notin \Omega \\
\left(\frac{x_i^2 - x_j^2}{x_1 \cos(\alpha x_n) - x_2 \sin(\alpha x_n)}\right) & i = j \\
\left(\frac{x_1^2 x_j + x_2^2 x_j}{x_1 \cos(\alpha x_n) - x_2 \sin(\alpha x_n)}\right) / (x_1 \cos(\alpha x_n) - x_2 \sin(\alpha x_n))^2 & i = n \end{cases}\)

\(\begin{cases} 
-x_j^2 \cos(\alpha x_n) / (x_1 \cos(\alpha x_n) - x_2 \sin(\alpha x_n))^2 & i = 1 \\
x_j^2 \sin(\alpha x_n) / (x_1 \cos(\alpha x_n) - x_2 \sin(\alpha x_n))^2 & i = 2 \\
0 & i \notin \Omega \\
2 x_j / (x_1 \cos(\alpha x_n) - x_2 \sin(\alpha x_n)) & i = j \\
\left(\frac{\alpha x_1 x_j^2 \sin(\alpha x_n) + \alpha x_2 x_j^2 \cos(\alpha x_n)}{x_1 \cos(\alpha x_n) - x_2 \sin(\alpha x_n)}\right) / (x_1 \cos(\alpha x_n) - x_2 \sin(\alpha x_n))^2 & i = n \end{cases}\)

for \(3 \leq k \leq n - 1, k \neq j\) we have

\(\begin{cases} 
-x_k x_j \cos(\alpha x_n) / (x_1 \cos(\alpha x_n) - x_2 \sin(\alpha x_n))^2 & i = 1 \\
x_k x_j \sin(\alpha x_n) / (x_1 \cos(\alpha x_n) - x_2 \sin(\alpha x_n))^2 & i = 2 \\
0 & i \notin \Omega \cup \{k\} \\
x_k / (x_1 \cos(\alpha x_n) - x_2 \sin(\alpha x_n)) & i = j \\
x_j / (x_1 \cos(\alpha x_n) - x_2 \sin(\alpha x_n)) & i = k \\
\left(\frac{\alpha x_1 x_k x_j \sin(\alpha x_n) + \alpha x_2 x_k x_j \cos(\alpha x_n)}{x_1 \cos(\alpha x_n) - x_2 \sin(\alpha x_n)}\right) / (x_1 \cos(\alpha x_n) - x_2 \sin(\alpha x_n))^2 & i = n \end{cases}\)

and
\[(u_n)_{x_1} = \frac{(K^2 - 1) \left[ (x_1 \sin^2 x_j) \left( x_1^2 + \cdots + x_{n-1}^2 \right)^{-1} - x_1^3 \sin^2 x_j \left( x_1^2 + \cdots + x_{n-1}^2 \right)^{-2} \right]}{K^2 + (1 - K^2) \frac{x_1^2 \sin^2 x_j}{x_1^2 + \cdots + x_{n-1}^2}},\]

\[(u_n)_{x_i} = \frac{(1 - K^2)x_1^2 x_i \sin^2 x_j}{(x_1^2 + \cdots + x_{n-1}^2) \left( K^2 + (1 - K^2) \frac{x_1^2 \sin^2 x_j}{x_1^2 + \cdots + x_{n-1}^2} \right)},\]

for \(2 \leq i \leq n - 1, i \neq j,\)

\[(u_n)_{x_j} = \frac{(K^2 - 1) \left[ (x_1 \sin x_j \cos x_j) \left( x_1^2 + \cdots + x_{n-1}^2 \right)^{-1} - x_1^2 x_j \sin^2 x_j \left( x_1^2 + \cdots + x_{n-1}^2 \right)^{-2} \right]}{K^2 + (1 - K^2) \frac{x_1^2 \sin^2 x_j}{x_1^2 + \cdots + x_{n-1}^2}},\]

and,

\[(u_n)_{x_n} = 1.\]

Note that the partial derivatives of \(u_n\) are bounded using similar calculations as in case I. Using the fact that \((x_1, \ldots, x_n) \in A_j\) and (3.10) we have the following bounds for \(l = 2, j\)

\[|(u_l)_{x_i}| \leq \begin{cases} 1 & i = 1, 2 \\ 0 & i \neq 1, 2, j, n \\ 6 & i = j \\ 4|\alpha| & i = n \end{cases},\]

\[|(u_1)_{x_i}| \leq \begin{cases} 1 & i = 1 \\ 0 & i \neq 1 \end{cases}.\]
by similar methods from Case I sub-case a) the bounds for partial derivatives of $u_n$ are

\[
|(u_n)_{x_i}| \leq \begin{cases} 
3|K^2 - 1| & i = 1, j \\
2|K^2 - 1| & 2 \leq i \leq n - 1, i \neq j \\
1 & i = n
\end{cases},
\]

and for $k \neq 1, 2, j, n$ we have

\[
|(u_k)_{x_i}| \leq \begin{cases} 
1 & i = 1, 2 \\
0 & i \neq 1, 2, j, n, k \\
4 & i = j \\
2 & i = k \\
4|\alpha| & i = n
\end{cases}.
\]

To compute the Jacobian of $\tilde{R}_s$ for this case, first let

\[
M = \begin{pmatrix} 
(u_2)_{x_1} & \cdots & (u_2)_{x_{j-1}} & (u_2)_{x_{j+1}} & \cdots & (u_2)_{x_n} \\
\vdots & \ddots & \ddots & \vdots & \ddots & \vdots \\
(u_n)_{x_{n-1}} & \cdots & (u_n)_{x_j} & (u_n)_{x_{j+1}} & \cdots & (u_n)_{x_n}
\end{pmatrix}.
\]

Taking the determinate first row, we have

\[
J_{\tilde{R}_s} = (-1)^{j+1} \det M.
\]
Define $M_i$ to be the square matrix of order $n - 2$ derived from removing the $(i - 1)$th row, $2 \leq i \leq n$ and $(n - 1)$th column from $M$. Taking the determinate of $M$ first along the column where the partial derivatives are taken with respect to $x_n$, we have that

$$J_{\tilde{R}_s} = \left( \prod_{\substack{3 \leq i \leq n-1 \atop i \neq j}} (u_i)_{x_i} \right) ((u_j)_{x_2})_1 - (u_j)_{x_1} (u_2)_{x_2} + \sum_{i=2}^{n-1} [(-1)^{n+i+j+1} (u_n)_{x_i} \det(M_i)],$$

so that the non-alpha term in $J_{\tilde{R}_s}$ is

$$Q = \frac{-(-x_j^2) \cos(\alpha x_n)(x_1 x_j) x_j^{n-4} (x_1 \cos(\alpha x_n) - x_2 \sin(\alpha x_n))^{n-4}}{(x_1 \cos(\alpha x_n) - x_2 \sin(\alpha x_n))^{2n-2}} \frac{x_j^2 \sin(\alpha x_n)(-x_2 x_j)x_j^{n-4} (x_1 \cos(\alpha x_n) - x_2 \sin(\alpha x_n))^{n-4}}{(x_1 \cos(\alpha x_n) - x_2 \sin(\alpha x_n))^{2n-2}}.$$

Using (3.10) and simplifying equations we have the following lower bound for $Q$,

$$Q = \frac{x_j^{n-1} (x_1 \cos(\alpha x_n) - x_2 \sin(\alpha x_n))^{n-3}}{(x_1 \cos(\alpha x_n) - x_2 \sin(\alpha x_n))^{2n-4}} \frac{x_j^{n-1}}{(x_1 \cos(\alpha x_n) - x_2 \sin(\alpha x_n))^{n-1}} \geq \frac{x_j^{n-1}}{x_j^{n-1}} = 1.$$

Then we need $\alpha$ to be sufficiently small so that

$$J_{\tilde{R}_s} > \frac{1}{2} Q \geq \frac{1}{2}.$$

Then $J_{\tilde{R}_s}$ is bounded below and the norms of $\tilde{R}_s'$ and $\left(\tilde{R}_s'\right)^{-1}$ are bounded above.

**Sub-case b:** The case when

$$m = \frac{1}{x_1 \sin(\alpha x_n) + x_2 \cos(\alpha x_n)}$$
is very similar to sub-case a. Using similar calculations we have that $J_{\tilde{R}_s}$ is bounded below, and that $\|\tilde{R}_s\|$ is bounded from above.

**Sub-case c:** Finally, we are left with our last case when we let

$$m = \frac{1}{x_j}.$$  

We have that

$$u_1 = x_1 \cos(\alpha x_n) - x_2 \sin(\alpha x_n),$$

$$u_2 = x_2 \sin(\alpha x_n) + x_2 \sin(\alpha x_n),$$

$$u_i = x_i \text{ for } 3 \leq i \leq n - 1 \text{ and,}$$

$$u_n = x_n + \ln(K) - \frac{1}{2} \ln \left( K^2 + (1 - K^2) \frac{x_j^2 \sin^2 x_j}{x_1^2 + \cdots + x_{n-1}^2} \right).$$

We have the corresponding derivative matrix

$$\tilde{R}_s' = \begin{pmatrix}
(u_1)_x & (u_2)_x & (u_3)_x & (u_4)_x & (u_5)_x & \cdots & (u_{n-1})_x & (u_n)_x \\
(u_2)_x & (u_2)_x & (u_3)_x & (u_4)_x & (u_5)_x & \cdots & (u_{n-1})_x & (u_n)_x \\
0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \cdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 \\
(u_n)_x & (u_n)_x & (u_n)_x & (u_n)_x & (u_n)_x & \cdots & (u_n)_{x_{n-1}} & 1
\end{pmatrix},$$

where
\[(u_1)_i = \begin{cases} 
\cos(\alpha x_n) & i = 1 \\
-\sin(\alpha x_n) & i = 2 \\
0 & 3 \leq i \leq n - 1 \\
-\alpha x_1 \sin(\alpha x_n) - \alpha x_2 \cos(\alpha x_n) & i = n 
\end{cases},
\]

\[(u_2)_i = \begin{cases} 
\sin(\alpha x_n) & i = 1 \\
\cos(\alpha x_n) & i = 2 \\
0 & 3 \leq i \leq n - 1 \\
\alpha x_1 \cos(\alpha x_n) - \alpha x_2 \sin(\alpha x_n) & i = n 
\end{cases},
\]

and the partial derivatives for \(u_n\) are the same as in the sub-case a which we already remarked were all bounded from above.

