Implementation of a 2D Bond Based Peridynamic Fatigue Model in FEM

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ABSTRACT

IMPLEMENTATION OF A 2D BOND-BASED PERIDYNAMIC FATIGUE MODEL IN FEM

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A relatively new fracture theory known as peridynamics has been developed which uses a nonlocal theory to describe crack growth, resolving many of the issues associated to the classical continuum theory. Internal material interactions are described via a family of bonds capable of exerting force over a macro-scale region of influence known as the horizon. Bond damage accumulates naturally due to repeated cyclic or critical loading and damage precipitates as a growing crack as bonds reach a critical damage and are severed. In the present work a modified compact tension specimen subjected to cyclic loading is modeled with the 2D peridynamic theory implemented in FEM software. A bond-based fatigue damage model is incorporated via user subroutine to model bond damage and crack growth. A constant stress intensity factor (SIF) at the crack tip is maintained via a feedback loop which evaluates the SIF via the nonlocal J-integral. Results of the crack path spatial position and damage accumulation rate will be validated against empirical data. The method and subroutines established will provide a basis for further validation of the theory and its applications.
IMPLEMENTATION OF A 2D BOND-BASED PERIDYNAMIC FATIGUE MODEL IN FEM

BY

KYLE MANSFIELD
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A THESIS SUBMITTED TO THE GRADUATE SCHOOL IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE MASTER OF SCIENCE

DEPARTMENT OF MECHANICAL ENGINEERING

Thesis Director:
Jenn Terng-Gau
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Chapter 1. INTRODUCTION

Numerous advances have occurred in the field of structural mechanics over the last century but the field of fracture mechanics has been plagued by limitations in describing the material response near a crack tip. Linear elastic fracture mechanics (LEFM) is hindered by stress singularity at a crack tip or other discontinuity. Utilizing traditional finite element approach leads to the requirement of continuous remeshing and specialized elements at the crack tip. Enhancements to traditional FEM such as cohesive zone elements and the extended finite element method (XFEM) are satisfactory in certain cases but not all.

The use of cohesive zone elements in FEM, first proposed by Hillerborg [1] builds off of the cohesive zone method introduced by Dugdale [2] and Barenblatt [3]. This method utilizes a traction-separation law which guides the crack propagation process. The method has shown positive results for both a FEM approach [4] and for meshless schemes [5]. However, this method is best suited for crack paths which are known in advance such as at material interfaces or layer boundaries. The simulated crack path is highly sensitive to the mesh pattern [6] as cracks are only able to propagate along element boundaries.

A second approach which eliminates the need for remeshing and allows cracks to propagate through elements is XFEM. XFEM was first introduced in 1999 [7, 8] as an adaptation of the finite element method. In this method a property known as partition of unity allows for cracks to grow through elements rather than along boundaries by utilizing displacement enrichment functions [9]. As was shown in [10] the accuracy of the results for this method are
dependent upon the choice of propagation criteria. A relatively new theory known as peridynamics has been developed which uses a nonlocal theory to describe crack growth, resolving many of the issues associated to the classical continuum theory including the need for remeshing or external growth criteria.

Peridynamics is a nonlocal theory in which equilibrium equations are formulated in terms of integro-differential equations as opposed to the differential equations of the classical theory. The theory is traditionally a meshless numerical scheme in which a material body is subdivided into collocation points representing differential material volumes. Material volumes exert forces on all neighboring material volumes within a boundary known as the horizon. These internal forces are represented as “bonds” which are capable of accumulating damage due to a critical load or repeated cyclic load. Bonds are removed as damage accumulates allowing for a natural progression of macro damage (visible as a crack).

The peridynamic theory has been applied to a large number of fields in structural mechanics, heat transfer, and diffusion processes. As peridynamics is a nonlocal theory it does not have the same efficiency of solution as does the traditional finite element method which lends itself to be well-suited to studying fracture and damage modeling. One of the methods to improve the economy of peridynamic solutions has been to pair the peridynamic theory with the finite element method. It was first shown in [11] that the “bonds” which transmit force between collocation points can be suitably modeled in a commercial FEM framework using truss elements. This method termed has been successfully utilized in multiple cases. Beckmann, et. al utilized the method to model the damage that occurs from thermal loading at the interface of a bimetal strip [12]. Yolum, et. al who described the method as PDIFEA (peridynamics implemented in finite element analysis) incorporated peridynamics into the commercial finite
element software Abaqus to model ductile failure of a notched plate [13]. The flexibility of PDIFEA was shown by Rokkam, et. al. [14] who utilized user subroutines in Abaqus/Explicit to create user-defined elements for peridynamics. PDIFEA has also been utilized to model highly dynamic events as was done for shock and vibration simulations of electronics [15].

Much of the early work in peridynamics focused on dynamic brittle fracture [16-18]. However, more recently some focus has been applied to quasi-static failure processes such as fatigue and incremental crack growth [19-21]. However, multiple fatigue models have been proposed which allow complex crack growth to be modeled with peridynamics. Fatigue models have been proposed which directly degrade peridynamic material properties [22, 23] as well as introduce a ‘remaining life’ variable which allows the initiation, growth, and final failure phase to be modeled [24]. Zhang, et. al utilized the remaining life approach to model crack paths in modified CT specimens and simulation results correlated very well with experimental results [20]. In [21], Gharehbagh utilized the remaining life fatigue model and the PDIFEA approach to model crack initiation in rails. The peridynamic approach has also been utilized successfully for complex material response [25, 26].

The present research focuses on the implementation of a peridynamic fatigue model into an existing commercial FEM software. As the theory is relatively new, using FEM as a solution framework allows acceleration of the validation process to occur. FEM solutions have previously been undertaken for dynamic brittle fracture [10] and ductile fracture [13]. In the present work a compact tension specimen subject to repeated cyclic loading is modeled using the peridynamic theory implemented in FEM. A fatigue damage parameter known as remaining life is implemented at the bond level via user subroutine. A feedback loop to maintain constant stress intensity factor throughout the test is implemented via evaluation of the non-local J-integral.
Results are compared against empirical data to evaluate the trajectory of the growing crack and the damage accumulation rate.
Chapter 2. BASICS OF THE PERIDYNAMIC THEORY

2.1 The Peridynamic Equation of Motion

Peridynamics is a nonlocal approach to solution of the equilibrium equations of motion. Traditional equations of motion are limited in fracture mechanics solutions by the spatial derivative of the stress tensor, i.e. the second term in equation (1).

\[
\rho a_i = \frac{\partial T_{ij}}{\partial x_j} + b_i \tag{1}
\]

The peridynamic equation of motion, first proposed by Silling [27] given as equation (2) below uses an integral equation to establish the internal force vectors. The left hand side of the peridynamic equation is of the same form as the classical continuum mechanics form involving an inertial vector. The right hand side involving the internal and external forces on the body does not include a spatial derivative but rather an integral equation.

\[
\rho(x)\ddot{u}(x,t) = \int_{Hx} f(q,x,t)dV_q + b(x,t) \tag{2}
\]

The first term on the right is the bond force density which is a corollary to the traction vector acting on a subdomain in the classical theory. In the peridynamic theory however, nonlocal interactions exist giving rise to a force summation. The integration represents the net force acting on a material point as described by all interacting material points exerting influence over a finite distance. These interacting material points make up a region known as the horizon. The last term in the peridynamic equation is a body force density which is identical to the body
force density in the classical theory with units of force/volume. Multiple reviews in the literature have shown that the peridynamic equation of motion satisfies the basic conservation laws [9, 28].

2.2 Peridynamic Bond Force

A body modeled using the peridynamic theory consists of a group of material points subdividing the body and a network of ‘bonds’ describing the interactions between points in the body. Each bond vector is described by its spatial orientation, $\xi$, where $\xi$ is the vector pointing from a node $x$ to an arbitrary node $q$ whose magnitude is the distance between $x$ and $q$. As the body deforms the spatial description of node $x$ and node $q$ will be described by the initial position vector and the displacements of each material point. The new description of the bond between $x$ and $q$ is described as $\eta + \xi$ where $\eta$ is the displacement vector and the sum is the deformed position vector (see Figure 1). Written in the more conventional notation of continuum mechanics,

$$\eta = u(q) - u(x)$$

$$\xi = x(q) - x(x)$$

where $u$ represents the displacement vector and $x$ the position vector.
Figure 1: Reference ($\xi$) and Deformed ($\xi + \eta$) Lengths of the Peridynamic Bond Vector

To describe the force function make note of the fact that the force developed in each bond is a function of the position and displacement of only the attached material points. Thus, the force in a bond is independent of the force in all the other bonds. For elastic materials, a linear relationship is then established between bond force density and bond stretch although the relation between bond force density and displacement is nonlinear. Using the previously defined parameters, the stretch, $s$, can be written as [11]

$$ s = \frac{|\xi + \eta| - |\xi|}{|\xi|} \quad (4) $$

A definition can be written for the magnitude of the bond force density according to equation (5) [29]

$$ f = cs \quad (5) $$
The bond force density is a linear function of the stretch, $s$, occurring in a bond and a constitutive material parameter, $c$ which will be defined in Chapter 3. An underlying assumption of the bond-based theory is that the force a node $q$ exerts on node $x$ is equal but opposite to the force that $x$ exerts on $q$. Then the forces are colinear and the bond force density may be described in vector notation as [20]

$$f = cs \frac{\eta + \xi}{|\eta + \xi|}$$  (6)

where the use of boldface type denotes vector quantities.

### 2.3 The Peridynamic Horizon

Previously, the bond force density was provided for nodes arbitrarily close to a node $x$ which exert influence over node $x$. Here a formal definition will be given for how “close” nodes must be to define influence. Peridynamics is commonly described as a link between continuum theory and molecular dynamics [30]. Interactions exist beyond directly adjacent points but these interactions cease beyond a boundary known as the horizon. As the limit of the horizon becomes smaller and smaller, results in the peridynamic theory converge to the local theory. The size of the horizon can represent a physical material length scale such as grain size or it can be non-physical. Suggested limitations on the horizon size are discussed in [31, 32].

To define the horizon it is first necessary to discretize a body into a group of subdomains at the center of which lies a material point (also termed collocation point). This material point represents a discrete volume of material. The volumes associated with material points $x$ and $q$ for instance are $V_x$ and $V_q$. A horizon region is then defined for each material point which defines the family of nodes capable of exerting a force on that material point. Choosing an arbitrary node
x which is subject to forces from neighboring material points, establish a sphere of radius $\delta$ (disk in 2D) and allow all points within this sphere to exert force on node $x$. For all nodes whose distance is greater than $\delta$ from node $x$, assume these nodes do not contribute to the force on $x$. Then all nodes interacting with node $x$ fall within a sphere of radius $\delta$ known as the family of $x$ also known as the horizon of $x$, $H_x$. Identify an arbitrary node within the family of $x$ as node $q$. Node $q$ exerts a force on node $x$ described by the bond force density function and node $x$ exerts a force on node $q$ of equivalent magnitude but opposite vector direction as seen in Figure 2.

![Figure 2: The Peridynamic Horizon and Division of Body into Subdomains](image)

It is now possible to return to the peridynamic equation of motion to describe the internal force vector that develops between material points. The bond force density is described as a force
per unit volume squared. Thus, it is a function of the material volume of node $x$ as well as the material volume of node $q$. Peridynamic bond force density is described as the force that an arbitrary material volume $q$, within the family of $x$, exerts on $x$.

### 2.4 The Micropotential Function

In the bond-based peridynamic theory, interactions between material points exist as parallel force vectors whose magnitude is dependent upon the “stretch” occurring between the points in the deformed state. Because of this the behavior is commonly described as that of a linear elastic spring [29]. In the spring analogy, a peridynamic material constitutive behavior known as the micromodulus function is comparable to a spring constant due to its linear, and completely recoverable, behavior and the extension of the bond is akin to the change in length of a spring. As the distance between points increases under deformation, energy is stored in the “bond” between material points. A relation between this stored energy and the bond force was first developed by Silling and is described by the micropotential [27].

$$f(\eta, \xi) = \frac{\partial w(\eta, \xi)}{\partial \eta}$$ (7)

In equation (7), $w$ represents the micropotential existing between two material points and is a function of displacement of the material points and the initial positions described by $\eta$ and $\xi$, respectively.

The micropotential can also be described by the more familiar strain energy density whose relation is given by equation (8) where $W_x$ is the strain energy density at node $x$, $w$ is the micropotential, $H_x$ is the peridynamic horizon, and $dV_q$ is the material volume of a point $q$ in the horizon of $x$ [29].
\[ W_x = \frac{1}{2} \int_{H_x} w \, dV_q \] (8)

The integration is taken over all nodes within the horizon of node x. The bond-based peridynamic model exhibits a force density function which is linearly proportional to bond stretch. A form of the micropotential function which satisfies this condition is given as equation (9) below [13,20,33].

\[ w = \frac{1}{2} cs^2 |\xi| \] (9)

By establishing a relation between the strain energy density which has definition in the classical theory and the material parameter c, a relation between the classical material parameters and the peridynamic parameter c can be established.
Chapter 3. PERIDYNAMIC CONSTITUTIVE MATERIAL MODELS

3.1 Material Behavior in the Peridynamic Theory

A force function has been established which relates the bond force density between two material points to the relative deformation vector between these points and a material model herein referenced as parameter c. The peridynamic theory was developed assuming an equivalent deformation map for the peridynamic theory as the one occurring in the classical theory. This is accomplished by requiring the strain energy density match the classical theory for all points in the body. By enforcing equivalence of the strain energy function, the parameter c can be developed in terms of existing material parameters, i.e. Poisson’s ratio, tensile modulus, and bulk modulus.

