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# Variants of Fine transformations

Sarah Wesley

## ABSTRACT

### VARIANTS OF FINE TRANSFORMATIONS



Sarah Wesley, Ph.D.
Department of Mathematical Sciences
Northern Illinois University, 2018
Douglas Bowman, Director



This dissertation studies the method of iteration introduced by Nathan J. Fine for the function $F(a, b; t) = \sum_{n\geq 0} \frac{(a)_n}{(bq)_n} t^n$, where $q$ is a fixed complex number with $|q| < 1$, $|t| < 1$, and $(z)_n = (1 - z)(1 - zq)(1 - zq^2) \cdots (1 - zq^{n-1})$ for $n > 0$ and $(z)_0 = 1$ (for $z \in \mathbb{C}$). Generalizing Fine's methods yields new basic hypergeometric identities. Certain identities have partition theory interpretations and are proved combinatorially using the method of overpartitions. Among other basic hypergeometric identities, generalizations of the Rogers-Fine identity are given.

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# VARIANTS OF FINE TRANSFORMATIONS

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SARAH WESLEY


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# TABLE OF CONTENTS

# CHAPTER 1

# INTRODUCTION AND PRELIMINARIES

## 1.1 Fine's Method

This dissertation makes an intensive investigation of a particular set of transformations of the function $F$ defined by the series

$$F(a, b; t) = \sum_{n \geq 0} \frac{(a)_n}{(bq)_n} t^n, \tag{1}$$

where $q$ is a fixed complex number with $|q| < 1$, $|t| < 1$, and

$$(z)_n = (z; q)_n = \begin{cases} (1-z)(1-zq)(1-zq^2)\cdots(1-zq^{n-1}) & n > 0 \\ 1 & n = 0, \end{cases}$$

for $z \in \mathbb{C}$. We employ the notation $(z)_\infty = \prod_{i=0}^{\infty}(1-zq^i)$. It is not difficult to show that this product converges and defines an entire function of $z$. Factors of this form $(z)_n$ are known as *$q$-factorials*, or *$q$-products*.

It is clear that this series for $F$ converges absolutely for $|t| < 1$ and thus is analytic in the unit disk. As a function of $a$ $F$ is entire away from singular values of $b$ and $t$. As a function of $b$ $F$ is regular except for the simple poles of $b = q^{-n}$, $n \geq 1$. Further details of the analytic behavior of $F$ are discussed in Section 1.3. The parameters $a$ and $b$ are taken to be fixed complex numbers, $b \neq q^{-n}$, $n > 0$. $F$ was intensively studied by Nathan Fine in [9] where he introduced a method to find a number of new expansions. He also gave numerous number-theoretic applications. In this dissertation we consider a generalization of Fine's method. We also give partition-theoretic proofs of several identities along with number-theoretic applications for a few of our results.

Our notation for $F$ differs slightly from that used by Fine; his $F(a, b; t)$ being our $F(aq, b; t)$. This change in notation was found to simplify the statements of most results. Throughout, we consistently use our notation, even when quoting results from Fine.

Power series whose coefficients consist of quotients of products of $q$-products (with different parameters) are called *basic hypergeometric series*, or *q-series*. (This can be seen to be equivalent to the ratio of successive terms of the series being rational functions of $q^n$, $n$ being the index of summation.) It can be shown that such series are meromorphic in the power series variable ($t$ above). *Basic hypergeometric functions* are defined as the meromorphic continuations of basic hypergeometric series. While $q$-series were considered by Euler and Gauss, they were first studied systematically by Heine in [13] and [14], as well as Thomae in [19] and [20]. In fact, $F(a, b; t)$ as defined above, is a special case of a function defined by Heine, namely

$$ {}_2\phi_1(a, b; c; t) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (q)_n} t^n, $$

for $|t| < 1$. (Here semicolons are used to delimit numerator and denominator parameters, as well as the power series variable. The dependence on $q$, which is taken as a constant, is suppressed.)

Transformations between basic hypergeometric series are important in proofs of theorems in number theory. However, the connection goes in both directions. Often, combinatorial number theoretic arguments can help in understanding transformations of basic hypergeometric series. We will see examples of both phenomena with the function $F(a, b; t)$. For example, the divisor function $d(N)$, counting the number of divisors of a positive integer $n$, is given by the coefficients of $qF(q, q; q)/(1 - q)$:

$$\frac{q}{1-q}F(q,q;q) = \sum_{n\geq 1} \frac{q^n}{1-q^n} = \sum_{n\geq 1} d(n)q^n. \tag{2}$$

Another example comes from the special case of $F$:

$$F(0, 1, q) = \sum_{n\geq 0} \frac{q^n}{(q)_n} = \sum_{n\geq 0} p(n)q^n,$$

where $p(n)$ is the number of integer partitions of $n$. In fact, each term on the right side is the generating function for integer partitions with largest part $n$. (A brief introduction to integer partitions is given in Section 1.4.) Many other partition functions and other functions of number theoretic interest are special cases of $F$. Fine's monograph [9] contains numerous examples.

## 1.2 Transformations of Basic Hypergeometric Functions

Transformations of basic hypergeometric series have found applications in many areas of mathematics, including number theory, enumerative combinatorics, modular forms, statistical mechanics, and quantum groups. A particularly famous and useful transformation of this kind is the following formula due to Heine:

$$ {}_2\phi_1(a, b; c; t) = \frac{(b)_\infty (at)_\infty}{(c)_\infty (t)_\infty} \, {}_2\phi_1(c/b, t; at; b), \tag{3} $$

for $|t|, |b| < 1$, [12]. Other transformations of basic hypergeometric functions have been found and further studied by F. H. Jackson, L. J. Rogers, S. Ramanujan, J. J. Sylvester, A. Cauchy, L. Euler, C. F. Gauss, C. Jacobi, G. N. Watson, J. Thomae, W.N. Bailey, D. B. Sears, G. Gasper, R. P. Agarual, L. J. Slater, D. Bressoud, R. Askey, G. Andrews, M. Ismail, N. J. Fine, and many others [10, p.10].

What is the context of transformations like Heine's (3) above? One answer to this question lies in the somewhat vaguely defined structure called the *q-hypergeometric hierarchy*. This hierarchy is a finite sequence of successively more general transformations, in the sense of having more free parameters, beginning with Heine's transformation. Next in the hierarchy are the ${}_3\phi_2$ transformations:

$$ {}_3\phi_2\left(a, b, c; d, e; \frac{de}{abc}\right) = \begin{cases} \dfrac{(b)_\infty (\frac{de}{ab})_\infty (\frac{de}{bc})_\infty}{(d)_\infty (e)_\infty (\frac{de}{abc})_\infty} \, {}_3\phi_2\left(\frac{d}{b}, \frac{e}{b}, \frac{de}{abc}; \frac{de}{ab}, \frac{de}{bc}; b\right) \\ \dfrac{(\frac{e}{a})_\infty (\frac{de}{bc})_\infty}{(e)_\infty (\frac{de}{abc})_\infty} \, {}_3\phi_2\left(a, \frac{d}{b}, \frac{d}{c}; d, \frac{de}{bc}; \frac{e}{a}\right), \end{cases} $$

due to Hall [11] and Sears [17], respectively. In fact, the whole hierarchy can be found in Appendix III of Gasper and Rahman's bible [10].

At the top of the hierarchy is Bailey's four-term ${}_{10}\phi_9$ transformation [6] involving seven free parameters. We do not display it here to avoid eye strain for the reader. Frequently, when identities arise in research, their proofs can be obtained by specializing (and possibly combining) identities from the hierarchy. However, some identities seem to elude the hierarchy. For example, consider Fine's transformation (13.1) of [9]:

$$\frac{1-t}{1-b}F(a,b;t) = \sum_{n\geq 0} \frac{(a)_n(at/b)_{2n}}{(tq)_n(a/b)_n} b^n - a\sum_{n\geq 0} \frac{(a)_n(at/b)_{2n+1}}{(tq)_n(a/b)_{n+1}} (bq)^n. \tag{4}$$

George Andrews, the author of the comments in Fine's book, writes [9, p. 34]:

> *"This section seems more than any in Chapter 1 to be begging for further study. Identities (13.1), (13.2), and (13.5) do not seem to fit into the known q-hypergeometric hierarchy of results."*

This quote on its own begs for further study into Fine's method and suggests that extensions of his method may yield new transformations. In this dissertation, we systematically generalize Fine's method and uncover a large family of new transformations that do not seem to fit into the known $q$-hypergeometric hierarchy of transformations.

Before leaving this identity we note that for the remainder of this dissertation we exclude cases for which the denominators vanish. This is possible since all the terms of the series are given in factored form making cases of vanishing denominators clear. For example, (4) cannot have $b = a$ and requires $|b| < 1$ for convergence. Note that in (4), $b = 0$ is possible if $q$-products are expanded and the $b$ power is distributed over the $q$-products with coefficient $at/b$ and $a/b$ in the numerator and denominator, respectively. Results given throughout this dissertation exclude conditions of invalidity while allowing for limiting cases in a similar way.

## 1.3 Fine's Method of Iteration

How does Fine obtain (4)? To explain Fine's method, it is helpful to start with a simpler example than (4). Such a transformation is (7.2) of [9]:

$$F(a,b;t) = -\frac{b}{at}\sum_{n\geq 0}\frac{(bq/a)_n}{(bq)_n(bq/at)_{n+1}}q^{n+1} + \frac{(bq/a)_\infty}{(bq)_\infty(bq/at)_\infty}\sum_{n\geq 0}(a)_n t^n. \tag{5}$$

Fine obtains this transformation by iterating what we refer to as a "seed" identity. The seed from which (5) is derived is the following:

$$F(a,b;t) = -\frac{bq}{(1-bq/at)at} + \frac{(1-bq/a)}{(1-bq)(1-bq/at)}F(a,bq;t). \tag{6}$$

More generally, Fine's seeds are identities of the form $F = B + AF(aq^{n_1}, bq^{n_2}; tq^{n_3})$, where $n_i = \{0,1\}$ and $A, B \in \mathbb{Q}(a,b,t,q)$. Fine denotes a transformation of the above form by the symbols $(a,b,t) \mapsto (aq^{n_1}, bq^{n_2}, tq^{n_3})$. Equations involving only a single parameter being $q$-shifted are written without parentheses; thus (6) is denoted by $b \mapsto bq$. Also, parameters that are not shifted are not listed in the parentheses. (We defer the elementary derivation of the seed identities until later in this section.) The transformation $b \mapsto bq^2$ can be found by replacing $b$ with $bq$ in (6), and then substituting back into (6). Thus,

$$\begin{aligned}F(a,b,t) = &-\frac{bq}{(1-bq/at)at} - \frac{(1-bq/a)bq^2}{(1-bq)(1-bq/at)(1-bq^2/at)at}\\ &+ \frac{(1-bq/a)(1-bq^2/a)}{(1-bq)(1-bq^2)(1-bq/at)(1-bq^2/at)}F(a,bq^2;t)\end{aligned}$$

$$= -\frac{b}{at}\sum_{i=0}^{1}\frac{(bq/a)_i}{(bq)_i(bq/at)_{i+1}}q^{i+1} + \frac{(bq/a)_2}{(bq)_2(bq/at)_2}F(a,bq^2;t). \tag{7}$$

Continuing in this way one can compute $b \mapsto bq^n$:

$$F(a,b;t) = -\frac{b}{at}\sum_{i=0}^{n-1}\frac{(bq/a)_i}{(bq)_i(bq/at)_{i+1}}q^{i+1} + \frac{(bq/a)_n}{(bq)_n(bq/at)_n}F(a,bq^n;t). \tag{8}$$

Letting $n \to \infty$, re-indexing, as well as noting that since $|q| < 1$, then $bq^n \to 0$ as $n \to \infty$ gives (5).

In [9] Fine shows that special cases of (5) yield famous results of Gauss and Jacobi, such as

$$\sum_{n\geq 0} q^{(n^2+n)/2} = \prod_{n\geq 1}\frac{1-q^{2n}}{1-q^{2n-1}},$$

$$\sum_{n=-\infty}^{+\infty}(-1)^n q^{n^2} = \prod_{n\geq 1}\frac{1-q^n}{1+q^n} = \prod_{n\geq 1}(1-q^{2n})(1-q^{2n-1})^2,$$

and

$$\sum_{n=-\infty}^{+\infty} q^{n^2} = \prod_{n\geq 1}(1-q^{2n})(1+q^{2n-1})^2.$$

Fine obtains six non-trivial identities by iterating seeds such as (6). The seed $(a,b) \mapsto (aq,bq)$ follows from the series definition of $F$,

$$F(a,b;t) = 1 + \frac{(1-a)}{(1-bq)}tF(aq,bq;t). \tag{9}$$

(Iterating this simply recovers (1).)

The seed associated with $t \mapsto tq$ is

$$F(a,b;t) = \frac{1-b}{1-t} + \frac{b-at}{1-t} F(a,b;tq). \tag{10}$$

Iterating and letting $n \to \infty$ for $|b| < 1$ yields (6.3) of [9]:

$$F(a,b;t) = (1-b) \sum_{n=0}^{\infty} \frac{(at/b)_n}{(t)_{n+1}} b^n. \tag{11}$$

Both (10) and (11) reveal the analytic nature of $F$. In Section 1.1 we discussed the analyticity of $F$ inside the unit disk. This region can be extended. The function on the right side of (10) is regular in $|t| < |q|^{-1}$ except for the simple pole at $t = 1$. Thus applying this transformation to $F$ gives a continuation to a larger disk. Applying this transformation a second time gives the continuation for $|t| < |q|^{-2}$. For $b \neq q^{-n}$ and $n \geq 1$, the only singularities of $F$ are simple poles and occur when $t \neq q^{-n}$, $n \geq 0$. This gives an analytic continuation for $F$ as a function of $t$.

Notice that (11) is equivalent to

$$(1-t)F(a,b;t) = (1-b)F(at/b, t; b). \tag{12}$$

Identity (12) shows that for $0 < |b| < 1$, $(1-t)F(a,b;t)$ is invariant under the change of variables $a \mapsto at/b$, $b \mapsto t$ and $t \mapsto b$. Throughout this dissertation, we refer to (12) as *the involution* since applying the change of variables another time recovers $(1-t)F(a,b;t)$. One common application of (12), which is used systematically by Fine, is to apply the involution to identities derived using Fine's method to obtain new transformations. We call these new identities generated by the involution *conjugate* identities. This terminology is motivated by the fact that the resulting identities result from conjugation in associated Ferrers diagrams. We apply this same technique to identities throughout this dissertation. Indeed, in [1], Alladi shows that the involution (12) arises combinatorially by applying conjugation to Ferrers diagrams of integer partitions. (Conjugation of integer partitions is defined in Section 1.4.) In Chapter 3 we show that the involution is also manifested as conjugation applied to *overpartitions*, also defined in Section 1.4.

Another application of (11) is the $q$-Binomial Theorem [10],

$$\frac{(at)_\infty}{(t)_\infty} = \sum_{n\geq 0} \frac{(a)_n}{(q)_n} t^n. \tag{13}$$

To derive this theorem from (11), let $b \to 1^-$ on both sides and apply Abel's Theorem [4]. The $q$-Binomial Theorem is also a special case of Heine's Transformation (3), as is (12). This theorem is used in applications in Chapter 5.

We continue by giving the seeds and iterations associated with other variables. The seed associated with $a \mapsto aq$ is

$$F(a,b;t) = -\frac{(1-b)a}{(1-a/b)b} + \frac{(1-a)(1-at/b)}{(1-a/b)} F(aq,b;t), \tag{14}$$

which by iterating and letting $n \to \infty$ yields (11.1) of [9]:

$$F(a,b;t) = -\frac{a(1-b)}{b} \sum_{n\geq 0} \frac{(a)_n (at/b)_n}{(a/b)_{n+1}} q^n + \frac{(a)_\infty (at/b)_\infty}{(a/b)_\infty} F(0,b;t). \tag{15}$$

The seed associated with $(a,t) \mapsto (aq,tq)$ is

$$F(a,b;t) = \frac{1-b}{1-t} - \frac{(1-b)(1-at/b)a}{(1-t)(1-a/b)} + \frac{(1-a)(1-at/b)(1-atq/b)}{(1-t)(1-a/b)} bF(aq,b;tq), \quad (16)$$

which by iterating and letting $n \to \infty$ with $|b| < 1$ yields (13.1) of [9] (this is (4) above).