We have the following bounds for the partial derivatives corresponding to \(l = 1, 2\)

\[| (u_l)_i | \leq \begin{cases} 
1 & i = 1, 2 \\
0 & 3 \leq i \leq n - 1 \\
6|\alpha| & i = n 
\end{cases}.
\]

The term without \(\alpha\) in \(J_{\tilde{R}_s}\) is

\[\cos^2(\alpha x_n) + \sin^2(\alpha x_n) = 1.\]

We can find an \(\alpha\) sufficiently small so that

\[J_{\tilde{R}_s} \geq \frac{1}{2}.\]
Therefore, the norm of $\tilde{R}_s'$ is bounded from above in the regions where $\tilde{R}_s$ is differentiable.

From cases I, II and III, we have that the linear distortion of $\tilde{R}_s$ is bounded from above where $\tilde{R}_s$ is differentiable.

### 3.2 Proof of Theorem 1.2.24

In this section we prove Theorem 1.2.24, that given a non-empty, compact, connected subset of $\mathbb{R}^n \setminus \{0\}$, we can realize it as an orbit space for a quasiregular, and in fact quasi-conformal, map. Before doing so, we introduce a couple of results that are necessary.

Let $f : D \to \mathbb{R}^n$ be a quasiregular mapping defined on $D \subset \mathbb{R}^n$ and let $x_0 \in D$. By Theorem 1.2.5, we can find $r_0 > 0$ small enough so that if $0 < r < r_0$ then

$$\frac{L_f(x_0, r)}{l_f(x_0, r)} \leq C_1,$$

where $C_1 = 2C$ depends only on $n$, $K_O(f)$ and $i(x_0, f)$. For $x \in \mathbb{R}^n$ fixed and $0 < t \leq r_0/|x|$, consider the curve

$$\gamma_x = \frac{f(x_0 + tx) - f(x_0)}{\rho_f(t)}. \quad (3.12)$$

We know that the curve $t \mapsto \gamma_x(t)$ is continuous for $0 < t < r_0/|x|$, [9, Lemma 3.1].

Let us define $h_{(K, \sigma, A)}$ to be a composition where we first stretch radially by a factor of $K$ in the $x_1$ direction using $R$, then followed by a composition of a rotation so that the stretch is in the direction of $\sigma \in S^{n-1}$, and then by an orthogonal map $A$ that fixes the line through $\sigma$ and the origin. In two dimensions there is a single way to radially stretch by a factor of $K$ in direction $\sigma$, whereas there are many ways to radially stretch by a factor of $K$ in direction $\sigma$ when $n \geq 3$. Whenever we introduce an orthogonal map, it is meant to give us the exact
ellipsoid to match with the paths described later on. In particular, \( h_{(K,\sigma,A)} \) is the family of all maps that stretch by a factor of \( K \) in the \( \sigma \) direction.

**Lemma 3.2.1.** Let \( K > 0, \sigma \in S^{n-1}, A \) an orthogonal map that fixes the line through \( \sigma \) and the origin, and let \( h_{(K,\sigma,A)} \) to be defined as mentioned. Then for \( r > 0 \), we have

\[
\frac{h_{(K,\sigma,A)}(rx_1)}{\rho(r)} = K^{1-1/n}\sigma,
\]

where \( x_1 = (1, 0, \ldots, 0) \in \mathbb{R}^n \).

**Proof.** The volume of the image of a closed ball of radius \( r \) under \( h_{(K,\sigma,A)} \) is an ellipsoid a semi axis of length \( Kr \) and the other semi axes of length \( r \). We have that

\[
\rho(r) = \left( \frac{\omega_n K r^n}{\omega_n} \right)^{1/n} = K^{1/n} r,
\]

where \( \omega_n \) is the volume of the unit ball in \( \mathbb{R}^n \). Therefore,

\[
\frac{h_{(K,\sigma,A)}(rx_1)}{\rho(r)} = \frac{Kr \sigma}{K^{1/n} r} = K^{1-1/n}\sigma.
\]

\( \square \)

When we allow \( x_0 = 0, x_1 = (1, 0, \ldots, 0) \in \mathbb{R}^n \), and recalling (3.12), for any \( r > 0 \) we have

\[
\gamma_{x_1}(rx_1) = K^{1-1/n}\sigma.
\]

Let us define some maps that we are using. First note that we can write any point \( x \in \mathbb{R}^n \) as \( u\sigma \) where \( u > 0 \) and \( \sigma \in S^{n-1} \). Let \( h_{(K,L,\sigma,A)} \) be \( R_I \) where we stretch by a factor \( K \) and \( L \) as described in Section 3.1.2, but followed by a composition of a rotation so that the stretch
is in the direction of $\sigma \in S^{n-1}$, and then by an orthogonal map $A$ that fixes the line through $\sigma$ and the origin. Note that the domain of $R_I$ is

$$\{ x \in \mathbb{R}^n : e_s \leq |x| \leq e_t \},$$

where $t$ and $s$ are constants such that $|\ln(K/L)| < (t - s)/2$. Note that if

$$x \in \{ x \in \mathbb{R}^n : |x| = e^s \}$$

then $h_{(K,L,\sigma,A)}(x) = h_{(L,\sigma,A)}(x)$, and if

$$x \in \{ x \in \mathbb{R}^n : |x| = e^s \}$$

then $h_{(K,L,\sigma,A)}(x) = h_{(K,\sigma,A)}(x)$. For the sphere of radius $|x| \in (e^s, e^t)$ centered at the origin, we have that the image of the sphere is an ellipsoid like shape but not necessarily an ellipsoid. Let $g_{(K,\sigma_1,\sigma_2,A,B)}$ be $R_s$ where we stretch by a factor of $K$, then composed with an orthogonal map $A$, which match us with the ellipsoid corresponding to $h_{(K,\sigma_1,A)}$, followed by another orthogonal map $B$ so that we start at a point on the radial line through $\sigma_1$ and end our spiraling at a point on the radial line through $\sigma_2$. Once we finish the rotation, we want the image of the map $g_{(K,\sigma_1,\sigma_2,A,B)}$ to correspond to the image of an ellipsoid corresponding to $h_{(K,\sigma_2,A')}$, where $A'$ is the corresponding orthogonal map that matches the directions we want the ellipsoid “turned” about a line through the origin in the $\sigma_2$ direction. Also, in the function $g_{(K,\sigma_1,\sigma_2,A,B)}$, $B$ counteracts $A$ and “turn” the ellipsoid about the radial line so that the spiraling is occurring in the direction we desire. Note that the image of any sphere of radius $r > 0$ under $g_{(K,\sigma_1,\sigma_2,A,B)}$ is an ellipsoid by construction.
Proof of Theorem 1.2.24. Let $X \subset \mathbb{R}^n \setminus \{0\}$ be compact and connected. For $k \in \mathbb{N}$, let $U_k$ be an open $1/k$-neighborhood of $X$. We can find $K \in \mathbb{N}$ and $C > 1$ so that for $k \geq K$, $U_k \subset \{x : 1/C \leq |x| \leq C\}$. For $k \geq K$, find a path $\Gamma_k \subset U_k$ starting and ending at (possibly different) points of $X$ so that:

- $\Gamma_k$ is made up of finitely many radial line segments and arcs of great circles,
- for every $x \in U_k$, there exists $u \in \Gamma_k$ with $|x - u| < 1/k$,
- the endpoint of $\Gamma_k$ coincides with the starting point of $\Gamma_{k+1}$.

Our aim is to construct a quasiconformal map $f$ so that, recalling (3.12), the curve $\gamma_{x_1}$ is the concatenation of $\Gamma_k$ for $k \geq K$. If this is so, then since by construction $\gamma_{x_1}$ accumulates exactly on $X$, we are done. In the parts of $f$ that take on $g(K,\sigma_1,\sigma_2,A,B)$, $f$ sends a ball of radius $r$ to an ellipsoid centered at the origin with appropriate eccentricity and orientation, and in the parts of $f$ that take on $h(K,L,\sigma,A)$ $f$ sends a ball of radius $r$ to an ellipsoid centered at the origin with appropriate eccentricity and orientation at least on the boundary of a spherical shell, so that $\gamma_{x_1}(rx_1)$ has the required value. Note, if we have a spherical shell with outer radius $r_k > 0$ and inner radius $r_{K+1}$, then for points on a radial line segment between the two boundaries create a radial line segment under the generalized derivative of $h(K,L,\sigma,A)$ even though the image of spheres in the interior of the spherical shell under $h(K,L,\sigma,A)$ may not be an ellipsoid. Recall that Lemma 3.2.1 and (3.14) says what ellipsoid we need to obtain a required value for $\gamma_{x_1}(rx_1)$.