3.2 2D Form of the Bulk Modulus

It is at this point that it is necessary to distinguish between the definition of the bulk modulus assuming three dimensional continuum theory and the bulk modulus for the two dimensional planar assumption. Bulk modulus is defined as the ratio of hydrostatic stress relative to unit volume change or [34]

\[ \kappa = \frac{\text{pressure}}{\text{dilation}} = \frac{\sigma_{\text{hyd}}}{\theta} \]  

(10)

where \( \sigma_{\text{hyd}} \) represents a uniform pressure and \( \theta \) represents volume dilation as depicted in Figure 3.
In two dimensions, the bulk modulus is the ratio of the pressure applied to the plane faces and the area dilation. As was shown in [29], the bulk modulus in the case of plane stress and plane stress is defined as

\[
κ_{\text{ptn, Stress}} = \frac{E}{2(1 - ν)}
\]

\[
κ_{\text{ptn, Strain}} = \frac{E}{2(1 - ν - 2ν^2)}
\]

3.3 Strain Energy Density and Constitutive Parameters in Peridynamics

Strain energy density will now be calculated according to peridynamic theory. Once complete, the strain energy density from peridynamics will be equivalenced with the strain energy density from classical continuum mechanics. It will be shown through determination of the peridynamic parameter known as the micromodulus, that an object modeled using the peridynamic theory will exhibit the same material response as an object modeled using classical continuum mechanics.
Peridynamics formulates the constitutive models which govern material response in terms of currently available material parameters. In order to determine the relation between material parameters for an isotropic material in classical continuum mechanics and in peridynamics, assume a section of material as given in Figure 4.

Figure 4: Evaluation of Strain Energy at Differential Volume dA in CCM and at Material Point i (coincident with dA) in Peridynamics – Biaxial Strain (Redrawn from [36])

Assume the part of the structure isolated in Figure 4 is far from the boundaries. Taking a differential area, dA, in the bulk of body B and equating strain energy density to the material point i in the peridynamic body S at the same spatial location will allow the peridynamic constitutive model to be determined. The two theories must predict an identical response under the same kinematic boundary conditions. To ensure the material model is valid for any possible stress state consider the cases of pure shear and biaxial stretch. All possible states of stress can be described as a combination of these two fundamental states.
3.4 Strain Energy in Classical Continuum Mechanics (CCM)

3.4.1 Body Subject to Biaxial Strain – Plane Strain Assumption

Strain energy density at a point in the classical theory is given as [35]

\[
U = \frac{1}{2} \left( \sigma_x e_x + \sigma_y e_y + \sigma_z e_z + \tau_{xy} \gamma_{xy} + \tau_{xz} \gamma_{xz} + \tau_{yz} \gamma_{yz} \right)
\]  \hspace{1cm} (13)

where \( \gamma \) defines engineering shear strain. In terms of a 2D plane assumption (plane stress or plane strain), this simplifies to

\[
U = \frac{1}{2} \left( \sigma_x e_x + \sigma_y e_y + \tau_{xy} \gamma_{xy} \right)
\]  \hspace{1cm} (14)

Apply biaxial strain to the 2D body where \( e_x = e_y \) and \( \tau_{xy} = 0 \). Noting that Hooke’s Law may be given in terms of strain as [35]

\[
\sigma_x = \frac{E}{(1 + \nu)(1 - 2\nu)} \left[ (1 - \nu)e_x + \nu(e_x + e_z) \right]
\]  \hspace{1cm} (15)

\[
\sigma_y = \frac{E}{(1 + \nu)(1 - 2\nu)} \left[ (1 - \nu)e_y + \nu(e_x + e_z) \right]
\]

\[
\sigma_z = \frac{E}{(1 + \nu)(1 - 2\nu)} \left[ (1 - \nu)e_z + \nu(e_x + e_y) \right]
\]

\[
\tau_{xy} = \frac{E}{1 + \nu} e_{xy}
\]

\[
\tau_{xz} = \frac{E}{1 + \nu} e_{xz}
\]

\[
\tau_{yz} = \frac{E}{1 + \nu} e_{yz}
\]

A substitution may be made to provide
\[ U = \frac{1}{2} \left\{ \left( \frac{E}{(1 + \nu)(1 - 2\nu)} \left[ (1 - \nu)\varepsilon_y + \nu(\varepsilon_z + \varepsilon_x) \right] \right) \varepsilon_x \right. \\
\left. + \left( \frac{E}{(1 + \nu)(1 - 2\nu)} \left[ (1 - \nu)\varepsilon_y + \nu(\varepsilon_z + \varepsilon_x) \right] \right) \varepsilon_y \right\} \quad (16) \]

Making use of the fact that \( \varepsilon_x = \varepsilon_y = \psi \) and \( \varepsilon_z = 0 \),

\[ U = \frac{1}{2} \left\{ \frac{E}{(1 + \nu)(1 - 2\nu)} \left[ (1 - \nu)\psi + \nu\psi \right] \psi + \frac{E}{(1 + \nu)(1 - 2\nu)} \left[ (1 - \nu)\psi + \nu\psi \right] \psi \right\} \]

\[ U = \frac{E}{2(1 + \nu)(1 - 2\nu)} \left[ (1 - \nu)\psi + \nu\psi \right] \psi + (1 - \nu)\psi + \nu\psi \right] \psi \]

\[ U = \frac{E}{2(1 + \nu)(1 - 2\nu)} \left[ \psi^2 + \psi^2 \right] = \frac{E\psi^2}{(1 + \nu)(1 - 2\nu)} \quad \text{[CCM Plane Strain]} \quad (17) \]

Equation (17) is the final form of the strain energy density at differential area, \( dA \), for a body subject to biaxial stretch under the assumption of plane strain.

**3.4.2 Body Subject to Biaxial Strain – Plane Stress Assumption**

A similar approach is taken to resolve the strain energy density associated to a body subject to biaxial stretch modeled with plane stress behavior. Note that for plane stress the form of strain energy is identical to equation (14). However, for plane stress \( \sigma_z = \tau_{xz} = \tau_{yz} = \tau_{xy} = 0 \). Derivation in terms of plane stress will utilize an alternate form of Hooke’s Law written in terms of strain [35].
\[ e_x = \frac{1}{E} \left[ \sigma_x - \nu (\sigma_y + \sigma_z) \right] \] (18)

\[ e_y = \frac{1}{E} \left[ \sigma_y - \nu (\sigma_x + \sigma_z) \right] \]

\[ e_z = \frac{1}{E} \left[ \sigma_z - \nu (\sigma_x + \sigma_y) \right] \]

\[ e_{xy} = \frac{1 + \nu}{E} \tau_{xy} \]

\[ e_{xz} = \frac{1 + \nu}{E} \tau_{xz} \]

\[ e_{yz} = \frac{1 + \nu}{E} \tau_{yz} \]

Rewriting equation (14) in terms of stress only

\[ U = \frac{1}{2} \left\{ \sigma_x \left( \frac{\sigma_x - \nu (\sigma_y + \sigma_z)}{E} \right) + \sigma_y \left( \frac{\sigma_y - \nu (\sigma_x + \sigma_z)}{E} \right) \right\} \] (19)

Which simplifies to

\[ U = \frac{1}{2} \left( \frac{\sigma_x^2 - \nu \sigma_x \sigma_y}{E} + \frac{\sigma_y^2 - \nu \sigma_x \sigma_y}{E} \right) = \frac{1}{2E} \left( \sigma_x^2 + \sigma_y^2 - 2\nu \sigma_x \sigma_y \right) \] (20)

Noting here that due to the presence of equal biaxial strain, \( \sigma_x = \sigma_y = \sigma \)

\[ U = \frac{1}{2E} (\sigma^2 + \sigma^2 - 2\nu \sigma) = \frac{1}{2E} (2\sigma^2 - 2\nu \sigma^2) = \frac{1 - \nu}{E} \sigma^2 \] (21)
Now from Hooke’s Law and the assumption of plane stress

\[ e_x = \frac{1}{E} [-\nu(2\sigma)] \]  

(22)

Substituting the relation for \( e_x \) into the form of Hooke’s Law in terms of stress gives

\[ \sigma = \frac{E}{(1 + \nu)(1 - 2\nu)} \left[ (1 - \nu)e_x + \nu \left( e_y + \left( \frac{-1}{E} \right) \nu(2\sigma) \right) \right] \]  

(23)

And replacing with the applied strain

\[ \sigma = \frac{E}{(1 + \nu)(1 - 2\nu)} \left[ \psi - \frac{1}{E} \nu(2\sigma) \right] = \frac{E}{(1 + \nu)(1 - 2\nu)} \psi - \nu \psi + \frac{1}{E} \nu^2 (2\sigma) \]  

(24)

Moving \( \sigma \) to the left side

\[ \sigma + \frac{\nu^2 2\sigma}{(1 + \nu)(1 - 2\nu)} = \sigma \left( 1 + \frac{2
u^2}{(1 + \nu)(1 - 2\nu)} \right) = \frac{Es}{(1 + \nu)(1 - 2\nu)} \]

\[ \sigma \left( \frac{(1 + \nu)(1 - 2\nu)}{(1 + \nu)(1 - 2\nu)} + \frac{2\nu^2}{(1 + \nu)(1 - 2\nu)} \right) = \frac{E \psi}{(1 + \nu)(1 - 2\nu)} \]

\[ \sigma \left( \frac{1 - 2\nu + \nu - 2\nu^2 + 2\nu^2}{(1 + \nu)(1 - 2\nu)} \right) = \sigma \left( \frac{1 - \nu}{(1 + \nu)(1 - 2\nu)} \right) = \frac{E \psi}{(1 + \nu)(1 - 2\nu)} \]  

(25)

And solving for \( \sigma \)

\[ \sigma = \frac{E \psi}{(1 + \nu)(1 - 2\nu)} \frac{(1 + \nu)(1 - 2\nu)}{1 - \nu} = \frac{E \psi}{1 - \nu} \]  

(26)
Substituting the equation (26) into equation (21) gives

\[
U = \frac{1 - \nu}{E} \left( \frac{E \psi}{1 - \nu} \right)^2 = \frac{1 - \nu}{E} \frac{E^2 \psi^2}{(1 - \nu)^2} = \frac{E \psi^2}{1 - \nu} \quad \text{(CCM Plane Stress)}
\]  \hspace{1cm} (27)

For the case of a body subject to biaxial strain, equation (17) provides the final form of strain energy density under plane strain assumption and equation (27) provides the final form of strain energy density under plane stress assumption.

### 3.4.3 Body Subject to Simple Shear

Consider the same body this time subjected to simple shear (see Figure 5). The engineering strain is \( \zeta \). In classical continuum mechanics the strain energy density is given by equation (13). In the case of 2D simple shear all terms are zero with the exception of \( \tau_{xy} \) and \( \gamma_{xy} \). Thus, the equation for strain energy density simplifies to

\[
U = \frac{1}{2} \tau_{xy} \gamma_{xy} = \frac{1}{2} \mu \gamma_{xy}^2
\]  \hspace{1cm} (28)

where \( \mu \) is the shear modulus. Noting that the shear modulus may be written in terms of Poisson’s ratio and elastic modulus as [35]

\[
\mu = \frac{E}{2(1 + \nu)}
\]  \hspace{1cm} (29)

The equation for strain energy density in classical continuum mechanics for the case of simple shear is given as equation (30).

\[
U = \frac{1}{2} \frac{E}{2(1 + \nu)} \gamma_{xy}^2 = \frac{E}{4(1 + \nu)} \zeta^2
\]  \hspace{1cm} (30)
3.5 Strain Energy in Peridynamic Theory

3.5.1 Body Subject to Biaxial Strain

Such that direct equivalence of the strain energy density at material point i (differential area dA) is possible the same kinematic boundary conditions are applied to body S. The deformed bond between points i and j has length \(|y_j - y_i| = |\eta + \xi|\). To establish \(|y_j - y_i|\) first the displacement gradient is found as

\[
\frac{\partial u}{\partial x} = e_x = \psi \tag{31}
\]

\[
\frac{\partial v}{\partial y} = e_y = \psi \tag{32}
\]
Which implies that

\[ u = \psi x + c \]  

\[ v = \psi y + c \]  

where \( c \) is an arbitrary integration constant. By definition the bond displacement may be written as

\[ \eta = \begin{bmatrix} u_j - u_i \\ v_j - v_i \end{bmatrix} = \begin{bmatrix} \psi x_j - \psi x_i \\ \psi y_j - \psi y_i \end{bmatrix} = \psi \begin{bmatrix} \xi \cos \theta \\ \xi \sin \theta \end{bmatrix} \]  

where it is noted that \( \xi \) is the reference bond length and the components of \( \xi \) may be written in terms of \( \xi \) and trigonometric functions of \( \Theta \). Noting that the vector \( \xi \) may be written as

\[ \xi = |\xi| \begin{bmatrix} \cos \Theta \\ \sin \Theta \end{bmatrix} \]  