The seed associated with $(b,t) \mapsto (bq,tq)$ is

$$F(a,b;t) = \frac{1}{1-t} + \frac{(b-aq^{-1})tq}{(1-bq)(1-t)} F(a,bq;tq), \quad (17)$$

which by iterating and letting $n \to \infty$ yields (12.2) of [9]:

$$(1-t)F(a,b;t) = \sum_{n \geq 0} \frac{(bq/a)_n}{(bq)_n (tq)_n} (-at)^n q^{(n^2-n)/2}. \quad (18)$$

Note that (18) directly shows that $F$ is entire in $a$ and has simple poles at $b = q^{-n}$ for $n \geq 1$ and $t = q^{-n}$ for $n \geq 0$. In [3], Andrews outlines a combinatorial proof of (18) using the odd and even parts of integer partitions. The convergence of the right side of (18) makes it useful for numerical evaluations of $F$. This is employed in the appendix to give numerical checks of new identities found in this dissertation. Note that (15) and (18) give applications to mock theta functions, as discussed in Chapter 3 of [9].

Finally, the seed associated with $(a,b,t) \mapsto (aq,bq,tq)$ is

$$F(a,b;t) = \frac{1-at}{1-t} + \frac{(1-a)(1-at/b)}{(1-t)(1-bq)} btqF(aq,bq;tq), \quad (19)$$

which by iterating and letting $n \to \infty$, yields the Rogers-Fine identity, first proved in [16] and derived again as (14.1) of [9],

$$(1-t)F(a,b;t) = \sum_{n\geq 0} \frac{(a)_n(at/b)_n}{(bq)_n(tq)_n}(1-atq^{2n})(bt)^n q^{n^2}. \tag{20}$$

This transformation has been studied and given several combinatorial proofs. Interestingly enough, the right side is invariant under the involution (12), thus giving another proof of (12). This identity is useful in proving many results in Ramanujan's "Lost" Notebook.

We conclude this section by showing how some of the seed identities are derived, with the remaining identities derived in the appendix in Section A.3.

First, if one denotes the coefficient of $t^n$ in the series definition of $F$ as $c_n$, then it is clear from (1) that $(1-bq^n)c_n = (1-aq^{n-1})c_{n-1}$, for $n \geq 1$. Multiplying this equation by $t^n$ and summing for $n \geq 1$ yields

$$\begin{aligned}\sum_{n\geq 1} c_n t^n - b\sum_{n\geq 1} c_n(tq)^n &= \sum_{n\geq 1} c_{n-1}t^n - at\sum_{n\geq 1} c_{n-1}(tq)^{n-1} \\ &= t + t\sum_{n\geq 1} c_n t^n - at - at\sum_{n\geq 1} c_n(tq)^n.\end{aligned}$$

Thus,

$$(1-t)\sum_{n\geq 1} c_n t^n = (t-at) + (b-at)\sum_{n\geq 1} c_n(tq)^n,$$

or

$$\sum_{n\geq 1} c_n t^n = \frac{t-at}{1-t} + \frac{b-at}{1-t}\sum_{n\geq 1} c_n(tq)^n.$$

Hence,

$$1+\sum_{n\geq 1} c_n t^n = \frac{t-at}{1-t} + \frac{b-at}{1-t}\left(1+\sum_{n\geq 1} c_n (tq)^n\right) - \frac{b-at}{1-t} + 1,$$

so that

$$\sum_{n\geq 0} c_n t^n = \frac{1-b}{1-t} + \frac{b-at}{1-t}\sum_{n\geq 0} c_n (tq)^n,$$

which is precisely (10).

Next, $(1-t)F(a,b;t) = 1+\sum_{n\geq 1}(c_n - c_{n-1})t^n$. Observe that for $n \geq 1$,

$$\begin{aligned} c_n - c_{n-1} &= c_n - \frac{1-bq^n}{1-aq^{n-1}} c_n \\ &= \left(1 - \frac{1-bq^n}{1-aq^{n-1}}\right) c_n \\ &= \frac{bq-a}{1-aq^{n-1}} q^{n-1} c_n. \end{aligned}$$

This gives

$$\begin{aligned} (1-t)F(a,b;t) &= 1+\sum_{n\geq 1}(c_n - c_{n-1})t^n \\ &= 1+\sum_{n\geq 1}\frac{bq-a}{1-aq^{n-1}} q^{n-1} c_n t^n \\ &= 1+\sum_{n\geq 1}\frac{bq-a}{1-aq^{n-1}} q^{n-1}\cdot\frac{(a)_n}{(bq)_n} t^n \\ &= 1+(b-aq^{-1})\sum_{n\geq 1}\frac{(a)_{n-1}}{(bq)_n}(tq)^n \\ &= 1+\frac{b-aq^{-1}}{1-bq} tq \sum_{n\geq 1}\frac{(a)_{n-1}}{(bq^2)_{n-1}}(tq)^{n-1} \end{aligned}$$

$$= 1 + \frac{b - aq^{-1}}{1 - bq} tqF(a, bq; tq).$$

Dividing both sides by $(1 - t)$, gives (17).

The remaining seed transformations are derived from (6), (14), (16), and (19), and are found in the appendix in Section A.3.

Next, we give an overview of partition theory as a background for combinatorial proofs of identities generated from Fine's method of iteration.

## 1.4 Partition Theory Preliminaries

**Definition 1.** An *integer partition* $\pi$ of $n$ (or a partition of $n$) is a representation of $n$ as the sum of one or more positive integers $\pi = a_1 + a_2 + \cdots + a_k$ with $a_{i-1} \geq a_i$ for $i = 2, 3, \ldots, k$. The integers $a_i$ in the sum are referred to as *parts*, or *summands*, of the partition.

For example, the integer partitions of 5 are given below.

$$5$$

$$4 + 1$$

$$3 + 2$$

$$3 + 1 + 1$$

$$2 + 2 + 1$$

$$2 + 1 + 1 + 1$$

$$1 + 1 + 1 + 1 + 1$$

The partition function $p(n)$ is defined to be the number of integer partitions of $n$. From the example above, $p(5) = 7$. By convention, $p(0)$ is defined to be 1. We refer to the number of parts by the variable $\nu$, the largest part by the variable $\lambda$, and the number of different parts by $\nu_d$. For example, in the partition $8 + 6 + 6 + 6 + 5 + 3 + 3 + 1 + 1$, $\nu = 9$, $\lambda = 8$, and $\nu_d = 5$. Other partition functions are defined by placing restrictions on the summands. Following Andrews and Eriksson [5], we write $p(n|[\text{condition}])$. Common examples are $p(n|\text{distinct parts})$, $p(n|\text{even parts})$, $p(n|\text{odd parts})$, and $p(n|\lambda = m)$. For example, $p(5|\text{distinct parts}) = 3$, $p(6|\text{even parts}) = 3$, $p(5|\text{odd parts}) = 3$, and $p(5|\lambda = 2) = 2$.

We employ these partition functions in later sections, but first we require the concept of a generating function.

**Definition 2.** If $S = \{s_n\}_{n=a}^{\infty}$ is a sequence of real numbers, then

$$f_S(x) = \sum_{n=a}^{\infty} s_n x^n$$

is called the *generating function* for the sequence $S$.

As an example, $f_S(x) = x/(1-x)^2$ is the generating function for the sequence $s_n = n$, $n \geq 0$, since

$$\sum_{n \geq 0} n x^n = \frac{x}{(1-x)^2}.$$

We use generating functions in this dissertation in the context of partition theory. Two important partition generating functions are provided by the following theorem.

**Theorem 1** (Euler)**.** *Let $A_m$ be the set of positive integers where all elements are of size at most $m$. Then,*

$$\sum_{n=0}^{\infty} p(n|\textit{parts from } A_m)q^n = \prod_{i\in A_m} \frac{1}{1-q^i}, \tag{21}$$

*and*

$$\sum_{n=0}^{\infty} p(n|\textit{distinct parts from } A_m)q^n = \prod_{i\in A_m} (1+q^i). \tag{22}$$

*Proof.*

For $|q| < 1$,

$$\prod_{i\in A_m} \frac{1}{1-q^i} = \prod_{i\in A_m} \sum_{n_i\geq 0} q^{n_i\cdot i} = \sum_{\substack{n_i\geq 0\\ i\in A_m}} q^{n_1\cdot 1+n_2\cdot 2+\cdots n_m\cdot m}$$

$$= 1 + \sum_{n\geq 1} \left( \sum_{\substack{n_1\cdot 1+n_2\cdot 2+\cdots+n_m\cdot m=n\\ n_i\geq 0\\ i\in A_m}} q^n \right) = \sum_{n\geq 0} p(n|\text{parts from } A_m)q^n.$$

Similarly,

$$\prod_{i\in A_m} (1+q^i) = \prod_{i\in A_m} \sum_{n_i=0,1} q^{n_i\cdot i} = \sum_{\substack{n_i=0,1\\ i\in A_m}} q^{n_1\cdot 1+n_2\cdot 2+\cdots n_m\cdot m}$$

$$= 1 + \sum_{n\geq 1}\left(\sum_{\substack{n_1\cdot 1+n_2\cdot 2+\cdots+n_m\cdot m=n\\ n_i=0,1\\ i\in A_m}} q^n\right) = \sum_{n\geq 0} p(n|\text{distinct parts from } A_m)q^n.$$

□

Letting $m \to \infty$ in (21) and (22) gives the following theorem.

**Theorem 2** (Euler). *Let $A$ be a set of positive integers. Then,*

$$\sum_{n\geq 0} p(n|\textit{parts from } A)q^n = \prod_{n\in A}\frac{1}{(1-q^n)}, \tag{23}$$

*and*

$$\sum_{n\geq 0} p(n|\textit{distinct parts from } A)q^n = \prod_{n\in A}(1+q^n). \tag{24}$$

Frequently we use generating functions with more than one variable. For example, $p(n|\lambda \leq m)$ is written as the two variable function $p(n,m)$.

Similar to the above proofs, it is easy to show that

$$\prod_{i=0}^{m}\frac{1}{1-aq^i} = \sum_{n,m=0}^{\infty} p(n,m)a^m q^n.$$

To show the utility of generating functions, we use them to give a proof of the following theorem of Euler [8].

**Theorem 3.** *The number of partitions of $n$ into distinct parts is equal to the number of partitions of $n$ into odd parts.*

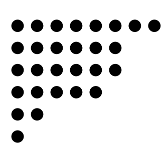
*Proof (Euler).*

$$\begin{aligned}\sum_{n=0}^{\infty} p(n|\text{distinct parts})q^n &= (1+q)(1+q^2)(1+q^3)\cdots \\ &= \frac{(1-q^2)}{(1+q)} \cdot \frac{(1-q^4)}{(1-q^2)} \cdot \frac{(1-q^6)}{(1-q^3)} \cdots \\ &= \frac{1}{(1-q)(1-q^3)(1-q^5)\cdots} \\ &= \sum_{n=0}^{\infty} p(n|\text{odd parts})q^n.\end{aligned}$$

□

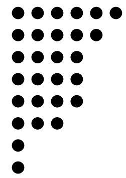
### 1.4.1 Algebraic and Combinatorial Proofs

Another way to represent a partition is as a Ferrers (or Young) diagram. This graphical representation of a partition is created by listing the parts largest to smallest and representing each part $j$ by $j$ dots. For example, below is the Ferrers diagram for the partition $\pi = 8+6+6+5+2+1$.



**Definition 3.** The *conjugate* of a partition $\pi$, denoted $\pi^*$, is given by reflecting the Ferrers diagram so that rows become columns and columns become rows.

For example, it is clear from the diagram below that the conjugate of $\pi = 8{+}6{+}6{+}5{+}2{+}1$ is $\pi^* = 6+5+4+4+4+3+1+1$.



**Theorem 4.** *The number of partitions of $n$ into parts no larger than $m$ is equal to the number of partitions of $n$ into at most $m$ parts.*

*Proof.*

Conjugation of the Ferrers diagram gives a bijection between these sets of partitions.

□

The *Durfee square* of a partition is the largest square which can be placed in the upper left corner of the Ferrers diagram of a given partition. For example $\pi = 8+6+6+5+2+1$ has a Durfee square of size 4. A more formal definition of Durfee square is the following.

**Definition 4.** A partition $\pi = a_1 + a_2 + a_3 + \cdots + a_k$ is said to have *Durfee square* of size $s$ if $s$ is the largest integer such that $a_s \geq s$.

Durfee squares, as well as the analogous Durfee rectangles, are abundant in modern research on integer partitions.

**Definition 5.** The *Durfee rectangle* of size $m \times n$ is the Ferrers diagram for the partition of $mn$ into $m$ parts of size $n$.

We use the notation $m \times n$ to denote a Durfee rectangle with $m$ parts of size $n$ throughout this dissertation. In [2], Andrews presents the concept of the *ambient* rectangle as the largest base of a Durfee rectangle that can be fit into a Ferrers diagram. In the case that more than one maximal base exists, choose the one with the maximal height. For example, the ambient rectangle in the partition $12 + 10 + 8 + 8 + 8 + 6 + 3 + 1$ has base length of 8 and height 5.

In the theory of partitions identities are typically proved in one of two ways: combinatorially or algebraically. The proof of Theorem 3 is an example of an algebraic proof. One may also give a combinatorial proof of an algebraic identity. Proofs of either type are an active part of partition theory research today. In most cases, combinatorial proofs are elegant, but frequently they are the more difficult path of proving an identity. Indeed there are many $q$-series identities for which combinatorial proofs are not known. To illustrate the beauty of combinatorial proofs, we prove an algebraic identity given by Euler.

**Theorem 5** (Euler)**.**

$$\frac{1}{(q)_\infty} = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q)_n^2}. \tag{25}$$

*Proof (Sylvester)* [18].

The left side of (25) generates unrestricted partitions. We show the right side enumerates the same. Consider the $N^{\underline{th}}$ term of the right side of (25),

$$\frac{q^{N^2}}{(q)_N^2}.$$

We construct the set of partitions in three steps using Ferrers diagrams.

1. The factor $q^{N^2}$ generates a Ferrers diagram with Durfee square of size $N$.

2. By Theorem 2 with $A$ equal to the set of integers less than or equal to $N$, the factor $1/(q)_N$ generates partitions into parts of size no larger than $N$. By Theorem 4, these

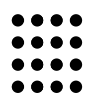
partitions are conjugated and become partitions into at most $N$ parts. These partitions are placed on the right side of the Durfee square.

We now have partitions into $N$ parts of size greater than or equal to $N$.

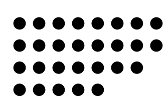
3. Again, the factor $1/(q)_N$ generates partitions into parts of size no larger than $N$. These partitions are placed under the partitions generated at the end of the previous step.

Thus the left side generates all partitions sorted by the size of their largest Durfee square. $\square$

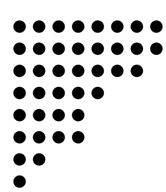
The following is an example of the proof given above. Begin with a Durfee square of size 4, which generates a partition enumerated by $q^{4^2}$ or $\pi_1 = 4 + 4 + 4 + 4$.



The factor $1/(q)_4$ generates a partition into parts of size at most 4, for example, $\lambda = 4 + 3 + 3 + 2$. These partitions are conjugated and placed on the right of the Durfee square $\pi_1$. Conjugating $\lambda$ yields $\lambda^* = 4 + 4 + 3 + 1$. Placing $\lambda^*$ to the right of the Durfee square $\pi_1$ gives $\pi_2 = 8 + 8 + 7 + 5$.



Finally, the factor $1/(q)_4$ generates partitions into at most 4 parts. For example, $\rho = 4 + 4 + 2 + 1$. These partitions are placed below $\pi_2$. Placing $\rho$ below $\pi_2$ yields $\pi_3 = 8 + 8 + 7 + 5 + 4 + 4 + 2 + 1$.



Note the size of the Durfee square remains 4 after the construction is complete. More generally, all partitions with Durfee square of size four have generating function $q^{4^2}/(q)_4^2$.

The Rogers-Fine identity (20) has been given several combinatorial proofs similar to the one above. Using only the simple idea above, we give combinatorial proofs of more general identities in Chapters 3 and 4.

