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ABSTRACT

IMPROVING THE ADELIC VERSION OF THE MINKOWSKI-HLAWKA THEOREM

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Northern Illinois University, 2023
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In this dissertation we take a method developed by Schmidt to improve the classical version of the Minkowski-Hlawka Theorem and extend that to certain number fields. This leads to an analogous improvement to the adelic Minkowski-Hlawka Theorem in certain cases. We also demonstrate that the adelic version, even in the case where the number field is the field of rational numbers, is more general than the classical version.

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IMPROVING THE ADELIC VERSION OF THE MINKOWSKI-HLAWKA THEOREM

BY

FATMAH ALNASER
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DEDICATION

To my Great Family.

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CHAPTER 1

INTRODUCTION

The German mathematician Hermann Minkowski developed the geometry of numbers at the end of the 19th and beginning of the 20th centuries. He published his treatise "Geometrie der Zahlen" (the Geometry of Numbers) in 1896 which laid the foundations for the subject. Minkowski started using geometric approaches to solve number theory problems. Presently, the geometry of numbers is widely applied in many diverse areas of mathematics and other sciences, such as combinatorics, algebraic geometry, optimization, chemistry, and physics.

The fundamental question in the geometry of numbers is: when do we have a (non-zero) lattice point in a given region, and when do we not? Minkowski's first convex bodies theorem is the major result in this area of study. It guarantees the existence of a non-zero lattice point in any convex body of relatively large volume. In the other direction Minkowski made the conjecture that for any star body of small enough volume we can find a lattice of determinant 1 such that the intersection of the star body and the lattice is just the origin. Here a convex body is a compact and convex subset of $\mathbb{R}^n$ that is symmetric about the origin and has a non-empty interior. A star body is a subset $S \subset \mathbb{R}^n$ such that for all $\mathbf{x} \in S$, $r\mathbf{x}$ is an interior point of S whenever $|r| \leq 1$. (Unlike a convex body, a star body may be unbounded). It is known that any convex body is Jordan measurable, meaning the characteristic function of the region is Riemann integrable, whence the region has a volume in the sense of Riemann/Jordan. A (full) lattice $\Lambda \subset \mathbb{R}^n$ is any discrete subgroup of $\mathbb{R}^n$ that spans all of $\mathbb{R}^n$. It is known that every lattice has a basis $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ as a $\mathbb{Z}$-module; the absolute value of the determinant of the $n \times n$ matrix formed by such a basis is called the *determinant* of the lattice and denoted $\det(\Lambda)$. This quantity is independent of the particular basis chosen.

Minkowski made the assertion, later proven by Hlawka, that for any star body $S \subset \mathbb{R}^n$ and any Δ_1 greater than the volume of S divided by $2\zeta(n)$ there is a lattice Λ of determinant Δ_1 with $S \cap \Lambda = \{\mathbf{0}\}$ [4].

The generalization of the Minkowski-Hlawka Theorem to adeles and general number fields is discussed in chapter 4. In the next chapter we will present background materials on number fields and lattices. In chapter 3 we will demonstrate background about absolute values and background material on the geometry of numbers over number fields.

1.1 Some Improvements to the Minkowski-Hlawka Theorem

Many scholars have made improvements to the Minkowski-Hlawka Theorem given above. In this section we will describe some of those improvements and methods used.

Siegel took the space of all lattices of determinant 1 and put a Haar measure on that space. He then considered the characteristic function of a star body on that space of lattices. He calculated the mean value of that function, which is the average number of lattice points in the star body. He concluded that if the average is less than two, then there are lattices of determinant 1 where the intersection with the star body is just the origin. This mean value argument improved slightly on the Minkowski-Hlawka Theorem as proven by Hlawka [1].

Rogers [6, 7] noted that rather than just taking a simple mean value argument, one can take the same characteristic function of a star body and calculate higher moments. He started with a function defined for all lattices of determinant 1 by taking the mean value of some positive integral power of that function. Rogers deduced the following improvement: for any positive number Δ_1 that satisfies

$$\Delta_1 > \frac{3V(S)}{\sqrt{n}},$$

there exist a lattice of determinant Δ_1 such that $\mathbf{0}$ is the only point in the lattice, where $V(S)$ is the volume of the star body S in n dimensions. Rogers' improvements to the Minkowski-Hlawka Theorem have been a subject of much interest and led to further research in the field of the geometry of numbers [8, 9].

Schmidt made an improvement in the Minkowski-Hlawka Theorem in the following way. He started with the characteristic function $g(x)$ of a star body S in n-dimensions with Jordan-volume $V(S)$ and then considered the function $f(x) = g(x) + 2g(2x)$. Schmidt was able to obtain the following improvement: for any number Δ_1 such that $3\zeta(n)\Delta_1 > (1 + 2^{1-n})V(S)$ there exists an admissible lattice for S of determinant Δ_1 [1]. Here by an admissible lattice we mean the following. Let $S \subseteq \mathbb{R}^n$ and Λ be a lattice in $\mathbb{R}^n$. Then Λ is called admissible for S provided that it has no lattice point except possibly the origin in the interior of S. Schmidt made further improvements that are applicable to all Jordan-measurable bounded sets. To do so he considered more complicated combinations with the characteristic function $g(x)$ above. For example, he was able to prove that for any number Δ_1 such that $(1+2^{1-n})(1+3^{1-n})V(S) < 2\Delta_1$ there is a lattice of determinant Δ_1 admissible for S [1].

The aim of this research is to generalize the methods of Schmidt to certain number fields.

CHAPTER 2

PRELIMINARIES

This chapter will provide precise definitions of the basic objects under consideration and also state (and provide proofs for) some pertinent results.

2.1 Lattices

In this section we illustrate definitions, notation, and important results. Additional information on this fundamental material can be found on various sources, such as [11, 5, 1]

Definition 2.1.1 Let $\mathbf{b}_1, \ldots, \mathbf{b}_n \in \mathbb{R}^n$ be linearly independent. The set

$$\Lambda = \{z_1\mathbf{b}_1 + \cdots + z_n\mathbf{b}_n, z_i \in \mathbb{Z}, 1 \leq i \leq n\}$$

is called an n-dimensional lattice. The set of generating vectors $\{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ or the matrix $B = (\mathbf{b}_1, \ldots, \mathbf{b}_n)$ with columns $\mathbf{b}_i$ is called the basis of the lattice Λ.

In other words, a lattice in $\mathbb{R}^n$ may be defined as follows: a lattice $\Lambda \subset \mathbb{R}^n$ is a finitely generated $\mathbb{Z}$-module spanning $\mathbb{R}^n$. It thus has the shape $\Lambda = M(\mathbb{Z}^n)$, where M is non-singular linear transformation of $\mathbb{R}^n$. As indicated above, an alternative definition is that a lattice $\Lambda \subset \mathbb{R}^n$ is a discrete subgroup of $\mathbb{R}^n$ spanning $\mathbb{R}^n$.

Definition 2.1.2 The number $\det(\Lambda) = |\det(B)| = |\det(M)|$ is called the determinant of the lattice Λ.

This number is well-defined (i.e., the determinant of a lattice does not depend on the choice of the basis). Indeed, though the basis of such a lattice is not unique, we do have the following.

Theorem 2.1.1 Let $\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ and $\{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be two bases for a lattice Λ. Then there is an $M = (m_{ij}) \in \mathrm{GL}_n(\mathbb{Z})$ such that

$$\mathbf{b}_i = \sum_{j=1}^{n} m_{ij}\mathbf{a}_j, , \qquad i = 1, \ldots, n.$$

In particular, $\det(\Lambda)$ is well-defined.

Proof: Since $\mathbf{b}_i \in \Lambda$, and the lattice Λ has the basis $\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$, there are integers m_{ij} such that $\mathbf{b}_i = \sum_{j=1}^{n} m_{ij}\mathbf{a}_j$ by definition of lattice. Similarly, since $\mathbf{a}_i \in \Lambda$ and the lattice Λ has the basis $\{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$, there are integers n_{ij} such that $\mathbf{a}_i = \sum_{j=1}^{n} n_{ij}\mathbf{b}_j$. By substituting $\mathbf{b}_i = \sum_{j=1}^{n} m_{ij}\mathbf{a}_j$ in $\mathbf{a}_i = \sum_{j=1}^{n} n_{ij}\mathbf{b}_j$ and using the fact that $\mathbf{a}_i$ are linearly independent, we obtain:

$$\sum_{j=1}^{n} m_{ij}n_{jl} = \begin{cases} 1 & \text{if } i = l, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, the product $\det(m_{ij})\det(n_{il}) = 1$ and so $(m_{i,j}) \in \mathrm{GL}_n(\mathbb{Z})$. [1].

Definition 2.1.3 A fundamental parallelopiped for a lattice $\Lambda \subset \mathbb{R}^n$ is any subset $P \subset \mathbb{R}^n$ of the form

$$P = \{a_1\mathbf{b}_1 + \cdots + a_n\mathbf{b}_n : 0 \le a_i \le 1\},$$

where $\Lambda = \mathbf{b}_1\mathbb{Z} \oplus \cdots \oplus \mathbf{b}_n\mathbb{Z}$.

Note that the determinant of the lattice is the volume of a fundamental parallelopiped.

Definition 2.1.4 A star body is a subset $S \subset \mathbb{R}^n$ such that for all $\mathbf{x} \in S$, $r\mathbf{x}$ is an interior point of S whenever $|r| \le 1$.

Definition 2.1.5 Let $S \subseteq \mathbb{R}^n$ and Λ be a lattice in $\mathbb{R}^n$. Then Λ is called admissible for S, or S-admissible, provided that it has no lattice point except possibly the origin in the interior of S. If the intersection of S and the lattice is empty, then we say Λ is strictly admissible for S [4].