Now

\[ \eta + \xi = \begin{bmatrix} \psi \xi \cos \theta + \xi \cos \theta \\ \psi \xi \sin \theta + \xi \sin \theta \end{bmatrix} = \begin{bmatrix} \xi \cos \theta (\psi + 1) \\ \xi \sin \theta (\psi + 1) \end{bmatrix} \]  

Now equation (37) may be substituted into the equation for bond stretch giving

\[ s = \frac{|\eta + \xi| - |\xi|}{|\xi|} = \sqrt{\left((1 + \psi)(\xi \cos \theta)\right)^2 + \left((1 + \psi)(\xi \sin \theta)\right)^2 - |\xi|} \]  

And factoring the strain term

\[ s = \frac{\sqrt{(1 + \psi)^2 \xi^2 \cos^2 \theta + \xi^2 \sin^2 \theta} - |\xi|}{|\xi|} = \frac{\sqrt{(1 + \psi)^2 \xi^2 (1)} - |\xi|}{|\xi|} = \frac{|\xi|(1 + \psi) - |\xi|}{|\xi|} \]
Then $s$ is

$$s = (1 + \psi) - 1 = \psi \quad (40)$$

Returning to the equation for strain energy density and utilizing $s=\psi$ which is a constant

$$W = \frac{1}{2} \int_{Hx} \frac{1}{2} c s^2 |\xi| dV = \frac{1}{4} c \psi^2 \int_{Hx} |\xi| dV \quad (41)$$

Under the plane assumption the integration is completed in 2D polar coordinates by noting that $dV = t*dA$ where $t$ is the thickness

$$W = \frac{1}{4} c t \psi^2 \int_{Hx} |\xi| (t dA) = \frac{1}{4} c t \psi^2 \int_0^{2\pi} \int_0^\delta |\xi| |\xi| d\theta d\xi = \frac{1}{4} c t \psi^2 [\theta]_0^\delta \int_0^\delta \xi^2 d\xi$$

$$W = \frac{1}{4} c t \psi^2 (2\pi) \left[ \frac{1}{3} \xi^3 \right]_0^\delta = \frac{\pi}{2} c t \psi^2 \left( \frac{1}{3} \delta^3 \right)$$

$$W = \frac{\pi}{6} c t \psi^2 \delta^3 \quad (42)$$

The final form of the strain energy density at point $i$ evaluated from peridynamic theory is given as equation (42).

### 3.5.2 Body Subject to Simple Shear

In order to determine the peridynamic strain energy density at point $i$, it will again be necessary to calculate the stretch in an arbitrary bond $\xi$. The length of a deformed bond for the case of simple shear was derived in [9] as

$$|\eta + \xi| = [1 + (\sin \theta \cos \theta) \zeta] |\xi| \quad (43)$$
A graphical representation of the method presented in [9] is provided in Figure 6 in lieu of a rigorous proof. The stretch is calculated from the length of the bond in the initial and deformed states as

\[
s = \frac{|\eta + \xi| - |\xi|}{|\xi|} = \frac{[1 + (\sin \theta \cos \theta) \zeta]|\xi|}{|\xi|} = \zeta \sin \theta \cos \theta \quad (44)
\]

![Figure 6: Detailed View of Bond Stretch under Simple Shear from Figure 5](image)

The strain energy density at point i is found by substituting the stretch from equation (44) into equation (8).

\[
W = \frac{1}{2} \int_{H_x} \frac{1}{2} c \sin^2 |\xi| (t \, dA)
\]

\[
= \frac{1}{2} \int_{H_x} \frac{1}{2} c (\xi \sin \theta \cos \theta)^2 |\xi| (t \, dA) = \frac{1}{4} c t \zeta^2 \int_0^{2\pi} \int_0^\delta \xi \sin^2 \theta \cos^2 \theta (\xi \, d\xi \, d\theta)
\]

\[
W = \frac{1}{4} c t \zeta^2 \left[ \frac{\theta}{8} - \frac{\sin(4\theta)}{32} \right] 2\pi \int_0^\delta \xi^2 d\xi = \frac{1}{4} c t \zeta^2 \left[ \frac{\pi}{4} - 0 \right] \int_0^\delta \xi^2 d\xi = \frac{\pi}{16} c t \zeta^2 \left[ \frac{1}{3} \xi^3 \right]_0^\delta
\]
The strain energy density for the case of simple shear is given as equation (45) above.

### 3.6 Evaluation of a Peridynamic Constitutive Parameter from Strain Energy Equivalence

Table 1 provides a summary of the values of strain energy under the conditions of biaxial strain and simple shear. It can be seen that for plane stress and plane strain the values of strain energy under simple shear are identical while the two differ for biaxial strain.

### Table 1: Table of Strain Energy

<table>
<thead>
<tr>
<th>Theory</th>
<th>Biaxial Strain</th>
<th>Simple Shear</th>
</tr>
</thead>
<tbody>
<tr>
<td>CCM – Plane Strain</td>
<td>( \frac{E\psi^2}{(1 + \nu)(1 - 2\nu)} )</td>
<td>( \frac{E\zeta^2}{4(1 + \nu)} )</td>
</tr>
<tr>
<td>CCM – Plane Stress</td>
<td>( \frac{E\psi^2}{1 - \nu} )</td>
<td>( \frac{E\zeta^2}{4(1 + \nu)} )</td>
</tr>
<tr>
<td>Peridynamic Theory</td>
<td>( \frac{\pi ct\psi^2 \delta^3}{6} )</td>
<td>( \frac{\pi ct\zeta^2 \delta^3}{48} )</td>
</tr>
</tbody>
</table>

NOTE: The derivations above are in agreement with the Literature. See, for example, [9,36]

### 3.6.1 Equate Energy for Plane Stress

Utilizing Table 1 above, the parameter c for use in plane stress theory can be determined. First, equating strain energy density between the classical and peridynamic theories under biaxial strain gives

\[
U_N = W_N \Rightarrow \frac{E\psi^2}{1 - \nu} = \frac{\pi ct\psi^2 \delta^3}{6}
\] (46)
where $U_N$ is the strain energy density from the classical theory and $W_N$ is the energy density from the peridynamic theory. Solving for the tensile modulus yields

$$E = \frac{(1 - \nu)\pi ct\delta^3}{6}$$  \hspace{1cm} (47)$$

Repeating the same procedure for the case of simple shear provides

$$U_S = W_S \Rightarrow \frac{E\zeta^2}{4(1 + \nu)} = \frac{\pi ct\zeta^2\delta^3}{48}$$  \hspace{1cm} (48)$$

where $U_S$ is the strain energy density from the classical case and $W_S$ from the peridynamic case. Again solving for the modulus yields

$$E = \frac{4(1 + \nu)\pi ct\delta^3}{48} = \frac{(1 + \nu)\pi ct\delta^3}{12}$$  \hspace{1cm} (49)$$

Equating the form of $E$ from each stress state gives

$$\frac{(1 - \nu)\pi ct\delta^3}{6} = \frac{(1 + \nu)\pi ct\delta^3}{12}$$  \hspace{1cm} (50)$$

$$\frac{1 - \nu}{6} = \frac{1 + \nu}{12} \Rightarrow 12(1 - \nu) = 6(1 + \nu)$$

$$2 - 2\nu = 1 + \nu$$

$$\nu = \frac{1}{3}(Plane\ Stress)$$  \hspace{1cm} (51)$$

Equation (51) proves that regardless of the choice of horizon size and independent of the elastic modulus, for the case of plane stress, the Poisson’s ratio for a peridynamic bond-based constitutive model is fixed at 1/3. To solve for $c$, rewrite equation (47) as
\[
\frac{E}{2(1-\nu)} = \frac{\pi ct\delta^3}{12} \tag{52}
\]

To describe the parameter c in terms of the bulk modulus recall the definition of bulk modulus for 2D plane stress from equation (11). Upon substitution into equation (52)

\[
\kappa'_{\text{pH strain}} = \frac{\pi ct\delta^3}{12} \tag{53}
\]

Rewriting in terms of c provides the desired form [29].

\[
c = \frac{12\kappa'_{\text{pH strain}}}{\pi t\delta^3} \tag{54}
\]

Equation (54) provides the value of the peridynamic function, c, for the case of plane stress in terms of known material parameters from the classical continuum mechanics theory. Substitution into equation (5) allows the peridynamic bond force density to be calculated from the constitutive behavior provided by the parameter c, and the bond stretch, s, between two adjacent material points.

### 3.6.2 Equate Energy for Plane Strain

A similar procedure is adopted for evaluation of c for the case of plane strain. The strain energy density from the classical theory, \(U_N\), is again equated with the term from the peridynamic theory, \(W_N\).

\[
U_N = W_N \Rightarrow \frac{E\psi^2}{(1+\nu)(1-2\nu)} = \frac{\pi ct\psi^2\delta^3}{6} \tag{55}
\]

The solution is again written in terms of E as

\[
E = \frac{(1+\nu)(1-2\nu)\pi ct\delta^3}{6} \tag{56}
\]
For the case of simple shear, the Young’s modulus is computed as below.

\[
U_S = W_S \Rightarrow \frac{E \zeta^2}{4(1 + \nu)} = \frac{\pi c t \zeta^2 \delta^3}{48} \tag{57}
\]

\[
E = \frac{(1 + \nu)\pi c t \delta^3}{12} \tag{58}
\]

Eliminating \(E\) from equations (56) and (58) again places a restriction on Poisson’s ratio for bond-based peridynamics.

\[
\nu = \frac{1}{4} \quad (Plane\ Strain) \tag{59}
\]

Similar to equation (51) for Plane Stress, equation (59) requires that Poisson’s ratio is fixed for all systems modeled under plane strain assumption regardless of the value of elastic modulus or the choice of horizon size, \(\delta\). Parameter \(c\) can be determined for plane strain by first rearranging equation (56).

\[
\frac{E}{2(1 + \nu)(1 - 2\nu)} = \frac{\pi c t \delta^3}{12} \tag{60}
\]

Substituting the definition of the 2D plane strain bulk modulus into equation (60) provides a relation between \(c\) and the bulk modulus [29].

\[
c = \frac{12 \kappa'_{\text{Plane Strain}}}{\pi t \delta^3} \tag{61}
\]

Note that equation (61) for plane strain is identical to equation (54) for Plane Stress with the exception of the 2D bulk modulus.
3.7 The Micromodulus Function

In section 3.6 the material parameter $c$ was determined for 2D peridynamic solutions under the plane stress and plane strain assumptions. The parameter $c$ is termed the peridynamic micromodulus (also called bond constant). In the present work the micromodulus function $c(\xi)$ is constant for all $\xi$. Other non-constant forms of the micromodulus function are available in the literature [18].
Chapter 4. IMPLEMENTATION OF PERIDYNAMIC THEORY IN COMMERCIAL FEM SOFTWARE

4.1 Discretization of the Equation of Motion

Analytical solutions of the peridynamic equation of motion, equation (2), are very limited (see [37-40]). Often, a numerical approximation is used in lieu of a closed form solution. Equation (2) may be discretized into n time steps as in equation (62) [11]. Here the integral equation becomes a sum over all nodes q in the family of x.

\[ \rho \ddot{u}_x^n = \sum_{(q)} f(u_{(q)}^n - u_{(x)}^n, x_{(q)} - x_{(x)}) \Delta V_{(q)} + b_{(x)}^n \]  \hspace{1cm} (62)

As the behavior of the crack growth is assumed to be quasi-static, the inertial terms are zero and thus equation (62) simplifies to the static solution given by equation (63).

\[ 0 = \sum_{(q)} f(u_{(q)}^n - u_{(x)}^n, x_{(q)} - x_{(x)}) \Delta V_{(q)} + b_{(x)}^n \]  \hspace{1cm} (63)

4.2 Characterization of Peridynamic Force

The peridynamic theory utilizes a meshless discretization for material characterization. A body is subdivided into domains at the center of which exists a collocation, or material, point. Each of these collocation points interacts with neighboring points within the horizon via an axial “bond” force that is coincident with the position vector in the current configuration.

The interactions of these “bonds” can be described in an existing Finite Element software by truss elements and the material points can be represented by nodes. Truss elements are
capable of transmitting only axial (purely compressive or purely tensile forces) with no
capability for shear or bending loads. This makes these elements well suited to represent the
peridynamic force between neighboring collocation points.

Consider a tension/compression member of arbitrary cross section subject to an end load
as shown in Figure 7. The strains are given by

\[ e = \begin{bmatrix} \frac{C}{E} & 0 & 0 \\ 0 & \frac{-\nu C}{E} & 0 \\ 0 & 0 & \frac{-\nu C}{E} \end{bmatrix} \]  

Integration of the strain in the x direction for this simple case provides the familiar equation

\[ \delta = \frac{PL}{AE} \]  

where P is the applied load, L the length of the member, A the cross section, and E the elastic
modulus.