To this end, we give a parameterization $p_k : [r_{k+1},r_k] \to \Gamma_k$ for $k \geq K$, where $r_k$ is given and $r_{k+1}$ is to be determined, with the requirements that $r_{k+1} < r_k$ and $r_k \to 0$ as $k \to \infty$. Suppose $k \geq K$, we have the open set $U_k$ and a point $p_k(r_k) \in X$. We can find a path $\Gamma_k$ with the required properties, made up of $\Gamma_k^1, \ldots, \Gamma_k^m$ where $m = m(k)$ and each $\Gamma_k^j$ is either a radial line segment or an arc of a great circle. We must have $r_k^m = r_{k+1}^1$. The
parameterization for $\Gamma^j_k$ is given by $p^j_k : [r^j_k, r^{j+1}_k] \to \Gamma^j_k$, where we are given $r^j_k$ and have to determine $r^{j+1}_k$.

**Case (i):** $\Gamma^j_k$ is an arc of a great circle, say from $u_1\sigma$ to $u_2\sigma$ with $1/C \leq u \leq C$ and the appropriate orientation. By (3.14) and our earlier discussion, on $|x| = r^j_k$ we have $f(x) = h_{u^{n/(n-1)},\sigma_1,A}(x)$ and $\gamma_x(r^j_kx_1) = u_1\sigma$.

From Section 3.1.4, we can let $K = u^{n/(n-1)}$ and $\alpha$ chosen with parity to give the correct direction of spiraling commensurate with the orientation of our piece of great circle, and $|\alpha|$ chosen small enough so that $J_{g(K,\sigma_1,\sigma_2,A,B)}$ is bounded from below, by $2^{-(n+1)/2}$. We then choose $r^{j+1}_k$ so that on $\{x : r^{j+1}_k \leq |x| \leq r^j_k\}$,

$$f(x) = r^j_k g_{K,\sigma_1,\sigma_2,A,B} \left( \frac{x}{r^j_k} \right),$$

and $f(r^{j+1}_kx_1) = u^{n/(n-1)}\sigma_2$. Recall that $B$ is the orthogonal map chosen that guarantees we are spiraling in the correct direction. Then by (3.14) and earlier discussion, we have $\gamma_x(r^{j+1}_kx_1) = u_2\sigma$. Also note that we an choose $\alpha$ small enough so that $f$ has bounded distortion of a constant depending on $C$, by construction of $g_{K,\sigma_1,\sigma_2,A,B}$.

**Case (ii):** $\Gamma^j_k$ is a radial line segment, say from $u_1\sigma$ to $u_2\sigma$ with $u_1, u_2 \in [1/C, C]$. By (3.14) and earlier discussion, on $|x| = r^j_k$ we have $f(x) = h_{u^{n/(n-1)},\sigma,A}(x)$ and $\gamma_x(r^j_kx_1) = u_1\sigma$.

Looking back at our discussion in Section 3.1.2, we can let $K = u_1^{n/(n-1)}$ and $L = u_2^{n/(n-1)}$, and we can choose $s$ and $t$ so that $h_{K,L,\sigma,A}$ is quasiconformal. Choosing $r^{j+1}_k = (r^j_k e^s)/e^t$, we have

$$f(x) = \frac{r^j_k}{e^t} h_{K,L,\sigma,A} \left( \frac{xe^t}{r^j_k} \right),$$
with \( f(r_{k}^{j+1}x_1) = u_2^{n/(n-1)} \sigma \). Then by (3.14) and earlier discussion, we have \( \gamma_{x_1}(r_{k}^{j+1}x_1) = u_2 \sigma \). Also, we have chosen \( s \) and \( t \) so that the distortion depends on a constant in terms of \( C \).

These two cases show how to parameterize each sub-arc of \( \Gamma_k \) and hence inductively how to define a parameterization for \( \gamma_{x_1} \) from \((0, r_K]\). By construction, the obtained map \( f \) has uniformly bounded distortion and hence is quasiconformal.
CHAPTER 4
GENERALIZING KOENIGS LINEARIZATION THEOREM TO
QUASIREGULAR MAPS

Recall that following definition for a fixed point of a quasiregular map to be geometrically attracting:

**Definition 4.0.1.** Suppose $f : \mathbb{R}^n \to \mathbb{R}^n$ is quasiconformal in a neighborhood of 0 with $f(0) = 0$. Then 0 is called a geometrically attracting fixed point of $f$ if there exist constants $r > 0$ and $0 < \lambda < 1$ such that $|f(x)| < \lambda |x|$ whenever $|x| < r$. Clearly $f^m(x) \to 0$ as $m \to \infty$ for $|x| < r$.

Throughout this chapter, we are working under the assumptions that we have a quasiregular mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ with $i(0, f) = 1$, that $T(0, f) = \{g\}$ is simple, and $\tilde{f} \in \text{BLD}$. The condition of $\tilde{f} \in \text{BLD}$ is related to Theorem 2.2.3. Here we are constructing a quasiconformal map $\psi$ that conjugates $f$ to the asymptotic representation of $f$, $\mathcal{D}$, defined in (1.8). First define $E(x) = f(x) - \mathcal{D}(x)$. Also define $\psi_1 = \mathcal{D}^{-1} \circ f$, $\psi_{k+1} = \mathcal{D}^{-1} \circ \psi_k \circ f$, $1 \leq k \in \mathbb{Z}$, and $E_k(x) = \psi_k(x) - x$ for each $k$. Note that $\tilde{E}_k$ is not the Zorich transform of $E_k$ but is the corresponding error term defined by $\tilde{E}_k(X) = \tilde{\psi}_k(X) - X$.

Then, as mentioned in Section 1.2.4.1 we need to make some assumptions to generalize Koenigs Linearization Theorem. In particular assume:

a) 0 is a geometrically attracting fixed point of $f$, with constant $r > 0$ and $0 < \lambda < 1$, ...
b) $|E_1(x)| < c|x|^\alpha$, where $c$ is a positive constant and $\alpha > 1$ such that $L^2\lambda^{\alpha-1} < 1$, and

c) the derivative $[\tilde{D}^{-1}]'(X)$ has the condition

$$\|\tilde{D}^{-1}(X) - \tilde{D}^{-1}(Y)\| \leq L_3|X - Y| \text{ a.e.}$$  \hspace{1cm} (4.1)

where $L_3 \geq 1$, and that

$$\|\tilde{E}'(X)\| \leq c_1 e^{\beta x_n} \text{ a.e.},$$  \hspace{1cm} (4.2)

where $\tilde{E} = \tilde{f} - \tilde{D}$, $c_1$ is a positive constant, and $\beta > 0$ is a constant such that $L^2\lambda^\beta < 1$. We prove a generalization of Koenigs Linearization Theorem in Section 4.5,

**Theorem 1.2.29.** Let $f$ be quasiconformal in a neighborhood of 0 where 0 is a geometrically attracting fixed point of $f$, $T(0, f) = \{g\}$ is simple and $\tilde{f} \in \text{BLD}$, where $D(x) = \rho_f(|x|)g(x/|x|)$, $|E_1(x)| < c|x|^\alpha$, where $c$ is a positive constant and $\alpha > 1$ such that $L^2\lambda^{\alpha-1} < 1$, and the derivative $[\tilde{D}^{-1}]'(X)$ has the condition

$$\|\tilde{D}^{-1}(X) - \tilde{D}^{-1}(Y)\| \leq L_3|X - Y| \text{ a.e.}$$

where $L_3 \geq 1$, and that

$$\|\tilde{E}'(X)\| \leq c_1 e^{\beta x_n} \text{ a.e.},$$

where $c_1$ and $\beta$ are positive constants such that $L^2\lambda^\beta < 1$. Then there exists a quasiconformal map $\psi$ where

$$\psi \circ f = D \circ \psi,$$

in a neighborhood of 0.

In Section 4.4 we show that the maximal dilatations of $\tilde{\psi}_k$ are uniformly bounded, where the four subsections are bounds for infinite series that appears in calculating bounds for
\[ \| \tilde{E}_{k+1}(X) \| \text{ for } k \geq 1. \] We calculate a uniform bound for \( |\tilde{E}_{k+1}(X)| \) in Section 4.3 and transition \( \psi_k \) and \( E_k, k \geq 1 \), to the Zorich transform forms in Section 4.2. Occasionally we will need a linear algebra result with regards to matrix norms which is given and proved in Section 4.1.

### 4.1 Preliminary Linear Algebra Result

The following result is necessary in calculations related to bounding the maximal dilatation of quasiconformal mappings.

**Lemma 4.1.1.** Let \( A \) be an \( n \times n \) matrix acting on \( \mathbb{R}^n \). For all \( \epsilon > 0 \) there exists a \( \delta > 0 \) such that if \( \| A \| < \delta \) then

\[
\frac{\| \text{Id} + A \|^n}{\text{J}_{\text{Id} + A}} < 1 + \epsilon,
\]

where \( \text{Id} \) is the \( n \times n \) identity matrix.