Using the language above we may state the Minkowski-Hlawka Theorem as follows:

Theorem 2.1.2 ***Minkowski-Hlawka Theorem***

For any positive Δ_1 and star body $S \subset \mathbb{R}^n$ with volume $V(S)$ satisfying $2\zeta(n)\Delta_1 > V(S)$, there is an S-admissible lattice Λ with $\det(\Lambda) = \Delta_1$.

2.2 Number Fields

Definition 2.2.1 A number field K is any finite algebraic extension of the field of rational numbers $\mathbb{Q}$. For any $\alpha \in K$ that is a root of a monic polynomial with coefficients in $\mathbb{Z}$, the number α is called an algebraic integer.

Theorem 2.2.1 Let $\alpha \in K$. If α is an algebraic integer, then α is a root of a monic polynomial with coefficients in $\mathbb{Z}$ that is irreducible over $\mathbb{Q}$.

Proof: We have the fact that $\mathbb{Z}[X]$ is a unique factorization domain and the irreducible elements in $\mathbb{Z}[X]$ are irreducible over $\mathbb{Q}$. Let $R(X) \in \mathbb{Z}[X]$ be a monic polynomial with root α and factor R into a product of powers of irreducible polynomials (note that the units of $\mathbb{Z}[X]$ are ± 1):

$$R(X) = \pm P_1^{e_1}(X) \times \ldots \times P_n^{e_n}(X).$$

Clearly, we may assume that each $P_i(X)$ is monic since R is. Since α is a root of R, it must be a root of some $P_i(X)$, which is irreducible over $\mathbb{Q}$.

Notation: Given a number field K, the subset $\mathfrak{O}_K \subset K$ consisting of algebraic integers in K is called the ring of integers of the number field K.

The set of all algebraic integers $\mathfrak{O}_K$ in a given number field K is a ring. This fact is not obvious since if we take two algebraic integers it is not straightforward to find a monic polynomial with integer coefficients that has the sum of these two as a root.

Proposition 2.2.1 Let K be a number field. For $\alpha \in K$ the following are equivalent:

1. α is an algebraic integer;

2. there is a non-zero finitely generated $\mathbb{Z}$-module $M \subset K$ with $\alpha M \subseteq M$.

Proof: Suppose α is an algebraic integer with minimal polynomial $P(X) \in \mathbb{Z}[X]$ and assume $P(X)$ is monic. If n is the degree of P, then we readily see that for the $\mathbb{Z}$-module M generated by $1, \alpha, \ldots \alpha^{n-1}$, $\alpha M \subseteq M$.

Now suppose there is some non-zero finitely generated $\mathbb{Z}$-module $M \subset K$ with $\alpha M \subset M$. Let M be generated by $\beta_1, \ldots, \beta_n \in K$. Then for all $i = 1, \ldots, n$ we may write $\alpha\beta_i = \sum_{j=1}^{n} a_{i,j}\beta_j$, where $a_{i,j} \in \mathbb{Z}$. We now set A to be the $n \times n$ matrix with diagonal entries $\alpha - a_{i,j}$ and off-diagonal entries $-a_{i,j}$ so that our system of equations implies that the determinant of A is zero. This determinant of A gives us our monic polynomial $P(X) \in \mathbb{Z}[X]$ of degree n where α is a root.

Corollary 2.2.1 For any number field K, $\mathfrak{O}_K$ is a subring of K.

Proof: We show that the set of algebraic integers in K is closed under addition and multiplication. Let $\alpha, \beta \in K$ be algebraic integers. Obtain finitely generated $\mathbb{Z}$-modules M and N with $\alpha M \subset M$ and $\beta N \subset N$. Then one readily sees that $(\alpha \pm \beta)MN \subseteq MN$ and $(\alpha\beta)MN \subseteq MN$.

We note that $\mathbb{Z}$ is the ring of integers in $\mathbb{Q}$.

Lemma 2.2.1 If $\alpha \in K$ for some number field K, then $z\alpha \in \mathfrak{O}_K$ for some non-zero $z \in \mathbb{Z}$. In particular, K is the quotient field of $\mathfrak{O}_K$.

Proof: Since α is necessarily algebraic over $\mathbb{Q}$ there is a non-zero polynomial $P(X) = a_nX^n + \cdots + a_0 \in \mathbb{Z}[X]$ with $P(\alpha) = 0$. Multiplying $P(X)$ by a_n^{n-1}, we see that $a_n\alpha$ is an algebraic integer. Since $\mathbb{Z} \subseteq \mathfrak{O}_K$, its quotient field is K.

Suppose K is a number field. Since $\mathbb{R}$ is a topological completion of $\mathbb{Q}$ and $\mathbb{C}$ is an algebraic closure of $\mathbb{R}$, it is an algebraically closed field containing $\mathbb{Q}$. Since K is axiomatically a separable extension of $\mathbb{Q}$, we obtain $[K : \mathbb{Q}]$ distinct embeddings of K into $\mathbb{C}$. Since $[\mathbb{C} : \mathbb{R}] = 2$, there are r_1 real embeddings of K into $\mathbb{R}$ and r_2 pairs of complex conjugate embeddings of K into $\mathbb{C}$, where $[K : \mathbb{Q}] = r_1 + 2r_2$. We denote these embeddings by σ_i and we arrange them so that the first r_1 are real and $\overline{\sigma_{i+r_2}} = \sigma_i$ for $i = r_1 + 1, \ldots, r_1 + r_2$, where the overline denotes complex conjugation. Set $n = [K : \mathbb{Q}]$ and let $\mathbb{E}^n$ be the subspace of $\mathbb{C}^n$ where the first r_1 coordinates real and $\overline{c_{i+r_2}} = c_i$ for $i = r_1 + 1, \ldots, r_1 + r_2$. Then $\mathbb{E}^n \cong \mathbb{R}^n$ and we have a $\mathbb{Q}$-linear embedding of K into $\mathbb{E}^n$ via

$$\alpha \mapsto (\sigma_1(\alpha), \ldots, \sigma_n(\alpha)).$$

We denote this embedding by ρ.

Theorem 2.2.2 Let K be a number field. The ring of integers $\mathfrak{O}_K$ is a free $\mathbb{Z}$-module of rank $[K : \mathbb{Q}] = n$. In other words there are $\alpha_1, \ldots, \alpha_n \in \mathfrak{O}_K$ such that $\mathfrak{O}_K = \alpha_1\mathbb{Z} \oplus \ldots \oplus \alpha_n\mathbb{Z}$.

Proof : Consider the embedding ρ above. It is clear that ρ is $\mathbb{Q}$-linear and one-to-one, so the image $\rho(\mathfrak{O}_K)$ is an additive subgroup of $\mathbb{E}^n$ isomorphic to $\mathfrak{O}_K$ as an additive group. To show that $\rho(\mathfrak{O}_k)$ is a lattice, we need to show that it is discrete. Note that the (non-zero) constant term of the unique monic irreducible polynomial for which $\alpha \in \mathfrak{O}_K$ is a root is of the form $\pm\sigma_1(\alpha)\cdots\sigma_n(\alpha)$. Since it is non-zero, this constant term has an absolute value of at least 1. This shows that for any non-zero $\alpha \in \mathfrak{O}_K$, the image $\rho(\alpha)$ is bounded away from the origin by an amount independent of α. Thus, $\rho(\mathfrak{O}_K)$ is discrete.

The ring $\mathfrak{O}_K$ is a Dedekind domain[5], i.e., all non-zero ideals factor uniquely into a product of powers of prime ideals.

Definition 2.2.2 A non-zero fractional ideal of a number field K is a non-zero $\mathfrak{O}_K$-module $\mathfrak{I} \subset K$ such that $\alpha\mathfrak{I} \subseteq \mathfrak{O}_K$ for some non-zero $\alpha \in \mathfrak{O}_K$.

Definition 2.2.3 A principal fractional ideal is any fractional ideal of the form $\alpha\mathfrak{O}_K$ for some non-zero $\alpha \in K$.

Theorem 2.2.3 *[11]* Every non-zero fractional ideal $\mathfrak{I}$ can be uniquely written as a product of integer powers of prime (maximal) ideals $\mathfrak{P}$ of $\mathfrak{O}_K$:

$$\mathfrak{I} = \mathfrak{P}_1^{e_1} \times \ldots \times \mathfrak{P}_r^{e_r},$$

where the $\mathfrak{P}_i$s are maximal ideals and the e_is are non-zero rational integers.

Definition 2.2.4 Let K be an algebraic number field with ring of integers $\mathfrak{O}_K$. Let $\{a_1, \cdots, a_n\}$ be an integral basis of $\mathfrak{O}_K$ and let $\{\sigma_1, \cdots, \sigma_n\}$ be the set of distinct embeddings of K into the complex numbers. The discriminant of K, $\Delta(K)$, is the square of the determinant of the the matrix $(\sigma_i(b_j))$, i.e., $\Delta(K) = [\det(\sigma_i(b_j))]^2$.

Theorem 2.2.4 ***Dirichlet's Unit Theorem****:[5]* Let K be a number field. Let r_1 denote the number of real embeddings and r_2 the number of pairs of complex conjugate embeddings. Set $r = r_1 + r_2 - 1$. There are units $u_1, \cdots, u_r \in \mathfrak{O}_K^\times$ such that all units of $\mathfrak{O}_K$ are uniquely expressible in the form

$$u = \zeta \prod_{i=1}^{r} u_i^{a_i}$$

for some root of unity ζ and $a_i \in \mathbb{Z}$. Moreover, there are only finitely many roots of unity $\zeta \in \mathfrak{O}_K^\times$.

Any set of units $u_1, \cdots, u_r$ as above is called a system of fundamental units.