### 1.4.2 Overpartition Preliminaries

Another technique used in later chapters is the idea of "overpartitions", which is a concept explored by Corteel and Lovejoy in [7].

**Definition 6.** An *overpartition* is a partition for which the first occurrence of a part (or equivalently, the last part) may be overlined. We use the phrase *initial overpartition* when the first occurrence of a part is overlined and the phrase *terminal overpartition* when the last occurrence of a part is overlined.

There is an obvious bijection between initial and terminal overpartitions, and for this reason, we omit the adjectives initial and terminal unless required. As an example, the overpartitions of 3 are given below.

$$3$$

$$\overline{3}$$

$$2+1$$

$$\overline{2}+1$$

$$2 + \overline{1}$$

$$\overline{2} + \overline{1}$$

$$1 + 1 + 1$$

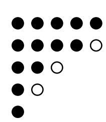
$$\overline{1} + 1 + 1$$

The number of overpartitions of $n$ is denoted by $\overline{p}(n)$ and has generating function

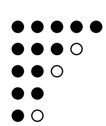
$$\sum_{n=0}^{\infty} \overline{p}(n) q^n = \frac{(-q)_\infty}{(q)_\infty}, \tag{26}$$

since $(-q)_\infty$ generates partitions into distinct parts (which are taken to be overlined parts) and since $1/(q)_\infty$ generates unrestricted partitions.

For Ferrers diagrams, we first employ the bijection to convert to a terminal overpartition. Then, in the Ferrers diagram, we display an overlined part as having the last dot of a part appear as an open circle. For example, the overpartition $\pi = \overline{5} + 5 + \overline{3} + \overline{2} + 1$ has the Ferrers diagram given below.



Conjugation of a Ferrers diagram for an overpartition is accomplished in the same way as conjugation of a Ferrers diagram for general partitions, where one reads the columns of a diagram instead of the rows. For example, the conjugate of $\pi = \overline{5} + 5 + \overline{3} + \overline{2} + 1$ is $\pi^* = 5 + \overline{4} + \overline{3} + \overline{2} + 2$. Again, the bijection between initial and terminal overpartitions is used.



Overpartitions are used in Chapters 3 and 4 to give combinatorial proofs of some of the new identities given in this dissertation. With these preliminaries behind us, the next section gives an outline of the rest of this dissertation and announces some of the results.

## 1.5 Overview

In Chapter 2, we begin with a systematic extension of Fine's method and generate sixteen new identities. Some highlights include Theorem 14, which is the identity

$$\begin{aligned} F(a,b;t) = &\sum_{n=0}^{\infty} \frac{(a)_n (at/b)_n (bq/a)_n}{(bq)_{2n} (t)_{2n+1}} (1 - atq^{3n})(-abt^2)^n q^{n(7n-1)/2} \\ &+ bt \sum_{n=0}^{\infty} \frac{(a)_{n+1} (at/b)_{n+1} (bq/a)_n}{(bq)_{2n+1} (t)_{2n+2}} (-abt^2)^n q^{(7n^2+7n+2)/2}. \end{aligned}$$

We give a combinatorial proof of Theorem 14 in Chapter 3. Other identities of particular interest include Theorem 17: for $|b| < 1$,

$$\begin{aligned} F(a,b;t) = (1-b) &\sum_{n=0}^{\infty} \frac{(a)_{2n} (at/b)_{3n}}{(a/b)_{2n} (t)_{n+1}} b^n - a(1-b) \sum_{n=0}^{\infty} \frac{(a)_{2n} (at/b)_{3n+1}}{(a/b)_{2n+1} (t)_{n+1}} (bq^2)^n \\ &- a(1-b) \sum_{n=0}^{\infty} \frac{(a)_{2n+1} (at/b)_{3n+2}}{(a/b)_{2n+2} (t)_{n+1}} b^n q^{2n+1}, \end{aligned}$$

and Theorem 15:

$$F(a,b;t)=\sum_{n=0}^{\infty}\frac{(a)_{2n}(at/b)_{3n}}{(t)_{2n+1}(a/b)_n(bq)_n}(1-atq^{4n})(b^2t)^nq^{2n^2}$$
$$+\frac{1}{b}\sum_{n=0}^{\infty}\frac{(a)_{2n+1}(at/b)_{3n+1}}{(t)_{2n+2}(a/b)_n(bq)_n}(b^2t)^{n+1}q^{(n+1)(2n+1)}$$
$$-\frac{a}{b}\sum_{n=0}^{\infty}\frac{(a)_{2n+1}(at/b)_{3n+2}}{(t)_{2n+2}(a/b)_{n+1}(bq)_n}(b^2t)^{n+1}q^{(2n+1)(n+2)}.$$

In Chapter 3, we employ overpartitions to give combinatorial proofs of three identities from Chapter 2.

Chapter 4 contains two general results of great interest. The first, one of the highlights in this dissertation, is a two parameter generalization of the Rogers-Fine identity (20). It is Theorem 29: valid for $k+l>0$ and $|b|,|t|<1$,

$$F(a,b;t)=\sum_{i=0}^{k-1}\sum_{n=0}^{\infty}\frac{(a)_{kn+i}(at/b)_{ln}}{(bq)_{kn+i}(t)_{ln}}b^{ln}t^{kn+i}q^{ln(kn+i)}$$
$$+\sum_{j=0}^{l-1}\sum_{n=0}^{\infty}\frac{(at/b)_{ln+j}(a)_{k(n+1)}}{(bq)_{k(n+1)}(t)_{ln+j+1}}(1-bq^{k(n+1)})b^{ln+j}t^{k(n+1)}q^{k(n+1)(ln+j)}.$$

When $k=1$ or $l=1$, we obtain two other identities. The case $l=1$ yields Corollary 30, which is a generalization of Theorem 11, proved in Chapter 2. Similarly, the case $k=1$ yields Corollary 31 which is a generalization of Corollary 12, proved in Chapter 2. Both corollaries extend the Rogers-Fine identity. Theorem 29 is proved with Fine's method algebraically and also combinatorially in Chapter 4.

The second general identity proved in Chapter 4 is Theorem 32:

$$F(a,b;t)=\sum_{n=0}^{\infty}\frac{(bq/a)_n(at/b)_{(k-1)n}}{(bq)_n(t)_{kn+1}}(-ab^{k-1}t)^n q^{(2k-1)\frac{n(n-1)}{2}+(k-1)n}$$
$$+\sum_{i=0}^{k-2}\sum_{n=0}^{\infty}\frac{(bq/a)_{n+1}(at/b)_{(k-1)n+i}}{(bq)_n(t)_{nk+i+2}}(-at)^{n+1}b^{(k-1)n+i}q^{(2k-1)\frac{n(n+1)}{2}+i(n+1)}.$$

This theorem is a generalization of (18) as well as Corollary 10 from Chapter 2. This identity is proved algebraically and combinatorially.

Chapter 5 begins with three general identities contained in Theorem 34. For $k+m>0$, $l\geq 0$, and $|b|,|t|<1$:

$$F(a,b;t)=\sum_{i=0}^{k-1}t^i\sum_{n=0}^{\infty}\frac{(a)_{(k+l)n+i}(at/b)_{(l+m)n}}{(bq)_{kn+i}(a/b)_{ln}(t)_{mn}}b^{mn}t^{kn}q^{mn(kn+i)}$$
$$-\frac{at^k}{b}\sum_{i=0}^{l-1}q^i\sum_{n=0}^{\infty}\frac{(a)_{(k+l)n+k+i}(at/b)_{(l+m)n+i}}{(bq)_{k(n+1)}(aq/b)_{ln+i+1}(t)_{mn}}(1-bq^{k(n+1)})b^{mn}t^{kn}q^{kmn(n+1)+ln}$$
$$+t^k\sum_{i=0}^{m-1}b^i\sum_{n=0}^{\infty}\frac{(a)_{(k+l)(n+1)}(at/b)_{(l+m)n+l+i}}{(bq)_{k(n+1)}(a/b)_{l(n+1)}(t)_{mn+i+1}}(1-bq^{k(n+1)})b^{mn}t^{kn}q^{k(mn+i)(n+1)}.$$

For $k+l>0$, $m\geq 0$, and $|b|,|t|<1$:

$$F(a,b;t)=\sum_{i=0}^{k-1}t^i\sum_{n=0}^{\infty}\frac{(a)_{kn+i}(bq/a)_{ln}(at/b)_{mn}}{(bq)_{(k+l)n+i}(t)_{(l+m)n}}$$
$$\cdot(-a)^{ln}b^{mn}t^{(k+l)n}q^{in(l+m)+\binom{ln}{2}+kln^2+mn^2(k+l)}$$
$$+t^k\sum_{i=0}^{l-1}(-a)^it^iq^{\binom{i}{2}+ik}\sum_{n=0}^{\infty}\frac{(a)_{k(n+1)}(bq/a)_{ln+i}(at/b)_{mn}}{(bq)_{(k+l)n+k+i}(t)_{(l+m)n+i+1}}$$
$$\cdot(-a)^{ln}b^{mn}t^{(k+l)n}q^{ikn+(k+i)(l+m)n+\binom{ln}{2}+kln^2+mn^2(k+l)}$$
$$+(-a)^lt^{k+l}q^{\binom{l}{2}+lk}\sum_{i=0}^{m-1}b^iq^{i(k+l)}\sum_{n=0}^{\infty}\frac{(a)_{k(n+1)}(bq/a)_{l(n+1)}(at/b)_{mn+i}}{(bq)_{(k+l)(n+1)}(t)_{(l+m)n+l+i+1}}(1-bq^{(k+l)(n+1)})$$
$$\cdot(-a)^{ln}b^{mn}t^{(k+l)n}q^{lkn+n(k+l)(i+l+m)+\binom{ln}{2}+kln^2+mn^2(k+l)}.$$

For $k+l>0$, $m\geq 0$, and $|t|<1$:

$$F(a,b;t)=\sum_{i=0}^{k-1}t^i\sum_{n=0}^{\infty}\frac{(a)_{kn+i}(bq/a)_{(l+m)n}}{(bq)_{(k+l+m)n+i}(t)_{ln}(bq/at)_{mn}}(-a)^{ln}t^{(k+l)n}q^{iln+\binom{ln}{2}+kln^2}$$
$$+t^k\sum_{i=0}^{l-1}(-a)^it^iq^{\binom{i}{2}+ik}\sum_{n=0}^{\infty}\frac{(a)_{kn+k}(bq/a)_{(l+m)n+i}}{(bq)_{(k+l+m)n+k+i}(t)_{ln+i+1}(bq/at)_{mn}}$$
$$\cdot t^{(k+l)n}(-a)^{ln}q^{kni+ln(k+i)+\binom{ln}{2}+kln^2}$$

$$+b(-a)^{l-1}t^{k+l-1}q^{1+\binom{l}{2}+lk}\sum_{i=0}^{m-1}q^i\sum_{n=0}^{\infty}\frac{(a)_{kn+k}(bq/a)_{(l+m)n+l+i}}{(bq)_{(k+l+m)n+k+l+i}(t)_{ln+l}(bq/at)_{mn+i+1}}$$
$$\cdot t^{(k+l)n}(-a)^{ln}q^{(k+l)ln+lkn+mn+\binom{ln}{2}+kln^2}.$$

Conjugation yields three further identities given in Corollary 35. Applications of Theorem 34 and Corollary 35 are given in Section 5.1. Section 5.1.1 considers cases where $a$, $b$, and $t$ are specialized to give identities for the divisor function $d(n)$. Specializing $k$, $l$, and $m$

yields Theorems 36 and 38, which expresses the divisor function as a linear combination of certain partition functions with restrictions on the parts. Section 5.1.2 gives a proof expressing partitions into distinct parts as a linear combination of three partition functions with restrictions on parts. Section 5.2 gives the proof of Theorem 34 and Corollary 35.

A numerical check of Theorem 34 is given in the appendix in Section A.5. In fact, numerical checks for all theorems are found in this section of the appendix. For the convenience of the reader, identities of Fine and all new identities given in this dissertation are listed in the appendix in Sections A.1 and A.2.

## 1.6 Future Directions

While combinatorial proofs are provided for some identities derived from Fine's methods, most proofs remain to be found. In addition, we only considered a few applications of the general transformations; many more remain to be explored. Which special cases yield new partition, number theoretic, or combinatorial results? Chapter 2 of Fine [9] considers applications of his transformations to Ramanujan's celebrated mock theta functions. Do our theorems have similar applications? The general transformations of Chapter 5 were selected to have attractive forms, however, these transformations are not the only of their kind. Other general expressions can be given that will be of similar interest. Finally, is there a natural extension of this method to more continuous parameters, perhaps to other well known functions in the $q$-hypergeometric hierarchy?

# CHAPTER 2

# ITERATIONS WITH $\mathbf{n_i \in \{0, 1, 2\}}$

In this chapter, we consider periodic iterations of transformations of the form $(a, b, t) \mapsto (aq^{n_1}, bq^{n_2}, tq^{n_3})$ with $n_i \in \mathbb{Z}_{\geq 0}$, $\max(n_i) \leq 2$, and $\gcd(n_1, n_2, n_3) = 1$. In later chapters, we consider other values of $n_1, n_2$, and $n_3$ where $\max(n_i) > 2$. From these periodic transformations, taking the limit $n \to \infty$ yields new identities. In the following chapters we build on our work here by giving combinatorial proofs along with generalizations.

The condition $\gcd(n_1, n_2, n_3) = 1$ is necessary to avoid redundant equations. For example, the iteration $b \mapsto bq^2$, or (7), generates the same transformation as $b \mapsto bq$ (6).

In addition to new identities obtained by extending Fine's method, using the involution (12) we obtain corollaries using the involution. Setting $a' = at/b$, $b' = t$, and $t' = b$ in $(a, b, t) \mapsto (aq^{n_1}, bq^{n_2}, tq^{n_3})$, notice that $(a', b', t') \mapsto (a'q^{n_1+n_3-n_2}, b'q^{n_3}, t'q^{n_2})$. This defines a function $\iota$ on $\mathbb{Z}^3$ where $\iota : (n_1, n_2, n_3) \mapsto (n_1 + n_3 - n_2, n_3, n_2)$. Notice that $(n_1, n_2, n_3)$ is fixed under the involution if and only if $n_2 = n_3$. We note the identities fixed under the involution as they occur.

This chapter begins with a generalization of Fine's method. Following this, new identities and their corollaries are presented. This chapter concludes with the algebraic proofs.

## 2.1 Extending Fine's Method

As described in Chapter 1, Fine's seed identities are the seven transformations $(a, b, t) \mapsto (aq^{n_1}, bq^{n_2}, tq^{n_3})$, with $n_i \in \{0, 1\}$, not all zero. From these it is clear that one can obtain any transformation with $n_i \in \mathbb{Z}$. Multiplying through such a relation to make the affine term 1, and subtracting any two such normalized transformations gives that the vector space $\mathcal{F}$ of functions spanned by $F(aq^{n_1}, bq^{n_2}; tq^{n_3})$ over the field $\mathbb{Q}(a, b, t, q)$ has dimension 2. We begin with a treatment of Fine's method; after this we give a generalization.

Consider the group $\mathcal{S}$ (isomorphic to $\mathbb{Z}^3$) of $q$-shift operators generated by $\alpha : a \mapsto aq$, $\beta : b \mapsto bq$, and $\tau : t \mapsto tq$. General elements of this group are of the form $\sigma = \alpha^{n_1}\beta^{n_2}\tau^{n_3}$. This group acts on $\mathcal{F}$ in the following way: given $\sigma \in \mathcal{S}$, Fine's transformations $(a, b, t) \mapsto (aq^{n_1}, bq^{n_2}, tq^{n_3})$ imply

$$F = B_\sigma + A_\sigma \sigma F$$

for some $A_\sigma, B_\sigma \in \mathbb{Q}(a, b, t, q)$. The next proposition shows how to compute the action of $\mathcal{S}$ on the rational functions $A_\sigma$ and $B_\sigma$.