**Proof.** Let \( \epsilon > 0 \) be fixed. Let \( A_{ij} \) denote the entry in the \( i \)th row and \( j \)th column of matrix \( A \). Let us define \( b = n + 2 + \frac{\ln(n!)}{\ln 2} \) and

\[
c(n) = \sum_{k=1}^{n} \binom{n}{k},
\]

then choose \( \delta = c^{-1}(n)2^{-b}\delta^* < 2^{-b}\delta^* < \delta^* \) where \( \delta^* = \min\{\frac{2\epsilon}{3}\epsilon, 1\} \). Since all matrix norms are equivalent norms on finite dimensional linear spaces, for example we can assume we are using either the standard 2-norm or the maximal sum of a column norm, so that when \( \| A \| < \delta \) we have \( |A_{ij}| < \delta \) for all \( 1 \leq i, j \leq n \).
Let $S_n$ be the symmetric group on $n$ elements. Let $\sigma \in S_n$ be a permutation, we use $\sigma_i$ to denote the element in the $i$th position of the permutation. Let $\text{sgn}(\sigma)$ be the sign of the permutation $\sigma$ defined by

$$\text{sgn}(\sigma) = \begin{cases} 1 & \text{if } \sigma \text{ is even} \\ -1 & \text{if } \sigma \text{ is odd} \end{cases}.$$ 

Also let

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases},$$

and $\sigma' = (1, 2, 3, ..., n - 1, n)$. We can define the Jacobian of $\text{Id} + A$ to be

$$J_{\text{Id} + A} = \sum_{\sigma \in S_n} \left( \text{sgn}(\sigma) \prod_{i=1}^{n} (\delta_{i\sigma_i} + A_{i\sigma_i}) \right)$$

$$= \prod_{i=1}^{n}(1 + A_{ii}) + \sum_{\sigma \in S_n \setminus \{\sigma'\}} \left( \text{sgn}(\sigma) \prod_{i=1}^{n} (\delta_{i\sigma_i} + A_{i\sigma_i}) \right)$$

$$= \prod_{i=1}^{n}(1 + A_{ii}) + \sum_{\sigma \in S_n \setminus \{\sigma'\}} \left( \text{sgn}(\sigma) \prod_{i=1}^{n} (\delta_{i\sigma_i} + A_{i\sigma_i}) \right).$$

Then, using the reverse triangle inequality, we can see that the Jacobian of $\text{Id} + A$ has the lower bound

$$J_{\text{Id} + A} \geq 1 - \left| \prod_{i=1}^{n}(1 + A_{ii}) - 1 \right| - \left| \sum_{\sigma \in S_n \setminus \{\sigma'\}} \left( \text{sgn}(\sigma) \prod_{i=1}^{n} (\delta_{i\sigma_i} + A_{i\sigma_i}) \right) \right|.$$

We show that

$$\left| \sum_{\sigma \in S_n \setminus \{\sigma'\}} \left( \text{sgn}(\sigma) \prod_{i=1}^{n} (\delta_{i\sigma_i} + A_{i\sigma_i}) \right) \right| < \frac{\delta^*}{4},$$

where $\delta^*$ is...
and

\[ |\prod_{i=1}^{n}(1 + A_{ii}) - 1| < \frac{\delta^*}{4}, \]

so that

\[ J_{kl+A} > 1 - \frac{\delta^*}{4} - \frac{\delta^*}{4} = \frac{2 - \delta^*}{2}. \]

The sum

\[ \prod_{i=1}^{n}(1 + A_{ii}) - 1 \]

has \(2^n - 1\) terms. Since each \(|A_{ii}| < 2^{-b}\delta^*\) where \(b = n + 2 + \frac{\ln(n!)}{\ln 2}\) for \(1 \leq i \leq n\), then any product of such terms has absolute value less than \(2^{-b}\delta^*\). We have that

\[
\left| \prod_{i=1}^{n}(1 + A_{ii}) - 1 \right| \leq \sum_{i=1}^{2^n-1} \frac{\delta^* 2^{n-1}}{2^b} \leq \frac{\delta^* 2^n}{n! 2^b} \leq \frac{\delta^*}{4}.
\]

Now, note that

\[
\left| \sum_{\sigma \in S_n \setminus \{\sigma'\}} \left( \text{sgn}(\sigma) \prod_{i=1}^{n}(\delta_{i\sigma_i} + A_{i\sigma_i}) \right) \right| \leq \sum_{\sigma \in S_n \setminus \{\sigma'\}} \prod_{i=1}^{n}(\delta_{i\sigma_i} + |A_{i\sigma_i}|).
\]

For a given \(\sigma \in S_n \setminus \{\sigma'\}\),

\[
\prod_{i=1}^{n}(\delta_{i\sigma_i} + |A_{i\sigma_i}|)
\]

has at most \(n - 2\) terms where \(\delta_{i\sigma_i} = 1\). We conclude that

\[
\prod_{i=1}^{n}(\delta_{i\sigma_i} + |A_{i\sigma_i}|) \leq (1 + 2^{-b})^{n-2} 2^{-2b}(\delta^*)^2 < (1 + 2^{-b})^{n-1} 2^{-b}\delta^*.
\]
Also, $S_n$ has $n!$ elements so that $S_n \setminus \{\sigma'\}$ has $n! - 1$ elements, which leads to

$$\sum_{\sigma \in S_n \setminus \{\sigma'\}} \prod_{i=1}^{n} (\delta_{i\sigma_i} + |A_{i\sigma_i}|) \leq \sum_{\sigma \in S_n \setminus \{\sigma'\}} (1 + 2^{-b})^{n-1}2^{-b}\delta^*$$

$$= \delta^*(n! - 1) \frac{(1 + 2^{-b})^{n-1}}{2^b}$$

$$\leq \delta^*n! \left(\frac{2^b + 1}{2^b}\right)^{n-1}2^{-b}$$

$$\leq \delta^*n! \left(\frac{2^b + 2^b}{2^b}\right)^{n-1}2^{-b}$$

$$= \delta^*n!(2)^{n-1}2^{-b}$$

$$= \delta^* \frac{n!2^{n-1}}{2^{n+2+\frac{\ln(n!)}{\ln 2}}} < \frac{\delta^*}{4}. $$

Now that we have

$$J_{Id+A} > 1 - \frac{\delta^*}{4} - \frac{\delta^*}{4} = \frac{2 - \delta^*}{2},$$

then using the triangle inequality and the binomial theorem we have

$$\frac{\|Id + A\|^n}{J_{Id+A}} < \frac{(1 + \|A\|)^n}{\frac{2-\delta^*}{2}}$$

$$\leq \frac{2 \left(1 + \sum_{k=1}^{n} \binom{n}{k} \delta^k\right)}{2 - \delta^*}$$

$$\leq \frac{2(1 + c(n)\delta)}{2 - \delta^*}$$

$$< \frac{2 + 2\delta^*}{2 - \delta^*}$$

$$\leq \frac{2 + \frac{4\epsilon}{3+\epsilon}}{2 - \frac{2\epsilon}{3+\epsilon}} = 1 + \epsilon. $$

$\square$
4.2 Transition to Zorich Coordinates

In working towards convergence of the sequence \((\psi_k)\), let us begin with the behavior of \(\psi_1\) in a neighborhood of 0.

**Lemma 4.2.1. Let us assume we have a). Then there exists an \(m_1 < 0\) such that when \(X_n < m_1, |\tilde{E}_1(X)| < \omega(X)\), where**

\[
\omega(X) = L_1\left|\frac{E_1(Z(X))}{e^{X_n}}\right|,
\]

**recalling that \(L_1\) is the bi-Lipschitz constant related to \(G\).**

**Proof.** From Theorem 1.2.17 and (1.8), we know that since \(f \sim D\), then \(D^{-1} \circ f \sim \text{Id}\), where we have

\[
|E_1(x)| = |D^{-1}(f(x)) - x| = o(|D^{-1}(f(x))| + |x|),
\]
as \(x \to 0\). We can interpret this as for \(\epsilon > 0\) there is a \(\delta > 0\) such that if \(|x| < \delta\), then

\[
\frac{|D^{-1}(f(x)) - x|}{|x| + |D^{-1}(f(x))|} < \epsilon.
\]

This implies that

\[
|D^{-1}(f(x)) - x| < \epsilon|x| + \epsilon|D^{-1}(f(x))|,
\]

which by the reverse triangle-inequality used twice gives us

\[
\frac{1 - \epsilon}{1 + \epsilon}|x| < |D^{-1}(f(x))| < \frac{1 + \epsilon}{1 - \epsilon}|x|.
\]

Hence, \(D^{-1}(f(x)) = x + E_1(x)\), where \(E_1(x) = o(|x|)\).
Let $X = (X_1, ..., X_n)$, and $\bar{X} = (X_1, ..., X_{n-1}, 0)$. The Zorich transform of $D^{-1} \circ f$ is

$$(\tilde{D}^{-1} \circ \tilde{f})(X) = Z^{-1} ((D^{-1} \circ f)(Z(X)))$$

$$= Z^{-1} (Z(X) + E_1(Z(X)))$$

$$= Z^{-1} \left(e^{X_n} \left(\mathcal{G}(\bar{X}) + \frac{E_1(Z(X))}{e^{X_n}}\right)\right),$$

where $|E_1(Z(X))| = o(|e^{X_n}|)$ as $X_n \to -\infty$. Note that as $X_n \to -\infty$ we have that $Z(X) \to 0$.

We also have

$Z^{-1} \left(e^{X_n} \left(\mathcal{G}(\bar{X}) + \frac{E_1(Z(X))}{e^{X_n}}\right)\right) = \mathcal{G}^{-1} \left(\left(\mathcal{G}(\bar{X}) + \frac{E_1(Z(X))}{e^{X_n}}\right)\right)$

$$+ \left(0, ..., 0, \ln e^{X_n} \left|\mathcal{G}(\bar{X}) + \frac{E_1(Z(X))}{e^{X_n}}\right|\right).$$

The $n$th coordinate above is

$$\ln(e^{X_n} |\mathcal{G}(\bar{X}) + E_1(Z(X))/e^{X_n}|) = X_n + \ln |\mathcal{G}(\bar{X}) + E_1(Z(X))/e^{X_n}|.$$

Let $\nu = \nu(X) = |E_1(Z(X))/e^{X_n}|$, where $\nu \to 0$ as $X_n \to -\infty$. By the triangle inequality we have

$$1 - \nu \leq |\mathcal{G}(\bar{X}) + E_1(Z(X))/e^{X_n}| \leq 1 + \nu,$$

so that

$$\ln(1 - \nu) \leq \ln |\mathcal{G}(\bar{X}) + E_1(Z(X))/e^{X_n}| \leq \ln(1 + \nu).$$
The image of the first \(n - 1\) coordinates of (4.3) is given by \(G^{-1}(G(\bar{X}) + E_1(Z(X))/e^{X_n})\).