The force density vector in bond-based peridynamics is of the form

\[ f = cs \frac{\xi + \eta}{|\xi + \eta|} \]
Here the stretch, defined in equation (4), is identical to the value of axial strain in a truss member. Also note that the peridynamic force density vector is a force per unit volume squared. The net force acting in the bond may be written as

\[ F = |\mathbf{f}| * V_i V_j \]  \hspace{1cm} (67)

where \( F \) is the force in the bond, \( \mathbf{f} \) is the bond force density vector, \( V_i \) is the volume associated to collocation point \( i \), and \( V_j \) is the volume associated with collocation point \( j \). Substitution of equation (66) into equation (67) allows for strain in the peridynamic theory to be equated to classical continuum mechanics.

\[ \varepsilon = \frac{\delta}{L} = \frac{P}{A E_{FE}} = s = \frac{F}{c * V_i * V_j} \]  \hspace{1cm} (68)

In the equation above a subscript was added to the elastic modulus to distinguish the measured elastic modulus of the material and the elastic modulus utilized in the FE model. For traditional FEM solutions the two will match but in this case the value of \( E \) used for simulation will not be characterizing the material parameter of Young’s modulus but rather will be a numerical value intended to describe the peridynamic micromodulus, \( c \). Noting that both \( P \) and \( F \) correspond to the same force, \( c \) may be solved for as [11]

\[ c = \frac{AE_{FE}}{V_i V_j} \]  \hspace{1cm} (69)

Equation (69) is simplified by noting that in the present work, a square 2D discretization is used resulting in \( V_i = V_j \). Noting that the distance between adjacent nodes can be described by the grid spacing, \( \Delta x \), \( V_i \) and \( V_j \) are both equal to \( \Delta x^2 * t \) where \( t \) is the thickness. Then equation (69) becomes
As the peridynamic bond force is described by the micromodulus \( c \), the individual choice of values for \( A \) and \( E \) is arbitrary. In this paper, the method proposed by [11] will be followed to maintain physical units for \( A \) and \( E \)

\[
c = \frac{AE_{FE}}{\Delta x^2 t^2}
\]

Note that the relation between \( E_{FE} \) and \( c \) is not the same as the relation between \( E \) and \( c \). \( E_{FE} \) is a numerical value which is chosen for convenience to describe the bond stiffness, \( c \), while \( E \) is the measurable material parameter determined from a tensile test. In the present work \( E_{FE} \) is formulated in terms of thickness and grid spacing resulting in differences in notation with [11]. By proper selection of the boundary conditions the two forms are equivalent.

### 4.3 The Skin Effect and Surface Corrections

It is evident in the integrations from Chapter 3, that the value of the micromodulus, \( c \), was computed assuming a collocation point has a horizon completely embedded within the boundaries of the body. For collocation points near the surface, the volume (area in 2D) of integration will be less (see Figure 8) and the resulting value of \( c \) will not match the value calculated previously. This discrepancy is known as the “skin effect” in peridynamics. If the same value of the micromodulus is applied throughout the body, including to those bonds near free surfaces, the results will be that the stiffness of the body will be less near the boundaries. To visualize this, consider a small perturbation load applied to a node near a free surface of a body and the same load applied at a location in the bulk. For the same magnitude of load, a node near the free surface will displace a greater amount because there are less bonds contained within the
(incomplete) horizon to resist the load when compared to a node in the center of the body.

Various methods have been proposed to stiffen bonds near the free surfaces to restore the bulk elastic properties near the boundaries of the body. Nearly all methods seek to apply a correction factor to the bulk micromodulus calculated above to bonds near the boundaries. A review of these methods was presented in [31]. The volume method for surface correction will be utilized here as this method was found to be one of the more successful.

4.4 Volume Method for Surface Correction.

As described in [33] the Volume Method stiffens bonds which do not have horizons completely within the bounds of the body by adjusting $c$ according to

$$c_{corrected} = \lambda * c$$  \hspace{1cm} (72)
where $\lambda$ is defined in equation (73) below and $c$ is the micromodulus calculated in the bulk material from chapter 3.

$$\lambda = \frac{2V_0}{V_x + V_q}$$  \hspace{1cm} (73)

The volumes above may be described as $V_0$: volume of the horizon of an arbitrary point in the bulk completely enclosed within the body, $V_x$: volume of the horizon at point $x$ which lies within the material boundaries, $V_q$: volume of the horizon at point $q$ which lies within the material boundaries. The above volumes must be distinguished from the volumes $V_i$ and $V_j$ which correspond to the material volumes associated to collocation points $i$ and $j$, respectively.

For all collocation points away from the boundaries, $V_x + V_q = 2V_0$ and $\lambda=1$ implying that $c_{\text{corrected}} = c$ for locations within the bulk as intended. For collocation points which do not have a horizon completely enclosed by the body, the values of $V_x$ and $V_q$ will be less than $V_0$, and $\lambda$ will be greater than 1. Thus, the applied value of the micromodulus is greater for points near the surface than in the bulk.

4.5 Implementation of the Surface Correction Factor

Implementation of the peridynamic theory in an FEM software package does not allow for the micromodulus, or the corrected micromodulus, to be input directly. The solution is formulated in terms of the stiffness of truss elements representing bonds with characteristics of length, area, and elastic modulus. The correction factor for the micromodulus of bonds near a free surface may be implemented from inspection of Equation (72). Substituting $c_{\text{corrected}}$ into equation (71) formulates a corrected Young’s modulus, $E$, in terms of a corrected micromodulus. The volumes in equation (73) can be evaluated by numerically integrating the volume of
material for a point in the bulk, for a point \( x \), and for a point \( q \). However, noting that each collocation point represents an identical volume in the case of a uniform mesh a more efficient method in the current application is to determine the number of collocation points within each family and use this value to compute \( \lambda \). By evaluating \( N_0 \) as the number of nodes associated to a family in the bulk, \( N_x \) as the number of nodes in the family at \( x \), and \( N_q \) as the number of nodes in the family at \( q \), equation (73) can be rewritten as equation (74).

\[
\lambda = \frac{2(\Delta x^3)N_0}{(\Delta x^3)(N_x + N_q)} = \frac{2N_0}{N_x + N_q}
\]  

(74)

This procedure does not apply a correction for material volumes bisected by the horizon which have only partial volume contained inside the horizon. Methods for such volume correction have been established in [9].

4.6 Sample Geometry and Material

The peridynamic theory was implemented for the present research to characterize the crack growth in a compact tension (CT) specimen with an initial precrack. The present study is modeled after the experiments by Miranda, et. al [40] and peridynamic simulations by Zhang, et. al [20]. In [40], the authors had previously conducted research into the fracture behavior of a standard CT specimen used for determining fracture toughness material parameters but modified with the addition of a hole. Four unique variations of this “modified” CT specimen which exhibit unique behavior in the presence of a growing crack. Two of the samples had a crack which propagated into the added hole and two more samples had a crack which migrated towards the added hole but bypassed it. The work by [20] utilized an algorithm developed in Fortran to solve
the peridynamic equations of motion and simulate the crack growth of the Miranda samples using peridynamics.

The present research seeks to implement the peridynamic theory for solution of the crack growth problem from [40] in a commercially available finite element package. At present most solutions for the governing equation of peridynamics has been via Matlab or Fortran codes developed specifically for the purpose or via open-source peridynamic codes [20,22,24,41,42]. Implementing a peridynamic solution in a finite element framework will utilize the inherent efficiencies of the nonlinear solvers contained in commercial FE code and will allow for a wider array of problems to be solved. Peridynamics has been successfully implemented into Abaqus software for the case of dynamic brittle fracture [10] and for the case of ductile fracture [13]. The present work will utilize a similar approach to implement peridynamics into Abaqus for a high cycle fatigue analysis.

The specimen CT1 was selected from the available configurations of [40] for the present work. Geometry of the specimen is as shown in Figure 9. The problem is formulated according to the 2D bond-based theory for plane strain. In order to allow comparison to the work by [20], the same value of horizon was utilized. In this case the horizon size is \( \delta = 4\Delta x \) where \( \Delta x \) is the spacing between adjacent nodes. The experimental sample was an SAE 1020 steel with modulus \( E = 205 \) GPa and Poisson’s ratio of 0.29 [40]. The sample contained a precrack of length 0.9mm.
Due to the nonlocal nature of the peridynamic solution, a much greater number of elements are present than in a typical finite element solution. Depending on the horizon size, a single node will contain tens or possibly hundreds of elements which poses a difficulty for manual generation of the mesh. Therefore, a script was developed in Fortran to automate mesh generation.

A subprogram first discretizes the geometry into smaller subdomains consisting of collocation points (nodes). While it is possible to spatially vary the density of the nodes, the
program developed for this application utilizes a uniform grid. Following the definition of nodes, a second subprogram generates the truss element assemblage by identifying the family of nodes for each node given a horizon size of $4\Delta x$. Truss elements are created once the family of each node has been determined. The subprogram also contains an algorithm to form a precrack by deleting elements crossing a specified plane (line in 2D). A third subprogram groups elements based upon family data. The node family is determined for each node and used in equation (74) to compute the required correction factor. This allows elements to be grouped based on specific material cards which account for varying stiffness near the boundaries.

### 4.8 Mesh Convergence

For peridynamic modeling convergence studies are necessary for (a) the overall size of the horizon and (b) the number of grid points contained within the horizon. The first of these convergence studies is known as $\delta$-convergence. The second is known as $m$-convergence as it is common to define the horizon as $\delta=m(\Delta x)$ where $m$ is an integer value describing the ratio of horizon size to grid size. As described in [31] it has been shown that $m$ values between 4 and 6 provide accurate results for most cases. An $m$ value of 4 was utilized in [20] with good correlation to experimental results and this choice of $m$ will be repeated in the present analysis.

A convergence study was completed for $\delta$ convergence. Horizon sizes of 1.2mm, 0.6mm, and 0.5mm were utilized for the present work which correspond to model sizes of 15,000, 60,000 and 90,000 nodes. Plots of the crack path in chapter 8 show that for the coarsest mesh the crack propagates into the hole rather than bypassing it. Good agreement is achieved between the crack paths for horizon sizes of 0.6mm and 0.5mm. Fatigue life predictions are reported for the model with horizon size of 0.5mm.
4.9 Boundary Conditions for Static Solution

A multi-step analysis was used to model the crack growth over a large number of cycles. Crack growth was assumed to be quasi-static such that the crack growth could be divided into a series of linear static load steps. Note that a linear static solution implies use of the small displacement (linear geometry) assumption. In each step, loads were applied at the outermost nodes of the upper and lower pin holes. In the experiments, the authors update the loading to maintain a near-constant stress intensity factor at the crack tip and this method is repeated in simulation so as to establish similar boundary conditions [40]. Loading was updated multiple times throughout the analysis so as to maintain a near constant stress intensity factor at the crack tip (see Chapter 7).
Chapter 5. FATIGUE DAMAGE MODEL

5.1 Peridynamic Fatigue Model

One of the primary advantages of the peridynamic theory is the ability to simulate discontinuous behavior such as crack growth. A large portion of research into the peridynamic theory to date has focused on dynamic brittle fracture [11,16-18,15]. Unlike FEM and XFEM, peridynamics has the capability to model multiple cracks which interact, branching cracks, and crack coalescence all of which lends itself to the study of brittle materials.

Some research has been undertaken to study fatigue modeling using peridynamics where cyclic loads are simulated under quasi-static loading and the growth of a crack is tracked over many numbers of cycles. Three distinct fatigue model theories have been proposed to date. The first two fatigue models assume that cyclic loading causes a degradation in some material property allowing for eventual failure. The last fatigue model does not modify any material property but assigns a remaining life parameter which forces a bond to break once the remaining life reaches zero.

Material weakening models have been proposed by Zaccariotto, et. al [22] and Oterkus, et. al [23]. Both models allow a critical stretch value to define when a bond breaks. The model proposed by Zaccariotto, et. al presumes that the micromodulus of a bond weakens as a function of cyclic loading. After a sufficient number of cycles, the bond stretch reaches the critical value breaking the bond allowing for extension of the crack surface. An alternative model by Oterkus,
et. al surmises that the critical stretch of a bond rather than the micromodulus degrades as a function of increasing numbers of cyclic loading.

Both of these models have shown correlation with experimental results for specific examples. However, both of these models are limited in that they are capable of predicting the crack growth stage alone and not initiation. The model proposed by Silling and Askari develops a remaining life parameter associated to each bond within a body which tracks usable number of cycles to failure [24]. In this way the remaining life parameter behaves similar to Miner’s rule where failure occurs once a sufficient amount of damage is accrued. The later fatigue model is used in the present work.

5.2 Damage Parameter and Remaining Life

It is first necessary to define the state of damage at a material point before establishing the fatigue model. The damage at a point is defined as the ratio of the number of bonds broken relative to the total number of bonds in the reference configuration. Designate the status of a bond (broken or intact) with the parameter \( \mu \) as in equation (75) below.

\[
\mu = \begin{cases} 
1 & \text{for intact bonds} \\
0 & \text{for broken bonds}
\end{cases}
\quad (75)
\]

As outlined in [29], the local damage at a node is evaluated as in equation (76) below.

\[
\phi(i) = \frac{\int_{H_i} (1 - \mu(\xi)) dV_\xi}{\int_{H_i} dV_\xi} = \frac{n_{\text{broken}}}{n_{\text{reference}}}
\quad (76)
\]

Here \( n_{\text{broken}} \) represents the number of failed bonds and \( n_{\text{reference}} \) represents the number of bonds in the initial configuration i.e. where no damage has accumulated. Damage may take on any real
value between 0, in which all bonds at a material point are intact, and 1 at which time all bonds initially associated to the material point have irreversibly broken.

