Spectral Analysis of Cyclotron Radiation for Electron Beam Diagnostics

Brendan Leung
bleung7.11@gmail.com

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The spectral distribution of the cyclotron radiation emitted by non-ultra-relativistic electrons traveling through solenoidal lenses can be used to characterize ensemble-averaged properties of a beam. In this paper we explore the potential use of cyclotron radiation to measure the energy spread and transverse emittance of a beam while remaining unintrusive. We specifically discuss the relation between the spectral properties of cyclotron radiation and the beam statistical properties and perform first principle particle-in-cell simulation to validate our findings.
SPECTRAL ANALYSIS OF CYCLOTRON RADIATION FOR ELECTRON BEAM DIAGNOSTICS

BY
BRENDAN LEUNG
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A THESIS SUBMITTED TO THE GRADUATE SCHOOL IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE MASTER OF SCIENCE

DEPARTMENT OF PHYSICS

Thesis Director:
Philippe Piot
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DEDICATION

This work is dedicated to my parents for supporting me throughout my entire life.
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CHAPTER 1
INTRODUCTION

The Fermilab Integrable Optics Test Accelerator (IOTA) is a storage ring that can operate with both electrons and protons. One of the things that sets IOTA apart from other accelerators is its flexibility, as it can store electrons with a kinetic energy of 150 MeV or protons at 2.5 MeV and it has a magnetic lattice that can be precisely controlled. One of the experiments that will be performed at IOTA will be the electron lens, which is based on low-energy (typically .5 to 10 keV), magnetically confined electron beams overlapping with the circulating beam in a straight region of the storage ring. Such an electron lens can provide a versatile control over the circulating beam. The layout of the IOTA electron lens setup can be seen in Fig. 1.1.[9, 7, 8].

When working with beams of particles such as those that will be circulating in IOTA, it is useful to be able to discern properties of said beam. Determining properties of a beam is a necessity that allows one to adjust the beam such that its properties will be more desirable. Finding the properties of a beam is rather simple with direct observation, however that process would change the system which would also change results, and thus it is not an appealing way to gain data. Electron beams in a magnetic field are no exception in this regard, and here we propose that we can find properties such as the energy spread and transverse emittance while remaining unintrusive. In order to do this we propose that we need not observe the beam itself, but the radiation emitted from said beam. Measuring the radiation emitted will not have any effect on the system, as radiation is a naturally occurring phenomenon.
Figure 1.1: A picture of the setup at IOTA is shown above, where the electron beam is generated in the electron gun to the right, then transported through solenoid channels and a toroidal section to the overlap region, where it interacts with the circulating beam. Afterwards, it is directed towards the collector on the left in a similar manner.
In order to do this, we will attempt to relate the radiation emitted with some beam properties, namely the energy spread and the transverse emittance of said beam. We will also attempt to simulate a beam of electrons to compare with our mathematical results.

1.1 Motion of an electron in a magnetic field

Before discussing the analysis of electron beams in a magnetic field, it is useful to remember the motion of an electron in a magnetostatic field is governed by the Lorentz force

\[ \vec{F} = q(\vec{E} + \vec{v} \times \vec{B}) \]  
\[ (1.1) \]

where \( q \) is the charge of the particle, \( \vec{E} \) is the electric field, \( \vec{v} \) is the velocity of the particle, and \( \vec{B} \) is the magnetic field. In a magnetostatic field this simplifies to

\[ \vec{F} = q\vec{v} \times \vec{B} \]  
\[ (1.2) \]

We set \( \vec{B} = B\hat{z} \) for simplicity. This leads to the result

\[ \vec{F} = \vec{p} = q(v_y B\hat{x} - v_x B\hat{y}) \]

from which we can infer

\[ \dot{v}_x = \frac{qB}{\gamma m} v_y \]

\[ \dot{v}_y = -\frac{qB}{\gamma m} v_x \]

\[ \dot{v}_z = 0 \]
where $m$ is the mass of the particle and $\gamma$ is the relativistic Lorentz factor. We define $\omega = \frac{qB}{\gamma m}$ and so we require the solution of the equation

$$\frac{d}{dt}(v_x + iv_y) = -i\omega(v_x + iv_y)$$

which is of the form

$$v_x + iv_y = v_\perp e^{-i(\omega t + a)}$$

where $t$ is time, $v_\perp = \sqrt{v_x^2 + v_y^2}$ is the velocity component perpendicular to the B-Field and $a$ is just a constant. We let $v_z = v_\parallel$, the velocity component perpendicular to the B-Field, and thus we can write

$$\begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = \begin{pmatrix} v_\perp \cos(\omega t + a) \\ -v_\perp \sin(\omega t + a) \\ v_\parallel \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_0 + R\sin(\omega t + a) \\ y_0 + R\cos(\omega t + a) \\ z_0 + v_\parallel t \end{pmatrix}$$

where $R = \frac{v_\perp}{\omega} = \frac{\gamma m v_\perp}{qB}$. We see from this that the electron will have a helical trajectory with an axis along $\hat{z}$, a gyroradius $R$, and a rotation frequency $\omega$.

### 1.2 Ensemble-averaged description of a beam

As we are working not with single particles, but a beam of particles, it is useful to discuss how a beam is simply an ensemble of individual particles. These particles all have individual position and energy values, among other variables. Thus for a beam we have
distributions of different variables, and when we consider distributions, we take particular interest in their moments. The most important of these in our case are the first moment, the expected value or mean, and the second moment, the variance, which is the square of the standard deviation. These two values allow one to visualize the distribution without a more complicated equation. The energy spread is a useful beam property that can be easily found from the second moment of the beam’s energy distribution.

The other property we are looking for is the transverse emittance. Emittance is a beam property that is invariant under linear forces and it is a useful value as it helps one to determine a beam’s brightness.

Emittance is interesting in that there is no one definition of emittance that is consistent across all existing literature. Some define the emittance as the trace space area

\[ A_x = \int \int dx \, dx', \]  

(1.3)

where \( x \) is the horizontal position and \( x' = dx/dz \) the horizontal divergence. Defining the emittance as this presents some problems though as it does not allow one any distinction between a “well-behaved” beam and a beam with a more distorted shape as long as they have the same trace space area. Because of this, a definition that measures beam quality instead of the trace-space area is preferable.