**Proposition 6.** For $\sigma, \sigma_i \in \mathcal{S}$, $1 \leq i \leq n$,

$$A_{\sigma^{-1}} = \frac{1}{\sigma^{-1} A_\sigma}, \tag{27}$$

$$B_{\sigma^{-1}} = -A_{\sigma^{-1}} \sigma^{-1} B_\sigma, \tag{28}$$

$$A_{\sigma_1\sigma_2} = A_{\sigma_1} \sigma_1 A_{\sigma_2}, \tag{29}$$

$$B_{\sigma_1\sigma_2} = B_{\sigma_1} + A_{\sigma_1} \sigma_1 B_{\sigma_2}. \tag{30}$$

By induction,

$$A_{\sigma_1\cdots\sigma_n} = A_{\sigma_1}(\sigma_1 A_{\sigma_2})(\sigma_1\sigma_2 A_{\sigma_3})\cdots(\sigma_1\cdots\sigma_{n-1}A_{\sigma_n}), \tag{31}$$

and

$$B_{\sigma_1\cdots\sigma_n} = B_{\sigma_1} + A_{\sigma_1}\sigma_1 B_{\sigma_2} + A_{\sigma_1\sigma_2}\sigma_1\sigma_2 B_{\sigma_3} + \cdots + A_{\sigma_1\cdots\sigma_{n-1}}\sigma_1\cdots\sigma_{n-1}B_{\sigma_n}. \tag{32}$$

*Proof of Proposition 6.*

By definition, $F = B_{\sigma_{-1}} + A_{\sigma^{-1}}\sigma^{-1}F$. Applying $\sigma^{-1}$ to $F = B_\sigma + A_\sigma \sigma F$, yields

$$\sigma^{-1}F = \sigma^{-1}B_\sigma + \sigma^{-1}(A_\sigma \sigma F).$$

Substituting the expression for $\sigma^{-1}F$ into the first equation, and matching terms on both sides yields (27) and (28).

Now we prove (29) and (30) using the equations

$$F = B_{\sigma_1} + A_{\sigma_1}\sigma_1 F,$$

and

$$F = B_{\sigma_2} + A_{\sigma_2}\sigma_2 F.$$

Composing these functions and matching terms with

$$F = B_{\sigma_1\sigma_2} + A_{\sigma_1\sigma_2}\sigma_1\sigma_2 F$$

gives both equations.

The proof of (31) and (32) follow by induction on (29) and (30), respectively. □

Thus, given an infinite sequence $\sigma_i \in \mathcal{S}$, we have a method to compute $F$ in terms of $\sigma_1\sigma_2\cdots\sigma_n F$. Then letting $n \to \infty$ potentially gives a new transformation of $F$.

In this dissertation we consider periodic products of $\alpha$, $\beta$, and $\tau$. To this end, let $\rho_i \in \{\alpha, \beta, \tau\}$, for $1 \le i \le k$. Put $\sigma := \rho_1 \cdots \rho_k$. Then we can compute $F$ in terms of $\sigma^n$ by using (31) and (32) with $\sigma_1 = \sigma_2 = \cdots = \sigma_n$. This yields

$$A_{\sigma^n} = A_\sigma(\sigma A_\sigma)(\sigma^2 A_\sigma)\cdots(\sigma^{n-1}A_\sigma) = \prod_{i=0}^{n-1} \sigma^i A_\sigma, \tag{33}$$

and

$$\begin{aligned} B_{\sigma^n} &= B_\sigma + A_\sigma \sigma B_\sigma + A_{\sigma^2}\sigma^2 B_\sigma + \cdots + A_{\sigma^{n-1}}\sigma^{n-1}B_\sigma \\ &= \sum_{i=0}^{n-1} A_{\sigma^i}\sigma^i B_\sigma \qquad (34) \\ &= \sum_{i=0}^{n-1}\left(\prod_{j=0}^{i-1}\sigma^j A_\sigma\right)\sigma^i B_\sigma. \qquad (35) \end{aligned}$$

Note that while (33) and (34) are actually equivalent to Fine's method of iteration when applied to a the seed relation $F = B_\sigma + A_\sigma \sigma F$, equations (31) and (32) are more general. Since Fine's results are confined to seeds derived from $(a, b, t) \mapsto (aq^{n_1}, bq^{n_2}; tq^{n_3})$ with $n_i \in \{0, 1\}$, our main approach to finding new results is to consider seeds of the form $n_i \in \mathbb{Z}_{\ge 0}$ with $\gcd(n_1, n_2, n_3) = 1$ and some $n_i \notin \{0, 1\}$.

## 2.2 Periodic Iterated Sequences

Using the notation of the last section, Fine's seed identities (6), (9), (10), (14), (16), (17), and (19) from Section 1.3 are put into Table 2.1 below.

Table 2.1: Fine's Seeds

| $\sigma$ | $A_\sigma$ | $B_\sigma$ |
|---|---|---|
| $\alpha$ | $\dfrac{(1-a)(1-at/b)}{1-a/b} \quad (36)$ | $-\dfrac{(1-b)a}{(1-a/b)b} \quad (37)$ |
| $\beta$ | $\dfrac{1-bq/a}{(1-bq)(1-bq/at)} \quad (38)$ | $\dfrac{-bq}{(1-bq/at)at} \quad (39)$ |
| $\tau$ | $\dfrac{1-at/b}{1-t}b \quad (40)$ | $\dfrac{1-b}{1-t} \quad (41)$ |

| | | |
|---|---|---|
| $\beta\tau$ | $-\dfrac{1-bq/a}{(1-bq)(1-t)}at \quad (42)$ | $\dfrac{1}{1-t} \quad (43)$ |
| $\alpha\beta$ | $\dfrac{1-a}{1-bq}t \quad (44)$ | $1 \quad (45)$ |
| $\alpha\tau$ | $\dfrac{(1-a)(1-at/b)(1-atq/b)}{(1-t)(1-a/b)}b \quad (46)$ | $\dfrac{1-b}{1-t}-\dfrac{(1-b)(1-at/b)a}{(1-t)(1-a/b)} \quad (47)$ |
| $\alpha\beta\tau$ | $\dfrac{(1-a)(1-at/b)}{(1-bq)(1-t)}btq \quad (48)$ | $\dfrac{1-at}{1-t} \quad (49)$ |

Iterating these seeds gives the following lemmas. We only prove (50) and (57); the remaining proofs are found in the appendix in Section A.4. Note, the results in the following two lemmas are not new. Results equivalent to these iterations are given and proved in Fine [9] in Chapter 1, Sections 6, 7, and 11-14.

**Lemma 7.** *For* $n \geq 0$,

$$A_{\alpha^n} = \frac{(a)_n (at/b)_n}{(a/b)_n}, \tag{50}$$

$$A_{\beta^n} = \frac{(bq/a)_n}{(bq)_n (bq/at)_n}, \tag{51}$$

$$A_{\tau^n} = \frac{(at/b)_n}{(t)_n} b^n, \tag{52}$$

$$A_{(\alpha\beta)^n} = \frac{(a)_n}{(bq)_n} t^n, \tag{53}$$

$$A_{(\beta\tau)^n} = \frac{(bq/a)_n}{(bq)_n (t)_n} (-at)^n q^{n(n-1)/2}, \tag{54}$$

$$A_{(\alpha\tau)^n} = \frac{(a)_n (at/b)_{2n}}{(t)_n (a/b)_n} b^n, \tag{55}$$

$$A_{(\alpha\beta\tau)^n} = \frac{(a)_n (at/b)_n}{(bq)_n (t)_n} (bt)^n q^{n^2}. \tag{56}$$

**Lemma 8.** *For* $n \geq 0$,

$$B_{\alpha^n} = -\frac{a(1-b)}{b} \sum_{i=0}^{n-1} \frac{(a)_i (at/b)_i}{(a/b)_{i+1}} q^i, \tag{57}$$

$$B_{\beta^n} = -\frac{b}{at} \sum_{i=0}^{n-1} \frac{(bq/a)_i}{(bq)_i (bq/at)_{i+1}} q^{i+1}, \tag{58}$$

$$B_{\tau^n} = (1-b) \sum_{i=0}^{n-1} \frac{(at/b)_i}{(t)_{i+1}} b^i, \tag{59}$$

$$B_{(\alpha\beta)^n} = \sum_{i=0}^{n-1} \frac{(a)_i}{(bq)_i} t^i, \tag{60}$$

$$B_{(\beta\tau)^n} = \sum_{i=0}^{n-1} \frac{(bq/a)_i}{(bq)_i (t)_{i+1}} (-at)^i q^{i(i-1)/2}, \tag{61}$$

$$B_{(\alpha\tau)^n} = (1-b) \sum_{i=0}^{n-1} \frac{(a)_i (at/b)_{2i}}{(t)_{i+1} (a/b)_i} b^i - a(1-b) \sum_{i=0}^{n-1} \frac{(a)_i (at/b)_{2i+1}}{(t)_{i+1} (a/b)_{i+1}} (bq)^i, \tag{62}$$

$$B_{(\alpha\beta\tau)^n} = \sum_{i=0}^{n-1} \frac{(a)_i (at/b)_i}{(bq)_i (t)_{i+1}} (1 - atq^{2i}) (bt)^i q^{i^2}. \tag{63}$$

*Proof of* (50) *and* (57).

Use (33) to get

$$A_{\alpha^n} = \prod_{i=0}^{n-1} \alpha^i A_\alpha = \prod_{i=0}^{n-1} \alpha^i \left( \frac{(1-a)(1-at/b)}{(1-a/b)} \right).$$

Then, applying the operator $\alpha^i : a \mapsto aq^i$ yields

$$A_{\alpha^n} = \prod_{i=0}^{n-1} \frac{(1-aq^i)(1-atq^i/b)}{(1-aq^i/b)}.$$

Noting that this product yields

$$A_{\alpha^n} = \frac{(a)_n (at/b)_n}{(a/b)_n}.$$

Next, use (34) to get

$$B_{\alpha^n} = \sum_{i=0}^{n-1} A_{\alpha^i} \alpha^i B_\alpha = \sum_{i=0}^{n-1} \frac{(a)_i (at/b)_i}{(a/b)_i} \alpha^i \left( - \frac{(1-b)a}{(1-a/b)b} \right).$$

Applying the operator $\alpha^i$ yields

$$= - \sum_{i=0}^{n-1} \frac{(a)_i (at/b)_i}{(a/b)_i} \cdot \frac{(1-b)aq^i}{(1-aq^i/b)b}.$$

Finally,

$$= - \frac{a(1-b)}{b} \sum_{i=0}^{n-1} \frac{(a)_i (at/b)_i}{(a/b)_{i+1}} q^i.$$

□

We now begin the new results of this chapter by considering different cases of $n_i \in \{0,1,2\}$ for $\sigma = \alpha^{n_1}\beta^{n_2}\tau^{n_3}$. $A_{\sigma^n}$ and $B_{\sigma^n}$ are computed using (33) and (34) to get a new transformation of $F$ of the form $F = B_{\sigma^n} + A_{\sigma^n}\sigma^n F$. Finally, sending $n \to \infty$ gives the resulting theorems. Note, all theorems have been checked numerically. These numerical checks can be found in the appendix in Section A.5.

Note that $\sigma = \alpha^2\beta$ under the involution (12) is conjugate to $\sigma = \alpha\tau$, given by Fine [9]. As Fine considered the involution of $\alpha\tau$, he in effect computed the iteration $\alpha^2\beta$.

The following theorem is proved in Chapter 4 combinatorially using a synthesis of methods outlined by Andrews in [3] as well as Corteel and Lovejoy in [7].

**Theorem 9.** *Iterating $\sigma = \alpha\beta^2\tau$ gives*

$$F(a,b;t) = \sum_{n=0}^{\infty} \frac{(a)_n(bq/a)_n}{(t)_{n+1}(bq)_{2n}}(1-atq^{2n})(-at^2)^n q^{\frac{n(3n-1)}{2}} + bt\sum_{n=0}^{\infty} \frac{(a)_{n+1}(bq/a)_n}{(t)_{n+1}(bq)_{2n+1}}(-at^2)^n q^{\frac{(3n+2)(n+1)}{2}}.$$

The following corollary is referenced again in Chapter 4 as a special case of Theorem 32. In Chapter 4 we derive the identity coming from iterating the seed generated by $\sigma = \beta\tau^n$. In this identity only the case $n = 2$ gives the corollary below. We also give a combinatorial proof of the general case.

**Corollary 10.** *Iterating $\sigma = \beta\tau^2$ gives*

$$F(a,b;t) = \sum_{n=0}^{\infty} \frac{(at/b)_n(bq/a)_n}{(t)_{2n+1}(bq)_n}(-abt)^n q^{\frac{n(3n-1)}{2}} + \frac{1}{b}\sum_{n=0}^{\infty} \frac{(at/b)_n(bq/a)_{n+1}}{(t)_{2n+2}(bq)_n}(-abt)^{n+1} q^{3\binom{n+1}{2}}.$$

The following theorem and corollary are proved in Chapter 3 combinatorially using the method of overpartitions. The proof of the corollary below for $\sigma = \alpha\beta\tau^2$ is given by observation between the Ferrers diagrams for the partitions generated by the right side of these two identities. In Chapter 4, we give a more general identity in Theorem 29, which specializes to the two given below and also generalizes the Rogers-Fine identity (20).

**Theorem 11.** *Iterating* $\sigma = \alpha^2\beta^2\tau$ *gives*

$$F(a,b;t) = \sum_{n=0}^{\infty} \frac{(a)_{2n}(at/b)_n}{(bq)_{2n}(t)_{n+1}}(1-atq^{3n})(bt^2)^n q^{2n^2} + \frac{1}{t}\sum_{n=0}^{\infty} \frac{(a)_{2n+1}(at/b)_{n+1}}{(bq)_{2n+1}(t)_{n+1}}(bt^2)^{n+1}q^{(2n+1)(n+1)}.$$

The following corollary is conjugate to Theorem 11.

**Corollary 12.** *Iterating* $\sigma = \alpha\beta\tau^2$ *gives*

$$F(a,b;t) = \sum_{n=0}^{\infty} \frac{(a)_n(at/b)_{2n}}{(bq)_n(t)_{2n+1}}(1-atq^{3n})(b^2t)^n q^{2n^2} + bt\sum_{n=0}^{\infty} \frac{(a)_{n+1}(at/b)_{2n+1}}{(bq)_n(t)_{2n+2}}(b^2t)^n q^{(n+1)(2n+1)}.$$

The following two theorems are self-conjugate.

**Theorem 13.** *Iterating* $\sigma = \alpha^2\beta\tau$ *gives*

$$F(a,b;t) = \sum_{n=0}^{\infty} \frac{(a)_{2n}(at/b)_{2n}}{(bq)_n(t)_{n+1}(a/b)_n}(1-atq^{3n})(bt)^n q^{n^2} - \frac{a}{b}\sum_{n=0}^{\infty} \frac{(a)_{2n+1}(at/b)_{2n+1}}{(bq)_n(t)_{n+1}(a/b)_{n+1}}(bt)^{n+1}q^{n^2+3n+1}.$$

**Theorem 14.** *Iterating* $\sigma = \alpha\beta^2\tau^2$ *gives*

$$F(a,b;t) = \sum_{n=0}^{\infty} \frac{(a)_n(at/b)_n(bq/a)_n}{(bq)_{2n}(t)_{2n+1}}(1-atq^{3n})(-abt^2)^n q^{\frac{n(7n-1)}{2}} + btq\sum_{n=0}^{\infty} \frac{(a)_{n+1}(at/b)_{n+1}(bq/a)_n}{(bq)_{2n+1}(t)_{2n+2}}(-abt^2)^n q^{\frac{n(7n+1)}{2}}.$$

Theorem 14 is the final combinatorial proof given in Chapter 3 using both the method of overpartitions as well as methods from Andrews [3].

After applying the involution (12) to the following three theorems, we obtain identities with $\sigma = \alpha^{n_1}\beta^{n_2}\tau^{n_3}$ where $\max(n_1, n_2, n_3) > 2$. We present the resulting conjugate identities as corollaries.