Definition 2.2.5 Given a system of fundamental units $u_1, \cdots, u_r$, the regulator $R(K)$ of K is defined to be the absolute value of the determinant

$$\det \begin{pmatrix} e_1 \log|\sigma_1(u_1)| & \dots & e_r \log|\sigma_r(u_1)| \\ \vdots & \ddots & \\ e_1 \log|\sigma_1(u_r)| & \dots & e_r \log|\sigma_r(u_r)| \end{pmatrix}.$$

When $r = 0$ we set $R(K) = 1$.

Definition 2.2.6 Given an integral ideal I in a number field K, the norm of I is defined as the cardinality of the finite quotient ring $\mathfrak{O}_K/I$, where $\mathfrak{O}_K$ is the ring of integers of K. In other words, the norm of an integral ideal I is the number of elements in the set of residue classes $x + I : x \in \mathfrak{O}_K$.

Definition 2.2.7 For a positive integer n, let a_n denote the number of integral ideals with norm n. The zeta function of K is defined for $\mathcal{R}(s) > 1$ by

$$\zeta_K := \sum_{n=1}^{\infty} \frac{a_n}{n^s}.$$

Definition 2.2.8 Let $\mathfrak{O}_k$ be the ring of integers for the number filed K. The non-zero ideals I, J in $\mathfrak{O}_k$ are equivalent, if there exist non-zero $a, b \in \mathfrak{O}_K$, such that $(a)I = (b)J$. This is an equivalence relation. Each equivalence class is called an ideal class.

Theorem 2.2.5 *[2] Let K be a number field, the set of ideal classes in $\mathfrak{O}_K$ form an abelian group.*

Definition 2.2.9 Let $\mathfrak{O}_K$ be the ring of integers of the number field K. The group of ideal classes in $\mathfrak{O}_K$ under multiplication is called the ideal class group of K. The order of the ideal class group is called the class number of K and denoted by $h(K)$.

CHAPTER 3

GEOMETRY OF NUMBERS OVER NUMBER FIELDS

This chapter illustrates basic definitions and results in the geometry of numbers. The chapter includes a presentation of the adelic Minkowski-Hlawka Theorem, as well as a prove of the existence of adelic star bodies in $\mathbb{Q}_{\mathbb{A}}^n$ that are not of the form one can obtain from a star body together with a lattice in $\mathbb{R}^n$. See the discussion following the statement of Theorem 3.1.2 below.

3.1 Absolute Values

Definition 3.1.1 An absolute value on a field K is a function $|\cdot| : K \to \mathbb{R}$ satisfying these properties, for all $\alpha, \beta \in K$.

1. $|\alpha| \geq 0$ with equality if and only if $\alpha = 0$.

2. $|\alpha\beta| = |\alpha| \cdot |\beta|$.

3. $|\alpha + \beta| \leq |\alpha| + |\beta|$ (triangle inequality).

   If $|\cdot|$ satisfies the following strong triangle inequality:

4. $|\alpha + \beta| \leq \max\{|\alpha|, |\beta|\}$,

   then we say the absolute value is non-archimedean. Otherwise, it is called archimedean.

The trivial absolute value is the absolute value with $|x| = 0$ when $x = 0$ and $|x| = 1$ otherwise.

Definition 3.1.2 Let $\alpha = p^n b/a$ be non-zero rational number, where $p \nmid a$, $p \nmid b$ and $a, b, n \in \mathbb{Z}$. We define the order of α at p to be $\mathrm{ord}_p(\alpha) = n$.

More generally, given any number field K, the maximal ideal $\mathfrak{P} \subset \mathfrak{O}_K$, and $\alpha \in K \setminus \{0\}$, we set $\mathrm{ord}_{\mathfrak{P}}(\alpha) = n$ where n is the exponent on $\mathfrak{P}$ occurring in the factorization of the principal fractional ideal $\alpha\mathfrak{O}_K$.

Definition 3.1.3 Let p be a prime number. We define the p-adic absolute value for a rational number α as follows:

$$|\alpha|_p = p^{-\mathrm{ord}_p(\alpha)}.$$

We denote the usual absolute value on $\mathbb{Q}$ by $|\cdot|_\infty$ and call it the absolute value associated with the prime at infinity.

Definition 3.1.4 Two absolute values are equivalent if they produce the same topology.

Definition 3.1.5 A place of K is defined to be an equivalence class of non-trivial absolute values on K. The set of all places of K is denoted $M(K)$.

Theorem 3.1.1 ***Ostrowski's Theorem:*** All non-trivial archimedean absolute values on $\mathbb{Q}$ are equivalent to the usual absolute value. All non-archimedean absolute values are equivalent to some p-adic absolute value. Thus, the set of places $M(\mathbb{Q})$ of $\mathbb{Q}$ corresponds to the set of primes and the prime at infinity.

Let $K \subset F$ be number fields and given an absolute value in a place $v \in M(F)$, the restriction of the absolute value to K is a non-trivial absolute value, which is in a unique place $w \in M(K)$. In this situation, we say v lies above w (w lies below v) and we write it as $v|w$. For the field, $\mathbb{Q}$ and a number field K, the places $v \in M(K)$ lying above the prime at infinity of $\mathbb{Q}$ correspond to the r_1 real embeddings of K into $\mathbb{C}$ and the r_2 pairs of complex conjugate embeddings. For a rational prime $p \in M(\mathbb{Q})$, the places $v \in M(K)$ that lie above p correspond to the maximal ideals $\mathfrak{P} \subset \mathfrak{O}_K$ containing p.

The topological completion of $\mathbb{Q}$ with respect to any p-adic absolute value is denoted $\mathbb{Q}_p$ and the p-adic integers $\mathbb{Z}_p \subset \mathbb{Q}_p$ is the subring of $a \in \mathbb{Q}_p$ with $|a|_p \leq 1$. Of course, $\mathbb{R}$ is the topological completion of $\mathbb{Q}$ with respect to the usual absolute value $|\cdot|_\infty$. More generally, for any place $v \in M(K)$ we can construct the topological completion K_v of K via the usual method using Cauchy sequences. If $v \mid \infty$, then K_v is either the field of real numbers $\mathbb{R}$ or the field of complex numbers $\mathbb{C}$. When $v \in M(K)$ is a finite place ($v \nmid \infty$) we have the maximal compact subring $\mathfrak{O}_v \subset K_v$ consisting of those elements $\alpha \in K_v$ with $|\alpha|_v \leq 1$. This $\mathfrak{O}_v$ will contain $\mathbb{Z}_p$ where $v \mid p$.

For any rational prime p the principal ideal in $\mathfrak{O}_K$ generated by p is a product of powers of prime ideals:

$$p\mathfrak{O}_K = \mathfrak{P}_1^{e_1} \cdots \mathfrak{P}_r^{e_r}.$$

Set $e_{\mathfrak{P}}$ to be the exponent of $\mathfrak{P}$ in this product for p lying below $\mathfrak{P}$ and set $f_{\mathfrak{P}}$ to be the degree of the extension $[\mathfrak{O}_K/\mathfrak{P} : \mathbb{Z}/p\mathbb{Z}]$. For an archimedean place $v \in M(K)$ we set $e_v = 1$. We set $f_v = 1$ if v corresponds to real embedding of K into $\mathbb{C}$ and $f_v = 2$ if v corresponds to a pair of complex embeddings of K into $\mathbb{C}$. We note that for all places $w \in M(\mathbb{Q})$ [3, 11]

$$\sum_{v \in M(K),\ v \mid w} e_v f_v = [K : \mathbb{Q}].$$

Proposition 3.1.1 Let K be a number field. For all $v \in M(K)$ let $|\cdot|_v$ be the unique absolute value on K that extends $|\cdot|_w$, where $w \in M(\mathbb{Q})$ is the place of $\mathbb{Q}$ lying below v. Then for all non-zero $\alpha \in K$ we have the product formula [11]:

$$\prod_{v \in M(K)} |\alpha|_v^{e_v f_v} = 1.$$

Definition 3.1.6 Suppose $P \subset M(K)$ is a finite set of places containing all infinite places of K. Set

$$K_{\mathbb{A}}(P) = \prod_{v \in P} K_v \times \prod_{v \notin P} \mathfrak{O}_v.$$

The adele ring $K_{\mathbb{A}}$ is defined to be the union of all $K_{\mathbb{A}}(P)$, where the union is taken over all finite subsets $P \subset M(K)$ where P contains all infinite places of K[11].

The topology on the adele ring is completely determined by the system of "open balls" centered at the origin. Such an open ball is of the form

$$\prod_{v \in M(K)} a_v \mathfrak{O}_v,$$

where $a_v \in K_v^{\times} = \mathrm{GL}_1(K_{\mathbb{A}})$. In particular, we note that $|a_v|_v = 1$ for almost all places v. Such an element is called an idele. Given an idele (a_v) we set

$$\Lambda(a_v) = \{(b_v) \in K_{\mathbb{A}} : |a_v b_v|_v \leq 1, \text{for all } v \in M(K)\}.$$

These $\Lambda(a_v)$ are thus the open balls of the adele ring centered at the origin. There is a diagonal embedding of K into $K_{\mathbb{A}}$, $K \hookrightarrow K_{\mathbb{A}}$, given by $\alpha \mapsto (\alpha, \alpha, \alpha, \cdots)$.

Definition 3.1.7 The module on $K_{\mathbb{A}}^{\times}$ is defined for ideles $a = (a_v)$ by

$$|a|_{\mathbb{A}} = \prod_{v \in M(K)} |a_v|_v^{e_v f_v}.$$

We let μ be the product measure on $K_{\mathbb{A}}^n$

$$\mu = \prod_{v \in M(K)} \mu_v,$$

where for finite v, μ_v is the Haar measure on K_v^n given by $\mu_v(\mathfrak{O}_v^n) = 1$, μ_v is the usual Lebesgue measure on $\mathbb{R}^n$ when $K_v = \mathbb{R}$, and μ_v is 2^n times the usual Lebesgue measure on $\mathbb{C}^n$ when $K_v = \mathbb{C}$.