In general the statistical emittance is defined as

\[ \tilde{\epsilon}_x = \left( \langle x^2 \rangle \langle x'^2 \rangle - \langle xx' \rangle^2 \right)^{1/2} \]  

(1.4)

where

\[ \langle x^2 \rangle = \int \int dx \, dx' \, x^2 \, f(x, x') \]
and \( f(x, x') \) is the phase space density function.

We note that we can define a relation

\[
x' = x'_{th} + ax
\]  

where \( x'_{th} \) is the divergence, which measures the increase of a beam’s radius with distance from the beam’s origin, and \( a \) is a proportionality constant.

Thus we see we can rewrite the statistical emittance as

\[
\tilde{\epsilon}_x = \left( \langle x^2 \rangle \langle (x'_{th} + ax)^2 \rangle - \langle x x'_{th} + ax^2 \rangle \right)^{1/2} = \left( \langle x^2 \rangle \langle x'^2_{th} \rangle \right)^{1/2} = \frac{\ddot{x} x'_{th}}{v_0},
\]  

\( \ddot{x} \) is the RMS width, \( x'_{th} \) is the RMS divergence, \( \ddot{v}_{x,th} = \langle v^2_{x,th} \rangle \) is the RMS thermal velocity, and \( v_0 \) is the mean axial velocity of the particle distribution. We can also define the RMS thermal velocity as

\[
\ddot{v}_{x,th} = \frac{(v^2_{\perp})^{1/2}}{\sqrt{2}}
\]  

which makes it simple to see the relationship between the RMS emittance and \( v_{\perp} \) with the reworked equation

\[
\tilde{\epsilon}_x = \ddot{x} \left( \frac{v^2_{\perp}}{\sqrt{2} v_0} \right)^{1/2} = \ddot{x} \frac{\tilde{\beta}_\perp c}{\sqrt{2} v_0},
\]  

where \( \tilde{\beta}_\perp = \left( \frac{v^2_{\perp}}{c} \right)^{1/2} \) and \( c \) is the speed of light.

We can find a distribution for \( \beta \) using the cathode temperature. The beam source of the electron lens is a planar diode consisting of a thermionic cathode, which is heated to a temperature \( T \) in order to stimulate emission of electrons, and an anode, toward which
Figure 1.2: Gaussian distribution of particles in phase space. Here \( x' \) is labeled as \( x_p \).

The particles are accelerated by a potential difference[5]. The generated particles will have a Maxwellian velocity distribution:

\[
f(v) = \left( \frac{m}{2\pi kT} \right)^{3/2} \exp \left[ \frac{-mv^2}{2kT} \right]
\]  

(1.8)

and they will have a mean velocity \( \bar{v} \) of the distribution is characterized by the formula

\[
\langle \bar{v} \rangle = \int f(v)vdv = \left( \frac{8kT}{\pi m} \right)^{1/2}
\]

(1.9)

Where \( k \) is the Boltzmann constant. With a probability distribution for \( v \), finding one for \( \beta \) is simple as \( \beta \) is simply \( \frac{v}{e} \).
1.3 Previous Research

The idea of using cyclotron radiation as a diagnostic tool was first discussed in Ref. [6]. In the paper Rubbia explains that in order to use high intensity electron beams to damp the size and momentum spread of heavy particles circulating in a storage ring the transverse component of the electron velocity $v_\perp$ must be as small as possible to make the damping force effective. The electrons in the beam undergo helical motion around the field lines from a radius of $\rho = \frac{v_\perp}{\omega_c}$ where the cyclotron radian frequency is $\omega_c = (e/m)B_0$, $e$ being the elementary charge, $m$ being the mass of the electron and $B_0$ being the magnetic field strength. In Rubbia’s projects a field strength of $B_0 = 0.3$ T was common which leads to cyclotron frequency of $5.277 \times 10^{10}$ rad/sec for electrons. This would mean a small radius such as $\rho = 0.2 mm$ would imply $v_\perp \approx 10^7 m/s$. Carefully applied perturbations could be used to make the final resultant radial velocity negligible but one must be precise in order to avoid these large amounts of transverse motion. Because of the precision required Rubbia thought of a diagnostic device capable of measuring radial velocities in order to allow for the correction of beam parameters to the optimal values. The experiment required motion to be kept to within a few hundred microns of the equilibrium position, so determining the beam profile mechanically was unfavorable due to how delicate the system was. In order to solve this problem he proposed a method whereby one observed the radiative losses from the electrons due to radial motion, just as we plan to do.

The total radiation emitted by the electrons starting is given by Larmor’s formula

$$P = \frac{2\varepsilon^2}{3c} \gamma^6 [(\hat{\beta})^2 - (\vec{\beta} \times \hat{\beta})^2],$$

(1.10)
where \( \beta = \frac{v}{c} \). After further derivation one arrives the equation

\[
P = \frac{2}{3} \frac{mr_e}{c} \gamma^4 \omega_c^2 v_\perp^2
\]

(1.11)

where \( r_e = \frac{e^2}{mc^2} \) is the classical electron radius.