**Theorem 15.** *Iterating* $\sigma = \alpha^2\beta\tau^2$ *gives*

$$F(a,b;t) = \sum_{n=0}^{\infty} \frac{(a)_{2n}(at/b)_{3n}}{(t)_{2n+1}(a/b)_n(bq)_n}(1-atq^{4n})(b^2t)^n q^{2n^2} + \frac{1}{b}\sum_{n=0}^{\infty} \frac{(a)_{2n+1}(at/b)_{3n+1}}{(t)_{2n+2}(a/b)_n(bq)_n}(b^2t)^{n+1} q^{(n+1)(2n+1)} - \frac{a}{b}\sum_{n=0}^{\infty} \frac{(a)_{2n+1}(at/b)_{3n+2}}{(t)_{2n+2}(a/b)_{n+1}(bq)_n}(b^2t)^{n+1} q^{(2n+1)(n+2)}.$$

**Corollary 16.** *Iterating* $\sigma = \alpha^3\beta^2\tau$ *gives*

$$F(a,b;t) = \sum_{n=0}^{\infty} \frac{(at/b)_{2n}(a)_{3n}}{(bq)_{2n}(a/b)_n(t)_{n+1}}(1-atq^{4n})(bt^2)^n q^{2n^2} + \frac{1}{t}\sum_{n=0}^{\infty} \frac{(at/b)_{2n+1}(a)_{3n+1}}{(bq)_{2n+1}(a/b)_n(t)_{n+1}}(bt^2)^{n+1} q^{(n+1)(2n+1)} - \frac{a}{b}\sum_{n=0}^{\infty} \frac{(at/b)_{2n+1}(a)_{3n+2}}{(bq)_{2n+1}(a/b)_{n+1}(t)_{n+1}}(bt^2)^{n+1} q^{(2n+1)(n+2)}.$$

**Theorem 17.** *Iterating $\sigma = \alpha^2\tau$ with $|b| < 1$ gives*

$$F(a,b;t) = (1-b)\sum_{n=0}^{\infty} \frac{(a)_{2n}(at/b)_{3n}}{(a/b)_{2n}(t)_{n+1}} b^n - a(1-b)\sum_{n=0}^{\infty} \frac{(a)_{2n}(at/b)_{3n+1}}{(a/b)_{2n+1}(t)_{n+1}} (bq^2)^n$$
$$- a(1-b)\sum_{n=0}^{\infty} \frac{(a)_{2n+1}(at/b)_{3n+2}}{(a/b)_{2n+2}(t)_{n+1}} b^n q^{2n+1}.$$

**Corollary 18.** *Iterating $\sigma = \alpha^3\beta$ with $|t| < 1$ gives*

$$F(a,b;t) = \sum_{n=0}^{\infty} \frac{(at/b)_{2n}(a)_{3n}}{(a/b)_{2n}(bq)_n} t^n - at/b\sum_{n=0}^{\infty} \frac{(at/b)_{2n}(a)_{3n+1}}{(a/b)_{2n+1}(bq)_n} (tq^2)^n$$
$$- at/b\sum_{n=0}^{\infty} \frac{(at/b)_{2n+1}(a)_{3n+2}}{(a/b)_{2n+2}(bq)_n} t^n q^{2n+1}.$$

**Theorem 19.** *Iterating $\sigma = \alpha\tau^2$ with $|b| < 1$ gives*

$$F(a,b;t) = (1-b)\sum_{n=0}^{\infty} \frac{(a)_n(at/b)_{3n}}{(t)_{2n+1}(a/b)_n} b^{2n} - a(1-b)\sum_{n=0}^{\infty} \frac{(a)_n(at/b)_{3n+1}}{(t)_{2n+1}(a/b)_{n+1}} (b^2q)^n$$
$$+ (1-b)\sum_{n=0}^{\infty} \frac{(a)_{n+1}(at/b)_{3n+2}}{(t)_{2n+2}(a/b)_{n+1}} b^{2n+1}.$$

**Corollary 20.** *Iterating $\sigma = \alpha^3\beta^2$ with $|t| < 1$ gives*

$$F(a,b;t) = \sum_{n=0}^{\infty} \frac{(at/b)_n(a)_{3n}}{(bq)_{2n}(a/b)_n} t^{2n} - at/b\sum_{n=0}^{\infty} \frac{(at/b)_n(a)_{3n+1}}{(bq)_{2n}(a/b)_{n+1}} (t^2q)^n$$
$$+ \sum_{n=0}^{\infty} \frac{(at/b)_{n+1}(a)_{3n+2}}{(bq)_{2n+1}(a/b)_{n+1}} t^{2n+1}.$$

After applying the involution (12) to the two theorems given below, we obtain identities with $\sigma = \alpha^{n_1}\beta^{n_2}\tau^{n_3}$ where $\min(n_1, n_2, n_3) < 0$. We present the resulting conjugate identities as corollaries.

**Theorem 21.** *Iterating $\sigma = \alpha\beta^2$ with $|t| < 1$ gives*

$$F(a,b;t) = \sum_{n=0}^{\infty} \frac{(a)_n (bq/a)_n}{(bq)_{2n}(bq/at)_n} t^n - \frac{b}{at} \sum_{n=0}^{\infty} \frac{(a)_{n+1}(bq/a)_n}{(bq)_{2n+1}(bq/at)_{n+1}} (tq)^{n+1}.$$

This theorem relates to Fine's identity (4). In (4) there is a $q$-product in the numerator with $2n$ factors. In Theorem 21, a $q$-product $2n$ factors appears in the denominator. Furthermore, applying the involution (12) to this theorem reveals that this is equivalent to the identity generated from $\alpha^{-1}\tau^2$.

**Corollary 22.** *Iterating $\sigma = \alpha^{-1}\tau^2$ with $|b| < 1$ gives*

$$F(a,b;t) = (1-b) \sum_{n=0}^{\infty} \frac{(at/b)_n (bq/a)_n}{(t)_{2n+1}(q/a)_n} b^n - \frac{1-b}{a} \sum_{n=0}^{\infty} \frac{(at/b)_{n+1}(bq/a)_n}{(t)_{2n}(q/a)_{n+1}} (tq)^{n+1}.$$

**Theorem 23.** *Iterating $\sigma = \beta^2\tau$ gives*

$$\begin{aligned} F(a,b;t) = &\sum_{n=0}^{\infty} \frac{(bq/a)_{2n}}{(bq)_{2n}(bq/at)_n (t)_{n+1}} (-at)^n q^{\binom{n}{2}} \\ &+ b \sum_{n=0}^{\infty} \frac{(bq/a)_{2n+1}}{(bq)_{2n+1}(bq/at)_{n+1}(t)_{n+1}} (-at)^n q^{\binom{n+2}{2}}. \end{aligned}$$

**Corollary 24.** *Iterating $\sigma = \alpha^{-1}\beta\tau^2$ gives*

$$\begin{aligned} F(a,b;t) = &\sum_{n=0}^{\infty} \frac{(bq/a)_{2n}}{(t)_{2n+1}(q/a)_n (bq)_n} (-at)^n q^{\binom{n}{2}} \\ &+ b \sum_{n=0}^{\infty} \frac{(bq/a)_{2n+1}}{(t)_{2n+2}(q/a)_{n+1}(bq)_n} (-at)^n q^{\binom{n+2}{2}}. \end{aligned}$$

## 2.3 Algebraic Proofs

*Proof of Theorem 9.*

Using (29), (33), (38), and (48),

$$
\begin{aligned}
A_{(\alpha\beta^2\tau)^n} &= \prod_{i=0}^{n-1}(\alpha\beta^2\tau)^i A_{\alpha\beta^2\tau} \\
&= \prod_{i=0}^{n-1}(\alpha\beta^2\tau)^i A_{\alpha\beta\tau}(\alpha\beta\tau)A_\beta \\
&= \prod_{i=0}^{n-1}(\alpha\beta^2\tau)^i \left(\frac{(1-a)(1-at/b)}{(1-bq)(1-t)}btq \cdot \frac{(1-bq/a)}{(1-bq^2)(1-b/at)}\right) \\
&= \prod_{i=0}^{n-1}(\alpha\beta^2\tau)^i \left(-\frac{(1-a)(1-bq/a)}{(1-bq)(1-bq^2)(1-t)}at^2q\right) \\
&= \prod_{i=0}^{n-1} -\frac{(1-aq^i)(1-bq^{i+1}/a)}{(1-bq^{2i+1})(1-bq^{2i+2})(1-tq^i)}at^2q^{3i+1} \\
&= \frac{(a)_n(bq/a)_n}{(bq)_{2n}(t)_n}(-at^2)^n q^{(3n-1)n/2}.
\end{aligned}
$$

Next, using (30), (39), and (49),

$$
\begin{aligned}
B_{\alpha\beta^2\tau} &= B_{\alpha\beta\tau} + A_{\alpha\beta\tau}(\alpha\beta\tau)B_\beta \\
&= \frac{1-at}{1-t} + \frac{(1-a)(1-at/b)}{(1-bq)(1-t)}btq(\alpha\beta\tau)\left(-\frac{bq}{(1-bq/at)at}\right) \\
&= \frac{1-at}{1-t} - \frac{(1-a)(1-at/b)}{(1-bq)(1-t)}btq \cdot \left(\frac{b}{(1-b/at)at}\right) \\
&= \frac{1-at}{1-t} + \frac{(1-a)}{(1-bq)(1-t)}btq.
\end{aligned}
$$

Thus by (34),

$$
\begin{aligned}
B_{(\alpha\beta^2\tau)^n} &= \sum_{i=0}^{n-1} A_{(\alpha\beta^2\tau)^i}(\alpha\beta^2\tau)^i B_{\alpha\beta^2\tau} \\
&= \sum_{i=0}^{n-1} \frac{(a)_i(bq/a)_i}{(bq)_{2i}(t)_i}(-at^2)^i q^{(3i-1)i/2}(\alpha\beta^2\tau)^i \left(\frac{1-at}{1-t} + \frac{(1-a)}{(1-bq)(1-t)}btq\right) \\
&= \sum_{i=0}^{n-1} \frac{(a)_i(bq/a)_i}{(bq)_{2i}(t)_i}(-at^2)^i q^{(3i-1)i/2} \cdot \left(\frac{1-atq^{2i}}{1-tq^i} + \frac{(1-aq^i)}{(1-bq^{2i+1})(1-tq^i)}btq^{3i+1}\right) \\
&= \sum_{i=0}^{n-1} \frac{(a)_i(bq/a)_i}{(bq)_{2i}(t)_{i+1}}(-at^2)^i q^{(3i-1)i/2}(1-atq^{2i}) \\
&+ bt\sum_{i=0}^{n-1} \frac{(a)_{i+1}(bq/a)_i}{(bq)_{2i+1}(t)_{i+1}}(-at^2)^i q^{(3i+2)(i+1)/2}.
\end{aligned}
$$

The $n^{\underline{th}}$ iteration is

$$
\begin{aligned}
F(a,b;t) = &\sum_{i=0}^{n-1} \frac{(a)_i(bq/a)_i}{(bq)_{2i}(t)_{i+1}}(-at^2)^i q^{(3i-1)i/2}(1-atq^{2i}) \\
&+ bt\sum_{i=0}^{n-1} \frac{(a)_{i+1}(bq/a)_i}{(bq)_{2i+1}(t)_{i+1}}(-at^2)^i q^{(3i+2)(i+1)/2} \\
&+ \frac{(a)_n(bq/a)_n}{(bq)_{2n}(t)_n}(-at^2)^n q^{(3n-1)n/2} F(aq^n, bq^{2n}; tq^n).
\end{aligned}
$$

Letting $n \to \infty$ and re-indexing gives Theorem 9. □

*Proof of Theorem 11.*

Using (29), (33), (44), and (48),

$$
\begin{aligned}
A_{(\alpha^2\beta^2\tau)^n} &= \prod_{i=0}^{n-1} (\alpha^2\beta^2\tau)^i A_{\alpha^2\beta^2\tau} \\
&= \prod_{i=0}^{n-1} (\alpha^2\beta^2\tau)^i A_{\alpha\beta\tau}(\alpha\beta\tau) A_{\alpha\beta}
\end{aligned}
$$

$$= \prod_{i=0}^{n-1} (\alpha^2\beta^2\tau)^i \left( \frac{(1-a)(1-at/b)}{(1-bq)(1-t)} btq \cdot \frac{(1-aq)}{(1-bq^2)} tq \right)$$
$$= \prod_{i=0}^{n-1} (\alpha^2\beta^2\tau)^i \left( \frac{(1-a)(1-aq)(1-at/b)}{(1-bq)(1-bq^2)(1-t)} bt^2q^2 \right)$$
$$= \prod_{i=0}^{n-1} \frac{(1-aq^{2i})(1-aq^{2i+1})(1-atq^i/b)}{(1-bq^{2i+1})(1-bq^{2i+2})(1-tq^i)} bt^2q^{4i+2}$$
$$= \frac{(a)_{2n}(at/b)_n}{(bq)_{2n}(t)_n}(bt^2)^n q^{2n^2}.$$

Next, using (30), (45), and (49),

$$B_{\alpha^2\beta^2\tau} = B_{\alpha\beta\tau} + A_{\alpha\beta\tau}(\alpha\beta\tau)B_{\alpha\beta}$$
$$= \frac{1-at}{1-t} + \frac{(1-a)(1-at/b)}{(1-bq)(1-t)} btq.$$

Thus by (34),

$$B_{(\alpha^2\beta^2\tau)^n} = \sum_{i=0}^{n-1} A_{(\alpha^2\beta^2\tau)^i}(\alpha^2\beta^2\tau)^i B_{\alpha^2\beta^2\tau}$$
$$= \sum_{i=0}^{n-1} \frac{(a)_{2i}(at/b)_i}{(bq)_{2i}(t)_i}(bt^2)^i q^{2i^2}(\alpha^2\beta^2\tau)^i \left( \frac{1-at}{1-t} + \frac{(1-a)(1-at/b)}{(1-bq)(1-t)} btq \right)$$
$$= \sum_{i=0}^{n-1} \frac{(a)_{2i}(at/b)_i}{(bq)_{2i}(t)_i}(bt^2)^i q^{2i^2} \cdot \left( \frac{1-atq^{3i}}{1-tq^i} + \frac{(1-aq^{2i})(1-atq^i/b)}{(1-bq^{2i+1})(1-tq^i)} btq^{3i+1} \right)$$
$$= \sum_{i=0}^{n-1} \frac{(a)_{2i}(at/b)_i}{(bq)_{2i}(t)_{i+1}}(bt^2)^i q^{2i^2}(1-atq^{3i})$$
$$+ \sum_{i=0}^{n-1} \frac{(a)_{2i+1}(at/b)_{i+1}}{(bq)_{2i+1}(t)_{i+1}} b^{i+1}t^{2i+1}q^{(2i+1)(i+1)}.$$

The $n^{\underline{th}}$ iteration is

$$F(a,b;t) = \sum_{i=0}^{n-1} \frac{(a)_{2i}(at/b)_i}{(bq)_{2i}(t)_{i+1}}(bt^2)^i q^{2i^2}(1-atq^{3i})$$

$$+\sum_{i=0}^{n-1}\frac{(a)_{2i+1}(at/b)_{i+1}}{(bq)_{2i+1}(t)_{i+1}}b^{i+1}t^{2i+1}q^{(2i+1)(i+1)}$$
$$+\frac{(a)_{2n}(at/b)_n}{(bq)_{2n}(t)_n}(bt^2)^n q^{2n^2}F(aq^{2n},bq^{2n};tq^n).$$

Letting $n\to\infty$ and re-indexing gives Theorem 11. □

*Proof of Theorem 13.*

Using (29), (33), (36), and (48),

$$\begin{aligned}
A_{(\alpha^2\beta\tau)^n} &= \prod_{i=0}^{n-1}(\alpha^2\beta\tau)^i A_{\alpha^2\beta\tau}\\
&= \prod_{i=0}^{n-1}(\alpha^2\beta\tau)^i A_{\alpha\beta\tau}(\alpha\beta\tau)A_\alpha\\
&= \prod_{i=0}^{n-1}(\alpha^2\beta\tau)^i\left(\frac{(1-a)(1-at/b)}{(1-bq)(1-t)}btq\cdot\frac{(1-aq)(1-atq/b)}{(1-a/b)}\right)\\
&= \prod_{i=0}^{n-1}(\alpha^2\beta\tau)^i\left(\frac{(1-a)(1-aq)(1-at/b)(1-atq/b)}{(1-bq)(1-t)(1-a/b)}btq\right)\\
&= \prod_{i=0}^{n-1}\frac{(1-aq^{2i})(1-aq^{2i+1})(1-atq^{2i}/b)(1-atq^{2i+1}/b)}{(1-bq^{i+1})(1-tq^i)(1-aq^i/b)}btq^{2i+1}\\
&= \frac{(a)_{2n}(at/b)_{2n}}{(bq)_n(t)_n(a/b)_n}(bt)^n q^{n^2}.
\end{aligned}$$