While the damage parameter above is a material point parameter, remaining life is a bond-specific parameter. The remaining life is most often cited in the literature as \( \lambda \), not to be confused with the correction factor for the skin effect. A value of \( \lambda \) is assigned to every bond in the model and can range between 0 and 1. A value of 1 indicates that no fatigue damage has occurred and is the initial value for all bonds prior to load application. A value of 0 occurs when the fatigue limit of the bond is reached and the bond has failed.

The change in remaining life is defined as [24]

\[
\frac{d\lambda}{dN}(N) = -A\varepsilon^m
\] (77)

where A and m are material specific parameters, \( \varepsilon \) is the cyclic bond strain, and N is the number of cycles applied. As the relation given by equation (77) suggests, the remaining life monotonically decays. The bond breaks for the first cycle \( N_{\text{critical}} \) at which \( \lambda(N_{\text{critical}}) \leq 0 \). Implementation of the remaining life parameter will be described further in the following sections.

5.3 Crack Initiation – Phase I

It will be assumed here that prior to any applied load cycles, unless otherwise specified, the body in question will be free of initial cracks. In this manner, a fracture surface must first nucleate before entering the crack growth phase. Remaining life of a bond may be computed using equation (77) assuming that appropriate values of the constants A and m can be selected. These parameters are derived from existing stress-life material data (i.e. an S-N curve). First note
that the constitutive model in use is brittle elastic and two restrictions are placed on the fatigue model inherent to the derivation. The first restriction is that all strains in the body even those at the crack tip are assumed to remain elastic. The second restriction is that stress (and strain) are linearly proportional to applied load. Note that these assumptions are very similar to the assumptions for the stress-life (S-N) theory. (In stress-life theory, the underlying assumption is that all strains, even at notches, remain elastic [43]).

Silling presents a procedure by which to derive the parameters A and m for the initiation phase in [24]. A brief review of that method is presented below. Consider equation (77) again and denote parameters for the initiation phase with subscript 1. Let \( N = N_1 \) be the number of cycles until the first bond breaks anywhere in the model. Based on the assumption of load proportionality, until the first bond breaks, the strain in all bonds will be constant assuming a constant applied load. Then the bond strain for that bond which is governing may be defined as \( \varepsilon = \varepsilon_1 \) where \( \varepsilon_1 \) is not a function of \( N \).

As was done in [24] proceed by integrating equation (77) as follows.

\[
d\lambda = -A_1 \varepsilon_1^{m_1} dN
\]

\[
\int_1^0 d\lambda = \int_0^{N_1} (-A_1 \varepsilon_1^{m_1}) dN
\]

\[
[\lambda]_1^0 = -A_1 \varepsilon_1^{m_1} \int_0^{N_1} dN
\]

\[-1 = -A_1 \varepsilon_1^{m_1} [N]_0^{N_1} = -A_1 \varepsilon_1^{m_1} N_1
\]

\[
N_1 = \frac{1}{A_1 \varepsilon_1^{m_1}}
\] (78)

Taking the logarithm of both sides of equation (78)
\[
\log(N_1) = \log\left(\frac{1}{A_1\varepsilon_1^{m_1}}\right) = \log(1) - \log(A_1\varepsilon_1^{m_1})
\]

\[
\log(N_1) = 0 - \left\{\log(A_1) + \log(\varepsilon_1^{m_1})\right\}
\]

\[
-\log(N_1) = \log(A_1) + m_1\log(\varepsilon_1)
\]

\[
-\frac{1}{m_1}\log(N_1) - \frac{\log(A_1)}{m_1} = \log(\varepsilon_1)
\]

Equation (79) represents the equation of a line on a log-log plot where \(\varepsilon_1\) is plotted on the ordinate and \(N_1\) plotted on the abscissa.

Equation (79) defines a relation between cyclic bond strain and number of fatigue cycles. Similarity is observed between equation (79) and the log-log form of the Basquin equation [43] given as equation (80) below.

\[
-\frac{1}{b}\log N + \frac{1}{b}\log a = \log S
\]

In the above equation, \(S\) is the stress amplitude, \(N\) the number of fatigue cycles, and \(a\) and \(b\) are material parameters. The form of equation (79) is a form of an SN diagram with the ordinate axis given in terms of strain as opposed to stress. By plotting SN material data in \(\log N - \log \varepsilon\) form, the parameters \(A_1\) and \(m_1\) can be determined as shown in Figure 10 and first posed in [24].
5.4 Crack Growth – Phase II

While equation (77) exclusively defines the remaining life parameter of a bond, values of $A$ and $m$ will differ for the crack initiation and crack growth phase as the two mechanisms are different. A precursor to the growth phase is that damage nucleation has previously occurred at one or multiple sites or a pre-crack is present. In either case, a crack tip has formed which will govern the growth stage. A summary of the procedure for determining the values of $A$ and $m$ for the growth phase is presented below. For additional details of the method, the reader is referenced to [24].

To apply the phase II crack growth model, limit the bonds subject to phase II to bonds within the horizon of the crack tip nodes. Bonds outside of the horizon remain subject to phase I.
Assume that local deformation behavior at the crack tip is constant in the frame of reference of the crack tip. Take $\frac{da}{dN}$ to be the incremental crack growth, constant for each cycle, where $a$ is the length of the crack and $N$ is the number of cycles. From [24] a spatial description of the crack tip location is

$$z = x - \frac{da}{dN} N$$  \hspace{1cm} (81)

where $z$, $x$, and $a$ all lie along the crack axis (see Figure 11).

![Figure 11: Coordinate Systems for Describing Deformation Local to the Crack Tip (Only Select Bonds Shown)](image)

The variable $z$ defines the horizontal axis of a local coordinate system with origin at the crack tip and $z$-axis oriented in the direction of the propagating crack. The parameter $x$ is chosen such that a constant condition is $z = 0$ at the crack tip. From the previous assumption of constant
deformation in the vicinity of the crack tip, the remaining life and cyclic bond strain can be described in terms of $z$ as

\[
\varepsilon(N) = \varepsilon(z) \\
\lambda(N) = \tilde{\lambda}(z)
\]

(82)

where the cyclic strain and remaining life behavior are now defined in terms of a relative position from the crack tip rather than as a function of the number of fatigue cycles. Calculate $\tilde{\lambda}(0)$, remaining life at $z=0$, by taking the derivative of $\tilde{\lambda}(z)$ and integrating.

\[
\frac{d}{dz} \left( \tilde{\lambda}(z) \right) \\
\int_0^\delta \frac{d}{dz} \tilde{\lambda} \, dz = [\tilde{\lambda}]_0^\delta = \tilde{\lambda}(\delta) - \tilde{\lambda}(0)
\]

(83)

After applying the chain rule

\[
\tilde{\lambda}(\delta) = \tilde{\lambda}(0) + \int_0^\delta \frac{d}{dN} \frac{d}{dz} \tilde{\lambda} \, dz 
\]

(84)

After substituting Equation (81) and equation (77),

\[
\begin{align*}
\tilde{\lambda}(\delta) &= \tilde{\lambda}(0) + \int_0^\delta (-A_2 e^{m_2}) \frac{d}{dz} \left( \frac{x - z}{d(\delta)} \right) \, dz \\
\tilde{\lambda}(\delta) &= \tilde{\lambda}(0) + \int_0^\delta (-A_2 e^{m_2}) \left( \frac{-1}{d(\delta)} \right) \, dz \\
\tilde{\lambda}(\delta) &= \tilde{\lambda}(0) + A_2 \frac{d}{dN} \int_0^\delta (\varepsilon(z))^{m_2} \, dz
\end{align*}
\]

(85)
From the previous assumption of constant deformation in the vicinity of the crack tip the cyclic strain function can be described by [24]

\[ \bar{\varepsilon}(z) = \bar{\varepsilon}(0)f(z) \] (86)

Above, the variable \( \bar{\varepsilon}(0) \) is the cyclic strain at the crack tip (i.e. \( z=0 \)) and \( f(z) \) is a function varying only with relative position from the crack tip. As defined in [24], remaining life at the crack tip is 0 as the bond is on the verge of breaking and remaining life at a distance of \( \delta \) or greater is equal to 1 since damage is only applied within the horizon of the crack tip. Denote the bond at the crack tip which has remaining life equal to 0 as the core bond. Then noting that \( \bar{\lambda}(\delta)=1 \) and \( \bar{\lambda}(0)=0 \), equation (85) can be written as

\[ 1 = \beta A_2(\bar{\varepsilon}(0))^{m_2} \frac{da}{dN} \] (87)

where

\[ \beta = \int_{0}^{\delta} (f(z))^{m_2} dz \] (88)

Rearranging gives

\[ \frac{da}{dN} = \beta A_2(\bar{\varepsilon}(0))^{m_2} \] (89)

Equation (89) provides a relation between the core bond strain and the crack growth rate.

In the traditional fracture mechanics, stable crack growth occurs in stage II and is described by the relation [44]

\[ \frac{da}{dN} = c\Delta K^m \] (90)
Here $\Delta K$ is the stress intensity factor at the crack tip, and $c$ and $M$ are material parameters determined empirically. This relation exhibits similarities to the crack growth rate defined by equation (89). By letting $\Psi = \beta A_2$, equation (89) can be written in the form

$$\frac{da}{dN} = \psi \varepsilon_0^{m_2} \quad (91)$$

For the fatigue model to be valid the crack growth rate, $da/dN$, must match the observed behavior, that is $da/dN$ described by the Paris Law is equivalent to $da/dN$ described by the damage model. Based on assumptions of material linearity and load proportionality, the cyclic strain at the core bond, $\varepsilon_0$ from equation (86), is directly proportional to the stress intensity factor at the crack tip from equation (90). For equation (91) and equation (90) to be consistent, it must be true that $m_2 = M$. Therefore, the parameter $m_2$ in equation (77) is found directly from the Paris Law plot (see Figure 12).
The remaining unknown, $\Psi$, cannot readily be determined by relating equation (91) to equation (90) meaning $A_2$ cannot be readily found. To find $A_2$, the damage model as currently defined without a precise value of $A_2$ is applied to an arbitrary simulation model which contains a pre-crack. A small number of cycles are simulated such that crack advance has occurred in the simulation. By taking the change in length of the crack from the model divided by the number of cycles that were applied, a simulated crack growth rate $(\frac{da}{dN})_{A_2}$ is obtained for an arbitrarily chosen value of $A_2$. From equation (89) crack growth rate, $\frac{da}{dN}$, is a linear function of $A_2$. By taking the ratio of the simulated value of $\frac{da}{dN}$ with the measured value of $\frac{da}{dN}$ obtained
directly from the Paris Law plot, the calibrated value of the material parameter $A_2$ is found from [24]

$$A_2 = A_2' \frac{da/dN}{(da/dN)_{A2'}}$$

(92)

Here $(da/dN)_{A2'}$ is the growth rate taken from the small, initial model, $da/dN$ is the growth rate from lab measurement fitted to the Paris Law and $A_2'$ is the arbitrary choice of the $A_2$ parameter in the remaining life equation from the initial model. With the value of $A_2$ found, definition of the phase II remaining life damage model is complete.

5.5 Change from Phase I to Phase II

A criterion must be specified governing when a crack transitions from the initiation phase to the growth phase. It has been proposed by Silling [24] to define the transition from initiation to growth at a node $x$ when the value of damage of any node $q$ in the family of $x$ has damage $\geq 0.5$. A damage value of 0.5 indicates that one half of all bonds have been severed, potentially along a continuous plane such that two distinct surfaces are formed indicating the presence of crack formation (see Figure 13). When a node transitions to phase II all bonds attached to that node switch from the phase I remaining life parameters to phase II and at the same time the remaining life is reset to 1.

$$\text{Phase I } \phi_q < 0.5 \text{ for all } q \text{ in } H_x$$

$$\text{Phase II } \phi_q \geq 0.5 \text{ for one or more } q \text{ in } H_x$$

(93)

In certain instances, a crack may already be present in the sample. In this case, the initiation phase is bypassed for all nodes within the horizon of the crack tip and these nodes are initialized to the phase II growth phase. Locations outside the horizon region of the crack tip
would be initialized to the phase I initiation phase similar to the case where no crack is present.

Figure 13: Bond Damage as a Method to Define a Crack in Peridynamics

5.6 Limitations of the Model

While not specifically addressed previously, it should be noted that for ductile materials, the present form of the fatigue damage model is restricted to high cycle fatigue. Many of the assumptions in deriving the peridynamic material parameters assumed small deformation theory, load proportionality, and linear material response. For example, the application of the load ratio (see Section 6.5) to reduce the number of simulated steps assumes linear response to the applied loading. The use of S-N data to calibrate the crack initiation phase of the fatigue damage model also assumes moderate to high cycles.
Chapter 6. IMPLEMENTATION OF THE DAMAGE MODEL IN FEM

6.1 Implementation of the Damage Model

The proposed fatigue damage models from Section 5 were each implemented in some form or another in either a user-created code scripted in Matlab/FORTRAN [20,22] or an open-source peridynamic code such as EMU/Peridigm [24,41,42]. In the present work a procedure is outlined, as shown in Figure 14, by which the remaining life fatigue damage model can be paired directly with an existing FEM software via use of user subroutines. This allows the full advantages of the efficiencies of a commercial FEM solver to be exploited and allows for pairing to existing finite elements.