For each place v we define $|\cdot|_v$ on K_v by $\mu_v(aM) = |a|_v \mu_v(M)$ for any measurable set $M \subset K_v$ and $a \in K_v$. Note that this definition is the same as the absolute value from Proposition 3.1.1. We have $\mu_v(\alpha M_v) = |\alpha|_v^{e_v f_v} \mu_v(M_v)$ for all $\alpha \in K_v$ and measurable subsets $M \subseteq K_v$. This gives us $\mu_v(A_v M_v) = |\det A_v|_v^{e_v f_v} \mu(M_v)$ for all $A_v \in \mathrm{GL}_n(K_v)$ and measurable $M_v \subseteq K_v^n$. Thus $\mu(AM) = |\det(A)|_{\mathbb{A}} \mu(M)$ for $A \in \mathrm{GL}_n(K_{\mathbb{A}})$. We remark that the compact quotient of $K_{\mathbb{A}}$ with respect to discrete subset $K \subset K_{\mathbb{A}}$ can be shown to have measure equal to the square root of the absolute value of the discriminant of K, i.e., $\mu(K_{\mathbb{A}}/K) = \sqrt{|\Delta(K)|}$ [12]. Let F be a fundamental domain for K^n in $K_{\mathbb{A}}^n$; we have $\mu(AF) = |\det(A)|_{\mathbb{A}} \mu(F) = |\det(A)|_{\mathbb{A}} (\sqrt{|\Delta(K)})^n$, where $\Delta(K)$ is the discriminant of the number field K.

Definition 3.1.8 A subset $S \subset K_{\mathbb{A}}^n$ is an adelic star body if, for every $\mathbf{x} \in S, a\mathbf{x}$ is an interior point of S whenever $a \in K_{\mathbb{A}}^{\times}$ satisfies $|a_v|_v \leq 1$ for all places v.

Similar to the above, we have an adelic version of the Minkowski-Hlawka Theorem.

Theorem 3.1.2 ***adelic Minkowski-Hlawka Theorem:****[10]* Let $n > 1$ and let $S \subseteq K_{\mathbb{A}}^n$ be a star body. Let Δ_1 be any number such that

$$\Delta_1 > \frac{\mu(S) n^{r_1+r_2-1} h(K) R(K)}{w(K)\zeta_K(n)}.$$

Then there exists an $A \in \mathrm{GL}_n(K_{\mathbb{A}})$ with $|\det(A)|_{\mathbb{A}} = \Delta_1$ and $S \cap A(K^n) = \{\mathbf{0}\}$.

This theorem in the case $K = \mathbb{Q}$ implies the classical case of the Minkowski-Hlawka Theorem in the introduction. Given a classical star body $S_\infty \subset \mathbb{R}^n$ we get an adelic star body $S \subset \mathbb{Q}_{\mathbb{A}}^n$ by setting $S = S_\infty \times \prod_p \mathbb{Z}_p^n$. This adelic star body has measure $\mu(S) = V(S_\infty)$. For any Δ_1 we have

$$\Delta_1 > \frac{V(S_\infty)}{2\zeta(n)} = \frac{\mu(S)h(\mathbb{Q})R(\mathbb{Q})}{w(\mathbb{Q})\zeta(n)} = \frac{\mu(S)}{2\zeta(n)}.$$

Therefore, there is an $A = (A_\infty, A_2, \cdots) \in \mathrm{GL}_n(\mathbb{Q}_{\mathbb{A}})$ such that

$$S \cap A(\mathbb{Q}^n) = \{\mathbf{0}\},$$

$$A^{-1}S \cap \mathbb{Q}^n = \{\mathbf{0}\}.$$

The lattice

$$\Lambda = \prod_p A_p^{-1}(\mathbb{Z}_p^n) \cap \mathbb{Q}^n,$$

has $\det(\Lambda) = \prod_p |\det A_p|_p$, and $A_\infty^{-1}(S_\infty)$ has volume $|\det A_\infty|^{-1} \cdot V(S_\infty)$ such that

$$A_\infty^{-1}(S_\infty) \cap \Lambda = \{\mathbf{0}\},$$

$$S_\infty \cap A_\infty(\Lambda) = \{\mathbf{0}\}.$$

And

$$\det(A_\infty(\Lambda)) = |\det A_\infty| \cdot \det(\Lambda) = |\det A_\infty| \cdot \prod_p |\det A_p|_p = |\det A|_{\mathbb{A}} = \Delta_1.$$

In fact there are adelic star bodies that do not correspond in this fashion to star bodies in $\mathbb{Q}^n$ and lattices. We will now show that Theorem 3.1.2 in the case where $K = \mathbb{Q}$ is more general than the classical Minkowski-Hlawka Theorem.

Lemma 3.1.1 A p-adic rectangle

$$R = \{(x, y) \in \mathbb{Q}_p^2 : |x|_p \le p^n,\ |y|_p \le p^m\}$$

is an open and closed set.

Proof: We show that the p-adic rectangle

$$R = \{(x, y) \in \mathbb{Q}_p^2 : |x|_p \le p^n,\ |y|_p \le p^m\}$$

is an open set.

Let

$$B = \{(a, b) \in \mathbb{Q}_p^2 : |z_1 - a|_p < p^n,\ |z_2 - b|_p < p^n\}$$

be an open ball centered at (z_1, z_2). We show that $B \subset R$. Let $(a, b) \in B$ be an arbitrary point. Now

$$|a|_p = |a - x + x|_p \le \max\{|a - x|_p, |x|_p\} = |x|_p < p^n,$$

and

$$|b|_p = |a - y + y|_p \le \max\{|b - y|_p, |y|_p\} = |y|_p < p^m.$$

Hence $(a, b) \in R$ and R is an open set.

Now we show that R is a closed set by showing the complement is open. Let $R^c = \{(x,y) \in \mathbb{Q}_p^2 : |x|_p > p^n,\ |y|_p > p^m\}$ be the complement and let U be an open ball centered at (s_1, s_2) in R^c. We show that $U \subseteq R^c$. Let $(a,b) \in U$ be an arbitrary point. Now

$$|a|_p = |a - x + x|_p \leq \max\{|a-x|_p, |x|_p\} = |x|_p > p^n,$$

and

$$|b|_p = |b - y + y|_p \leq \max\{|b-y|_p, |y|_p\} = |y|_p > p^m.$$

Hence $(a,b) \in R^c$ and R^c is an open set.

Lemma 3.1.2 A non-trivial p-adic rectangle is a p-adic star body.

Proof: Suppose n, and m are integers with $n \leq m$. Let R be a p-adic rectangle

$$R = \{(x_1, x_2) \in \mathbb{Q}_p^2 : |x_1|_p \leq p^n,\ |x_2|_p \leq p^m\}.$$

The origin is an interior point of R because there exists an open ball centered at the origin, which is completely contained in R. Let U be the open ball centered at $\mathbf{0}$.

$$U = \{(x_1, x_2) \in \mathbb{Q}_p^2 : |0 - x_1|_p < p^n, |0 - x_2|_p < p^n\}$$

$$= \{(x_1, x_2) \in \mathbb{Q}_p^2 : |x_1|_p < p^n, |x_2|_p < p^m\} \subseteq R$$

Let $a_p \in \mathbb{Q}_p$ such that $|a_p|_p \leq 1$ and let $\mathbf{x}_p \in R$. We show that $a_p\mathbf{x}_p$ is an interior point of R. So we show that there exists an open ball centered at $a_p\mathbf{x}_p$, which is completely contained in R. Let $B = \{(y_1, y_2) \in \mathbb{Q}_p^2 : |a_p x_1 - y_1|_p < p^n, |a_p x_2 - y_2|_p < p^n\}$ be an open ball. We show that $B \subset R$. Let $(y_1, y_2) \in B$,

$$|y_1|_p = |a_p x_1 - y_1 - a_p x_1|_p \leq \max\{|a_p x_1 - y_1|_p + |a_p x_1|_p\} = |a_p x_1|_p = |a_p| \cdot |x_1| = 1 \cdot p^n = p^n,$$

and similarly

$$|y_2|_p = |a_p x_2 - y_2 - a_p x_2|_p \leq \max\{|a_p x_2 - y_2|_p + |a_p x_2|_p\} = |a_p x_2|_p = |a_p|_p \cdot |x_2|_p = 1 \cdot p^m = p^m.$$

Hence $(y_1, y_2) \in R$ and $a_p \mathbf{x}_p$ is an interior point of R.

Lemma 3.1.3 Let I be any non-empty set and suppose $S_i \subseteq \mathbb{Q}_p^n$ is a p-adic star body for all $i \in I$. Then the union

$$S = \bigcup_{i \in I} S_i$$

is a p-adic star body.

Proof: Let $a_p \in \mathbb{Q}_p$ such that $|a_p|_p \leq 1$ and let $\mathbf{x}_p \in S$. Then the point $a_p \mathbf{x}_p$ is an interior point of S since $\mathbf{x}_p \in S_i$ for some $i \in I$, and so $a_p \mathbf{x}_p$ is an interior point of $S_i \subseteq S$. The origin is an interior point of S_i for all $i \in I$, so it is also an interior of S. The union S is symmetric about the origin since any $\mathbf{x}_p \in S$ is in some S_i, so that $-\mathbf{x}_p \in S_i \subseteq S$. Hence the union $S = \bigcup_{i \in I} S_i$ is a p-adic star body.