Next, using (30), (37), and (49),

$$\begin{aligned}
B_{\alpha^2\beta\tau} &= B_{\alpha\beta\tau}+A_{\alpha\beta\tau}(\alpha\beta\tau)B_\alpha\\
&= \frac{1-at}{1-t}+\frac{(1-a)(1-at/b)}{(1-bq)(1-t)}btq(\alpha\beta\tau)\left(-\frac{(1-b)a}{(1-a/b)b}\right)\\
&= \frac{1-at}{1-t}+\frac{(1-a)(1-at/b)}{(1-bq)(1-t)}btq\cdot\left(-\frac{(1-bq)a}{(1-a/b)b}\right)
\end{aligned}$$

$$= \frac{1-at}{1-t} - \frac{(1-a)(1-at/b)}{(1-t)(1-a/b)} atq.$$

Thus by (34),

$$\begin{aligned}
B_{(\alpha^2\beta\tau)^n} &= \sum_{i=0}^{n-1} A_{(\alpha^2\beta\tau)^i} (\alpha^2\beta\tau)^i B_{\alpha^2\beta\tau} \\
&= \sum_{i=0}^{n-1} \frac{(a)_{2i}(at/b)_{2i}}{(bq)_i(t)_i(a/b)_i} (bt)^i q^{i^2} (\alpha^2\beta\tau)^i \left( \frac{1-at}{1-t} - \frac{(1-a)(1-at/b)}{(1-t)(1-a/b)} atq \right) \\
&= \sum_{i=0}^{n-1} \frac{(a)_{2i}(at/b)_{2i}}{(bq)_i(t)_i(a/b)_i} (bt)^i q^{i^2} \left( \frac{1-atq^{3i}}{1-tq^i} - \frac{(1-aq^{2i})(1-atq^{2i}/b)}{(1-tq^i)(1-aq^i/b)} atq^{3i+1} \right) \\
&= \sum_{i=0}^{n-1} \frac{(a)_{2i}(at/b)_{2i}}{(bq)_i(t)_{i+1}(a/b)_i} (bt)^i q^{i^2} (1-atq^{3i}) - at \sum_{i=0}^{n-1} \frac{(a)_{2i+1}(at/b)_{2i+1}}{(bq)_i(t)_{i+1}(a/b)_{i+1}} (bt)^i q^{i^2+3i+1}.
\end{aligned}$$

The $n^{\underline{th}}$ iteration is

$$\begin{aligned}
F(a,b;t) &= \sum_{i=0}^{n-1} \frac{(a)_{2i}(at/b)_{2i}}{(bq)_i(t)_{i+1}(a/b)_i} (bt)^i q^{i^2} (1-atq^{3i}) - at \sum_{i=0}^{n-1} \frac{(a)_{2i+1}(at/b)_{2i+1}}{(bq)_i(t)_{i+1}(a/b)_{i+1}} (bt)^i q^{i^2+3i+1} \\
&\quad + \frac{(a)_{2n}(at/b)_{2n}}{(bq)_n(t)_n(a/b)_n} (bt)^n q^{n^2} F(aq^{2n}, bq^n; tq^n).
\end{aligned}$$

Letting $n \to \infty$ and re-indexing gives Theorem 13. □

*Proof of Theorem 14.*

Using (29), (33), (42), and (48),

$$\begin{aligned}
A_{(\alpha\beta^2\tau^2)^n} &= \prod_{i=0}^{n-1} (\alpha\beta^2\tau^2)^i A_{\alpha\beta^2\tau^2} \\
&= \prod_{i=0}^{n-1} (\alpha\beta^2\tau^2)^i A_{\alpha\beta\tau} (\alpha\beta\tau) A_{\beta\tau}
\end{aligned}$$

$$\begin{aligned}
&= \prod_{i=0}^{n-1} (\alpha\beta^2\tau^2)^i \left( \frac{(1-a)(1-at/b)}{(1-bq)(1-t)} btq \cdot - \frac{(1-bq/a)}{(1-bq^2)(1-tq)} atq^2 \right) \\
&= \prod_{i=0}^{n-1} (\alpha\beta^2\tau^2)^i \left( - \frac{(1-a)(1-at/b)(1-bq/a)}{(1-bq)(1-bq^2)(1-t)(1-tq)} abt^2q^3 \right) \\
&= \prod_{i=0}^{n-1} - \frac{(1-aq^i)(1-atq^i/b)(1-bq^{i+1}/a)}{(1-bq^{2i+1})(1-bq^{2i+2})(1-tq^{2i})(1-tq^{2i+1})} abt^2q^{7i+3} \\
&= \frac{(a)_n(at/b)_n(bq/a)_n}{(bq)_{2n}(t)_{2n}} (-abt^2)^n q^{n(7n-1)/2}.
\end{aligned}$$

Next, using (30), (43), and (49),

$$\begin{aligned}
B_{\alpha\beta^2\tau^2} &= B_{\alpha\beta\tau} + A_{\alpha\beta\tau}(\alpha\beta\tau)B_{\beta\tau} \\
&= \frac{1-at}{1-t} + \frac{(1-a)(1-at/b)}{(1-bq)(1-t)} btq(\alpha\beta\tau) \left( \frac{1}{1-t} \right) \\
&= \frac{1-at}{1-t} + \frac{(1-a)(1-at/b)}{(1-bq)(1-t)(1-tq)} btq.
\end{aligned}$$

Thus by (34),

$$\begin{aligned}
B_{(\alpha\beta^2\tau^2)^n} &= \sum_{i=0}^{n-1} A_{(\alpha\beta^2\tau^2)^i}(\alpha\beta^2\tau^2)^i B_{\alpha\beta^2\tau^2} \\
&= \sum_{i=0}^{n-1} \frac{(a)_i(at/b)_i(bq/a)_i}{(bq)_{2i}(t)_{2i}} (-abt^2)^i q^{i(7i-1)/2} (\alpha\beta^2\tau^2)^i \left( \frac{1-at}{1-t} \right. \\
&\left. + \frac{(1-a)(1-at/b)}{(1-bq)(1-t)(1-tq)} btq \right) \\
&= \sum_{i=0}^{n-1} \frac{(a)_i(at/b)_i(bq/a)_i}{(bq)_{2i}(t)_{2i}} (-abt^2)^i q^{i(7i-1)/2} \cdot \left( \frac{1-atq^{3i}}{1-tq^{2i}} \right. \\
&\left. + \frac{(1-aq^i)(1-atq^i/b)}{(1-bq^{2i+1})(1-tq^{2i})(1-tq^{2i+1})} btq^{4i+1} \right) \\
&= \sum_{i=0}^{n-1} \frac{(a)_i(at/b)_i(bq/a)_i}{(bq)_{2i}(t)_{2i+1}} (-abt^2)^i q^{i(7i-1)/2} (1-atq^{3i})
\end{aligned}$$

$$+\, bt\sum_{i=0}^{n-1}\frac{(a)_{i+1}(at/b)_{i+1}(bq/a)_i}{(bq)_{2i+1}(t)_{2i+2}}(-abt^2)^i q^{(7i^2+7i+2)/2}.$$

The $n^{\underline{th}}$ iteration is

$$\begin{aligned}F(a,b;t) &= \sum_{i=0}^{n-1}\frac{(a)_i(at/b)_i(bq/a)_i}{(bq)_{2i}(t)_{2i+1}}(-abt^2)^i q^{i(7i-1)/2}(1-atq^{3i})\\ &\quad + bt\sum_{i=0}^{n-1}\frac{(a)_{i+1}(at/b)_{i+1}(bq/a)_i}{(bq)_{2i+1}(t)_{2i+2}}(-abt^2)^i q^{(7i^2+7i+2)/2}\\ &\quad + \frac{(a)_n(at/b)_n(bq/a)_n}{(bq)_{2n}(t)_{2n}}(-abt^2)^n q^{n(7n-1)/2}F(aq^n,bq^{2n};tq^{2n}).\end{aligned}$$

Letting $n\to\infty$ and re-indexing gives Theorem 14. □

*Proof of Theorem 15.*

Using (29), (33), (46), and (48),

$$\begin{aligned}A_{(\alpha^2\beta\tau^2)^n} &= \prod_{i=0}^{n-1}(\alpha^2\beta\tau^2)^i A_{\alpha^2\beta\tau^2}\\ &= \prod_{i=0}^{n-1}(\alpha^2\beta\tau^2)^i A_{\alpha\beta\tau}(\alpha\beta\tau)A_{\alpha\tau}\\ &= \prod_{i=0}^{n-1}(\alpha^2\beta\tau^2)^i\left(\frac{(1-a)(1-at/b)}{(1-bq)(1-t)}btq\cdot\frac{(1-aq)(1-atq/b)(1-atq^2/b)}{(1-tq)(1-a/b)}bq\right)\\ &= \prod_{i=0}^{n-1}(\alpha^2\beta\tau^2)^i\left(\frac{(1-a)(1-aq)(1-at/b)(1-atq/b)(1-atq^2/b)}{(1-bq)(1-t)(1-tq)(1-a/b)}b^2tq^2\right)\\ &= \prod_{i=0}^{n-1}\frac{(1-aq^{2i})(1-aq^{2i+1})(1-atq^{3i}/b)(1-atq^{3i+1}/b)(1-atq^{3i+2}/b)}{(1-bq^{i+1})(1-tq^{2i})(1-tq^{2i+1})(1-aq^i/b)}b^2tq^{4i+2}\\ &= \frac{(a)_{2n}(at/b)_{3n}}{(bq)_n(t)_{2n}(a/b)_n}(b^2t)^n q^{2n^2}.\end{aligned}$$

Next, using (30), (47), and (49),

$$
\begin{aligned}
B_{\alpha^2\beta\tau^2} &= B_{\alpha\beta\tau} + A_{\alpha\beta\tau}(\alpha\beta\tau)B_{\alpha\tau} \\
&= \frac{1-at}{1-t} + \frac{(1-a)(1-at/b)}{(1-bq)(1-t)}btq(\alpha\beta\tau)\left(\frac{1-b}{1-t} - \frac{(1-b)(1-at/b)}{(1-t)(1-a/b)}a\right) \\
&= \frac{1-at}{1-t} + \frac{(1-a)(1-at/b)}{(1-bq)(1-t)}btq\cdot\left(\frac{1-bq}{1-tq} - \frac{(1-bq)(1-atq/b)}{(1-tq)(1-a/b)}aq\right) \\
&= \frac{1-at}{1-t} + \frac{(1-a)(1-at/b)}{(1-t)(1-tq)}btq - \frac{(1-a)(1-at/b)(1-atq/b)}{(1-t)(1-tq)(1-a/b)}abtq^2.
\end{aligned}
$$

Thus by (34),

$$
\begin{aligned}
B_{(\alpha^2\beta\tau^2)^n} &= \sum_{i=0}^{n-1} A_{(\alpha^2\beta\tau^2)^i}(\alpha^2\beta\tau^2)^i B_{\alpha^2\beta\tau^2} \\
&= \sum_{i=0}^{n-1} \frac{(a)_{2i}(at/b)_{3i}}{(bq)_i(t)_{2i}(a/b)_i}(b^2t)^i q^{2i^2}(\alpha^2\beta\tau^2)^i\left(\frac{1-at}{1-t} + \frac{(1-a)(1-at/b)}{(1-t)(1-tq)}btq\right. \\
&\left. - \frac{(1-a)(1-at/b)(1-atq/b)}{(1-t)(1-tq)(1-a/b)}abtq^2\right) \\
&= \sum_{i=0}^{n-1} \frac{(a)_{2i}(at/b)_{3i}}{(bq)_i(t)_{2i}(a/b)_i}(b^2t)^i q^{2i^2}\cdot\left(\frac{1-atq^{4i}}{1-tq^{2i}} + \frac{(1-aq^{2i})(1-atq^{3i}/b)}{(1-tq^{2i})(1-tq^{2i+1})}btq^{3i+1}\right. \\
&\left. - \frac{(1-aq^{2i})(1-atq^{3i}/b)(1-atq^{3i+1}/b)}{(1-tq^{2i})(1-tq^{2i+1})(1-aq^i/b)}abtq^{5i+2}\right) \\
&= \sum_{i=0}^{n-1} \frac{(a)_{2i}(at/b)_{3i}}{(bq)_i(t)_{2i+1}(a/b)_i}(b^2t)^i q^{2i^2}(1-atq^{4i}) \\
&+ \sum_{i=0}^{n-1} \frac{(a)_{2i+1}(at/b)_{3i+1}}{(bq)_i(t)_{2i+2}(a/b)_i} b^{2i+1}t^{i+1}q^{(2i+1)(i+1)} \\
&- a\sum_{i=0}^{n-1} \frac{(a)_{2i+1}(at/b)_{3i+2}}{(bq)_i(t)_{2i+2}(a/b)_{i+1}} b^{2i+1}t^{i+1}q^{(2i+1)(i+2)}.
\end{aligned}
$$

The $n^{\underline{th}}$ iteration is

$$
F(a,b;t) = \sum_{i=0}^{n-1} \frac{(a)_{2i}(at/b)_{3i}}{(bq)_i(t)_{2i+1}(a/b)_i}(b^2t)^i q^{2i^2}(1-atq^{4i})
$$

$$+\sum_{i=0}^{n-1}\frac{(a)_{2i+1}(at/b)_{3i+1}}{(bq)_i(t)_{2i+2}(a/b)_i}b^{2i+1}t^{i+1}q^{(2i+1)(i+1)}$$
$$-a\sum_{i=0}^{n-1}\frac{(a)_{2i+1}(at/b)_{3i+2}}{(bq)_i(t)_{2i+2}(a/b)_{i+1}}b^{2i+1}t^{i+1}q^{(2i+1)(i+2)}$$
$$+\frac{(a)_{2n}(at/b)_{3n}}{(bq)_n(t)_{2n}(a/b)_n}(b^2t)^nq^{2n^2}F(aq^{2n},bq^n;tq^{2n}).$$

Letting $n\to\infty$ and re-indexing gives Theorem 15. □

*Proof of Theorem 17.*

Using (29), (33), (36), and (46),

$$\begin{aligned}
A_{(\alpha^2\tau)^n} &= \prod_{i=0}^{n-1}(\alpha^2\tau)^iA_{\alpha^2\tau}\\
&= \prod_{i=0}^{n-1}(\alpha^2\tau)^iA_{\alpha\tau}(\alpha\tau)A_\alpha\\
&= \prod_{i=0}^{n-1}(\alpha^2\tau)^i\left(\frac{(1-a)(1-at/b)(1-atq/b)}{(1-t)(1-a/b)}b\cdot\frac{(1-aq)(1-atq^2/b)}{(1-aq/b)}\right)\\
&= \prod_{i=0}^{n-1}(\alpha^2\tau)^i\left(\frac{(1-a)(1-aq)(1-at/b)(1-atq/b)(1-atq^2/b)}{(1-t)(1-a/b)(1-aq/b)}b\right)\\
&= \prod_{i=0}^{n-1}\frac{(1-aq^{2i})(1-aq^{2i+1})(1-atq^{3i}/b)(1-atq^{3i+1}/b)(1-atq^{3i+2}/b)}{(1-tq^i)(1-aq^{2i}/b)(1-aq^{2i+1}/b)}b\\
&= \frac{(a)_{2n}(at/b)_{3n}}{(t)_n(a/b)_{2n}}b^n.
\end{aligned}$$

Next, using (30), (37), and (47),

$$\begin{aligned}
B_{\alpha^2\tau} &= B_{\alpha\tau}+A_{\alpha\tau}(\alpha\tau)B_\alpha\\
&= \frac{1-b}{1-t}-\frac{(1-b)(1-at/b)}{(1-t)(1-a/b)}a+\frac{(1-a)(1-at/b)(1-atq/b)}{(1-t)(1-a/b)}b(\alpha\tau)\left(-\frac{(1-b)a}{(1-a/b)b}\right)
\end{aligned}$$

$$= \frac{1-b}{1-t} - \frac{(1-b)(1-at/b)}{(1-t)(1-a/b)}a - \frac{(1-a)(1-at/b)(1-atq/b)(1-b)}{(1-t)(1-a/b)(1-aq/b)}aq.$$