6.2 Available User Subroutines in Abaqus

The commercial FEM software selected for the analysis was Abaqus. Abaqus offers multiple user subroutine structures capable of providing the necessary functionality to program the fatigue damage model algorithm. Two of the most flexible subroutines for this purpose are UEL and UMAT. UEL is a subroutine which allows for user-defined element creation and UMAT creates a user-defined material definition, but these subroutines are recommended for advanced users of Abaqus only. The subroutines USDFLD (user-defined field variables) and URDFIL (user-defined subroutine for reading the results {.fil} file) are capable of satisfying the requirements of the fatigue damage model and are more accessible to most Abaqus users. The function of the URDFIL subroutine allows access to nodal and element data at the end of every solution step. The USDFLD subroutine allows access to element integration point variables and
modification of solution dependent variables during a solution step. The URDFIL and USDFLD subroutines were used for execution of the fatigue damage model in the present work.

Figure 14: Algorithm for Fatigue Damage Model via User Subroutine

6.3 Peridynamic Variables and SDV’s

The remaining life fatigue damage model requires the remaining life be tracked for every bond in the model. Abaqus allows for unique variables to be defined via the solution dependent
variables (SDV) functionality. This method allowed variables to be stored in the Abaqus framework rather than in an external array which allowed for reduced computation time and enabled for results to be plotted readily. Two SDVs were defined for each element with one retaining the remaining life of the particular element and the other maintaining element status (intact or severed). Remaining life was initialized to one for all elements in the model in the first step of the analysis and was updated in each subsequent step. The element status parameter was initially a uniform value of one for all elements in the model. Once the remaining life of a particular element was less than or equal to zero, the status was changed to zero and the element was inactivated (deleted from the solution phase).

6.4 Locating the Crack Tip and Identifying the Damage Region

As described in Chapter 5, there are two phases associated to the remaining life damage model and it becomes necessary to define the current phase for every bond. As described in [24] phase II damage applies to (a) bonds within the horizon of a pre-crack tip or (b) those lying in a region which has transitioned from phase I to phase II. For all other regions, phase I would apply. A pre-crack was present in the CT specimen used for the present analysis and therefore, all elements within the horizon of the crack tip were initialized to the phase II growth phase bypassing the initiation phase. Due to the high concentrations of strain energy present at the crack tip, spurious crack development away from this location was deemed unlikely and therefore, phase I fatigue (crack initiation) was out of scope for the present work. However, implementation of phase I is feasible utilizing the present algorithm requiring only assignment of parameters $A_1$ and $m_1$ and a calculation of the nodal damage, $\phi$, as described in Section 5.2. By replacing $A_2$ and $m_2$ (phase II parameters) in the current algorithm with $A_1$ and $m_1$ derived as
described in Section 5.3, the damage model can be modified to compute the time to crack initiation. The current phase (I or II) at a given node is maintained by the damage variable, \( \phi \), calculated as the ratio of the number of broken bonds (elements with SDV element status = 0) to the number of intact bonds (elements with SDV element status = 1).

As the simulation progresses nodes along the length of the crack, excluding the tip, have damage greater than 0.5 due to complete separation along a 2D surface (1D line in the case of the 2D assumption). These nodes technically meet the transition criteria from phase I to phase II by consequence of the damage variable. However, as investigated in [19], continued phase II damage accumulation along these surfaces leads to highly unrealistic behavior such as crack thickening and crack growth in regions removed from locations of maximal strain energy density. The solution proposed in [19] was the addition of a threshold stress intensity factor (SIF) range to the fatigue model below which cracks do not advance. Implementation of a threshold SIF range to the damage model was out of scope for the present work and here the phase II region was limited to the horizon region of the crack tip. Regions on the crack surface away from the tip meeting the second of the phase II criteria (\( \phi > 0.5 \)) were not subject to degradation of remaining life.

It has already been noted that the core bond is the element under-going the largest amount of strain in the vicinity of the crack. It was necessary to track the core bond throughout the simulation as this element defines the crack tip and thus those elements that are susceptible to a reduction in remaining life. The URDFIL user subroutine was used to access element strain data and from a search for the maximum value, the coordinates of the crack tip were identified. The crack growth phase of the damage model requires that bonds on the edge of the horizon or beyond have remaining life equal to one and remaining life degrades only for those bonds within
the horizon of the crack tip. This subset of elements was identified based on element centroid values. The URDFIL subroutine allows access to centroid locations for all elements such that, once the core bond location is known, the remaining elements subject to damage accumulation can be found.

**6.5 Bond Damage and Deletion**

With the horizon region of the crack tip identified in the URDFIL subroutine, the USDFLD subroutine was used to compute the change in remaining life according to equation (77). Remaining life is a function defined by the number of cycles, the cyclic strain, and two material-dependent parameters established from either SN data (phase I) or Paris Law data (phase II). The subroutine operated by first computing the cyclic strain for each element. The experiment by Miranda, et. al [40] used a constant load ratio in each cycle of $R = 0.1$ where the load ratio is defined in equation (94) as the ratio of the minimum load to the maximum load.

\[ R = \frac{\text{Load}_{\text{MIN}}}{\text{Load}_{\text{MAX}}} \]  

(94)

Given the assumption of linearity, the load ratio combined with the maximum load condition allowed the cyclic strain range to be computed by only simulating the maximum load state. In this case, the cyclic strain range was calculated as [24]

\[ \varepsilon_{\text{cyclic}} = |\varepsilon_{\text{max}} - \varepsilon_{\text{min}}| = |(1 - R)\varepsilon_{\text{max}}| \]  

(95)

The simulation itself consisted of a series of static solutions (simulation steps) which, as a whole, was representative of quasi-static crack growth. To establish a relation between the series of discrete static simulations and the continuous fracture process, a time mapping was introduced which defined a relation between fatigue cycles and simulation steps. Both linear and
exponential time mapping are defined in [24]. The author defines a relation between a fictitious simulation ‘time’ and an equivalent number of cycles. A similar relation is developed in the present work but rather than define mapping between fatigue cycles and fictitious simulation time, simulation step number was used. As shown in equation (96) the number of fatigue cycles at any instance of the simulation, N, is given by the current time step, n, and a constant mapping coefficient, μ. For all horizon sizes, a value of μ=900 was utilized. While the choice of μ is somewhat arbitrary, a value should be selected which is large enough to allow economical solve times yet small enough so as to prevent an excessive amount of broken bonds in a single time step which can lead to unstable behavior and invalid crack growth direction.

\[ N = \mu(n_i - 1) \]  

(96)

Material dependent parameters A and m for use in the remaining life equation were found according to Chapter 5. The parameter m is found directly from a Paris Law relation. From the material data provided in Miranda, m=2.1 [40]. The remaining parameter, A2, was determined from a calibration procedure described in Section 6.6.

All elements were initialized to have remaining life equal to one. A conditional statement was included within the USDFLD subroutine which checked the remaining life value for all elements during each time step. The SDV used for monitoring element status was used as a deletion flag and was updated based on the value of remaining life. Any element with remaining life below 0 had the status SDV changed to inactive and the element was removed from the model.
6.6 Calibration Step

As described in Section 5.4, the parameter $A_2$ cannot be determined from material test data. The value of $A_2$ is found iteratively by first simulating crack growth in an arbitrary shape section in a model hereto referred to as the iteration model and then calculating the exact value to be used in the calibrated model (i.e. the model under study). While the geometry of the iteration model is arbitrary, the horizon size and material specification must match the calibrated model. Simulation of the iteration model will produce crack growth whose rate will be nearly constant and thus, the simulated crack growth rate can be determined by dividing the total length of the crack at the end of the simulation by the number of applied fatigue cycles (see discussion of time mapping above). This value of simulated crack growth rate in the iteration model is designated as $(da/dN)_{A_2'}$. A calibrated value of $A_2$ for use in the calibrated model is then found from equation (92), repeated below.

$$A_2 = A_2' \frac{da/dN}{(da/dN)_{A_2'}} \quad (97)$$

In equation (97) $(da/dN)_{A_2'}$ is the simulated crack growth rate from the iteration model, $da/dN$ is the measured growth rate available from Paris Law data for the material, $A_2'$ is the value of $A_2$ used in the iteration model and $A_2$ is the calibrated value of the material parameter of the same name from the remaining life equation, equation (77).

Due to the dependence of $A_2$ on horizon size, each of the models of varying mesh density used in the present study required an iteration model. For simplicity, the same mesh from the calibrated model was used in the iteration model. All iteration models were initially analyzed for a total of 30 time steps at the end of which crack length was plotted. For the iteration models,
each time step used a mapping coefficient $\mu=1750$. The calibrated value of $A_2$ was then calculated from equation (97) using the simulated crack growth rate $(da/dN)_{A2}$ from the iteration model together with the measured crack growth rate from the Miranda experiments, $da/dN = 3.9283 \times 10^{-5}$ mm/cyc [40]. The calibrated values of $A_2$ were then applied to the final simulations {i.e. the calibrated models} (see Table 2).

Table 2: Values of the Calibration Parameters for the Phase II Damage Model

<table>
<thead>
<tr>
<th>Horizon Size (mm)</th>
<th>Parameter $m_2$</th>
<th>Parameter $A_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.2</td>
<td>2.1</td>
<td>39.3</td>
</tr>
<tr>
<td>0.6</td>
<td>2.1</td>
<td>41.2</td>
</tr>
<tr>
<td>0.5</td>
<td>2.1</td>
<td>53.5</td>
</tr>
</tbody>
</table>
Chapter 7. CONSTANT SIF RANGE VIA THE J-INTEGRAL

7.1 J-Integral and Stress Intensity Factor (SIF)

In the work by Miranda, et al [40] experimental crack growth data for the modified CT specimen was obtained utilizing a constant stress intensity factor range at the crack tip. In [40] a loading program was created for the test apparatus which maintained a constant ΔK₁ value of 20 MPa √m. In order to simulate the experiment of [40] it was necessary to maintain the same constant stress intensity factor range. This was achieved by calculating the SIF range by means of the nonlocal J-integral (see Sec. 7.3-7.4) throughout the crack advance and updating the loading applied in the simulation accordingly. From classical fracture mechanics, the stress intensity factor (SIF) for mode I crack growth is related to the classical J-integral through equation (98) assuming the material is either linear elastic or nonlinear elastic [45].

\[
J_{local} = \frac{K_I^2}{E} (1 - \nu^2) \{Plane\ Strain\} \tag{98}
\]

It is important to note that the relation of equation (98) is founded on classical fracture mechanics and thus the values of E and ν are the classical values. It was shown in [33] that the nonlocal J-integral converges to the local (classical) J-integral as the horizon size approaches zero. The nonlocal J-integral may be used in lieu of the local J-integral to compute the stress intensity factor according to equation (99).

\[
K = \frac{J_{nonlocal}E}{\sqrt{1 - \nu^2}} \{Plane\ Strain\} \tag{99}
\]
7.2 Classical J-Integral

The classical J-Integral described by Rice [45] provides a way to describe the energy release rate in an elastic solid (linear or nonlinear) at a growing crack. The J-Integral is a contour integral enclosing the crack tip and is described by equation (100).

\[
J = \int_S W \, dy - T \cdot \frac{\delta u}{\delta x} \, dS
\]  

(100)

Here W represents the strain energy density, T represents the traction vector in the direction of the unit outward normal, n, and \( \frac{\delta u}{\delta x} \) represents the derivative of displacement with respect to the global x direction. The differentials dy and dS are evaluated in the vertical (global y) direction and along the integration path, respectively. The choice of integration path is arbitrary as shown in Figure 15 but the integration path must start at the lower face of the crack surface and terminate at the upper face. Note that the global coordinate axes are aligned parallel to (x-axis) and normal to (y-axis) the crack direction.
Figure 15: Contour Path for Evaluation of Classical J-Integral

7.3 Peridynamic J-Integral

The peridynamic J-integral is constructed in a similar manner where a contour path $\partial R$ traverses the crack tip from the bottom to the top face (see Figure 16). First proposed by Hu, et. al [33] for the 2D bond-based case, the J-integral is described by equation (101).

$$ J_{peri} = \oint_{\partial R} W(x; a)n_1 dS - \frac{1}{2} \int_{R_2} \int_{R_3} f(\eta, \xi) \cdot \left( \frac{\partial u_q}{\partial x_1} + \frac{\partial u_x}{\partial x_1} \right) dA_q dA_x $$  (101)

Similarities are observed with the classical J-integral with one key difference being that the second term of the equation is an integration over a domain rather than along a contour. This arises from the fact that the peridynamic theory is nonlocal with interactions occurring over a volume rather than across a surface.
The first integral is the same contour integration from the classical theory with $W$ representing the strain energy density function and $n_1$ representing the outward unit normal in the direction of the global x vector. The integration is performed over the path $\partial R$. The strain energy density function is evaluated using the relation of the bond micropotential and strain energy density given by equation (8). In Section 7.4.1, a relation is developed between the micropotential function and the strain energy in each of the truss elements defining the bonds.