Lemma 3.1.4 Suppose n, m, r, and s are all integers with $r < n < m < s$ and consider the two p-adic rectangles R_1 and R_2 as follows:

$$R_1 = \{(x_1, x_2) \in \mathbb{Q}_p^2 : |x_1|_p \leq p^n,\ |x_2|_p \leq p^m\},$$

$$R_2 = \{(x_1, x_2) \in \mathbb{Q}_p^2 : |x_1|_p \leq p^r,\ |x_2|_p \leq p^s\}.$$

The union $R_1 \cup R_2$ is a star body and is not a $\mathrm{GL}_2(\mathbb{Q}_p)$ transformation of any p-adic rectangle.

Proof: The p-adic rectangles R_1 and R_2 are p-adic star bodies by Lemma 3.1.2. Their union $R_1 \cup R_2$ therefore is a p-adic star body by Lemma 3.1.3.

The measure of R_1 is $\mu(R_1) = p^{n+m}$ and the measure of R_2 is $\mu(R_2) = p^{r+s}$. We note that $R_1 \cap R_2$ is the rectangle $\{(x_1, x_2) : |x_1|_p \leq p^r,\ |x_2|_p \leq p^m\}$ of measure p^{r+m} since $r < n$ and $m < s$ by hypothesis. Therefore the measure of the union $R_1 \cup R_2$ is

$$\mu(R_1 \cup R_2) = \mu(R_1) + \mu(R_2) - \mu(R_1 \cap R_2) = p^{n+m} + p^{r+s} - p^{r+m}. \tag{3.1}$$

Now if $R_1 \cup R_2$ were in the form $A_p(R)$ for some $A_p \in \mathrm{GL}_2(\mathbb{Q}_p)$ and some p-adic rectangle R, then the measure would be $|\det(A_p)|_p \times \mu(R) = p^t$ for some integer t. On the other hand the expression $p^{n+m} + p^{r+s} - p^{r+m}$ in (1) is not a power of p. To see why we write

$$p^{n+m} + p^{r+s} - p^{r+m} = p^{r+m}(p^{n-r} + p^{s-m} - 1).$$

By hypothesis $n > r$ and $s > m$, so that $p^{n-r} \equiv 0 \pmod p$ and $p^{s-m} \equiv 0 \pmod p$. Thus the factor $p^{n-r} + p^{s-m} - 1 \equiv -1 \pmod p$ is not a power of p, so that the expression $p^{n+m} + p^{r+s} - p^{r+m}$ is not a power of p. Hence the union $R_1 \cup R_2$ cannot be of the form $A_p(R)$.

Lemma 3.1.5 For every prime p, there is an unbounded p-adic star body of measure 2.

Proof: We will prove there is such a body S of the form

$$S = \bigcup_{n \geq 1} R_n,$$

where each R_n is a p-adic rectangle. Let R_1 be the 1×1 rectangle and the second p-adic rectangle R_2 be defined by

$$R_2 = \{(x_1, x_2) \in \mathbb{Q}_p^2 : |x_1|_p \leq p,\ |x_2|_p \leq p^{-1}\}.$$

The measure of R_1 is $\mu(R_1) = 1$ and the measure of R_2 is $\mu(R_2) = p \times p^{-1} = 1$. Note that the intersection $R_1 \cap R_2 = \{(x_1, x_2) \in \mathbb{Q}_p^2 : |x_1|_p \leq 1,\ |x_2|_p \leq p^{-1}\}$ is another rectangle and has measure $\mu(R_1 \cap R_2) = \frac{1}{p}$.

We take the union $R_1 \cup R_2$, and the measure of the union is calculated by using (3.1):

$$\mu(R_1 \cup R_2) = \mu(R_1) + \mu(R_2) - \mu(R_1 \cap R_2) = 1 + 1 - \frac{1}{p} = 2 - \frac{1}{p}.$$

For all $n \geq 2$ let

$$R_n = \{(x_1, x_2) \in \mathbb{Q}_p^2 : |x_1|_p \leq p^{n-1},\ |x_2|_p \leq p^{-(2n-3)}\},$$

a p-adic rectangle with measure $\mu(R_n) = p^{-n+2}$. Now we set S_n for $n \geq 2$ to be the union of the first n rectangles: $S_n = \bigcup_{i \leq n} R_i$. We prove by induction on n that the measures of these unions satisfy

$$\mu(S_n) = 2 - \frac{1}{p^{n-1}}. \tag{3.2}$$

For $n = 2$ we have $\mu(R_1 \cup R_2) = 2 - \frac{1}{p^{2-1}} = 2 - \frac{1}{p}$ from above. For the induction hypothesis, we assume that $n \geq 2$ and the measure

$$\mu(S_n) = \mu(\bigcup_{i \leq n} R_i) = 2 - \frac{1}{p^{n-1}}.$$

Now

$$R_{n+1} = \{(x_1, x_2) \in \mathbb{Q}_p^2 : |x_1|_p \leq p^n,\ |x_2|_p \leq p^{-(2n-1)}\}$$

and R_{n+1} has measure $\mu(R_{n+1}) = p^{-n+1}$. The intersection of R_{n+1} and S_n is a p-adic rectangle: $R_{n+1} \cap S_n = \{(x_1, x_2) \in \mathbb{Q}_p^2 : |x_1|_p \leq p^{n-1},\ |x_2|_p \leq p^{-(2n-1)}\}$, and the measure of the intersection is thus $\mu(R_{n+1} \cap S_n) = \frac{1}{p^n}$.

We apply (3.1) to calculate the measure of S_{n+1}:

$$\begin{aligned}\mu(S_{n+1}) = \mu(R_{n+1} \cup S_n) &= \mu(R_{n+1}) + \mu(S_n) - \mu(R_{n+1} \cap S_n) \\ &= \frac{1}{p^{n-1}} + 2 - \frac{1}{p^{n-1}} - \frac{1}{p^n} = 2 - \frac{1}{p^n}.\end{aligned}$$

This proves (3.2) by induction for all $n \geq 2$.

Now taking the limit as n goes to infinity yields

$$\mu(S) = \lim_{n\to\infty} \mu(S_n) = \lim_{n\to\infty} 2 - \frac{1}{p^{n-1}} = 2.$$

Further, S is a p-adic star body by Lemmas 3.1.2 and 3.1.3. This union is clearly unbounded by construction since given any bound p^m $(m \geq 0)$ there are $\mathbf{x} \in R_{m+2} \subset S$ with $\|\mathbf{x}\|_p = p^{m+1} > p^m$.

From the above Lemmas we conclude that there are adelic star bodies in $\mathbb{Q}_{\mathbb{A}}^n$ that do not correspond to a classical star body $S_\infty \subset \mathbb{R}^n$ and a lattice $\prod_p A_p(\mathbb{Z}_p^n)$ for any $A = (I_n, A_2, A_3, \ldots) \in \mathrm{GL}_n(\mathbb{Q}_{\mathbb{A}})$. For example, we could use the open ball of radius 1 for the place at infinity and the S from Lemma 3.1.5 at any finite set of finite places, with $\mathbb{Z}_p^n$ for all other places.

CHAPTER 4

AN IMPROVEMENT TO THE MINKOWSKI-HLAWKA THEOREM

This chapter illustrates the main result of this research. We extend Schmidt's ideas to general number fields and obtain analogous improvements to the Minkowski-Hlawka Theorem for number fields. First, we start with the necessary preliminary definitions and results for the rest of this research.

For $n \geq 1$ we define

$$G_n = \{A \in \mathrm{GL}_n(K_{\mathbb{A}}) : |\det(A)|_{\mathbb{A}} = 1\}$$

and $\Gamma_n = \mathrm{GL}_n(K)$. We note that Γ_n is a discrete subgroup of G_n and $\mathrm{GL}_1(K_{\mathbb{A}}) = K_{\mathbb{A}}^{\times}$. We write $K_{\mathbb{A}}^1$ for G_1 and $K^{\times}$ for Γ_1.

As in [12] we define the invariant measure (invariant under multiplication on the left or right) $w_{n,v}$ on $\mathrm{GL}_n(K_v)$ as the following:

$$dw_{n,v}(A) = |\det(A)|_v^{-1} \prod_{1 \leq i,j \geq n} d\mu_v(a_{i,j}), \ A = (a_{i,j}).$$

Then we have the Tamagawa measure w_n on $\mathrm{GL}_n(K_{\mathbb{A}})$ defined by

$$w_n = \prod_{v|\infty} w_{n,v} \times \prod_{v\nmid\infty} (1 - |\pi_v|_v)^{-1} w_{n,v},$$

where π_v generates the unique maximal ideal of $\mathfrak{O}_K$.

By Weil, [11](chapter 4, Theorem 6) the $K_{\mathbb{A}}^{\times}$ is isomorphic to the direct product of $K_{\mathbb{A}}^{1}$ and $\mathbb{R}_{+}^{\times}$, where $\mathbb{R}_{+}^{\times}$ is the multiplicative group of positive real numbers. For a Haar measure β on $\mathbb{R}_{+}^{\times}$ given by $d\beta(x) = x^{-1}dx$, the Tamagawa measure $w_n = \mu_n \times \beta$ for some measure μ_n on G_n. For the number field K the G_n/Γ_n is compact and $\mu_n(G_n/\Gamma_n)$ is finite, and can be expressed in terms of the field K and residues of the zeta function (see [12], chapter 3).

For $n > 1$ let P_n be the subgroup of $\mathrm{GL}_n(K_{\mathbb{A}})$ consisting of all $A = (a_{i,j}) \in \mathrm{GL}_n(K_{\mathbb{A}})$ where $a_{i,1} = 0$ for $i > 1$. Let P' be the subgroup of P_n where $a_{1,1} = 1$. We write a typical element of P_n as

$$\begin{pmatrix} a & \mathbf{b} \\ 0 & \\ \vdots & A \\ 0 & \end{pmatrix}$$

Where $a \in K_{\mathbb{A}}^{\times}, \mathbf{b} \in K_{\mathbb{A}}^{n-1}$ and $A \in \mathrm{GL}_{n-1}(K_{\mathbb{A}})$ and similarly for P'.