Thus by (34),

$$\begin{aligned}
B_{(\alpha^2\tau)^n} &= \sum_{i=0}^{n-1} A_{(\alpha^2\tau)^i}(\alpha^2\tau)^i B_{\alpha^2\tau} \\
&= \sum_{i=0}^{n-1} \frac{(a)_{2i}(at/b)_{3i}}{(t)_i(a/b)_{2i}} b^i (\alpha^2\tau)^i \left( \frac{1-b}{1-t} - \frac{(1-b)(1-at/b)}{(1-t)(1-a/b)}a \right. \\
&\left. - \frac{(1-a)(1-at/b)(1-atq/b)(1-b)}{(1-t)(1-a/b)(1-aq/b)}aq \right) \\
&= \sum_{i=0}^{n-1} \frac{(a)_{2i}(at/b)_{3i}}{(t)_i(a/b)_{2i}} b^i \cdot \left( \frac{1-b}{1-tq^i} - \frac{(1-b)(1-atq^{3i}/b)}{(1-tq^i)(1-aq^{2i}/b)}aq^{2i} \right. \\
&\left. - \frac{(1-aq^{2i})(1-atq^{3i}/b)(1-atq^{3i+1}/b)(1-b)}{(1-tq^i)(1-aq^{2i}/b)(1-aq^{2i+1}/b)}aq^{2i+1} \right) \\
&= (1-b)\sum_{i=0}^{n-1} \frac{(a)_{2i}(at/b)_{3i}}{(t)_{i+1}(a/b)_{2i}} b^i - (1-b)a \sum_{i=0}^{n-1} \frac{(a)_{2i}(at/b)_{3i+1}}{(t)_{i+1}(a/b)_{2i+1}} (bq^2)^i \\
&- (1-b)a \sum_{i=0}^{n-1} \frac{(a)_{2i+1}(at/b)_{3i+2}}{(t)_{i+1}(a/b)_{2i+2}} b^i q^{2i+1}.
\end{aligned}$$

The $n^{\underline{th}}$ iteration is

$$\begin{aligned}
F(a,b;t) &= (1-b)\sum_{i=0}^{n-1} \frac{(a)_{2i}(at/b)_{3i}}{(t)_{i+1}(a/b)_{2i}} b^i - (1-b)a \sum_{i=0}^{n-1} \frac{(a)_{2i}(at/b)_{3i+1}}{(t)_{i+1}(a/b)_{2i+1}} (bq^2)^i \\
&- (1-b)a \sum_{i=0}^{n-1} \frac{(a)_{2i+1}(at/b)_{3i+2}}{(t)_{i+1}(a/b)_{2i+2}} b^i q^{2i+1} + \frac{(a)_{2n}(at/b)_{3n}}{(t)_n(a/b)_{2n}} b^n F(aq^{2n}, b; tq^n).
\end{aligned}$$

Letting $n \to \infty$ with $|b| < 1$ and re-indexing gives Theorem 17. $\square$

*Proof of Theorem 19.*

Using (29), (33), (40), and (46),

$$\begin{aligned}
A_{(\alpha\tau^2)^n} &= \prod_{i=0}^{n-1}(\alpha\tau^2)^i A_{\alpha\tau^2} \\
&= \prod_{i=0}^{n-1}(\alpha\tau^2)^i A_{\alpha\tau}(\alpha\tau)A_\tau \\
&= \prod_{i=0}^{n-1}(\alpha\tau^2)^i \left(\frac{(1-a)(1-at/b)(1-atq/b)}{(1-t)(1-a/b)}b \cdot \frac{(1-atq^2/b)}{(1-tq)}b\right) \\
&= \prod_{i=0}^{n-1}(\alpha\tau^2)^i \left(\frac{(1-a)(1-at/b)(1-atq/b)(1-atq^2/b)}{(1-t)(1-tq)(1-a/b)}b^2\right) \\
&= \prod_{i=0}^{n-1}\frac{(1-aq^i)(1-atq^{3i}/b)(1-atq^{3i+1}/b)(1-atq^{3i+2}/b)}{(1-tq^{2i})(1-tq^{2i+1})(1-aq^i/b)}b^2 \\
&= \frac{(a)_n(at/b)_{3n}}{(t)_{2n}(a/b)_n}b^{2n}.
\end{aligned}$$

Next, using (30), (41), and (47),

$$\begin{aligned}
B_{\alpha\tau^2} &= B_{\alpha\tau} + A_{\alpha\tau}(\alpha\tau)B_\tau \\
&= \frac{1-b}{1-t} - \frac{(1-b)(1-at/b)}{(1-t)(1-a/b)}a + \frac{(1-a)(1-at/b)(1-atq/b)}{(1-t)(1-a/b)}b(\alpha\tau)\left(\frac{1-b}{1-t}\right) \\
&= \frac{1-b}{1-t} - \frac{(1-b)(1-at/b)}{(1-t)(1-a/b)}a + \frac{(1-a)(1-at/b)(1-atq/b)(1-b)}{(1-t)(1-tq)(1-a/b)}b.
\end{aligned}$$

Thus by (34),

$$\begin{aligned}
B_{(\alpha\tau^2)^n} &= \sum_{i=0}^{n-1} A_{(\alpha\tau^2)^i}(\alpha\tau^2)^i B_{\alpha\tau^2} \\
&= \sum_{i=0}^{n-1}\frac{(a)_i(at/b)_{3i}}{(t)_{2i}(a/b)_i}b^{2i}(\alpha\tau^2)^i\left(\frac{1-b}{1-t} - \frac{(1-b)(1-at/b)}{(1-t)(1-a/b)}a\right.
\end{aligned}$$

$$
\begin{aligned}
&+\frac{(1-a)(1-at/b)(1-atq/b)(1-b)}{(1-t)(1-tq)(1-a/b)}b\Bigg)\\
&=\sum_{i=0}^{n-1}\frac{(a)_i(at/b)_{3i}}{(t)_{2i}(a/b)_i}b^{2i}\cdot\Bigg(\frac{1-b}{1-tq^{2i}}-\frac{(1-b)(1-atq^{3i}/b)}{(1-tq^{2i})(1-aq^i/b)}aq^i\\
&+\frac{(1-aq^i)(1-atq^{3i}/b)(1-atq^{3i+1}/b)(1-b)}{(1-tq^{2i})(1-tq^{2i+1})(1-aq^i/b)}b\Bigg)\\
&=(1-b)\sum_{i=0}^{n-1}\frac{(a)_i(at/b)_{3i}}{(t)_{2i+1}(a/b)_i}b^{2i}-a(1-b)\sum_{i=0}^{n-1}\frac{(a)_i(at/b)_{3i+1}}{(t)_{2i+1}(a/b)_{i+1}}b^{2i}q^i\\
&+(1-b)\sum_{i=0}^{n-1}\frac{(a)_{i+1}(at/b)_{3i+2}}{(t)_{2i+2}(a/b)_{i+1}}b^{2i+1}.
\end{aligned}
$$

The $n^{\underline{th}}$ iteration is

$$
\begin{aligned}
F(a,b;t)=(1-b)\sum_{i=0}^{n-1}\frac{(a)_i(at/b)_{3i}}{(t)_{2i+1}(a/b)_i}b^{2i}-a(1-b)\sum_{i=0}^{n-1}\frac{(a)_i(at/b)_{3i+1}}{(t)_{2i+1}(a/b)_{i+1}}b^{2i}q^i\\
+(1-b)\sum_{i=0}^{n-1}\frac{(a)_{i+1}(at/b)_{3i+2}}{(t)_{2i+2}(a/b)_{i+1}}b^{2i+1}+\frac{(a)_n(at/b)_{3n}}{(t)_{2n}(a/b)_n}b^{2n}F(aq^n,b;tq^{2n}).
\end{aligned}
$$

Letting $n\to\infty$ with $|b^2|<1$ and re-indexing gives Theorem 19. □

*Proof of Theorem 21.*

Using (29), (33), (38), and (44),

$$
\begin{aligned}
A_{(\alpha\beta^2)^n}&=\prod_{i=0}^{n-1}(\alpha\beta^2)^iA_{\alpha\beta^2}\\
&=\prod_{i=0}^{n-1}(\alpha\beta^2)^iA_{\alpha\beta}(\alpha\beta)A_\beta\\
&=\prod_{i=0}^{n-1}(\alpha\beta^2)^i\left(\frac{(1-a)}{(1-bq)}t\cdot\frac{(1-bq/a)}{(1-bq^2)(1-bq/at)}\right)
\end{aligned}
$$

$$= \prod_{i=0}^{n-1} (\alpha\beta^2)^i \left( \frac{(1-a)(1-bq/a)}{(1-bq)(1-bq^2)(1-bq/at)} t \right)$$

$$= \prod_{i=0}^{n-1} \frac{(1-aq^i)(1-bq^{i+1}/a)}{(1-bq^{2i+1})(1-bq^{2i+2})(1-bq^{i+1}/at)} t$$

$$= \frac{(a)_n (bq/a)_n}{(bq)_{2n} (bq/at)_n} t^n.$$

Next, using (30), (39), and (45),

$$\begin{aligned} B_{\alpha\beta^2} &= B_{\alpha\beta} + A_{\alpha\beta}(\alpha\beta) B_\beta \\ &= 1 + \frac{(1-a)}{(1-bq)} t(\alpha\beta) \left( -\frac{bq}{(1-bq/at)at} \right) \\ &= 1 - \frac{(1-a)}{(1-bq)} t \cdot \frac{bq}{(1-bq/at)at} \\ &= 1 - \frac{(1-a)bq}{(1-bq)(1-bq/at)a}. \end{aligned}$$

Thus by (34),

$$\begin{aligned} B_{(\alpha\beta^2)^n} &= \sum_{i=0}^{n-1} A_{(\alpha\beta^2)^i} (\alpha\beta^2)^i B_{\alpha\beta^2} \\ &= \sum_{i=0}^{n-1} \frac{(a)_i (bq/a)_i}{(bq)_{2i} (bq/at)_i} t^i (\alpha\beta^2)^i \left( 1 - \frac{(1-a)bq}{(1-bq)(1-bq/at)a} \right) \\ &= \sum_{i=0}^{n-1} \frac{(a)_i (bq/a)_i}{(bq)_{2i} (bq/at)_i} t^i \cdot \left( 1 - \frac{(1-aq^i)bq^{2i+1}}{(1-bq^{2i+1})(1-bq^{i+1}/at)aq^i} \right) \\ &= \sum_{i=0}^{n-1} \frac{(a)_i (bq/a)_i}{(bq)_{2i} (bq/at)_i} t^i - \frac{b}{at} \sum_{i=0}^{n-1} \frac{(a)_{i+1} (bq/a)_i}{(bq)_{2i+1} (bq/at)_{i+1}} (tq)^{i+1}. \end{aligned}$$

The $n^{\underline{th}}$ iteration is

$$F(a,b;t) = \sum_{i=0}^{n-1} \frac{(a)_i (bq/a)_i}{(bq)_{2i} (bq/at)_i} t^i - \frac{b}{at} \sum_{i=0}^{n-1} \frac{(a)_{i+1} (bq/a)_i}{(bq)_{2i+1} (bq/at)_{i+1}} (tq)^{i+1}$$

$$+\frac{(a)_n(bq/a)_n}{(bq)_{2n}(bq/at)_n}t^nF(aq^n,bq^{2n};t).$$

Letting $n\to\infty$ with $|t|<1$ and re-indexing gives Theorem 21. □

*Proof of Theorem 23.*

Using (29), (33), (38), and (42),

$$\begin{aligned}
A_{(\beta^2\tau)^n} &= \prod_{i=0}^{n-1}(\beta^2\tau)^iA_{\beta^2\tau}\\
&= \prod_{i=0}^{n-1}(\beta^2\tau)^iA_{\beta\tau}(\beta\tau)A_\beta\\
&= \prod_{i=0}^{n-1}(\beta^2\tau)^i\left(-\frac{(1-bq/a)}{(1-bq)(1-t)}at\cdot\frac{(1-bq^2/a)}{(1-bq^2)(1-bq/at)}\right)\\
&= \prod_{i=0}^{n-1}(\beta^2\tau)^i\left(-\frac{(1-bq/a)(1-bq^2/a)}{(1-bq)(1-bq^2)(1-t)(1-bq/at)}at\right)\\
&= \prod_{i=0}^{n-1}-\frac{(1-bq^{2i+1}/a)(1-bq^{2i+2}/a)}{(1-bq^{2i+1})(1-bq^{2i+2})(1-tq^i)(1-bq^{i+1}/at)}atq^i\\
&= \frac{(bq/a)_{2n}}{(bq)_{2n}(t)_n(bq/at)_n}(-at)^nq^{n(n-1)/2}.
\end{aligned}$$

Next, using (30), (39), and (43),

$$\begin{aligned}
B_{\beta^2\tau} &= B_{\beta\tau}+A_{\beta\tau}(\beta\tau)B_\beta\\
&= \frac{1}{1-t}-\frac{(1-bq/a)}{(1-bq)(1-t)}at(\beta\tau)\frac{-bq}{(1-bq/at)at}\\
&= \frac{1}{1-t}-\frac{(1-bq/a)}{(1-bq)(1-t)}at\cdot\frac{-bq}{(1-bq/at)at}\\
&= \frac{1}{1-t}+\frac{(1-bq/a)bq}{(1-bq)(1-t)(1-bq/at)}.
\end{aligned}$$

Thus by (34),

$$\begin{aligned}
B_{(\beta^2\tau)^n} &= \sum_{i=0}^{n-1} A_{(\beta^2\tau)^i}(\beta^2\tau)^i B_{\beta^2\tau} \\
&= \sum_{i=0}^{n-1} \frac{(bq/a)_{2i}}{(bq)_{2i}(t)_i(bq/at)_i}(-at)^i q^{i(i-1)/2}(\beta^2\tau)^i \left(\frac{1}{1-t}\right. \\
&\left. + \frac{(1-bq/a)bq}{(1-bq)(1-t)(1-bq/at)}\right) \\
&= \sum_{i=0}^{n-1} \frac{(bq/a)_{2i}}{(bq)_{2i}(t)_i(bq/at)_i}(-at)^i q^{i(i-1)/2} \cdot \left(\frac{1}{1-tq^i}\right. \\
&\left. + \frac{(1-bq^{2i+1}/a)bq^{2i+1}}{(1-bq^{2i+1})(1-tq^i)(1-bq^{i+1}/at)}\right) \\
&= \sum_{i=0}^{n-1} \frac{(bq/a)_{2i}}{(bq)_{2i}(t)_{i+1}(bq/at)_i}(-at)^i q^{i(i-1)/2} \\
&+ b\sum_{i=0}^{n-1} \frac{(bq/a)_{2i+1}}{(bq)_{2i+1}(t)_{i+1}(bq/at)_{i+1}}(-at)^i q^{(i+1)(i+2)/2}.
\end{aligned}$$

The $n^{\underline{th}}$ iteration is

$$\begin{aligned}
F(a,b;t) &= \sum_{i=0}^{n-1} \frac{(bq/a)_{2i}}{(bq)_{2i}(t)_{i+1}(bq/at)_i}(-at)^i q^{i(i-1)/2} \\
&\quad + b\sum_{i=0}^{n-1} \frac{(bq/a)_{2i+1}}{(bq)_{2i+1}(t)_{i+1}(bq/at)_{i+1}}(-at)^i q^{(i+1)(i+2)/2} \\
&\quad + \frac{(bq/a)_{2n}}{(bq)_{2n}(t)_n(bq/at)_n}(-at)^n q^{n(n-1)/2} F(a, bq^{2n}; tq^n).
\end{aligned}$$

Letting $n \to \infty$ and re-indexing gives Theorem 23. □

# CHAPTER 3

# COMBINATORIAL PROOFS

In this chapter we employ a hybrid of methods from Andrews [3] and Corteel and Lovejoy [7] to give combinatorial proofs of some of the theorems from Chapter 2. The first three sections are preparatory. Among other techniques, we will adapt a combinatorial proof of (18) from Andrews [3] to the context of overpartitions. To facilitate this, Section 3.1 gives an exposition of Andrews' technique. Section 3.2 begins with a description of how Andrews' method is adapted to the context of overpartitions and gives a needed proposition from Corteel and Lovejoy [7]. This proposition is shown to be combinatorially equivalent to a result of Alladi [1]. Section 3.3 continues the process of adapting Andrews' ideas to overpartitions, and in particular, describes an adaptation of his concept of "Cauchy order" in the context of overpartitions. Finally, in Section 3.4 we give proofs of three theorems from Chapter 2. In this section, the proofs are presented in order of difficulty; each proof builds on ideas from the previous proof.