Since \(G\) is a \(L_1\)-bi-Lipschitz map, we have

\[
|G^{-1}(G(\bar{X}) + E_1(Z(X))/e^{X_n}) - G^{-1}(G(\bar{X}))| \leq L_1|E_1(Z(X))/e^{X_n}|.
\]

Define \(\omega(X) = \max\{L_1\nu(X), |\ln(1 - \nu(X))|, |\ln(1 + \nu(X))|\}\). Clearly \(\omega(X) \to 0\) as \(X_n \to -\infty\) and we have

\[
|(\hat{D}^{-1} \circ \hat{f})(X) - X| \leq \omega(X).
\]

This implies that \((\hat{D}^{-1} \circ \hat{f})(X) = X + \tilde{E}_1(X)\), where \(\tilde{E}_1(X)\) is our error term with the condition \(|\tilde{E}_1(X)| \leq \omega(X)\). Since \(L_1 > 1\) and \(\nu(X) \to 0\) as \(X_n \to -\infty\) we can find \(m_1 < 0\), so that when \(X_n < m_1\) we have

\[
\omega(x) = L_1\nu(x).
\]

Hence, the lemma follows.

\[\square\]

Recall that \(\tilde{\psi}_k = \hat{D}^{-1} \circ \tilde{\psi}_{k-1} \circ \hat{f}\). Our next aim is to find estimates for \(|\tilde{\psi}_k(X) - X|\).

\[\hfill\]

### 4.3 Uniform Bounds for Corresponding Zorich Transform Error Terms

**Lemma 4.3.1.** Suppose that a) holds, then for \(M_1 < 0\) from Theorem 1.2.28, when \(X_n < M_1\) we have \(|\tilde{E}(X)| \leq L|\tilde{E}_1(X)|\). Here \(L\) is the bi-Lipschitz constant related to \(\hat{D}\) from Theorem 1.2.28.
Proof. First note, since we have \( \tilde{f} \sim \tilde{D} \) and

\[
(\tilde{D}^{-1} \circ \tilde{f})(X) = X + \tilde{E}_1(X),
\]

Theorem 1.2.28 gives us

\[
|\tilde{E}(X)| = |\tilde{f}(X) - \tilde{D}(X)| = |(\tilde{D} \circ \tilde{D}^{-1} \circ \tilde{f})(X) - \tilde{D}(X)| \leq L|((\tilde{D}^{-1} \circ \tilde{f})(X) - X| = L|\tilde{E}_1(X)|,
\]

when \( X_n < M \) from Theorem 1.2.28. Hence, \( |\tilde{E}(X)| \leq L|\tilde{E}_1(X)|. \)

\[ \square \]

**Lemma 4.3.2.** Let \( a) \) and \( b) \) hold, and recall \( m_1 \) from Lemma 4.2.1. When \( X_n < m_1 \) we have

\[
|\tilde{E}_1(\tilde{f}^k(X))| \leq L_1 c \lambda^k (\alpha - 1) e^{(\alpha - 1) X_n},
\]

for all \( k \in \mathbb{N} \), where \( L_1 \) is the bi-Lipschitz constant related to \( \mathcal{G} \).

Proof. From Lemma 4.2.1, there is an \( m_1 < 0 \) such that when \( X_n < m_1 \) we have \( \omega(X) = L_1 \nu(X) \) where \( \nu(X) = |E_1(Z(X))/e^{X_n}|. \) Under the assumption of 0 being a geometrically attracting fixed point of \( f \), with \( 0 < \lambda < 1 \), we know that the \( n \)th coordinate of \( \tilde{f} \) is \( \tilde{f}_n(X) \leq \ln \lambda + X_n < m_1 \), so that the \( k \)th iterate of \( \tilde{f} \) has \( n \)th coordinate

\[
\tilde{f}_n^k(X) \leq k \ln \lambda + X_n < m_1.
\]

(4.4)
Since $|E_1(x)| < c|x|^{\alpha}$ by assumption b), we have that

$$\frac{|E_1(Z(X))|}{e^{X_n}} \leq \frac{c|Z(X)|^{\alpha}}{e^{X_n}} = c e^{\alpha X_n} e^{X_n} = c e^{X_n(\alpha - 1)}.$$  It follows from this, (4.4), and $\nu(X) = \frac{|E_1(Z(X))/e^{X_n}|}{e^{X_n}}$ that

$$|\tilde{E}_1(\tilde{f}^k(X))| \leq L_1 \frac{|E_1(Z(\tilde{f}^k(X)))|}{e^{\tilde{f}^k_n(X)}}$$

$$\leq L_1 c e^{\tilde{f}^k_n(X)}(\alpha - 1)$$

$$\leq L_1 c e^{(k \ln |\lambda| + X_n)(\alpha - 1)}$$

$$= L_1 c \lambda^{k(\alpha - 1)} e^{(\alpha - 1) X_n}.$$

\[\square\]

**Proposition 4.3.3.** Suppose that assumptions a), and b) hold. Then there is a constant $M_2 < 0$ such that when $X_n < M_2$ we have that $\tilde{\psi}_k(X) = X + \tilde{E}_k(X)$ with

$$|\tilde{E}_k(X)| \leq \frac{L^2 L_1 c e^{(\alpha-1) X_n}}{1 - L \lambda^{\alpha - 1}},$$

and that the sequence of quasiconformal mappings $(\tilde{\psi}_k)_{k=1}^\infty$ is uniformly convergent.

**Proof.** First we prove the uniform bound for $|\tilde{E}_k(X)|$, and then using the bound we show that $(\tilde{\psi}_k)_{k=1}^\infty$ is uniformly convergent. By hypothesis there is a constant $N$ so that when $|x| < N$ we have that $f$ is quasiconformal. There is a corresponding constant $M$ so that when $X_n < M$ we have that $\tilde{f}$ is quasiconformal. Let $m_1$ be the constant from Lemma 4.2.1, when $X_n < m_1$ we have

$$|\tilde{E}_1(\tilde{f}^k(X))| \leq L_1 c \lambda^{k(\alpha - 1)} e^{(\alpha - 1) X_n}.$$
Let $M_1$ be the constant from Theorem 1.2.28, so that when $X_n < M_1$ we have $\bar{D}$ is $L$-bi-Lipschitz. Let us define $M_2 = \min\{M_1, m_1, M\}$. In particular, since $\bar{D}$ is quasiconformal and $\tilde{f}$ is quasiconformal when $X_n < M_2$, then $\tilde{\psi}_1(X) = (\bar{D}^{-1} \circ \tilde{f})(X)$ is quasiconformal when $X_n < M_2$. Correspondingly, we have that $\tilde{\psi}_{k+1} = \bar{D}^{-1} \circ \tilde{\psi}_k \circ \tilde{f}$ is also quasiconformal when $X_n < M_2$. By the definition of $\tilde{\psi}_{k+1} = \bar{D}^{-1}(\tilde{\psi}_k(\tilde{f}(X))) = X + \tilde{E}_{k+1}(X)$ we have

$$\tilde{\psi}_{k+1}(X) = \bar{D}^{-1}(\tilde{f}(X) + \tilde{E}_k(\tilde{f}(X)))$$

$$= \bar{D}^{-1}(\bar{D}(X) + \tilde{E}(X) + \tilde{E}_k(\tilde{f}(X))).$$

Then by Lemma 4.3.1 and since $\bar{D}$ is $L$-bi-Lipschitz

$$|\tilde{E}_{k+1}(X)| = |\tilde{\psi}_{k+1}(X) - X|$$

$$= |\bar{D}^{-1}(\bar{D}(X) + \tilde{E}(X) + \tilde{E}_k(\tilde{f}(X))) - \bar{D}^{-1}(\bar{D}(X))|$$

$$\leq L|\tilde{E}(X) + \tilde{E}_k(\tilde{f}(X))|.$$

Using Lemmas 4.3.1 and 4.3.2, by induction we have

$$|\tilde{E}_{k+1}(X)| \leq L^2|\tilde{E}_1(X)| + L^2|\tilde{E}(\tilde{f}(X)) + \tilde{E}_{k-1}(\tilde{f}^2(X))|$$

$$\leq L^2 \sum_{i=0}^k L^i|\tilde{E}_1(\tilde{f}^i(X))|$$

$$\leq L^2 L_1 c^l(a^{-1})X_n \sum_{i=1}^k L^i|\lambda|^{i(a-1)}$$

$$= L^2 L_1 c^l(a^{-1})X_n \sum_{i=1}^k (L|\lambda|^{a-1})^i$$

$$\leq \frac{L^2 L_1 c^l(a^{-1})X_n}{1 - L|\lambda|^{a-1}}.$$


Since we have \( L\lambda^{\alpha-1} < 1 \), we conclude that each \( \tilde{E}_k \) is uniformly bounded for \( X_n < M_2 \).