Afterwards he derived the radiation that was radiated at the frequency \( \omega_c \) starting with the rate at which current performs work on the electromagnetic field

\[
P = - \int \vec{j} \cdot \vec{E}_{ret} dv
\]

where \( \vec{j} \) is the current density and \( \vec{E}_{ret} \) is the retarded electric field. After a lengthy derivation in which values corresponding to motion in the moving electron frame at a radius \( \rho = v_\perp / \omega_c \) are substituted in the power \( P_n \) radiated at the \( n \)th harmonic of \( \omega_c \) is found to be

\[
P_n = n \omega_c \frac{e^2}{\rho} \left[ 2 \beta_\perp^2 J_{2n}'(2n\beta) - (1 - \beta^2) \int_{0}^{2n\beta_\perp} J_{2n}(x) dx \right]
\]

where \( J_{2n} \) and \( J_{2n}' \) are Bessel functions and their derivative. When \( \beta_\perp \ll 1 \) the dominant term of \( P_n \) is

\[
P_n \approx n \omega_c \frac{e^2}{\rho} (n + 1) \frac{n^{(2n+1)}}{(2n + 1)!} \beta_\perp^{2n+1}
\]

It can be seen that \( n=1 \) will dominate and so radiation emitted at the frequency \( \omega_c \) is approximately

\[
P_n \approx \frac{2}{3} \omega_c \frac{e^2}{\rho} \beta_\perp^3
\]

(1.12)

Eq. 1.12 is then compared with Eq. 1.11 after some manipulation:
We can see the equations are identical except for the $\gamma^4$ term and so the fraction of power radiated at frequencies other than $\omega_c$ is negligible.

After this Rubbia goes on to calculate expected results in the laboratory frame and talk about detection of the electromagnetic radiation and comes to the conclusion that the technique seems feasible and offers a convenient way of monitoring and adjusting an electron beam [6]. This conclusion from one as well regarded as Rubbia is a large motivation for the start and continued work of this project.

Most recently Ref. [2] and [4] report observations on the cyclotron radiation of electrons from atomic decay, which is of course different than our case in which we work with an electron beam. Their objective in measuring the electron energy distribution is to find the mass of the electron neutrino. A schematic of their proposed experiment can be seen in Figure 1.3.
Though the experiment being done by Monreal and his team differs in many ways it also has many similarities that give us an opportunity to learn from. Though the emitted cyclotron radiation is narrowband and of a certain frequency $\omega$ the signal seen by the antenna will be different because of a variety of reasons such as a Doppler shift due to $v_\parallel$, some kind of dependence on the electron-antenna distance, and the differential angular power distribution inherent in the emission. Because the detected signal depends on the antenna configuration, there is discussion on the merits of both endcap antenna and transverse antenna arrays, but for our situation if we were to attempt to detect cyclotron radiation, we would need an antenna array similar to the transverse antenna array shown in Figure 1.3 [4].

Another paper by Monreal reports on the observation of the radiation spectrum associated with cyclotron radiation from a single electron. The temporal evolution of the radiation’s spectrum appears in Fig. 1.4 and demonstrates, for the case of a single electron, how the frequency of the radiation changes as a non-relativistic electron moves in the solenoidal field. The increase in frequency is the result of energy losses in a relativistic regime. We would expect a similar spectrum to this when observing a beam of electrons, though we would see bands instead of lines due to the energy spread inherent in a beam of particles.

Finally Ref. 3 derived the angular properties of cyclotron radiation. It specifically shows that a function $G(\beta_\parallel, \beta_\perp, \theta)$ analogous to the directivity gain of an antenna which is defined by

$$G(\beta_\parallel, \beta_\perp, \theta) = \frac{3}{4}(1 - \beta^2)^2 \frac{4g_\parallel^2(\theta)(1 + \beta_\parallel^2)(1 + \cos^2\theta) - 4\beta_\parallel\cos\theta] - (1 - \beta_\parallel^2 + 3\beta_\perp^2)\beta_\perp^2\sin^4\theta}{4[g_\parallel^2(\theta) - \beta_\perp^2\sin^2\theta]^{7/2}}$$ (1.13)

where $\beta$ is the velocity relative to the speed of light, $\theta$ is the angle between the observer and $\vec{B}$, the B-field, and $g_\parallel(\theta) = 1 - \beta_\parallel\cos\theta$. 
Figure 1.4: A plot of the cyclotron radiation emitted by a single electron from $^{83}\text{Kr}$ decays. It can be seen that as energy is radiated away, emission frequency increases because of relativistic effects. One can also see a series of jumps in the frequency due to collisions between electrons and residual gas atoms[2].
Figure 1.5: The function $G(\theta)$ when $\beta = 0.9$. The three curves all have different values of the ratio $\beta_\parallel/\beta_\perp$. As can be seen, there are curves where the ratio equals 2, 5 and 10, which are plotted with the colors blue, orange, and green, respectively.

A plot for $G(\beta_\parallel, \beta_\perp, \theta)$ can be seen in Figure 1.5. As can be seen from the plot, during both simulation and experiment we would expect most energy radiated away to be in the direction of propagation. A formula for the observed cyclotron frequency

$$\omega_0(\theta) = \frac{\omega_c(1 - \beta^2)^{1/2}}{1 - \beta_\parallel \cos \theta}$$

(1.14)

is also introduced, which was used by us extensively [3]. We note that this $\omega_c$ is the classical cyclotron frequency previously introduced, which was defined by $\omega_c = (e/m)B_0$, where $e$ was the elementary charge, $m$ was the mass of the electron and $B_0$ was the magnetic field strength.
CHAPTER 2
MATHEMATICAL FORMULATION

2.1 The relation between Cyclotron radiation, \(\beta_\parallel\), \(\theta\), and \(\gamma\)

We have stated we can use the cyclotron frequency in order to more easily find the transverse emittance and energy spread of a beam. We see from Eq 1.4, however, that a relation is not readily apparent between these parameters and \(\omega_0\). Because of Eq. 1.14 we know that \(\beta\), \(\theta\), and \(\beta_\parallel\) are variables that \(\omega_0\) is dependent on. We can modify Eq. 1.14 to explicitly depend on \(\gamma\) as

\[
\omega_0(\theta) = \frac{\omega_c (1 - \beta^2)^{1/2}}{1 - \beta_\parallel \cos \theta} \rightarrow \omega_0(\theta) = \frac{\omega_c}{\gamma} \frac{1}{1 - \beta_\parallel \cos \theta}
\]

(2.1)

In this form we can see if we have the distribution for \(\omega_0\) we should be able to find the distribution for \(\gamma\) and \(\beta_\parallel\). We require \(\gamma\) as it is related to energy and energy spread, but we must consider both variables in order to determine \(\beta_\perp\), which we need to find transverse emittance. This can be seen easily when we consider the relation \(\beta\) and \(\gamma\)

\[
\beta^2 = 1 - \frac{1}{\gamma^2}
\]

Thus we see

\[
\beta_\parallel^2 = 1 - \beta_\perp^2 - \frac{1}{\gamma^2}
\]
and

\[ 2d\beta_{\parallel}\beta_{\parallel} = -2d\beta_{\perp}\beta_{\perp} + \frac{2\gamma}{\gamma^3} \]

where we can see the dependence of both \( \beta_{\perp} \) and the spread in \( \beta_{\perp} \) on \( \gamma, \beta_{\parallel} \), and their respective spreads.