The second term evaluates the work done by the force “flux” across the integration contour, $\partial R$. Because the bonds in peridynamics act over a distance the line contour expands to an area integration. The integrand of the second term evaluates the scalar product of the bond force density and the displacement derivatives over this area domain. The integration is evaluated for bonds with one node lying inside the boundary of $dR$ and the second node lying on the exterior of $dR$. By identifying the node in the interior region as $x$ and the exterior node as $q$ it
is easily verified from the definition of the horizon that all nodes \( x \) fall within a band of thickness \( \delta \) on the inside of \( \partial R \) and all nodes \( q \) fall within a band of thickness \( \delta \) on the outside of \( \partial R \). If the interior band is defined as \( R_2 \) and the exterior band as \( R_3 \) as in Figure 16, the double integral will consist of an integration over all differential area \( dA_q \) in \( R_3 \) and all differential area \( dA_x \) in \( R_2 \).

For all bonds which do not have a single node contained in the region \( R_2 \) and the other node contained in \( R_3 \), the double domain integral is zero [33].

The numerical solution of the peridynamic J-integral is obtained by discretizing equation (101) as

\[
J_{peri} \approx \sum_{n=1}^{n_{contour}} W_{n1}(\Delta s) - \frac{1}{2} \sum_{x=1}^{n \in R_2} \sum_{q=1}^{n \in R_3} \left[ f_1(\eta, \xi) \left( \frac{\partial u_1^q}{\partial x_1} + \frac{\partial u_1^x}{\partial x_1} \right) + f_2(\eta, \xi) \left( \frac{\partial u_2^q}{\partial x_1} + \frac{\partial u_2^x}{\partial x_1} \right) \right] (\Delta A_x)(\Delta A_q)
\]

The first summation is found based on evaluating strain energy density at discrete intervals along the integration contour. By selecting a rectangular integration path \( n_1 = 0 \) on horizontal segments and \( n_1 = 1 \) for vertical segments. The incremental distance \( \Delta s \) is evaluated as the nodal spacing \( \Delta x \).

FE solution data is used to compute the second summation. It will be shown in Section 7.4.2 that the components of the bond force density vector can be calculated from truss element axial stress. The displacement derivatives are approximated in Section 7.4.3. A uniform mesh is used in the present work and \( \Delta A_x \) and \( \Delta A_q \) are both given as \( \Delta x^2 \) for all nodes \( x \) and \( q \).
It was previously mentioned that the peridynamic J-integral converges to the classical J-integral as the horizon size approaches zero [33]. By inspection of the first term of the peridynamic J-integral it is clear that the strain energy density must match in the peridynamic and classical formulations because equivalence of strain energy density was enforced in the development of the peridynamic constitutive model. Noting also that for the rectangular integration paths utilized the term \( n_1^* dS \) becomes \( dy \) as \( n_1 \) is equal to 1 on vertical integration paths and zero on horizontal paths. In the second term of the peridynamic J-integral note that for \( \delta > 0 \), the bond position vector, \( \xi \), approaches zero and the displacement derivative, \( \partial u_i / \partial x_1 \) for \( i = x \) approaches that for \( i = q \). The bands \( R_2 \) and \( R_3 \) of the double domain integral shrink to a line as the horizon approaches zero resulting in a contour integration over \( \partial R \), the same as the classical theory.

**7.4 Algorithm for Calculation of Peridynamic J-Integral**

**7.4.1 Calculation of Peridynamic J-Integral**

An algorithm was developed to compute the peridynamic J-integral within the framework of the Abaqus solver. The discretized form of the nonlocal J-integral is given in equation (102). Due to the path independence of the J-integral an arbitrary size and shape can be selected with some exceptions to the proximity to boundaries and the crack tip [33]. For simplicity of calculation a rectangular contour is chosen as seen in Figure 17. The contour path lies halfway between collocation points of the uniform mesh (along the edges of the material volumes depicted in Figure 2) and thus, the strain energy term must be averaged between adjacent nodes on each side of the contour. Strain energy density at each point is found by relating the
micropotential in the bonds to the strain energy density function in peridynamics through equation (8).

![Image](image.png)

Figure 17: Rectangular Integration Path used in Evaluation of Peridynamic J-Integral

### 7.4.2 Relation of SED to Micropotential

As previously mentioned, the method by which peridynamic bond forces are modeled within Abaqus is via truss elements. Strain energy density is accessible during solution for all trusses in the model. The strain energy term of equation (101) can be evaluated by relating the strain energy density in each truss element to the strain energy density at the collocation point.

The energy stored in a bond is described in peridynamics by the micropotential. To relate the micropotential to existing quantities within FEM, the FEM evaluation of micromodulus given as equation (69) is substituted into equation (9). The micropotential then has the form,

$$\omega = \frac{1}{2} \left( \frac{AE_{FE}}{V_s V_q} \right) s^2 |\xi|$$

(103)
Noting that $\xi = L$ and $s = \varepsilon$, where $L$ and $\varepsilon$ are truss length and truss strain, respectively, equation (103) is rewritten as

$$\omega = \frac{1}{2} \left( \frac{A E_{FE}}{V_x V_q} \right) \varepsilon^2 L = \frac{1}{2} A E_{FE} L \varepsilon^2$$  \hspace{1cm} (104)$$

Recall that for a single tension/compression member such as a truss subject to applied load the strain energy is given as equation (105) below.

$$U = \frac{1}{2} A E L \varepsilon^2$$  \hspace{1cm} (105)$$

Then upon substitution

$$\omega = \frac{U_{FE}}{V_x V_q}$$  \hspace{1cm} (106)$$

where $U_{FE}$ is the strain energy for each truss element. Recall the relation of strain energy density and micropotential,

$$W(x) = \frac{1}{2} \int_{H_x}^{V_x} \omega \, dV_q$$  \hspace{1cm} (107)$$

Noting that $\omega$ is already in a discretized form in equation (106) ($U_{FE}$ is evaluated from the truss elements and $V_x$ and $V_q$ are discrete forms of differential volume) equation (107) is rewritten in discrete form as

$$W(x) \approx \frac{1}{2} \sum_{q=1}^{\text{Family of } x} \omega \, V_q = \frac{1}{2} \sum_{q=1}^{\text{Family of } x} \left( \frac{U_{FE}}{V_x V_q} \right) V_q = \frac{1}{2} \sum_{q=1}^{\text{Family of } x} \frac{U_{FE}}{V_x}$$  \hspace{1cm} (108)$$
The parameter $U_{FE}$ is known from the strain energy density of the truss available as output data from the FEM solution and its volume. With the strain energy density for all points on the integration path evaluated according to the equation (108) above, the first term of the nonlocal $J$-integral is readily evaluated.

### 7.4.3 Relation of Element Force to PD Bond Force Density

In order to evaluate the second term of the $J$-integral, the peridynamic bond force must be known for all elements which have one node in region $R_2$ and one node in region $R_3$. A user subroutine was developed which identifies all nodes lying in region $R_2$ and all nodes lying in region $R_3$. Mesh connectivity data then allows elements to be identified which possess one node within each region. From nodal information the position components $\xi_x$ and $\xi_y$ can be determined for determining the components of the load force density function.

Peridynamic bond force is not directly available from the FE solution and thus it is necessary to calculate this value from available integration point variables. The micromodulus function was previously derived in equation (69) using terms directly obtained from the FE model. By noting that for truss elements the peridynamic parameter stretch is equivalent to strain, substitution of equation (69) into equation (5) allows the bond force density to be written as equation (109) below.

$$f = c\varepsilon = \left(\frac{AE_{FE}}{\Delta x^4 t^2}\right)\varepsilon = \left(\frac{E_{FE}}{(\Delta x)^2 t^2}\right)\varepsilon \quad (109)$$

Utilizing Hooke’s Law, bond force density can be written in terms of mesh dependent parameters and stress.
\[ f = \frac{\sigma}{(\Delta x)^2 t^2} \]  

In equation (110), \( f \) is the peridynamic bond force, \( \sigma \) is the stress in a truss element evaluated from the FE solution, \( \Delta x \) is the distance between collocation points and \( t \) is the thickness.

**7.4.4 Evaluation of Displacement Derivatives**

The displacement derivative with respect to global \( x \) is required for each node \( x \) and node \( q \) associated to a bond evaluated in the second integral. These derivatives can be evaluated numerically at node \( q \) as [33]

\[
\frac{\partial u_1}{\partial x_1} \approx \frac{u_1(x_1 + \Delta x, x_2^q) - u_1(x_1 - \Delta x, x_2^q)}{2(\Delta x)} \\
\frac{\partial u_2}{\partial x_1} \approx \frac{u_2(x_1 + \Delta x, x_2^q) - u_2(x_1 - \Delta x, x_2^q)}{2(\Delta x)}
\]

(111)

Here \( x \) defines a position vector in the reference configuration, \( u \) defines the displacement vector, and \( \Delta x \) is the spacing between nodes. The subscripts 1 and 2 correspond to the horizontal and vertical coordinates, respectively. The partial derivatives at node \( x \) are performed similarly.

**7.4.5 Peridynamic J-Integral Algorithm**

The peridynamic J-integral calculation can be broken into subsequent procedures where mesh and contour integration dependent data is obtained, the first and second terms of the J-integral are calculated, and finally the J-integral is calculated (see Figure 18). User-defined subroutines such as URDFIL and USDFLD do not have access to nodal position data or element connectivity data and thus it is necessary to read this data into arrays managed by the user. The most efficient method is to read portions of the solver file (the FE data file containing the mesh
and boundary conditions) and store this data into an array containing node data and an array containing element data. To complete the configuration portion of the J-integral algorithm coordinates of the corners of a rectangular integration path are specified.

Figure 18: Algorithm for Calculation of Peridynamic J-Integral within FEM Software

The first term of the J-integral (both classical and peridynamic) is the evaluation of the flux of strain energy density across the integration boundary. To define this value it is necessary
to establish the nodes along the integration contour. As shown in [33], the integration path is along the edges of the material subdomains so an average value of strain energy density must be computed from the values of the collocation points on either side of the path. Nodes encompassing the integration path are found from a search algorithm which uses the user-input coordinates from the corners of the integration path and the nodal array data. A relation was identified in Section 7.4.1 which allows the internal results for strain energy density in the individual truss elements to be related to the micropotential and thus to the strain energy density at the collocation points. Strain energy results are averaged from collocation points on either side of the integration path to obtain the strain energy density on the path. With strain energy density known at discrete locations on the contour the first term of the peridynamic $J$-integral is calculated by the product of strain energy density, the outward normal of the integration path and the grid spacing as shown in equation (102).

The second term of the $J$-integral involves integration over a region rather than along a contour in the classical sense due to the nonlocal attribute of peridynamics. To establish this region it is necessary to identify all bonds which transmit force across the integration contour. These bonds describe a domain of force flux akin to a traction surface (curve in 2D) inherent to the applied work term of the $J$-integral (second term). Hu, et. al. identified that all bonds which have non-zero contribution to the second term of the $J$-integral are contained within a band of $2\delta$ centered on the integration path [33]. Furthermore, only bonds which have one node on the inside of the integration path and the other node on the outside are non-zero. As mentioned previously, no collocation points lie on the integration boundary such that nodes are defined uniquely on the interior of the integration contour as well as on the exterior. Hu, et. al define the
interior band as the region $R_2$ and the exterior band as the region $R_3$ and the same terminology is retained in the present work.

Once truss elements meeting the aforementioned criteria for a non-zero contribution have been identified, the URDFIL subroutine is used to extract nodal displacements and truss element stresses pertinent to these elements. Displacement derivatives are calculated at each node owning the bond as described in Section 7.4.3. The bond force density is calculated using the relation in Section 7.4.2 between net section force and bond force density. Vector components of the bond force density in the global x and y coordinates are calculated based on nodal positions stored in the user array. With the components of the bond force density and the displacement derivatives known the second, term of the J-integral can be computed and combined with the first to establish the peridynamic J-integral.