Next let us define \( A = \{ X \in B : X_n < M_2 \} \). Also, since \( \mathbb{R}^n \) is a complete metric space, we know that the quotient space \( B \) is also a complete metric space with respect to the appropriate induced metric. To show that \( \tilde{\psi}_k : A \to B \) is uniformly convergent, we may just use the Cauchy Criterion for uniform convergence, that is \( (\tilde{\psi}_k)_{k=1}^{\infty} \) converges uniformly on a set \( A \) if and only if for each \( \epsilon > 0 \) there is an integer \( N \) such that

\[
\sup_{X \in A} |\tilde{\psi}_p(X) - \tilde{\psi}_q(X)| < \epsilon \text{ if } p, q \geq N.
\]

Given \( \epsilon > 0 \), then since \( L\lambda^{\alpha-1} < 1 \) choose \( N \) so that

\[
\left[ \frac{L^2L_1}{1 - L\lambda^{\alpha-1}} + 1 \right] ce^{(\alpha-1)M_2(L\lambda^{\alpha-1})N-1} < \epsilon.
\]

Let \( X \in A \), then for \( N \leq p \leq q \) we have by using the fact that \( \tilde{D} \) is \( L \)-bi-Lipschitz \( p - 1 \) times that

\[
|\tilde{\psi}_q(X) - \tilde{\psi}_p(X)| = \left| \tilde{D}^{-1} \left( \tilde{D}(X) + \tilde{E}(X) + \tilde{E}_{q-1}(\tilde{f}(X)) \right) - \tilde{D}^{-1} \left( \tilde{D}(X) + \tilde{E}(X) + \tilde{E}_{p-1}(\tilde{f}(X)) \right) \right|
\leq L \left| \tilde{E}_{q-1}(\tilde{f}(X)) - \tilde{E}_{p-1}(\tilde{f}(X)) \right|
= L \left| \tilde{\psi}_{q-1}(\tilde{f}(X)) - \tilde{\psi}_{p-1}(\tilde{f}(X)) \right|
\leq L^{p-1} \left| \tilde{E}_{q-p+1}(\tilde{f}^{p-1}(X)) - \tilde{E}_1(\tilde{f}^{p-1}(X)) \right|
\leq L^{p-1} \left( |\tilde{E}_{q-p+1}(\tilde{f}^{p-1}(X))| + |\tilde{E}_1(\tilde{f}^{p-1}(X))| \right).
\]
Since $L\lambda^{a-1} < 1$ and $N - 1 \leq p - 1$ we know that $(L\lambda^{a-1})^{p-1} \leq (L\lambda^{a-1})^{N-1}$. Using this, Lemma 4.3.2, assumption a), $X_n < M_2$, and

$$|\tilde{E}_k(X)| \leq \frac{L^2 L_1 c e^{(a-1)X_n}}{1 - L\lambda^{a-1}},$$

we have that

$$L^{p-1} \left( |\tilde{E}_{q-p+1}(\tilde{f}^{p-1}(X))| + |\tilde{E}_1(\tilde{f}^{p-1}(X))| \right) \leq \frac{L^{p-1} L^2 L_1 c e^{(a-1)M_2(p-1)(a-1)}}{1 - L\lambda^{a-1}} + L^{p-1} c \lambda^{(p-1)(a-1)} e^{(a-1)M_2} \leq \left( \frac{L^2 L_1}{1 - L\lambda^{a-1}} + 1 \right) c e^{(a-1)M_2(L\lambda^{a-1})^{N-1}}.$$

Since these calculations hold true for all $X \in A$, we have that

$$\sup_{X \in A} \left| \tilde{\psi}_q(X) - \tilde{\psi}_p(X) \right| \leq \left( \frac{L^2 L_1}{1 - L\lambda^{a-1}} + 1 \right) c e^{(a-1)M_2(L\lambda^{a-1})^{N-1}} < \epsilon.$$

Therefore, the sequence of functions $(\tilde{\psi}_k)_{k=1}^\infty$ is uniformly convergent on $A$ by the Cauchy criterion.

\[ \square \]

### 4.4 Maximal Dilatations of $\tilde{\psi}_k$ are Uniformly Bounded

**Proposition 4.4.1.** Suppose that a), b) and c) hold. Then there is an $M_4 < 0$ such that the maximal dilatation, $K(\tilde{\psi}_k)$, of $\tilde{\psi}_k$ is uniformly bounded when $X_n < M_4$. In particular, when $\tilde{\psi}_k$ is restricted to the set $A_t = \{ X : X_n < t \}$ then $K(\tilde{\psi}_k|_{A_t}) \to 1$ uniformly in $k$ as $t \to -\infty$. 

Proof. Our strategy is the following. From [20, Section I.2] we know that $K_I(\tilde{\psi}_k) \leq (K_O(\tilde{\psi}_k))^{n-1}$. We show that $K_O(\tilde{\psi}_k)$ is uniformly bounded for all $k$, so that the maximal dilatation $K(\tilde{\psi}_k) = \max\{K_O(\tilde{\psi}_k), K_I(\tilde{\psi}_k)\}$ is uniformly bounded. Recall that $K_O(\tilde{\psi}_k)$ is the smallest $K$ such that

$$\|\tilde{\psi}_k'(X)\|^n \leq K J_{\tilde{\psi}_k}^{-1}(X) \text{ a.e.},$$

or

$$\frac{\|\tilde{\psi}_k'(X)\|^n}{J_{\tilde{\psi}_k}^{-1}(X)} \leq K \text{ a.e.}$$

Then $K_O(\tilde{\psi}_k)$ is uniformly bounded if $\|\tilde{\psi}_k'(X)\|^n$ is bounded a.e. from above and $J_{\tilde{\psi}_k}(X)$ is bounded a.e from below for each $k$. In fact we show that

$$\frac{\|\tilde{\psi}_k'(X)\|^n}{J_{\tilde{\psi}_k}^{-1}(X)}$$

is bounded a.e. by the same function depending only on $X_n$ and the data associated to $\tilde{f}$ and $\tilde{D}$ for each $k$. For a fixed $k$, the set of measure zero where (4.5) does not exist may hold on a countable union of sets of measure zero which we know still has measure zero. By definition we know $\tilde{\psi}_k(X) = X + \tilde{E}_k(X)$. Then, wherever $\tilde{\psi}_k'$ exists we know that $\tilde{\psi}_k'(X) = \text{Id} + \tilde{E}_k'(X)$, where $\text{Id}$ is the $n \times n$ identity matrix. Using the triangle and reverse triangle inequality, we know that

$$\|\text{Id} - \|\tilde{E}_k'(X)\| \leq \|\tilde{\psi}_k'(X)\| \leq \|\text{Id}\| + \|\tilde{E}_k'(X)\|. $$

Step 1 is to show that for each $k$, $\|\tilde{E}_k'(X)\|$ is bounded a.e. In fact where it’s defined $\|\tilde{E}_k'(X)\| \to 0$ as $X_n \to -\infty$ so that $\|\tilde{\psi}_k'(X)\| \to 1$ as $X_n \to -\infty$. Step 2 is to invoke the matrix norm result, Lemma 4.1.1 to claim our conclusion.
Step 1: First, note that \( \bar{E}_k(X) = \bar{\psi}_k(X) - X \). Also note that \( [\bar{D}^{-1}]'(\bar{D}(X))\bar{D}'(X) = Id \), where the derivative exists. We know that \( \bar{f}(X) = \bar{D}(X) + \bar{E}(X) \). Using Theorem 1.2.28, we have an \( M_1 < 0 \) such that \( \bar{D}(X) \) is \( L \)-bi-Lipschitz when \( X_n < M_1 \). Using the fact that \( \bar{D}(X) \) is \( L \)-bi-Lipschitz and (1.15) from assumption \( c) \), we have the following calculations a.e.:

\[
\|\bar{E}'_1(X)\| = \|(\bar{D}^{-1} \circ \bar{f})'(X) - Id\|
\]
\[
= \|[\bar{D}^{-1}]'(\bar{D}(X) + \bar{E}(X))(\bar{D}'(X) + \bar{E}'(X)) - Id\|
\]
\[
\leq \|(\bar{D}^{-1})'(\bar{D}(X) + \bar{E}(X))\bar{D}'(X) - [\bar{D}^{-1}]'(\bar{D}(X))\bar{D}'(X)\|
\]
\[
+ \|(\bar{D}^{-1})'(\bar{D}(X) + \bar{E}(X))\||\bar{E}'(X)||
\]
\[
\leq \|(\bar{D}^{-1})'(\bar{D}(X) + \bar{E}(X)) - [\bar{D}^{-1}]'(\bar{D}(X))\||\bar{D}'(X)|| + L\|\bar{E}'(X)||
\]
\[
\leq L_3|\bar{D}(X) + \bar{E}(X) - \bar{D}(X)|L + L\|\bar{E}'(X)||
\]
\[
= LL_3|\bar{E}(X)| + L\|\bar{E}'(X)||.
\]

Lemma 4.3.1 tells us that \( |\bar{E}(X)| \leq L|\bar{E}_1(X)| \), so that

\[
\|\bar{E}'_1(X)\| \leq L^2L_3|\bar{E}_1(X)| + L\|\bar{E}'(X)|| \text{ a.e.} \tag{4.6}
\]

Similarly, using the fact that \( \bar{D}(X) \) is \( L \)-bi-Lipschitz and (1.15) from assumption \( c) \), we have the following calculations for \( \bar{E}'_{k+1}(X) \) a.e.:
\[
\| \tilde{E}_{k+1}^\prime(X) \| = \| (\tilde{D}^{-1} \psi_k \tilde{f})'(X) - (X)' \| \\
= \| [\tilde{D}^{-1}]'(\tilde{D}(X) + \tilde{E}(X) + \tilde{E}_k(\tilde{f}(X))) \tilde{D}'(X) - \text{Id} \\
+ [\tilde{D}^{-1}]'(\tilde{f}(X) + \tilde{E}_k(\tilde{f}(X))) \tilde{E}'(X) \\
+ [\tilde{D}^{-1}]'(\tilde{f}(X) + \tilde{E}_k(\tilde{f}(X))) \tilde{E}'_k(\tilde{f}(X)) [\tilde{D}'(X) + \tilde{E}'(X)] \| \\
\leq \| [\tilde{D}^{-1}]'(\tilde{D}(X) + \tilde{E}(X) + \tilde{E}_k(\tilde{f}(X))) \tilde{D}'(X) - [\tilde{D}^{-1}]'(\tilde{D}(X)) \tilde{D}'(X) \| \\
+ \| [\tilde{D}^{-1}]'(\tilde{f}(X) + \tilde{E}_k(\tilde{f}(X))) \| \| \tilde{E}'(X) \| \\
+ \| [\tilde{D}^{-1}]'(\tilde{f}(X) + \tilde{E}_k(\tilde{f}(X))) \| \| \tilde{E}'_k(\tilde{f}(X)) \| \| \tilde{D}'(X) + \tilde{E}'(X) \| \\
\leq \| [\tilde{D}^{-1}]'(\tilde{D}(X) + \tilde{E}(X) + \tilde{E}_k(\tilde{f}(X))) - [\tilde{D}^{-1}]'(\tilde{D}(X)) \| \| \tilde{D}'(X) \| \\
+ L \| \tilde{E}'(X) \| + L \| \tilde{E}'_k(\tilde{f}(X)) \| \| \tilde{D}'(X) + \tilde{E}'(X) \| \\
\leq LL_3 |\tilde{E}(X) + \tilde{E}_k(\tilde{f}(X))| + L(L + \| \tilde{E}'(X) \|) \| \tilde{E}'_k(\tilde{f}(X)) \| + L \| \tilde{E}'(X) \|. 
\]