Figures 2.1, 2.2, 2.4, and 2.6 are histograms of \( \omega_0 \) that were obtained while varying \( \theta, \gamma, \) and \( \beta \). In all these situations \( \beta_{\parallel} \) was approximated to be \( \beta \) and \( \vec{B} \) was taken to be 1 Tesla. Normal distributions were created for \( \gamma \) and \( \beta \) with a standard deviation equal to 10% of the distribution’s mean. This approximation, while not accurate, gives a general expected set of values and trends for more accurate simulations. Figures 2.3, 2.5, and 2.7 are, respectively, plots of the standard deviation of \( \omega_0 \) vs. the standard deviation of \( \theta, \gamma, \) and \( \beta \).

As expected from Fig. 1.3, we can see in Figures 2.1 and 2.2 that mean frequency decreases from \( \theta = 0 \) to \( \theta = \pi \). In Fig. 2.3 we can see that the standard deviation decreases from 0 to 60 degrees, then increases some before become mostly constant. The minimum at 60 degrees can be explained as we approximated \( \beta_{\parallel} \) to be 0. This causes the function \( \omega_0(\theta) \) to have a maximum at \( \beta_{\parallel} = \cos(\theta) \). In this simple model, the mean of \( \beta_{\parallel} \) was taken to be .5, is equal to the cosine of 60 degrees. At this maximum there is not much change within two standard deviations of \( \beta_{\parallel} \). All this amounts to a massive decrease in the standard deviation of \( \omega_0 \) at 60 degrees in Fig. 2.3. From Figures 2.4 and 2.5 we see \( \omega_0 \) and the standard deviation of \( \omega_0 \) increase as \( \gamma \) increases, and from Figures 2.6 and 2.7 we see \( \omega_0 \) and the standard deviation of \( \omega_0 \) increase as \( \beta \) increases as well.
Figure 2.1: Frequency spectra in steps of $\frac{\pi}{12}$ from $\theta=0$ to $\theta=\frac{5\pi}{12}$ degrees. Here the $\beta_\parallel$ is a normal distribution with 100000 values that have a mean of .5 and a standard deviation of .05 and $\gamma$ has a mean of 1.1547 and a standard deviation of .11547
Figure 2.2: Frequency spectra in steps of $\frac{\pi}{2}$ from $\theta=0$ to $\theta=\pi$ degrees. Here the $\beta_\parallel$ is a normal distribution with 100000 values that have a mean of .5 and a standard deviation of .05.
Figure 2.3: A plot of the standard deviations of the cyclotron frequency spectra in Figures 2.1 and 2.2 vs. their corresponding $\theta$ values.
Figure 2.4: Frequency spectra with $\gamma$ values corresponding to $\beta_\parallel$ values of .4, .45, .5, .55, and .6. Here the original $\gamma$ distributions are Gaussian, have 100,000 values, have a mean shown above their corresponding histograms, and have standard deviations equal to .1 times their mean. $\theta$ here is assumed to be 0.
Figure 2.5: A plot of the standard deviations of the cyclotron frequency spectra in Figure 2.4 vs. their corresponding gamma distribution’s standard deviations.
Figure 2.6: Frequency spectra with $\beta_\parallel$ values of .4, .45, .5, .55, and .6. Here the $\beta_\parallel$ distributions are Gaussian, have a mean shown above their corresponding histograms, and have standard deviations equal to .1 times their mean. $\theta$ here is assumed to be 0.
Figure 2.7: A plot of the standard deviations of the cyclotron frequency spectra in Figure 2.6 vs. their corresponding beta distribution's standard deviations.
2.1.1 Finding the Probability Distribution of Cyclotron Frequency

Looking at Eq. 2.1 again, while it is simple to see that $\omega_0$ is dependent on $\gamma$ and $\beta_\parallel$, we have a problem in that we are not working with single values of $\gamma$ and $\beta_\parallel$. We are working with distributions of values, and so we need to make sure we can connect the random variable $\omega_0$'s probability distribution with the probability distributions of $\omega$ and $\beta$. To help us in this regard we think of a set of three random variables, X, Y, and Z. The random variable Z is dependent on random variables X and Y, and we introduce the following equations:

$$g(z) = \int_{-\infty}^{\infty} f_1(x) f_2\left(\frac{z}{x}\right) \frac{dx}{|x|}$$  \hspace{1cm} (2.2)

$$g(z) = \int_{-\infty}^{\infty} f_1(yz) f_2(y)|y|dy$$  \hspace{1cm} (2.3)

Eq. 2.2 is used to find the probability distribution formed by the multiplication of two random variables while Eq 2.3 is used to find the distribution formed by the division of two random variables. In the case of Eq. 2.2 $z = xy$ and in the case of Eq. 2.3 $z = x/y$. Here $f_1$ and $f_2$ are the probability density functions of the variables, and they are multiplied by the absolute value of the Jacobian of the transformation before integrating over all space. The end result is $g(z)$, the probability density function of the random variable $z$ [1].

Referring back, we can split up Eq. 2.1 into two fractions so that we will have the multiplication of two distributions, which means we will be able to use Eq. 2.2. The two distributions would be
\[ \frac{\omega_c}{\gamma}, \quad \text{and} \quad \frac{1}{1 - \beta\| \cos(\theta)}. \tag{2.4} \]

The first one is simply a constant, \( \omega_c \) times the inverse of gamma, which is a random variable. This separates our two distributions of interest, \( \gamma \) and \( \beta\| \), though both will be needed to find the distribution of \( \beta\perp \).