Let g_n be the subgroup of G_n defined by

$$g_n = \left\{ \begin{pmatrix} a & \mathbf{b} \\ 0 & \\ \vdots & A \\ 0 & \end{pmatrix} \in P_n : a \in K_{\mathbb{A}}^{1}, \mathbf{b} \in K_{\mathbb{A}}^{n-1}, A \in G_{n-1} \right\}.$$

Let $\gamma_n = g_n \cap \Gamma_n$ and $g_n' = g_n \cap P_n'$, $\gamma_n' = \gamma_n \cap g_n'$. Similar to the above, we get a measure σ_n on g_n with

$$\sigma_n(g_n/\gamma_n) = \mu_1(K_{\mathbb{A}}/K^{\times})\mu_{n-1}(G_{n-1}/\Gamma_{n-1})\mu^{n-1}(K_{\mathbb{A}}^{n-1}/K^{n-1}).$$

Also, we get a measure σ'_n on g'_n with

$$\sigma'_n(g'_n/\gamma'_n) = \mu_{n-1}(G_{n-1}/\Gamma_{n-1})\mu^{n-1}(K_{\mathbb{A}}^{n-1}/K^{n-1}).$$

We have $\sigma'_n(g'_n/\gamma'_n)\mu_1(K_{\mathbb{A}}/K^{\times}) = \sigma_n(g_n/\gamma_n)$.

As in [12](chapter 2), the G_n/g_n is a homogeneous space. Let dv_n be the relatively invariant gauge form on that homogeneous space which satisfies $d\mu_n = dv_n d\sigma_n$, where μ_n and σ_n are the measures defined above. particularly, for an integrable function f on G_n,

$$\int_{G_n} f(A)d\mu_n(A) = \int_{G_n/g_n} dv_n(Ag_n)\int_{g_n} f(Aa)d\sigma_n(a).$$

Definition 4.0.1 Let S be a star body. For non-zero $\xi \in K_{\mathbb{A}}^n$ we define the distance function as the following:

$$\chi(\xi) = \inf_{a\in K_{\mathbb{A}}^{\times}}\{|a|_{\mathbb{A}} : \xi \in aS\}.$$

The distance function χ is well-defined since S contains an open neighborhood of the origin. Moreover, χ is invariant under scalar multiplication by ideles $a \in G_1$. Thus, we consider χ as a continuous function on G_n/g_n.

Remark: Note that rescaling the star body S at some infinite place (say) by a factor of $r > 0$ alters the measure of S by a factor of r^n and the function χ by a factor of r^{-1}.

Let κ be the measure of the set of $Ag_n \in G_n/g_n$ with $\chi(Ag_n) \leq 1$.

Lemma 4.0.1 [10] Let K be a number field, then

$$\kappa\frac{\sigma_n(g_n/\gamma_n)}{\mu_n(G_n/\Gamma_n)} = \frac{\mu(S)n^{r_1+r_2-1}h(K)R(K)}{w(K)\zeta_K(n)}.$$

Lemma 4.0.2 [10] If K is a number field then,

$$\int_{x \in FS} d\mu(\mathbf{x}) = \frac{h(K)R(K)n^{r_1+r_2-1}\mu^n(S)}{w(K)\zeta_K(n)},$$

where F is the fundamental set modulo $K^\times$ of $K^1_{\mathbb{A}}$.

Lemma 4.0.3 [10] Let $B > 0$ satisfy

$$\frac{B^n \kappa \sigma(g_n/\gamma_n)}{\mu_n(G_n/\Gamma_n)} < 1.$$

Then there is an $A \in G_n$ such that $\chi(A\xi) > B$ for all non-zero $\xi \in K^n$.

Proof: See [10].

We now show how one may prove Theorem 3.1.2. By the remark following Definition 4.0.1 we see that rescaling the star body S at any infinite place by $\Delta_1^{-1/n}$ shows that we may assume without loss of generality that $\Delta_1 = 1$.

By Lemma 4.0.1 and the hypotheses we have

$$\kappa \frac{\sigma_n(g_n/\gamma_n)}{\mu_n(G_n/\Gamma_n)} = \frac{\mu(S)n^{r_1+r_2-1}h(K)R(K)}{w(K)\zeta(n)} < 1,$$

say

$$\kappa \frac{\sigma_n(g_n/\gamma_n)}{\mu_n(G_n/\Gamma_n)} = \frac{\mu(S)n^{r_1+r_2-1}h(K)R(K)}{w(K)\zeta(n)} = 1 - \epsilon$$

for some $\epsilon > 0$. Thus by Lemma 8 with $B = 1 - \delta$ for suitable $\delta > 0$ (depending on ϵ above and n) there is an $A \in G_n$ such that $\chi\big(A(\xi)\big) \leq 1 - \delta$ for all non-zero $\xi \in \mathbb{Q}^n_{\mathbb{A}}$. By the definition of χ, this means $S \cap A(\mathbb{Q}^n) = \{\mathbf{0}\}$, completing the proof.

4.1 Schmidt's Improvements

This section outlines Schmidt's ideas for improving the Minkowski-Hlawka Theorem.

Lemma 4.1.1 *[1]*

Let Λ be an n-dimensional lattice and p be a prime number. Let $\mathbf{b}_1, \ldots, \mathbf{b}_k \in \Lambda$, which are not of the form $p\mathbf{b}$ for any $\mathbf{b} \in \Lambda$, and let $r_1, \ldots, r_k \in \mathbb{R}$. Then there is a sublattice $\Lambda' \subset \Lambda$ with index $[\Lambda : \Lambda'] = p$ such that

$$\sum_{\mathbf{b}_j \in \Lambda'} r_j \leq \frac{p^{n-1} - 1}{p^n - 1} \sum_{1 \leq j \leq k} r_j.$$

Lemma 4.1.2 *[1]*

Let p be a prime number and let $\mathbf{b}_1, \ldots, \mathbf{b}_p \in \Lambda$ not of the form $p\mathbf{a}$ for $\mathbf{a} \in \Lambda$. Then there is a sublattice $\Lambda' \subset \Lambda$ of index $[\Lambda : \Lambda'] = p$ such that $\mathbf{b}_i \notin \Lambda'$ for all $i = 1, \ldots, p$.

Schmidt shows that if S is a symmetric star body with Jordan-volume $V(S)$ and Δ_1 is such that

$$3\zeta(n)\Delta_1 > (1 + 2^{1-n})V(S),$$

then there is an S-admissible lattice Λ of determinant Δ_1. Specifically, starting with the characteristic function $g(x)$ of a star body S, Schmidt sets

$$f(x) = g(x) + 2g(2x),$$

so that

$$f(x) = \begin{cases} 3 & \text{if } x \in \frac{1}{2}S, \\ 1 & \text{if } x \in S, x \notin \frac{1}{2}S, \\ 0 & \text{otherwise.} \end{cases}$$

Integrating this function, we get

$$\int f(x)dx = (1 + 2^{1-n})V(S).$$

Now choose ϵ so small that

$$3\zeta(n)\Delta_1 > (1 + 2^{1-n})\big(V(S) + \epsilon\big).$$

Schmidt shows there is a lattice Λ with $\det(\Lambda) = \frac{1}{2}$ such that

$$\sum_{\mathbf{a}\in\Lambda}{}' f(\mathbf{a}) < 6,$$

Where the prime $'$ indicates that the summation is only over primitive lattice points. Here by primitive lattice point we mean a point $a \in \Lambda$ such that $\frac{1}{z}a$ is not a lattice point for any $z \in \mathbb{Z} \setminus \{\pm 1\}$.

Since $f(-x) = f(x)$ by the symmetry of S, there is no primitive point $\mathbf{a} \in \Lambda$ for which $f(\mathbf{a}) = 3$, and so no point of Λ at all in $\frac{1}{2}\Lambda$ except $\mathbf{0}$. Further, there are at most two pairs of primitive points, say $\pm\mathbf{a}_1,\ \pm\mathbf{a}_2$, of Λ in S. By Lemma 4.1.2 there is a sublattice Λ' of index 2 that contains neither $\mathbf{a}_1$ nor $\mathbf{a}_2$. Since neither $\mathbf{a}_1$ nor $\mathbf{a}_2$ are in $\frac{1}{2}\Lambda$, the points $2\mathbf{a}_1, 2\mathbf{a}_2$ of Λ' are not in S. Hence Λ' is S-admissible. Since $\det(\Lambda') = 2\det(\Lambda) = \Delta_1$, the lattice Λ' does what is required [1].

4.2 An Improvement to the Minkowski-Hlawka Theorem

Lemma 4.2.1 Suppose K has class number 1, so that every finitely generated $\mathfrak{O}_K$-module is a free $\mathfrak{O}_K$ module. Suppose p is a rational prime and $v \in M(K)$ lies above p. Then the maximal ideal corresponding to v is a principal ideal generated by some $\rho \in \mathfrak{O}_K$. Let Λ be a rank n module over $\mathfrak{O}_K$ and let $\mathbf{a}_1, \cdots, \mathbf{a}_R$ be any points of Λ which are not of the form $\rho\mathbf{b}$ for $\mathbf{b} \in \Lambda$. Let $k_1, \cdots, k_R$ be real numbers. Then there is a submodule Λ' of index $|N(\rho)|$ such that

$$\sum_{\mathbf{a}_r \in \Lambda'} k_r \leq \frac{|N(\rho)|^{n-1} - 1}{|N(\rho)|^n - 1} \sum_{1 \leq r \leq R} k_r.$$

Proof: Let $\mathbf{b}_1, \cdots, \mathbf{b}_n$ be a basis for Λ. Choose a set $\mathcal{S}$ of representatives of the quotient ring $\mathfrak{O}_K/\rho\mathfrak{O}_K$; the cardinality of S is $|N(\rho)|$. Let $c_1, \cdots, c_n \in S$ with $(c_1, \cdots, c_n) \neq (0, \cdots, 0)$. We may assume that $0 \in \mathcal{S}$. Let $\Lambda(c_1, \cdots .c_n)$ be the submodule of points $u_1\mathbf{b}_1 + \cdots + u_n\mathbf{b}_n$, where $u_1, \cdots, u_n \in \mathfrak{O}_K$ satisfy

$$u_1c_1 + \cdots + u_nc_n \equiv 0 \pmod{\rho}.$$

Clearly $\Lambda(c_1, \cdots, c_n)$ is a submodule of Λ; we will determine its index.