Then for \( k + 1 = 2 \) we have

\[
\| \tilde{E}_2'(X) \| \leq LL_3 |\tilde{E}(X) + \tilde{E}_1(\tilde{f}(X))| + L(L + \| \tilde{E}'(X) \|) \| \tilde{E}'_1(\tilde{f}(X)) \| + L \| \tilde{E}'(X) \|. 
\]
It follows by induction that

\[
\|\tilde{E}_{k+1}(X)\| \leq LL_3 \sum_{i=0}^{k-1} \left[ L^i \|\tilde{E}'(\tilde{f}^i(X)) + \tilde{E}_{k-i}(\tilde{f}^{i+1}(X))\| \prod_{j=0}^{i-1} \left( L + \|\tilde{E}'(\tilde{f}^j(X))\| \right) \right] \\
+ L^k \|\tilde{E}'_1(\tilde{f}^k(X))\| \prod_{j=0}^{k-1} \left( L + \|\tilde{E}'(\tilde{f}^j(X))\| \right) \\
+ L \sum_{i=0}^{k-1} \left[ L^i \|\tilde{E}'(\tilde{f}^i(X))\| \prod_{j=0}^{i-1} \left( L + \|\tilde{E}'(\tilde{f}^j(X))\| \right) \right] \\
\leq LL_3 \sum_{i=0}^{k-1} \left[ L^{2i} \|\tilde{E}'(\tilde{f}^i(X)) + \tilde{E}_{k-i}(\tilde{f}^{i+1}(X))\| \prod_{j=0}^{i-1} \left( 1 + \|\tilde{E}'(\tilde{f}^j(X))\|/L \right) \right] \\
+ L^{2k} \|\tilde{E}'_1(\tilde{f}^k(X))\| \prod_{j=0}^{k-1} \left( 1 + \|\tilde{E}'(\tilde{f}^j(X))\|/L \right) \\
+ L \sum_{i=0}^{k-1} \left[ L^{2i} \|\tilde{E}'(\tilde{f}^i(X))\| \prod_{j=0}^{i-1} \left( 1 + \|\tilde{E}'(\tilde{f}^j(X))\|/L \right) \right] \text{ a.e.}
\]

For \( i = 0 \), we let

\[
\prod_{j=0}^{-1} \left( L + \|\tilde{E}'(\tilde{f}^j(X))\| \right) = 1.
\]

Since \( \|\tilde{E}'(\tilde{f}^j(X))\|/L > 0 \) for all \( j \geq 0 \), we know that

\[
1 \leq \prod_{j=0}^{k-1} \left( 1 + \|\tilde{E}'(\tilde{f}^j(X))\|/L \right) \leq \prod_{j=0}^{\infty} \left( 1 + \|\tilde{E}'(\tilde{f}^j(X))\|/L \right).
\]
Also, \( \prod_{j=0}^{\infty} (1 + \| \tilde{E}'(\tilde{f}^i(X)) \|/L) \) is absolutely convergent a.e. to a finite value \( q \), if and only if \( \sum_{j=0}^{\infty} \| \tilde{E}'(\tilde{f}^i(X)) \|/L \) is absolutely convergent a.e. We refer to Knopp, [13, Section 28] for more information on the convergence of infinite products. By assumption \( c) \) we have

\[
\frac{1}{L} \sum_{j=0}^{\infty} \| \tilde{E}'(\tilde{f}^i(X)) \| \leq \frac{c_1}{L} \sum_{j=0}^{\infty} e^{\beta \tilde{f}_k(X)} \\
\leq \frac{c_1}{L} \sum_{j=0}^{\infty} e^{j \ln \lambda + \beta X_n} \\
= \frac{c_1 e^{\beta X_n}}{L} \sum_{j=0}^{\infty} (\lambda^j) 
\]

which is a convergent geometric series a.e. since \( 0 < \lambda < 1 \). Therefore, the corresponding infinite product is absolutely convergent a.e. to a constant \( q \geq 1 \) depending on \( X_n \). Pulling \( q \) out of the infinite sum, we now have

\[
\| \tilde{E}'_{k+1}(X) \| \leq q LL_3 \sum_{i=0}^{k-1} \left[ L^{2i} \| \tilde{E}'(\tilde{f}^i(X)) \| + \tilde{E}_{k-1}(\tilde{f}^{i+1}(X)) \right] \\
+ q L^{2k} \| \tilde{E}'_1(\tilde{f}^k(X)) \| + q L \sum_{i=0}^{k-1} \left[ L^{2i} \| \tilde{E}'(\tilde{f}^i(X)) \| \right] \text{ a.e.} \tag{4.7}
\]

Soon we need to invoke Proposition 4.3.3. In anticipation, we let \( M_2 \) be the constant from Proposition 4.3.3 such that when \( X_n < M_2 \) we have

\[
| \tilde{E}_{k+1}(X) | \leq \frac{L^2 L_1 c e^{(\alpha-1)X_n}}{1 - L \lambda^{\alpha-1}}.
\]

Recall from the proof of Proposition 4.3.3 that \( M_2 \leq M_1 \) and \( M_2 \leq m_1 \), where \( m_1 \) is the constant value from Lemma 4.3.2 so that when \( X_n < M_2 \) we have

\[
| \tilde{E}_1(\tilde{f}^k(X)) | \leq L_1 c \lambda^{k(\alpha-1)} e^{(\alpha-1)X_n}.
\]
Recall that $M_1$ is the bound where $\mathcal{D}$ is $L$-bi-Lipschitz when $X_n < M_1$. For now on, we have $X_n < M_2$. In our effort to bound (4.7), on the right hand side we bound $\sum_{i=0}^{k-1} L^{2i} |\tilde{E}(\tilde{f}^i(X))|$, $\sum_{i=0}^{k-1} \left[ L^{2i} |\tilde{E}_{k-i}(\tilde{f}^{i+1}(X))| \right]$, $qL^{2k} \|\tilde{E}_1(f^k(X))\|$, and $qL \sum_{i=0}^{k-1} \left[ L^{2i} \|\tilde{E}'(\tilde{f}^i(X))\| \right]$ individually, and then bring all the parts together for a final bound on (4.7).

### 4.4.1 Bound for $\sum_{i=0}^{k-1} L^{2i} |\tilde{E}(\tilde{f}^i(X))|$ 

**Lemma 4.4.2.** Suppose assumptions a) and b) hold. Then when $X_n < m_1$, where $m_1$ is the constant from Lemma 4.2.1, we have for all $k \geq 1$

$$\sum_{i=0}^{k-1} \left( L^{2i} |\tilde{E}(\tilde{f}^i(X))| \right) \leq \frac{LL_1ce^{(\alpha-1)X_n}}{1 - L^2 \lambda^{\alpha-1}}.$$ 

**Proof.** By using Lemma 4.3.1 we have

$$\sum_{i=0}^{k-1} \left( L^{2i} |\tilde{E}(\tilde{f}^i(X))| \right) \leq L \sum_{i=0}^{\infty} \left( L^{2i} |\tilde{E}_1(\tilde{f}^i(X))| \right).$$ 

Lemma 4.3.2 implies

$$L \sum_{i=0}^{\infty} \left( L^{2i} |\tilde{E}_1(\tilde{f}^i(X))| \right) \leq L \sum_{i=0}^{\infty} \left( L^{2i} L_1c \lambda^i e^{(\alpha-1)X_n} \right) = LL_1ce^{(\alpha-1)X_n} \sum_{i=0}^{\infty} (L^2 \lambda^{\alpha-1})^i.$$ 

Since we are assuming that $L^2 \lambda^{\alpha-1} < 1$, the series converges, that is

$$LL_1ce^{(\alpha-1)X_n} \sum_{i=0}^{\infty} (L^2 \lambda^{\alpha-1})^i = \frac{LL_1ce^{(\alpha-1)X_n}}{1 - L^2 \lambda^{\alpha-1}}.$$ 

\[\square\]
4.4.2 Bound for $\sum_{i=0}^{k-1} \left[ L^{2i} |\tilde{E}_{k-i}(\tilde{f}^{i+1}(X))| \right]$

**Lemma 4.4.3.** Suppose assumptions a), b), and c) hold. Then when $X_n < M_2$ we have for $k \geq 1$ that

$$ \sum_{i=0}^{k-1} \left[ L^{2i} |\tilde{E}_{k-i}(\tilde{f}^{i+1}(X))| \right] \leq \frac{\lambda L^2 L_1 c e^{(\alpha-1)X_n}}{(1 - L\lambda^{\alpha-1})(1 - L^2\lambda^{\alpha-1})}, $$

where $M_2$ is the constant from Proposition 4.3.3.