We now introduce the equation

\[ g(y) = f[w(y)]|J| \tag{2.5} \]

where \( J \) is the Jacobian, to find the probability distribution of \( Y \) as long as \( Y = u(X) \) defines a one-to-one correspondence between the values of \( X \) and \( Y \)[11]. We can now find the distributions of the two fractions in this manner, and we see (assuming \( \gamma \) and \( \beta\| \) are normally distributed)

\[
\frac{\omega_c}{\gamma} \rightarrow Y = \frac{\omega_c}{X} \rightarrow x = \frac{\omega}{y} = w(y) \rightarrow J = \frac{-\omega_c}{y^2}
\]

\[
g(y) = \frac{1}{\sigma_\gamma \sqrt{2\pi}} \exp\left[-\frac{1}{2} \left( \frac{\omega_c}{y\sigma_\gamma} - \frac{\mu_\gamma}{\sigma_\gamma}\right)^2 \right] \frac{\omega_c}{y^2}
\]

\[
g(\gamma) = \frac{\gamma^2}{\omega_c \sigma_\gamma \sqrt{2\pi}} \exp\left[-\frac{1}{2} \left( \frac{\gamma - \mu_\gamma}{\sigma_\gamma}\right)^2 \right]
\]

\[
\frac{1}{1 - \beta\| \cos(\theta)} \rightarrow Y = \frac{1}{1 - X\cos(\theta)} \rightarrow x = \frac{1}{\cos(\theta)} \left( 1 - \frac{1}{y} \right) \rightarrow J = \frac{1}{y^2 \cos(\theta)}
\]

\[
g(y) = \frac{1}{\sigma_{\beta\|} \sqrt{2\pi}} \exp\left[-\frac{1}{2} \left( \frac{1 - \frac{1}{y} - \mu_{\beta\|} \cos(\theta)}{\cos(\theta) \sigma_{\beta\|}}\right)^2 \right] \frac{1}{y^2 \cos(\theta)}
\]

\[
g(\beta\|) = \frac{1}{(1 - \beta\| \cos(\theta))^2 \cos(\theta)} \frac{1}{\sigma_{\beta\|} \sqrt{2\pi}} \exp\left[-\frac{1}{2} \left( \frac{\beta\| - \mu_{\beta\|}}{\sigma_{\beta\|}}\right)^2 \right]
\]
With the probability functions of the two fractions we can now utilize Eq. 2.2 and get the following result for the probability function of $\omega_0$

$$g(\omega_0) = g(z) = \frac{\omega_c}{2\pi\sigma_\gamma \sigma_{\beta||} \cos(\theta)} \int_{-\infty}^{\infty} \exp\left[-\frac{1}{2} \left( \frac{\omega_c}{x\sigma_\gamma} - \frac{\mu_{\gamma}}{\sigma_\gamma} \right)^2 - \frac{1}{2} \left( \frac{1 - \frac{\omega}{\omega_0} - \frac{\mu_{\beta||} \cos(\theta)}{\sigma_{\beta||}}}{\cos(\theta) \sigma_{\beta||}} \right)^2 \right] \frac{dx}{|x|}$$

To make the integral solvable numerically, we can rewrite this in terms of $x$ and $y$ instead of $x$ and $z$ and rearrange things to get

$$g(\omega_0) = \frac{\omega_c \exp\left(-\frac{1}{2} \left( 1 - \frac{\mu_{\beta||} \cos(\theta)}{\sigma_{\beta||}} \right)^2 \right)}{2\pi\sigma_\gamma \sigma_{\beta||} y^2 \cos(\theta)} \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2} \left( \frac{\omega_c}{x\sigma_\gamma} - \frac{\mu_{\gamma}}{\sigma_\gamma} \right)^2 \right) \frac{dx}{x^2 |x|}$$

We can rewrite this to see that the probability density function becomes

$$g(\omega_0) = \frac{\omega_c^2 \exp\left(-\frac{1}{2} \left( \frac{\omega_c}{x\sigma_\gamma} - \frac{\mu_{\gamma}}{\sigma_\gamma} \right)^2 \right)}{2\pi\sigma_\gamma \sigma_{\beta||} \omega_0^2 \gamma^2 \cos(\theta)} \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2} \left( \frac{\omega_c}{x\sigma_\gamma} - \frac{\mu_{\gamma}}{\sigma_\gamma} \right)^2 \right) \frac{dx}{x^2 |x|} \quad (2.6)$$

The integrand in Eq. 2.6 is plotted in Fig. 2.8 and it can be seen that the result of integrating from 0 to 50 kHz is about the same as integrating over all space.

After solving the integral numerically over this range we plugged in our result to Eq. 2.6, and after taking $\mu_{\gamma}$ to be 1.1957, which corresponds to a kinetic energy of 100 keV, $\mu_{\beta||}$ to be .5, $\theta$ to be .01, $\sigma_{\gamma}$ to be .1$\mu_{\gamma}$, and $\sigma_{\beta||}$ to be .1$\mu_{\beta||}$, we found an expression for $g(\omega_0)$ that depended only on $\omega_0$ and $\gamma$. Plugging in a few values for $\gamma$ we found what is shown in Fig. 2.9. Our result is interesting in that if we only look at the case $\gamma = \mu_{\gamma}$ it has a mean frequency in the range that we expected from our simpler histograms in Fig. 2.4. However, unlike in Fig. 2.4, as gamma increases the mean frequency and standard deviation of frequency, and vice versa. While the results here disagree with those of the simpler model discussed previously, that is not completely unexpected as the simple model considers the
Figure 2.8: The integrand in Eq. 2.6 plotted as a function of \( x \), which is defined by \( x = \omega_c / \gamma \). Here \( B \) was taken to be 1 T, \( \mu_\gamma \) was taken to be 1.1957, which corresponds to a kinetic energy of 100 keV, and the standard deviation \( \sigma_\gamma \) was set to be \( 0.1 \mu_\gamma \).
Figure 2.9: The final result of Eq. 2.6 plotted as a function of frequency after taking $\gamma = \mu_\gamma$, $\gamma = \mu_\gamma + \sigma_\gamma$, and $\gamma = \mu_\gamma - \sigma_\gamma$

case in which $\beta_\perp = 0$, which is a large simplification that is not truly representative of the expected experiments.