For $\mathbf{b} = u_1\mathbf{b}_1 + \cdots + u_n\mathbf{b}_n \in \Lambda$ set $\phi(\mathbf{b}) = u_1c_1 + \cdots + u_nc_n$. First we show that the function $\phi : \Lambda \to \mathfrak{O}_K$ is a homomorphism of $\mathfrak{O}_K$-modules. Note that for $\mathbf{b}$ as above and $\mathbf{a} = w_1\mathbf{b}_1 + \cdots w_n\mathbf{b}_n \in \Lambda$,

$$\begin{aligned}
\phi(\mathbf{b}+\mathbf{a}) &= \phi\big((u_1+w_1)\mathbf{b}_1 + \cdots + (u_n+w_n)\mathbf{b}_n\big) \\
&= (u_1+w_1)c_1 + \cdots + (u_n+w_n)c_n \\
&= u_1c_1 + w_1c_1 + \cdots + u_nc_n + w_nc_n \\
&= u_1c_1 + \cdots + u_nc_n + w_1c_1 + \cdots + w_nc_n \\
&= \phi(\mathbf{b}) + \phi(\mathbf{a}).
\end{aligned}$$

Let $c \in \mathfrak{O}_K$. Then

$$\begin{aligned}
\phi(c\mathbf{b}) &= \phi(cu_1b_1 + \cdots + cu_nb_n) \\
&= cu_1c_1 + \cdots + cu_nc_n \\
&= c(u_1c_1 + \cdots + u_nc_n) \\
&= c\phi(\mathbf{b}).
\end{aligned}$$

Now consider the natural map $\pi : \mathfrak{O}_K \to \mathfrak{O}_K/\rho\mathfrak{O}_K$. This map is also an $\mathfrak{O}_K$-module homomorphism; for $r_1, r_2 \in \mathfrak{O}_K$,

$$\begin{aligned}
\pi(r_1+r_2) &= (r_1+r_2) + \rho\mathfrak{O}_K \\
&= (r_1+\rho\mathfrak{O}_K) + (r_2+\rho\mathfrak{O}_K) \\
&= \pi(r_1) + \pi(r_2),
\end{aligned}$$

and

$$\begin{aligned}
\pi(r_1r_2) &= r_1r_2 + \rho\mathfrak{O}_K \\
&= r_1(r_2+\rho\mathfrak{O}_K) \\
&= r_2\pi(r_2).
\end{aligned}$$

This implies that the composition $\pi \circ \phi$ is an $\mathfrak{O}_K$-module homomorphism. The kernel of this $\mathfrak{O}_K$-module homomorphism is, by definition, the submodule $\Lambda(c_1, \ldots, c_n)$.

Note that the quotient $\mathfrak{O}_K/\rho\mathfrak{O}_K$ is irreducible. Since ρ is irreducible, $\rho\mathfrak{O}_K$ is a maximal ideal of $\mathfrak{O}_K$. Thus $\mathfrak{O}_K/\rho\mathfrak{O}_K$ is a field. If D is a non-zero submodule, then it contains non-zero element. We show that such a submodule D must be the entire field. Let $r + \rho\mathfrak{O}_K \neq \mathbf{0} + \rho\mathfrak{O}_K \in D$, then this element has a multiplicative inverse $s + \rho\mathfrak{O}_K$ where $r, s \in \mathfrak{O}_K$. It follows that

$$1 + \rho\mathfrak{O}_K = (r + \rho\mathfrak{O}_K)(s + \rho\mathfrak{O}_K) = rs + \rho\mathfrak{O}_K = s(r + \rho\mathfrak{O}_K) \in D.$$

Since $1 + \rho\mathfrak{O}_K \in D$ any element $r + \rho\mathfrak{O}_K \in \mathfrak{O}_K/\rho\mathfrak{O}_K$ is also in D. Thus we have $D = \mathfrak{O}_K/\rho\mathfrak{O}_K$.

Next we show that $\pi \circ \phi$ is not identically zero. Note that

$$(\pi \circ \phi)(\mathbf{b}) = \pi(\phi\mathbf{b}) = (u_1c_1 + \cdots + u_nc_n) + \rho\mathfrak{O}_K,$$

and this is identically $0 + \rho\mathfrak{O}_K$ only if each of $c_1, \ldots, c_n$ are congruent to zero modulo ρ, which is a contradiction to our hypotheses.

Finally, since the function $\pi \circ \phi : \Lambda \to \mathfrak{O}_K/\rho\mathfrak{O}_K$ is not identically zero it must be onto: $\pi \circ \phi(\Lambda) = \mathfrak{O}_K/\rho\mathfrak{O}_K$. Hence by The First Isomorphism Theorem for Modules we have

$$\Lambda/\Lambda(c_1, \cdots, c_n) \cong \mathfrak{O}_K/\rho\mathfrak{O}_K,$$

so that the index $[\Lambda : \Lambda(c_1, \ldots, c_n)] = [\mathfrak{O}_K : \rho\mathfrak{O}_K] = |N(\rho)|$.

Clearly there are $|N(\rho)|^n - 1$ submodules $\Lambda(c_1, \ldots, c_n)$. We now show that a point $\mathbf{a}_r$ belongs to precisely $|N(\rho)|^{n-1} - 1$ of them.

We have

$$\mathbf{a}_r = m_{r,1}\mathbf{b}_1 + \cdots + m_{r,n}\mathbf{b}_n,$$

where $m_{r,1}, \cdots, m_{r,n} \in \mathfrak{O}_K$ are not all divisible by ρ by the hypothesis. Without loss of generality $m_{r,1}$ is not divisible by ρ. The congruence

$$m_{r,1}c_1 + \cdots + m_{r,n}c_n \equiv 0 \pmod{\rho}$$

then determines c_1 uniquely if $c_2, \cdots, c_n \in S$ are given. In particular, the congruence

$$m_{r,1}c_1 + \cdots + m_{r,n}c_n \equiv 0 \pmod{\rho}$$

gives $c_1 = 0$ if already $c_2 = \cdots = c_n = 0$, which is contradiction since $(c_1, \cdots, c_n) \neq (0, \cdots, 0)$ by hypothesis. But $c_2, \cdots, c_n$ may be given any other of the $|N(\rho)|^{n-1} - 1$ possible sets of values since $|S| = |N(\rho)|$. Hence the average of the left-hand side of inequality over the lattices $\Lambda(c_1, \cdots, c_n)$ is given by the right-hand side of inequality above, and the inequality above must be true for at least one of them.

Lemma 4.2.2 Suppose the class number of K is 1 and let $\rho \in \mathfrak{O}_K$ be irreducible. Let $\Lambda \in K^n$ be an $\mathfrak{O}_K$-module of rank n and let $\mathbf{a}_1, \ldots, \mathbf{a}_p \in \Lambda$ none of which are of the form $\rho\mathbf{b}$ for $\mathbf{b} \in \Lambda$, where $p = |N(\rho)|$. Then there is a rank n submodule $\Lambda' \subset \Lambda$ with $[\Lambda : \Lambda'] = p$ such that $\mathbf{a}_i \notin \Lambda'$ for $i = 1, \ldots, p$.

Proof: Let $k_r = 1$ for all r above in Lemma 4.0.10. Then the submodule Λ' guaranteed by Lemma 9 satisfies

$$\sum_{\mathbf{a}_r \in \Lambda'} 1 \leq p\frac{p^{n-1} - 1}{p^n - 1} < 1,$$

whence no $\mathbf{a}_r \in \Lambda'$.

Lemma 4.2.3 Let $B > 0$ satisfy

$$\frac{B^n \kappa (pc^{-n} + 1)\sigma_n(g_n/\gamma_n)}{\mu_n(G_n/\Gamma_n)} < (p+1).$$

Then there is an $A \in G_n$ such that there are no ξ with $\chi(A\xi) \leq B/c$ and there are at most p ξ's with $B/c < \chi(A\xi) \leq B$ for $c > 0$.

We will use Lemma 4.2.3 in the same manner that Lemma 4.0.3 is used to prove Theorem 3.1.2.

Proof: Suppose $B, c > 0$ are parameters to be chosen later. Let

$$f(x) = \begin{cases} p+1 & \text{if } x \leq B/c, \\ 1 & \text{if } B/c < x \leq B \\ 0 & \text{if } x > B \end{cases}$$

The function $f(x)$ is chosen in a manner similar to the function $f(x)$ discussed in Schmidt's improvement of the Minkowski-Hlawka Theorem.

Then we have

$$\begin{aligned} \int_{G_n/g_n} f(\chi(Ag_n))dv_n(Ag_n) &= n\kappa \int_0^\infty x^{n-1} f(x)dx \\ &= (p+1)n\kappa \int_0^{B/c} x^{n-1}dx + n\kappa \int_{B/c}^{B} x^{n-1}dx \\ &= (p+1)\kappa \left(\frac{B}{c}\right)^n + \kappa \left(B^n - \left(\frac{B}{c}\right)^n\right) \\ &= (p+1)\kappa \left(\frac{B}{c}\right)^n + \kappa B^n - \kappa \left(\frac{B}{c}\right)^n \end{aligned}$$

(see [10] Lemma 3).