**Proof.** Proposition 4.3.3 tells us that when $X_n < M_2$ we have

$$ |\tilde{E}_{k+1}(X)| \leq \frac{L^2 L_1 c e^{(\alpha-1)X_n}}{1 - L\lambda^{\alpha-1}}. $$

This bound, assumption a) and the fact that $\tilde{f}_n(X) \leq \ln \lambda + X_n$ leads to the following calculations:

$$ \sum_{i=0}^{k-1} \left[ L^{2i} |\tilde{E}_{k-i}(\tilde{f}^{i+1}(X))| \right] \leq \sum_{i=0}^{k-1} \left[ L^{2i} L^2 L_1 c e^{(\alpha-1)\tilde{f}_n^{i+1}(X)} \right] \leq \frac{\lambda L^2 L_1 c e^{(\alpha-1)X_n}}{1 - L\lambda^{\alpha-1}} \sum_{i=0}^{k-1} (L^2\lambda^{\alpha-1})^i \leq \frac{\lambda L^2 L_1 c e^{(\alpha-1)X_n}}{1 - L\lambda^{\alpha-1}} \sum_{i=0}^{\infty} (L^2\lambda^{\alpha-1})^i \leq \frac{\lambda L^2 L_1 c e^{(\alpha-1)X_n}}{(1 - L\lambda^{\alpha-1})(1 - L^2\lambda^{\alpha-1})}. $$

\qed
4.4.3 Bound for \( qL^{2k}\|\tilde{E}'(\tilde{f}^k(X))\| \)

**Lemma 4.4.4.** Suppose assumptions a), b) and c) hold. Then when \( X_n < m_1 \) we have for all \( k \geq 1 \) that

\[
qL^{2k}\|\tilde{E}'(\tilde{f}^k(X))\| \leq qL^2L_3L_1e^{(\alpha-1)X_n} + qLc_1e^{\beta X_n} \text{ a.e.,}
\]

(4.8)

where \( m_1 \) is the constant from Lemma 4.2.1.

**Proof.** Using (4.6), we have that

\[
qL^{2k}\|\tilde{E}'(\tilde{f}^k(X))\| \leq qL^2 \left(L^2L_3|\tilde{E}_1(\tilde{f}^k(X))| + L\|\tilde{E}'(\tilde{f}^k(X))\|\right) \text{ a.e.}
\]

Also recall that since 0 is a geometrically attracting fixed point of \( f \), we have that the \( n \)th coordinate of \( \tilde{f}(X) \) is \( \tilde{f}_n(X) \leq \ln \lambda + X_n \). Using this, Lemma 4.3.2, and assumptions b) and c), give us

\[
qL^{2k} \left(L^2L_3|\tilde{E}_1(\tilde{f}^k(X))| + L\|\tilde{E}'(\tilde{f}^k(X))\|\right) \leq qL^{2k} \left(L^2L_3L_1c\lambda^{k\alpha}(\alpha-1)e^{(\alpha-1)X_n} + Lc_1e^{\beta \tilde{f}_n(X)}\right)
\]

\[
\leq qL^{2k}L^2L_3L_1c\lambda^{k\alpha}(\alpha-1)e^{(\alpha-1)X_n} + qL^{2k}Lc_1e^{\beta(k\ln \lambda + X_n)}
\]

\[
= qL^{2k}L_3L_1c(e^{(\alpha-1)X_n}L^2\lambda^{\alpha-1})^k + qLc_1e^{\beta X_n}(L^2\lambda^\beta)^k
\]

\[
\leq qL^{2k}L_3L_1c(e^{(\alpha-1)X_n} + qLc_1e^{\beta X_n} \text{ a.e.}
\]

\[
\square
\]
4.4.4 Bound for \( qL \sum_{i=0}^{k-1} \left[ L^{2i} \| \tilde{E}'(\tilde{f}^i(X)) \| \right] \)

**Lemma 4.4.5.** Suppose assumptions a) and c) hold. Then we have that

\[
qL \sum_{i=0}^{k-1} \left[ L^{2i} \| \tilde{E}'(\tilde{f}^i(X)) \| \right] \leq \frac{qLc_1e^{\beta X_n}}{1 - L^2\lambda^\beta} \text{ a.e.}
\]

**Proof.** Note that

\[
qL \sum_{i=0}^{k-1} \left[ L^{2i} \| \tilde{E}'(\tilde{f}^i(X)) \| \right] \leq qL \sum_{i=0}^{\infty} \left[ L^{2i} \| \tilde{E}'(\tilde{f}^i(X)) \| \right].
\]

Using assumptions c) and that \( \tilde{f}_n(X) < \ln \lambda + X_n \), give us

\[
qL \sum_{i=0}^{\infty} \left[ L^{2i} \| \tilde{E}'(\tilde{f}^i(X)) \| \right] \leq qLc_1 \sum_{i=0}^{\infty} \left[ L^{2i} e^{\beta \tilde{f}_n(X)} \right]
\]

\[
\leq qLc_1 \sum_{i=0}^{\infty} \left[ L^2 \lambda^\beta \right]^i
\]

\[
= \frac{qLc_1e^{\beta X_n}}{1 - L^2\lambda^\beta} \text{ a.e.}
\]

Putting Lemmas 4.4.2, 4.4.3, 4.4.4, and 4.4.5 together, (4.7) becomes

\[
\| \tilde{E}'_{k+1}(X) \| \leq qLL_3 \left( \frac{LL_1ce^{(\alpha-1)X_n}}{1 - L^2\lambda^{\alpha-1}} + \frac{\lambda L_1ce^{(\alpha-1)X_n}}{(1 - L\lambda^{\alpha-1})(1 - L^2\lambda^{\alpha-1})} \right)
\]

\[
+ qL^2L_3L_1ce^{(\alpha-1)X_n} + qLc_1e^{\beta X_n} + \frac{qLc_1e^{\beta X_n}}{1 - L^2\lambda^\beta} \text{ a.e.}
\]
Since $\alpha - 1, \beta > 0$ and the bound for $\|\tilde{E}_{k+1}'(X)\|$ depends only on $X_n$ and the data associated to $\tilde{f}$ and $\tilde{D}$, since $X_n < M_2$ and the exponential function is an increasing function, we conclude that

$$\|\tilde{E}_{k+1}'(X)\| \leq Ce^{dX_n} \text{ a.e.,}$$

for some constants $C, d > 0$. In other words, $\|\tilde{E}_{k+1}'(X)\|$ is bounded a.e. by the same function for each $k \geq 0$. Also, one can observe that $\|\tilde{E}_{k+1}'(X)\| \to 0$ a.e. as $X_n \to -\infty$ since the bound only depends on $X_n$.

**Step 2:** Recall that $\tilde{\psi}_k'(X) = \text{Id} + \tilde{E}_{k}'(X)$. Lemma 4.1.1, tells us for all $\epsilon > 0$ that there is $\delta > 0$ such that if $\|\tilde{\psi}_k'(X)\| < \delta$ a.e then

$$\frac{\|\tilde{\psi}_k'(X)\|^n}{J_{\tilde{\psi}_k(X)}} < 1 + \epsilon \text{ a.e.}$$

From Step 1 we know that $\|\tilde{E}_{k+1}'(X)\| \to 0$ a.e. as $X_n \to -\infty$ for each $k \geq 0$. Then there is an $M_3$ such that $X_n < M_3$ implies that $\|\tilde{E}_{k+1}'(X)\| < \delta$ a.e. Note that the same $M_3$ works for all $k$ since $\|\tilde{E}_{k+1}'(X)\|$ is bounded a.e. by the same function involving only $X_n$. Hence $K_O(\tilde{\psi}_k) < 1 + \epsilon$ for all $k \geq 0$ when $X_n < M_4 = \min\{M_2, M_3\}$.

Therefore, the maximal dilatation $K(\tilde{\psi}_k) < K_f(\tilde{\psi}_k)^{n-1} < (1 + \epsilon)^{n-1}$ for all $k \geq 0$ when $X_n < M_4 = \min\{M_2, M_3\}$. In fact, when $\tilde{\psi}_k(X)$ is restricted to the set $A_t = \{X : X_n < t\}$ then $K(\tilde{\psi}_k|_{A_t}) \to 1$ uniformly as $t \to -\infty$.

### 4.5 Proof of Theorem 1.2.29

Now we are ready to prove our generalization of Koenigs Linearization Theorem.
Proof. Let $M_2$ be the constant from Proposition 4.3.3, so that when $X_n < M_2$ we have a uniformly convergent sequence of quasiconformal mappings $(\tilde{\psi}_k)_{k=1}^\infty$. Recall that $B$ is the fundamental set of the quotient space related to the bi-Lipschitz map $g$ used to define $Z$. Using $B$ as the domain of $Z$ we have that $Z : B \to \mathbb{R}^n \setminus \{0\}$ and $Z^{-1} : \mathbb{R}^n \setminus \{0\} \to B$ are both continuous mappings. This along with $(\tilde{\psi}_k)_{k=1}^\infty$ being uniformly convergent implies that $(\psi_k)_{k}$ is uniformly convergent as well. Also, let $M_4$ be the constant from Proposition 4.4.1 so that the maximal dilatation of $\tilde{\psi}_k(X)$ is uniformly bounded when $X_n < M_4$. Let us define $M = \min\{M_2, M_4\}$. Proposition 4.4.1 tells us that $\tilde{\psi}_k$ are uniformly $K$-quasiconformal for some $K > 1$ when $X_n < M$, then since $\psi_k = Z \circ \tilde{\psi}_k \circ Z^{-1}$ there is a constant $m$ so that $(\psi_k)_{k}$ is uniformly $K'$ quasiconformal for $|x| < m$. By Theorem 1.2.7 we have that $\psi = \lim_{k \to \infty} \psi_k$ is also $K'$-quasiconformal. Therefore, $f$ is conjugate to $\mathcal{D}$ by a quasiconformal map $\psi$ in a neighborhood of 0.

$\square$
REFERENCES