### 2.1.2 Relating Cathode temperature, $\gamma$, and $\beta_\parallel$

To compare our results with another method we began working on a model that was only dependent on cathode temperature. This would be useful as we could easily measure the cathode temperature and from that find beam properties. We had previously discussed the Maxwell-Boltzmann Distribution and know the distribution is defined as

$$f(v) = \left( \frac{m}{2\pi kT} \right)^{3/2} \exp \left[ -\frac{mv}{2kT} \right]$$
and its mean velocity $\bar{v}$ is characterized by the formula

$$\langle v \rangle = \int f(v)vdv = \left(\frac{8kT}{\pi m}\right)^{1/2}$$

Where $k$ is the Boltzmann constant, $T$ is the temperature, which in our case is the cathode temperature, and $m$ is the mass of the particle in question. With this we can find the variance of the distribution to be

$$\sigma_v^2 = \int_{\mathbb{R}} \left[v - \left(\frac{8kT}{\pi m}\right)^{1/2}\right]^2 \left(\frac{m}{2\pi kT}\right)^{3/2} \exp\left[-\frac{mv}{2kT}\right] dv$$

After integrating and simplifying we find

$$\sigma_v^2 = \frac{3kT}{m} - \frac{8kT}{\pi m}$$

From this it follows that the standard deviation is

$$\sigma_v = \sqrt{\frac{3kT}{m} - \frac{8kT}{\pi m}}$$

And thus the standard deviation of $\beta$ is

$$\sigma_{\beta_0} = \frac{1}{c} \sqrt{\frac{3kT}{m} - \frac{8kT}{\pi m}}$$

Mean $\beta$ is simple enough to find from $\langle v \rangle$ and we see

$$\langle \beta_0 \rangle = \sqrt{\frac{8kT}{m\pi c^2}}$$

From $\beta$ we can find $\gamma$, and we assume that the total cathode temperature is an isotropic distribution so that $T_\parallel = \frac{1}{3} T$. Thus
\[ \langle \beta_\parallel |0 \rangle = \sqrt{\frac{8kT}{3m\pi c^2}} \]

and

\[ \sigma_{\langle \beta_\parallel \rangle} = \frac{1}{c} \sqrt{\frac{3kT}{3m} - \frac{8kT}{3\pi m}} \]

An accelerating voltage is also applied and so the mean value of \( \beta_\parallel \)

\[ \langle \beta_\parallel \rangle = \langle \beta_\parallel |0 \rangle + \beta_{av} \]

where \( \beta_{av} \) is a constant value from the accelerating voltage. Thus from the cathode temperature we should be able to find the values we are interested in. However, this model appears to have some shortcomings as can be seen in Figures 2.10 and 2.11. We attempted to see the relationship between frequency and temperature, setting \( \theta \) to be 30 degrees, having a temperature range of 0 to 3000 K and setting \( \beta_{av} \) to correspond with an energy of 50 keV. In these figures we can see the mean frequency and the standard deviation of frequency increasing with temperature, as expected, however both have ranges that are different than what we expected. As can be seen the mean frequency doesn’t get much higher than 200 GHz, even though in Fig. 2.6 we can see a mean frequency of about 270 GHz for \( \beta_\parallel = .4 \). The standard deviation is far lower than expected, being in the range of about 10 MHz, also far less than seen in Fig. 2.6 for \( \beta_\parallel = .4 \). This means we are predicting a very tight frequency distribution with a lower than expected mean, and so currently the method shown to relate the cathode temperature and beam parameters needs more work.
Figure 2.10: The projected mean cyclotron frequency as a function of cathode temperature, setting $\theta = 30$ degrees. Here $\beta_{av}$ is taken to be .412965, which corresponds with an energy of 50 keV, and $\beta$ is a normal distribution with mean and standard deviation as defined in section 2.1.2 and $\gamma = \frac{1}{(1-\beta^2)}$. 

![Graph showing the projected mean cyclotron frequency as a function of cathode temperature.](image-url)
Figure 2.11: The projected standard deviation of cyclotron frequency as a function of cathode temperature, setting $\theta = 30$ degrees
3.1 Warp-X and the Particle-In-Cell method

While working with statistics it is useful to be able to validate your findings through simulations. In order to do this we worked with the Particle-In-Cell (PIC) simulation code Warp-X. The PIC method works by looking at a collection of charged macro-particles that change self-consistently under the influence of their electromagnetic or electrostatic fields. During each time step, a PIC algorithm completes four operations. First, the velocity and position of the particles are changed using the Newton-Lorentz equations. Next, the charge and/or current densities are deposited onto the grid using interpolation from the distribution of the particles. Afterwards, the Maxwell’s wave equations are solved for electromagnetic cases or Poisson’s equation is solved for electrostatic cases. Finally, the fields are interpolated from the grid onto the particles for the next particle push. Besides these main four operations, supplementary operations can be inserted to account for additional physics or numerical effects. Some additional operations include the simulation of absorption and emission of particles, addition of external forces to account for accelerator focusing or an accelerating component, and filtering of the charge densities, current densities, and fields on the grid. A diagram outlining this algorithm is shown in Fig. 3.1 [10].
3.2 Rudimentary Simulations

We began with simulations of a single electron in a constant magnetic field in order to familiarize ourselves with the code, which can be seen in Figures 3.2 and 3.3. In this simulation and all following ones, $z$ is the direction of propagation while $x$ and $y$ are the horizontal and vertical axes of the plane orthogonal to $z$. The simulation was .05 by .05 by .4 meters with a mesh of 64 by 64 by 192 points. The results in Figures 3.2 and 3.3 confirmed the predicted helical path of an electron in a magnetostatic field.

After this we simulated a beam of electrons instead of a single particle. The positions of the particles are plotted below in Fig. 3.4.