Since $f \circ \chi$ is a positive and integrable function, by ([12] Lemma 2.4.2) this implies that

$$\int_{G_n/g_n} f(\chi(Ag_n))dv_n(Ag_n) = \frac{1}{\sigma_n(g_n/\gamma_n)} \int_{G_n/\Gamma_n} \sum_{\xi} f(\chi(A\xi))d\mu_n(A).$$

Whence

$$\frac{1}{\sigma_n(g_n/\gamma_n)} \int_{G_n/\Gamma_n} \sum_{\xi} f(\chi(A\xi))d\mu_n(A) = \kappa B^n(pc^{-n}+1).$$

Now suppose B and c are chosen such that

$$\kappa B^n(pc^{-n}+1) < \frac{(p+1)\mu_n(G_n/\Gamma_n)}{\sigma_n(g_n/\gamma_n)}. \tag{4.1}$$

Then there exists an $A \in G_n$ such that

$$\sum_{\xi \in \mathbb{P}^{n-1}(K)} f(\chi(A\xi)) < (p+1). \tag{4.2}$$

(Note that in the sum above, we may choose particular representatives $\xi \in K^n \setminus \{\mathbf{0}\}$ for the projective points.) This tells us in particular that there are no ξ with $\chi(A\xi) \leq B/c$ and there are at most p ξ's with $B/c < \chi(A\xi) \leq B$.

Theorem 4.2.1 Suppose K is a field with class number 1 and let

$$p = \min_{\rho \in \mathfrak{O}_K} \{|N(\rho)| : \rho \text{ is irreducible}\}.$$

Let $S \subset (K_{\mathbb{A}})^n$ be a star body of the form $S = S_\infty \times \prod_{v \in M(K),\ v \nmid \infty} S_v$ where $S_v = M_v(\mathfrak{O}_v^n)$ for each finite place v with $M_v \in \mathrm{GL}_n(K_v)$. Let Δ_1 be any number such that

$$\Delta_1 > \frac{p(1+p^{1-n})\mu(S)n^{r_1+r_2-1}R(K)}{(p+1)w(K)\zeta_K(n)}.$$

Then there exists an $A \in \mathrm{GL}_n(K_{\mathbb{A}})$ with $|\det(A)|_{\mathbb{A}} = \Delta_1$ and $S \cap A(K^n) = \{\mathbf{0}\}$.

This result implies Schmidt's improvement of The Minkowski-Hlawka Theorem in the previous section. In the case when $K = \mathbb{Q}$ and for prime $p = 2$, we get Schmidt's result.

Proof: As in the proof of Theorem 3.1.2 we may assume without loss of generality that $\Delta_1 = 1$ so that $\mu(S)$ satisfies

$$\frac{(p+1)w(K)\zeta(n)}{p(1+p^{1-n})n^{r_1+r_2-1}R(K)} > \mu(S). \tag{4.3}$$

With the above normalizations/assumptions we now set $c = p$ and $B = p^{1/n}$. Then by (4.3) and Lemma 4.0.1, we have (4.1), whence (4.2). Thus there is an $A = (A_v)_v \in G_n$ such that there are no non-zero projective points ξ with $\chi(A\xi) \leq \frac{p^{1/n}}{p}$ and there are at most p distinct projective points ξ with $\frac{p^{1/n}}{p} < \chi(A\xi) \leq p^{1/n}$. If there are none of the latter, then of course we are done. Let $\xi_1, \ldots, \xi_l$ be the projectively distinct points with $\chi(A\xi) \leq p^{1/n}$ and set $\xi_{l+1} = \cdots = \xi_p = \xi_1$ if $l < p$.

Set

$$\Lambda = K^n \cap \prod_{v \nmid \infty} A_v^{-1} M_v(\mathfrak{O}_v^n).$$

We show that Λ is a $\mathfrak{O}_K$-submodule. First, Λ is not empty since $\mathbf{0} \in \Lambda$. Let $\mathbf{a} = (a_1, \cdots, a_n), \mathbf{b} = (b_1, \cdots, b_n) \in \Lambda$ and let $\alpha, \beta \in \mathfrak{O}_K$.

$$\begin{aligned} \alpha\mathbf{a} + \beta\mathbf{b} =& \alpha(a_1, \cdots, a_n) + \beta(b_1, \cdots, b_n) \\ &= (\alpha a_1, \cdots, \alpha a_n) + (\beta b_1, \cdots, \beta b_n) \\ &= (\alpha a_1 + \beta b_1, \cdots, \alpha a_n + \beta b_n) \in \Lambda. \end{aligned}$$

Hence Λ is closed under linear combinations and is a $\mathfrak{O}_K$- submodule. Since Λ is an $\mathfrak{O}_K$-module, it is a $\mathbb{Z}$-module (cf. Theorem 2.2.2). Moreover, the proof of Theorem 2.2.2 shows that Λ is discrete under the embedding ρ there, so that it is finitely generated as a $\mathbb{Z}$-module. Thus it is a finitely generated $\mathfrak{O}_K$-module which is clearly of rank n since it spans K^n.

Since $\chi(A\xi_i) \leq p^{1/n}$ we see that none of the ξ_i are of the form $\rho\mathbf{x}$ for some $\mathbf{x} \in \Lambda$ since such an $\mathbf{x} \in K^n$ would necessarily have $\chi(A\mathbf{x}) = \chi(A\xi_i)/p$ for the appropriate ξ_i. We thus get a submodule $\Lambda' \subset \Lambda$ of index p where $\xi_i \notin \Lambda'$ for $i = 1, \cdots, p$ by Lemma 4.2.2. Let $A' = (A'_v) \in \mathrm{GL}_n(K_{\mathbb{A}})$ with $A'_v = I_n$ for all $v|\infty$ and

$$\Lambda' = K^n \cap \prod_{v\nmid\infty} A'^{-1}_v A^{-1}_v M_v(\mathfrak{O}^n_v),$$

so that $|\det(A')|_{\mathbb{A}} = [\Lambda : \Lambda'] = p$.

Then any non-zero $\mathbf{x} \in \Lambda'$ is equal to some $\mathbf{y} \in \Lambda$ but not any ξ_i so that

$$\chi\big((AA')\mathbf{x}) = \chi(A\mathbf{y}) > p^{1/n}. \tag{4.4}$$

Choose a $v_0|\infty$ and let $A''_{v_0} \in \mathrm{GL}_n(K_{v_0})$ be the diagonal matrix with determinant p^{-1} and set $A''_v = I_n$ for all other places. Then multiplication of S by $(A'')^{-1}$ is simply a rescaling by $p^{1/n}$, so that by the remark following Definition 4.0.1 above we have

$$\chi(A''\xi) = p^{-1/n}\chi(\xi) \tag{4.5}$$

for all $\xi \in K^n_{\mathbb{A}}$.

We have $|\det(A'')|_{\mathbb{A}} = p^{-1}$ since $|\det(A''_{v_0})|_{v_0} = p^{-1}$. In particular we see that the product $A^* := AA'A'' \in G_n$. Since $A'' = I_n$ for all places $v \nmid \infty$ we have

$$\Lambda' = K^n \cap \prod_{v\nmid\infty} (A^*)^{-1} M_v(\mathfrak{O}^n_v). \tag{4.6}$$

Suppose $\mathbf{x} \in K^n \setminus \{\mathbf{0}\}$ satisfies $A^*(\mathbf{x}) \in S$, i.e. $\mathbf{x} \in K^n \cap (A^*)^{-1}(S)$. Then $\mathbf{x} \in \Lambda'$ by (4.6), so that $\chi(A^*\mathbf{x}) > 1$ by (4.4) and (4.5). This completes the proof.

Now we give a generalization for the result of Schmidt stated at the end of section 1.1.

Theorem 4.2.2 Suppose K is a field with class number 1 and let

$$p_1 = \min_{\rho_1 \in \mathfrak{O}_K} \{|N(\rho)| : \rho_1 \text{ is irreducible}\},$$

$$p_2 = \min_{\rho_2 \in \mathfrak{O}_K} \{|N(\rho)| : \rho_2 \text{ is irreducible}\},$$

where ρ_1 and ρ_2 are distinct irreducible elements. Let $S \subset (K_{\mathbb{A}})^n$ be a star body of the form $S = S_\infty \times \prod_{v \in M(K),\ v \nmid \infty} S_v$ where $S_v = M_v(\mathfrak{O}_v^n)$ for each finite place v with $M_v \in \mathrm{GL}_n(K_v)$. Let Δ_1 be any number such that

$$\Delta_1 > \frac{p_1 p_2 (1 + p_1^{1-n})(1 + p_2^{1-n}) \mu(S) n^{r_1 + r_2 - 1} R(K)}{(p_1 + 1)(p_2 + 1) w(K) \zeta_K(n)}.$$

Then there exists an $A \in \mathrm{GL}_n(K_{\mathbb{A}})$ with $|\det(A)|_{\mathbb{A}} = \Delta_1$ and $S \cap A(K^n) = \{\mathbf{0}\}$.

The proof follows that of Schmidt in a manner entirely similar to our proof of Theorem 4.2.1 above with appropriate function $f(x)$ defined as the following:

$$f(x) = \begin{cases} (p_1+1)(p_2+1) & \text{if } x \le B/c, \\ 1 & \text{if } B/c < x \le B \ , \\ 0 & \text{if } x > B \end{cases}$$

In the case where $K = \mathbb{Q}$ we get the result of Schmidt stated at the end of section 1.1.

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