### 7.5 Calculating the Stress Intensity Factor Range and Load Adjustment

Equation (102) provides a means to calculate the peridynamic J-integral. The J-integral value may then be used to calculate the stress intensity factor range. A relation between the stress intensity factor and the J-integral was given for crack growth purely of the mode I type in equation (98). A relation between the J-integral and the stress intensity factor for all three fracture modes was provided in [45] for linear elastic solids as

$$J = \frac{1 - \nu^2}{E} (K_I^2 + K_{II}^2) + \frac{1 + \nu}{E} K_{III}^2$$  \hspace{1cm} (112)

Here $K$ denotes the stress intensity factor and the subscripts I, II, and III denote the particular fracture mode. For plane strain conditions $K_{III}$ is equal to zero and equation (112) simplifies to

$$J = \frac{1 - \nu^2}{E} (K_I^2 + K_{II}^2)$$  \hspace{1cm} (113)
For the CT specimen the loading is mode I dominated with small mode II contributions and the J-integral can be approximated as

\[ J \approx \frac{1 - \nu^2}{E} K_i^2 \] (114)

To establish the relation between the SIF and the SIF range, the definition of the load ratio is revisited. Previously, R was defined in equation (94) as the ratio of minimum and maximum loading. By noting that the stress intensity factor is proportional to load for materials with linear response, R can be redefined as

\[ R = \frac{K_{MIN}}{K_{MAX}} \] (115)

\( K_{MAX} \) is the mode I contribution of the SIF and is obtained from the FE solution of the J-integral and equation (114). For a specific load ratio R, the SIF range can be calculated without directly simulating the minimum load condition as [20]

\[ \Delta K_I = K_{MAX} - K_{MIN} = K_{MAX} - RK_{MAX} = K_{MAX}(1 - R) \] (116)

In order to maintain a constant \( \Delta K_I = 20 \text{ MPa} \sqrt{\text{m}} \), same as in the experiment, the stress intensity factor range is computed according to equation (117) and the applied load is adjusted [20].

\[ \Delta K_I \approx \sqrt{\frac{JE}{1 - \nu^2}} (1 - R) \] (117)

The load is adjusted according to equation (118). During the first cycle an initial load, \( P_0 \), is applied to produce \( \Delta K_I = 20 \text{ MPa} \sqrt{\text{m}} \). In subsequent steps the initial load is reapplied and the J-integral obtained and used to calculate the SIF range. By taking the product of the initial load \( P_0 \)
and the ratio of SIF ranges at the initial time step and at the \( n^{th} \) time step, the updated load \( P_n \) can be calculated.

\[
P_n = P_0 \times \frac{\Delta K_0(P_0)}{\Delta K_n(P_0)}
\]  

(118)
Chapter 8. RESULTS

8.1 Postprocessing

Postprocessing was completed in Altair Hyperview as this software is capable of plotting standard Abaqus variables as well as the user-defined solution dependent variables. Most custom codes which implement the peridynamic theory utilize a meshless scheme and thus, are not capable of deleting “elements” to establish the crack path. The preferred method of viewing the crack path is to plot the damage parameter, \( \varphi \), which displays the degree to which bonds from the reference configuration have been severed on a node by node basis. Locations where damage has not occurred will show a value of 0.0 while locations along the crack path will have \( 0 < \varphi \leq 1 \). Where damage exceeds 0.5 it can be assumed that a macroscale crack is present since \( \varphi=0.5 \) would imply that 50\% of the bonds at a collocation point have been eliminated creating two distinct surfaces.

Because the remaining life degrades to a certain extent for all elements within the horizon of the core bond (i.e. crack tip) having a radius of \( \delta \), bonds are broken over a width of \( 2\delta \) centered on the core bond (see Figure 19). The damage plot will exhibit a region with nonzero damage with width equal to twice the horizon and the centerline of this region defines the crack. This process can be supplemented by various software which can be utilized to shrink the damage plot with width \( 2\delta \) to a crack path of single pixels [20].
To allow the present study to be compared to existing research, crack path was also defined using the damage variable. Hyperview does not have an option to compress the damage plot to a single line but an alternate approach was used. A plot of element status (the second solution dependent variable defined) was captured with elemental quantities averaged at the nodes, producing a plot of $(1-\phi)$. Linear superposition was used to subtract this state from the reference state (where $\phi=1$ for the entire domain) to produce a plot of $\phi$. The resulting plot of $\phi$ could then be used to identify the crack path coordinates by taking the average of the spatial positions of nodes on either boundary of the damage region whose width was $2\delta$. To identify the crack tip the damage parameter was probed along the periphery of the non-zero damage region.
For complete separation of two surfaces the damage parameter must be greater than 0.5 and the crack tip was identified as the outward most node with $\varphi \geq 0.5$.

### 8.2 Crack Path Predictions

The direction of incremental crack propagation is predicted by many theories: maximum potential energy release rate, maximum stress, minimum strain energy density factor, etc. In [19] when studying the expected regions of crack growth, strain energy density plots were examined to evaluate the likelihood of potential crack paths when defining a new fatigue model. It was shown in [44] that for pure mode I loading a crack will propagate purely in the horizontal direction (normal to the applied tensile load). By introducing an auxiliary hole, the mode II stress intensity factor, $K_{II}$, is no longer zero and an unsymmetric crack path results.

It is plausible to assert that the crack will propagate towards the upper half of the sample given that the cross sectional area of the sample is reduced in this region by the addition of the tertiary hole. To evaluate this assumption, a plot of nodal values of strain energy density is given in Figure 20. The contour displays regions of greatest strain energy density in shades of red with regions of lower strain energy density in shades of blue and the legend scale fixed in all plots. A threshold value of strain energy density is applied to each of the four plots in the figure which shades all regions below a specified minimum value of strain energy density in black indicating, to a certain extent, regions which provide less contribution to the load path. The threshold value of strain energy density is gradually increased for the plots in the clockwise direction so that in the bottom left image, only bands of the highest strain energy density in a given region are shown in color. The plots indicate that strain energy density is asymmetric about the horizontal midplane of the specimen with greater strain energy density on the half with the added hole.
While this is a qualitative assessment only it leads one to predict a crack would propagate mostly horizontal from right to left as described in [44] with some bias toward the upper half of the specimen where strain energy density is greater.

Figure 20: Strain Energy Density in the CT Specimen with Initial Pre-crack under Static Load (Zero Fatigue Cycles Applied)

Damage plots are provided for all three mesh densities in Figure 21 to Figure 23. The crack propagation behavior observed experimentally for the geometry of the present CT specimen was such that the crack initially propagated towards the tertiary hole but bypassed it as the crack advanced. In the experiments the tertiary hole position was varied up to a maximum of
1.5mm from the present configuration. In two of these alternate configurations the crack propagated into the hole indicating significant sensitivity of crack path with placement of the added hole.

Figure 21: Crack Path as Determined from Damage Plot, $\delta = 1.2$mm
Figure 22: Crack Path as Determined from Damage Plot, $\delta=0.6\text{mm}$

Figure 23: Crack Path as Determined from Damage Plot, $\delta=0.5\text{mm}$
Figure 24 compares the simulated crack path to the experimental crack path. The path for the coarsest mesh ($\delta=1.2\text{mm}$) propagates into the tertiary hole and does not converge to the experimental path over any portion of the advance. Path results are very similar for horizon sizes of $\delta=0.6\text{mm}$ and $\delta=0.5\text{mm}$ providing confidence that mesh density is sufficient to obtain converged results. Further refinement of the horizon size was limited by solution time.

Figure 24: Spatial Position of Crack Tip

Correlation of both refined models to the experimental crack path is achieved for spatial position of the X coordinate in the range of 14mm to 22mm. Thus, the simulation accurately
predicts the crack path over the first 200,000 cycles. At X values less than 14mm the path flattens ahead of the “knee” in the experimental crack path. It was previously noted that for a mode I crack, the crack will grow purely in the horizontal direction. Deviation from this horizontal path occurs when mixed-mode loading occurs (i.e. introduction of $K_{II}$). When deviation occurs towards the end of the simulation the crack is growing purely in the horizontal direction indicating the $K_{II}$ component is under-represented. At this time the reduction in cross section from the initial condition is 38%. Additional rotation of the specimen about the pin-loaded holes will have occurred in the experiment such that the applied load will no longer be perpendicular to the crack plane. This effect will contribute to a larger magnitude $K_{II}$ which would tend to produce an inclined crack trajectory.

It is assumed that greater deviation in simulation results at high cycles occurs for two reasons. First, geometry nonlinearity is not considered in the present analysis due to increased computational costs. As the crack propagates reducing the cross sectional area of the sample, the sample may be viewed as an upper cantilevered beam section and a lower cantilevered beam section connected by a hinge which is represented by the remaining material connecting the two regions. As the crack grows and the moment arm of the load increases on the upper and lower beams, additional rotation of each beam occurs producing a small force component which acts tangential to the beams. With an assumption of linear geometry this smaller tangential force is not accounted for in the simulation.

The second source comes from limitations of the damage model. As the crack tip advances the remaining life of the bonds decreases and bonds are deleted. However, bonds oriented horizontally experience much less strain and, in some cases, are not broken. As these bonds accumulate, further opening of the crack is limited impacting the magnitude of the load
acting tangent to the crack (see Figure 25). This issue is resolved by introduction of the threshold $\Delta K$ value into the damage model as proposed by [19] (see Section 6.4) and the inclusion of nonlinear geometric effects. The threshold $\Delta K$ value would allow for remaining life to continually degrade for bonds along a fracture surface once beyond the horizon of the crack tip. The threshold $\Delta K$ ensures that remaining life degradation only occurs for those elements which are continually loaded (i.e. resisting the opening of the crack) and not for bonds along the crack surface with minimal strain. Methods to increase computational efficiency to allow for nonlinear geometric effects to be included are discussed in Section 8.4.

Figure 25: Horizontal Bonds Perpendicular to the Applied Load which do not Fail
8.3 Fatigue Life

As a precrack is present in the sample as an initial condition, fatigue life of the sample is modeled via an appropriate growth law. As shown in [46] the crack growth rate, \( \frac{da}{dN} \), under fixed environmental factors such as R-ratio, load frequency, etc. is independent of the specific geometry. The CT specimen modified with an auxiliary hole and subject to (primarily) mode I loading will exhibit behavior associated to the Paris Law. In the simulation as well as the experimental tests after which the simulation is modeled, a near constant stress intensity factor of 20 Mpa\( \sqrt{\text{m}} \) was maintained. As provided in [40] the threshold stress intensity factor for the particular SAE 1020 steel used in the experimental data is \( \Delta K_{th} = 11.6 \) Mpa\( \sqrt{\text{m}} \). For a SIF of 20 Mpa\( \sqrt{\text{m}} \) the crack growth will be in the stage II regime of a Paris Law plot allowing for the use of equation (90), repeated below.

\[
\frac{da}{dN} = c\Delta K^m
\]  

(119)

With \( c \) and \( m \) fixed based on material and a near constant \( \Delta K \), the value of the growth rate will be constant. Integrating \( \frac{da}{dN} = \text{constant} \) shows that the crack length will be a linear function of the number of applied cycles. Figure 26 plots the empirical crack growth data along with simulated results for horizon sizes of 0.6mm and 0.5mm. A linear trend is observed for the empirical data as well as simulation. The simulated fatigue life very closely matches the experimental data indicating that the assumption of a small Mode II contribution is valid.
Figure 26: Crack Length as a Function of Fatigue Cycles

8.4 Efficiency of the Numerical Method

The present research focuses on implementation of a peridynamic fatigue model within the structure of a finite element software. Inherent efficiencies are present in the solution of the static load step given the optimized solvers used in commercial code, but restrictions are placed upon efficient handling of PD-specific parameters contained in user established arrays. The most refined model simulated a total of 235,000 fatigue cycles for the approximately 2.1 million elements in the model. Solution time was 71 hours on an Intel Xeon E5-2637 v2 3.5GHz processor with 64GB of RAM. In the present research massively parallel processing (MPP) was not utilized due to the complexity of partitioning the user arrays among multiple compute nodes. However, parallel computing using user subroutines is possible with ABAQUS software. To
evaluate the potential reduction in solution time, a single static step without damage calculation was performed on the aforementioned hardware and on a high performance computing (HPC) cluster with two compute nodes each having 16 CPUs. Solution time for the HPC was 1/6 of the time for the local machine. Further efficiency improvements are viable by utilizing peridynamics only in regions of expected crack growth and utilizing traditional finite element modeling elsewhere. Various coupling strategies have been proposed in the literature [9].
Chapter 9. CONCLUDING REMARKS

9.1 Conclusion

In the present research, a peridynamic fatigue damage model was implemented in a commercial FEM software. To the best of the author’s knowledge this is the first such usage case of integration of a peridynamic fatigue model within a finite element software. User subroutines were implemented to execute an algorithm which utilizes solution data from the FEM software to compute damage at each step and update the model allowing for modeling of progressive damage. To maintain a constant stress intensity factor range at the crack tip an algorithm which computed the nonlocal peridynamic J-integral was also implemented in FEM via user-subroutine. Both algorithms were evaluated by simulating fatigue behavior of a modified CT specimen with a pre-crack. Simulation results were compared to experimental data and correlation was achieved over a large range of fatigue cycles.

9.2 Future Work

The theoretical capabilities and relatively recent nature of the peridynamic theory allow for many decades of continued work that will significantly advance the field. The implementation of a peridynamic fatigue model within an existing finite element framework allows broader access to the theory and its applications. Further dissemination is possible with updates such as parallel processing and implementation of FEM-PD coupling strategies which will both significantly reduce solution time allowing for greater complexity in the modeling and more economical solutions. Massively parallel processing (MPP) can be implemented in the
fatigue damage model presented by updating the user subroutines to allow for simultaneous solution across multiple compute nodes. Introducing a “coupling region” between the peridynamic truss elements and traditional triangular and rectangular elements of FEM would allow for larger structures to be modeled with the computationally expensive peridynamic zones limited to specific areas of interest making the peridynamic method more economical.
REFERENCES