In Fig. 3.5 we have the $E$ Field in $x$ as a function of time. This $E$ Field is measured by taking the electric field calculated for a portion of the mesh. In a simulation area of 64 by 64 by 192 in the $x$, $y$, and $z$ directions, this portion was located at 30, 1, 32. This puts the area about in the middle horizontally, at the bottom vertically, and in the first fifth of the simulation area along the axis of propagation. Afterwards we did a Fast Fourier Transform
Figure 3.2: The path of a single electron in the $xy$ plane under the influence of a B Field of strength $0.015$ T and an initial $\beta_z$ of $0.7$ and $\beta_x$ of $0.3$. 
Figure 3.3: The path of a single electron in the $yz$ plane under the influence of a B Field of strength $0.015$ T and an initial $\beta_z$ of $0.7$ and $\beta_x$ of $0.3$. 
Figure 3.4: The positions of a beam of electrons in the xy plane over time under the influence of a B Field of strength .15 T. The particles had a mean normalized momenta \((\gamma \beta_i)\) where \(i = x, y, z\) in the z direction of .57735 and 0 in the x and y directions. The standard deviation of the normalized momenta in x, y, and z is .06, .06, and .057735, respectively.
Figure 3.5: The electric field in x measured the simulation in Figure 3.4 over time. (FFT) of the $E_x$ and plotted it as a function of frequency. The results are shown below in Fig. 3.6. From an FFT of the the electric field we hope to see some peak in the frequency spectrum that corresponds to the cyclotron frequency. If we could find this, we would be able to find a distribution for $\omega_0$, and then attempt to find distributions for $\beta_\parallel$ and $\gamma$ from said distribution. This would then allow us to try and verify our findings in the previous chapter.

We believed that the peaks in Fig 3.6 may have been related to the cyclotron frequency, and in order to test this, we repeated the simulation with the same parameters except we changed the normalized momentum so that it corresponded with different energies. This normalized momentum is simply the product of $\gamma$ and $\beta$. We did this because we can see from Eq. 2.1 that $\omega_0$ depends on $\gamma$, which corresponds directly with the normalized momentum. The results are plotted in Fig 3.7.
Figure 3.6: An FFT of the $E_x$ in Fig. 3.5 plotted as a function of frequency.

As can be seen, even when the energy changes by an order of magnitude, the locations of the peaks do not change. If they were related to the cyclotron frequency we would expect a noticeable shift in corresponding frequency for the peaks due to the $\gamma$ dependence of the cyclotron frequency.

### 3.3 Further Simulations

In an attempt to find a frequency spectrum for $\omega_0$, we looked for inputs we could change to get new results from our simulations. When looking at Fig. 3.5, one sees that the simulation ends while $E_x$ is undergoing a large fluctuation. In order to see if we were missing some other frequency peak, we doubled the length of the simulation from about .7 nanoseconds to
Figure 3.7: FFT's of $E_x$ for simulations that were the same as in Fig. 3.6 except their normalized momentum corresponded with different energy values (100, 250, 500, 750 and 1000 keV)
Figure 3.8: The electric field in x over time of a simulation that is the same as the one in Fig 3.4 except the length of the simulation was doubled.

1.4 nanoseconds, keeping all other things constant. The results of this can be seen in Figures 3.8 and 3.9.

We can see from Fig 3.9 that there is another peak that has appeared in the FFT around 180 GHz. We decided to explore if this could be related to the cyclotron frequency by changing the value of the B-Field, as the B-Field has a direct correlation with cyclotron frequency even when gamma is irrelevant. Also, instead of limiting ourselves to $E_x$, we also chose to look at $E_y$, $E_z$ and $E_r$ and their corresponding FFTs as we might be able to see a peak corresponding to $\omega_0$ there. The results can be seen in Figures 3.10 through 3.13.

As can be seen, the peaks did not occur at different frequencies in any of the graphs despite the fact that Eq. 2.1 shows there should be dependence on $\omega_c$ which depends on $\vec{B}$.

In an attempt to find peaks corresponding to the cyclotron frequency we measured $E_x$ in different areas than before to see how placement of this sample area changed our results.
Figure 3.9: An FFT of the $E_x$ in Fig 3.8 plotted as a function of frequency.

Figure 3.10: FFT’s of $E_x$ for simulations in which the value of the B-Field is changed.
Figure 3.11: FFT’s of $E_y$ for simulations in which the value of the B-Field is changed.

Figure 3.12: FFT’s of $E_z$ for simulations in which the value of the B-Field is changed.
Figure 3.13: FFT’s of $E_r$ for simulations in which the value of the B-Field is changed.

Considering the simulation area to be a box ranging from -0.025 to 0.025 m in x, -0.025 to 0.025 m in y, and -0.2 to 0.2 m in z, we measured the electric field at, in the format x,y,z, the lower left corner (-0.025, -0.025, 0), the upper left corner (-0.025, 0.025, 0), the upper right corner (0.025, 0.025, 0), and the lower right corner (0.025, -0.025, 0). The results can be seen in Figures 3.14 through 3.21.

Looking at the FFTs in Figures 3.15, 3.17, 3.19, and 3.21, we can see no prominent peaks at the expected values of $\omega_0$ predicted by Eq. 2.1. We would expect peaks somewhere in the hundreds of GHz. Though one is not prominent, we attempted to find a frequency peak corresponding to $\omega_0$ by varying the B-Field from .15 T to 3.0 T. A portion of the spectrum affected by this change in B-Field should be more noticeable under such a large change.

As can be seen in Figures 3.22 through 3.37, we do see some changes as the B-Field is changed, however the peaks in the FFT’s do not shift by any significant amount like what we would expect, as the B-Field changes by a factor of 20, so the Cyclotron frequency should
Figure 3.14: $E_x$ over time for the same simulation as in Fig. 3.8 but with the sample area being in the middle and top right of the simulation’s space.

Figure 3.15: An FFT of the $E_x$ in Fig 3.14 plotted as a function of frequency.
Figure 3.16: $E_x$ over time for the same simulation as in Fig. 3.8 but with the sample area being in the middle and top left of the simulation’s space.

Figure 3.17: An FFT of the $E_x$ in Fig 3.16 plotted as a function of frequency.
Figure 3.18: $E_x$ over time for the same simulation as in Fig. 3.8 but with the sample area being in the middle and bottom left of the simulation’s space.

Figure 3.19: An FFT of the $E_x$ in Fig 3.18 plotted as a function of frequency.
Figure 3.20: $E_x$ over time for the same simulation as in Fig. 3.8 but with the sample area being in the middle and bottom right of the simulation’s space.

Figure 3.21: An FFT of the $E_x$ in Fig. 3.20 plotted as a function of frequency.
Figure 3.22: FFT’s of the $E_x$ taken in the middle and top right of the simulation’s space while varying the B-Field. Here only the positive frequencies are shown to make analysis clearer.
Figure 3.23: FFT’s of the $E_y$ taken in the middle and top right of the simulation’s space while varying the B-Field. Here only the positive frequencies are shown to make analysis clearer.
Figure 3.24: FFT’s of the $E_z$ taken in the middle and top right of the simulation’s space while varying the B-Field. Here only the positive frequencies are shown to make analysis clearer.
Figure 3.25: FFT’s of the $E_r$ taken in the middle and top right of the simulation’s space while varying the B-Field. Here only the positive frequencies are shown to make analysis clearer. The .15 T spectrum is purple instead of blue because of its overlap with the red 3.0 T spectrum.
Figure 3.26: FFT’s of the $E_x$ taken in the middle and top left of the simulation’s space while varying the B-Field. Here only the positive frequencies are shown to make analysis clearer. Change by about the same amount as well, as we can see in Eq. 2.1. While we did not find a noticeable frequency peak that seemed to correspond to $\omega_0$ in our simulations, the place we measure electric field does seem to make frequency peaks more or less pronounced, so a frequency peak corresponding to $\omega_0$ may be there, and we may have simply chosen to collect data at unfavorable points.
Figure 3.27: FFT’s of the $E_y$ taken in the middle and top left of the simulation’s space while varying the B-Field. Here only the positive frequencies are shown to make analysis clearer.
Figure 3.28: FFT’s of the $E_z$ taken in the middle and top left of the simulation’s space while varying the B-Field. Here only the positive frequencies are shown to make analysis clearer.
Figure 3.29: FFT’s of the $E_r$ taken in the middle and top left of the simulation’s space while varying the B-Field. Here only the positive frequencies are shown to make analysis clearer. The .15 T spectrum is purple instead of blue because of its overlap with the red 3.0 T spectrum.
Figure 3.30: FFT's of the $E_x$ taken in the middle and bottom left of the simulation's space while varying the B-Field. Here only the positive frequencies are shown to make analysis clearer.
Figure 3.31: FFT’s of the $E_y$ taken in the middle and bottom left of the simulation’s space while varying the B-Field. Here only the positive frequencies are shown to make analysis clearer.
Figure 3.32: FFT's of the $E_z$ taken in the middle and bottom left of the simulation’s space while varying the B-Field. Here only the positive frequencies are shown to make analysis clearer.
Figure 3.33: FFT’s of the $E_r$ taken in the middle and bottom left of the simulation’s space while varying the B-Field. Here only the positive frequencies are shown to make analysis clearer. The .15 T spectrum is purple instead of blue because of its overlap with the red 3.0 T spectrum.
Figure 3.34: FFT's of the $E_x$ taken in the middle and bottom right of the simulation's space while varying the B-Field. Here only the positive frequencies are shown to make analysis clearer.
Figure 3.35: FFT’s of the $E_y$ taken in the middle and bottom right of the simulation’s space while varying the B-Field. Here only the positive frequencies are shown to make analysis clearer.
Figure 3.36: FFT’s of the $E_z$ taken in the middle and bottom right of the simulation’s space while varying the B-Field. Here only the positive frequencies are shown to make analysis clearer.
Figure 3.37: FFT’s of the $E_r$ taken in the middle and bottom right of the simulation’s space while varying the B-Field. Here only the positive frequencies are shown to make analysis clearer. The .15 T spectrum is purple instead of blue because of its overlap with the red 3.0 T spectrum.
Previous experiments have indicated that cyclotron radiation could be used as a diagnostic tool. Such a work motivated the present study to attempt to find a relationship between the probability distribution of the cyclotron frequency and the probability distributions of $\gamma$ and $\beta_{\parallel}$ so that we could find them from a detected frequency distribution. This would allow us to find the energy spread and transverse emittance while remaining unintrusive on an electron beam like those that will be utilized in IOTA. Using the relationships between random variables and an existing equation, we were able to find an equation for $\omega_0$’s probability distribution that was dependent on the probability distributions of $\gamma$ and $\beta_{\parallel}$. After this we attempted to find beam properties by using the properties of the Maxwell-Boltzmann distribution to compare our results with another method which would be dependent only on cathode temperature. Ultimately, however, our results with this cathode temperature method were inconclusive.

In order to try and verify or refute our findings we worked with the Particle-in-Cell simulation code Warp-X in order to simulate beams of particles and attempt to measure the cyclotron radiation from these beams. We measured the electric field at various points in our simulation area and performed FFTs of these electric field measurements. These FFTs, plotted as a function of frequency, failed to show peaks that behaved the way we expected $\omega_0$ to, remaining mostly unchanged when varying energy and B-Field. As such these results have, so far, been inconclusive. Despite this, we saw changes in the frequency spectra depending on where we took measurements, so the cyclotron frequency may have been present but unnoticeable.
In the future we plan to find an effective way to extract the cyclotron radiation from Warp-X simulations and compare the results with our theoretical work. If the results do not agree we would have to find a conclusive formulation for $\omega_0$’s probability distribution in terms of $\omega$ and $\beta_\parallel$’s probability distributions in some other manner or find where we made a mistake in our current process.

After finding such a formulation that agrees with simulation we could move on to designing antenna arrays that detect the cyclotron radiation from electron beams at IOTA and test their effectiveness in experiments.
REFERENCES