## 3.1 Andrews' Method

Equation (4) (due to Fine [9]) does not seem to have a known combinatorial proof. However, other identities given by Fine have been given combinatorial proofs using partition interpretations. Andrews [3] gave a combinatorial proof of the Rogers-Fine identity

$$(1-t)\sum_{n=0}^{\infty}\frac{(a)_n}{(b)_n}t^n = \sum_{n=0}^{\infty}\frac{(a)_n(atq/b)_n}{(b)_n(tq)_n}b^nt^nq^{n^2-n}(1-atq^{2n}), \tag{64}$$

as well as an outline of a proof of another Fine identity (18), which we state again,

$$\sum_{n=0}^{\infty}\frac{(a)_n}{(bq)_n}t^n = \sum_{n=0}^{\infty}\frac{(bq/a)_n}{(bq)_n(t)_{n+1}}(-at)^nq^{(n^2-n)/2}.$$

We first give an exposition of Andrews' techniques for proving (18).

Andrews begins by employing a dilation $q \mapsto q^2$ followed by the $q$-shifts $a \mapsto -aq$ and $t \mapsto tq^2$. Then (18) becomes

$$\sum_{n=0}^{\infty}\frac{(-aq;q^2)_n}{(bq^2;q^2)_n}bt^nq^{2n} = \sum_{n=0}^{\infty}\frac{(-bq/a;q^2)_n}{(bq^2;q^2)_n(tq^2;q^2)_{n+1}}b^{n+1}t^nq^{2n(n+1)}. \tag{65}$$

Next, Andrews provides a partition interpretation for the summand on the left side of (65):

**Proposition 25** (Andrews [3])**.** Let $p(k,l,m,n)$ denote the number of partitions of $n$ with $k$ odd parts (all distinct), $l$ even parts, and largest part $2m$. Let $P$ denote the set of partitions counted by $p(k,l,m,n)$. Then,

$$\sum_{n,k,l\geq 0}p(k,l,m,n)a^kb^lt^mq^n = \frac{(-aq;q^2)_m}{(bq^2;q^2)_m}bt^mq^{2m}.$$

To see this, notice that the factor $bt^m q^{2m}/(bq^2;q^2)_m$ generates partitions into even parts, where the exponent of $b$ counts the number of even parts, and the exponent of $t$ counts half the size of the largest part (generated by the factor $q^{2m}$). The factor $(-aq;q^2)_m$ is the generating function for partitions into distinct odd parts no larger than $2m-1$. This gives a partition interpretation for the left side of (65). (Corteel and Lovejoy provide a different interpretation for the left side of (18) by letting the distinct odd parts become distinct overlined parts.)

We now focus our attention on the right side of (65) by first employing a different representation of the $q$-product $(-bq/a;q^2)_n = (1+bq/a)(1+bq^3/a)\cdots(1+bq^{2n-1}/a)$. Rewrite this product as

$$(1+aq^{-1}/b)(1+aq^{-3}/b)\cdots(1+aq^{-2n+1}/b)a^{-n}b^n q^{n^2}. \tag{66}$$

Then (65) becomes

$$\sum_{n=0}^{\infty} \frac{(-aq;q^2)_n}{(bq^2;q^2)_n} bt^n q^{2n} = \sum_{n=0}^{\infty} \frac{(1+aq^{-1}/b)(1+aq^{-3}/b)\cdots(1+aq^{-2n+1}/b)}{(bq^2;q^2)_n(tq^2;q^2)_{n+1}} b^{n+1}t^n q^{2n(n+1)}. \tag{67}$$

Andrews now considers the summand on the right side of (67). He indicates a combinatorial interpretation of the summand using a similar argument to Theorem 5: the term

$$\frac{b^{n+1}t^n q^{2n(n+1)}}{(bq^2;q^2)_n(tq^2;q^2)_{n+1}}$$

is the generating function for partitions consisting of all even parts with Durfee rectangle $(n+1)\times 2n$. In the product $(1+aq^{-1}/b)(1+aq^{-3}/b)\cdots(1+aq^{-2n+1}/b)$, the term $aq^{-2i+1}/b$ subtracts $2i-1$ from the $i+1^{\underline{st}}$ even part, resulting in the creation of an odd part. As there are at most $n$ odd parts being subtracted, the shape of the Durfee rectangle ensures the parts of the resulting partition are positive. Toward carrying out the converse operation, note that before subtraction we have a partition characterized by having a Durfee rectangle of size $(n+1) \times 2n$, that is, $n$ is the largest integer such that the number of even parts greater than or equal to $2n$ is at least $n+1$. As some of these even parts are converted to odd parts after subtraction, it must be true that the number of even parts greater than or equal to $2n$ plus the number of odd parts must be at least as large as $n+1$. This motivates Andrews' definition of *Cauchy order*.

**Definition 7.** The *Cauchy order* of a partition $\pi$ is the largest integer $\eta$ such that the following inequality is maximal: given a partition, the number of even parts greater than $2\eta - 1$ plus the number of distinct odd parts must be at least as large as $\eta + 1$.

For example, the Cauchy order of the partition $\pi = 11 + 10 + 8 + 4 + 3 + 2 + 2$ is 3, since there are two distinct odd parts and two even parts greater than 5 (since $5 = 2 \cdot 3 - 1$). This gives the maximal $\eta$ satisfying the desired inequality. Note, it is clear that the Cauchy order always exists.

We now give a proof of (18), using the representation (67), and we note that Andrews outlines a different version for the fourth step which is incorrect.

*Combinatorial Proof of* (18).

By Proposition 25, the coefficient of $a^k b^l t^m q^n$ on the left side enumerates partitions of $n$ with $k$ odd parts (all distinct), $l$ even parts, and largest part $2m$.

We show the $N^{th}$ term of the sum on the right side of (67)

$$\frac{\prod_{i=1}^{N}(1 + aq^{-2i+1}/b)}{(bq^2;q^2)_N(tq^2;q^2)_{N+1}} b^{N+1} t^N q^{2N(N+1)} \tag{68}$$

enumerates partitions of $n$ with $k$ odd parts (all distinct), $l$ even parts, largest part $2m$, and Cauchy order $N$. This enumeration gives a refinement of the left side. We construct this set of partitions in four steps using Ferrers diagrams.

1. Consider $b^{N+1}t^N q^{2N(N+1)}$, which represents a Durfee rectangle of size $(N+1) \times 2N$, that is, $N+1$ parts of size $2N$. The exponent of $b$ is the number of even parts and the exponent of $t$ is half the size of the largest part. The Cauchy order is $N$, since there are no odd parts and $N+1$ even parts larger than $2N-1$.

2. Consider $1/(tq^2;q^2)_{N+1}$. This is the generating function for partitions into even parts of size at most $2N+2$. We transform these partitions in the following way: take each even part $2d$ and create a Ferrers diagram consisting of two columns of height $d$ each. The resulting Ferrers diagram is for partitions have at most $N+1$ even parts. Now append each such Ferrers diagram to the right side of the Durfee rectangle from the previous step to yield new a Ferrers diagram. The resulting partitions have exactly $N+1$ even parts. The exponent of $t$ still equals half the size of the largest part under this transformation. The Cauchy order is unaltered, since the number of even parts has not changed and the smallest even part has not decreased.

3. Consider $1/(bq^2;q^2)_{2N}$. This is the generating function for partitions into even parts of size less than or equal to $2N$. The Ferrers diagram for such partitions are placed under the diagram from the previous step. Note the Cauchy order is unaltered from the previous step, since parts added increase the number of even parts and are less than or equal to $2N$. In this step, the number of even parts increases and the exponent of $b$ still counts the number of even parts.

We now have partitions into at least $N+1$ even parts greater than or equal to $2N$ with the remaining even parts smaller than $2N$.

4. Consider $\prod_{i=1}^{N}(1+aq^{-(2i-1)}/b)$. The exponent of $q$ in a typical term from the expansion of this product has the form $-[(2c_1-1)+(2c_2-1)+\cdots+(2c_k-1)]$, where $1 \leq c_1 < c_2 < \cdots < c_k \leq N$. Thus, this factor generates partitions into negative distinct odd parts from the set $\{-1,-3,\cdots,-(2N-1)\}$. Each odd part $2c_i-1$, where $1 \leq i \leq k$, is now subtracted from the $c_i + 1^{\underline{st}}$ even part from the partitions generated in step 3. (Note, that since there is no $t$ present in this generating function, we do not subtract from the largest part since the exponent of $t$ is half the size of the largest part and must be preserved.) The new odd parts do not change the Cauchy order of the partitions, since the odd parts replace even parts. In this step, the exponent of $a$ tracks the number of odd parts added to the partitions; the exponent of $b$ decreases by the number of even parts which become odd parts. *Notice that when subtracting the odds in increasing order from the even parts, the resulting odd parts will be distinct and in decreasing order.*

In [3], Andrews indicates to subtract starting from the first even part. This is an error due to the absence of the variable $t$ in the generating function above; the exponent of $t$ would no longer be equal to half the size of the largest even part if subtraction began with the first part.

Thus, we have generated terms of the form $q^{\pi}a^k b^l t^m$ where $\pi$ is a partition with $k$ distinct odd parts, $l$ even parts, largest part $2m$, and Cauchy order $N$.

Now we show that, given a partition described by $P(k,l,m,n)$ with Cauchy order $N$, there is exactly one generating term of the form (68). Note that the Cauchy order $N$ indicates the size of the Durfee rectangle before the odds were subtracted. $N$ is also an upper bound to the number of odds and gives the distinct odds that were subtracted are from the set $\{1,\cdots,2N-1\}$.

The following algorithm recovers the original partition of evens generated by the first three steps.

(i) Start with a partition with $l$ even parts, $k$ distinct odd parts, and Cauchy order $N$. Separate the partition into two tuples, one $l$-tuple consisting of the even parts $(e_1, \cdots, e_i, \cdots e_l)$, and one $k$-tuple of the distinct odd parts $(o_1, \cdots, o_j, \cdots, o_k)$. Here, both tuples are ordered largest to smallest.

(ii) Add the odd numbers $\{1, \cdots, 2N-1\}$ in order to the first odd, $o_1$, to obtain an increasing sequence of even numbers $o_1+1, o_1+3, \cdots, o_1+2N-1$. One of these evens must have been the even that was transformed into an odd in step 4. This even is uniquely determined by comparing our increasing even sequence against the decreasing sequence $e_1, e_2, \cdots, e_l$. In particular, there exists an odd $2i-1 \in \{1, \cdots, 2N-1\}$ such that $o_1 + 2i - 1$ is between the $i^{\underline{th}}$ even and the $i+1^{\underline{st}}$ even. When this is the case, then shift down all evens less than or equal to $e_{i+1}$ (in other words, let $e_{i+1} \mapsto e_{i+2} \mapsto e_{i+3} \mapsto \cdots$) and let $o_1 + 2i - 1$ be the $i+1^{\underline{st}}$ even. Note, we now have $l+1$ evens in our sequence.

(iii) Repeat this process for all distinct odds $o_2, o_3, \cdots, o_k$.

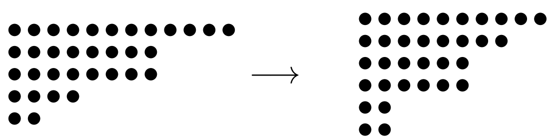
This algorithm recovers the odds that were subtracted as well as the original even parts from the partition generated at the end of step 3. Thus, finding the Cauchy order $N$ yields the term on the right side of (67) which generates any partition from $P$. □

We now consider an example where $N = 5$; the $5^{\underline{th}}$ term on the right side of (67) is:

$$\frac{(1+aq^{-1}/b)(1+aq^{-3}/b)\cdots(1+aq^{-9}/b)}{(bq^2;q^2)_5(tq^2;q^2)_6} b^6 t^5 q^{10\times 6}.$$

The factor $b^6t^5q^{10\times 6}$ generates a Durfee rectangle of size $6 \times 10$, giving $\pi_1 = 10 + 10 + 10 + 10 + 10 + 10$. Note that the exponent of $b$ equals the number of even parts and the exponent of $t$ equals half the size of the largest part. Also note the Cauchy order of this partition is 5, since there are 6 even parts greater than or equal to 10, giving $\eta = 5$ as maximal.

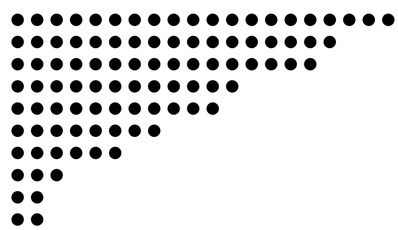
Now consider $1/(tq^2;q^2)_6$. This is the generating function for partitions with even parts ranging from size 2 to 12. We transform these partitions as described in the proof above. For example, say we have $\lambda = 12 + 8 + 8 + 4 + 2$, which corresponds to a term of the form $q^{12+8+8+4+2}t^5$. Using the transformation described in the proof, divide each part into two columns of equal height. For example, 12 becomes two columns of 6 and 8 becomes two columns of 4. The transformation of $\lambda$ is $\hat{\lambda} = 10 + 8 + 6 + 6 + 2 + 2$.



Adding $\hat{\lambda}$ to the right side of $\pi_1$ gives $\pi_2 = 20 + 18 + 16 + 16 + 12 + 12$. The exponent of $t$ tracks half the size of the largest part. Note, the Cauchy order is unchanged in this new partition, since the addition of parts only increases the size of the evens considered in $\pi_1$ for a Cauchy order of 5.

Next, consider $1/(bq^2;q^2)_5$. This is the generating function for partitions with even parts of size between 2 and 10. These partitions are placed under $\pi_2$. An example is the even partition $\rho = 8 + 6 + 2 + 2$ (which corresponds to a term of the form $q^{8+6+2+2}b^4$). Adding $\rho$ to $\pi_2$ gives $\pi_3 = 20 + 18 + 16 + 16 + 12 + 12 + 8 + 6 + 2 + 2$. Note, the Cauchy order is still unchanged, since the addition of parts maintains the inequality desired from the definition of Cauchy order.

Finally, consider the factor $(1 + aq^{-1}/b)(1 + aq^{-3}/b)(1 + aq^{-5}/b)(1 + aq^{-7}/b)(1 + aq^{-9}/b)$. This generating function provides the possibility to create distinct odd parts by subtracting none, some, or all of the odds 1 through 9. Say we want to subtract $1, 5$, and 9. Note, this choice of odds creates the sequence $c_i$ with $c_1 = 1$, since 1 is the first odd from the set $\{1, 3, 5, 7, 9\}$, $c_2 = 3$, since 5 is the third odd from the set, and $c_3 = 5$, since 9 is the fifth odd from the set. We begin subtraction from the $c_1 + 1^{\underline{st}}$ part, which is the second part of $\pi_3$. Thus, we subtract $2(c_1) - 1 = 1$ from the second part. Next, subtract $2(c_2) - 1 = 5$ from the fourth part of $\pi_3$, since $c_2 + 1 = 4$. Finally, subtract 9 from the sixth part, since $2(c_3) - 1 = 9$ and $c_3 + 1 = 6$. After subtraction, the partition is $\pi_4 = 20+17+16+11+12+3+8+6+2+2$. Rearrangement of parts yields $\pi_4 = 20 + 17 + 16 + 12 + 11 + 8 + 6 + 3 + 2 + 2$.



To illustrate the inverse of the mapping, first count the number of odds, which is 3, and then determine the Cauchy order of $\pi_4$, which is 5. Then, begin the algorithm by separating the evens as $e_1 = 20$, $e_2 = 16$, $e_3 = 12$, $e_4 = 8$, $e_5 = 6$, $e_6 = 2$ and $e_7 = 2$, and odds as $o_1 = 17$, $o_2 = 11$, and $o_3 = 3$. We place the evens to the left, ordered largest to smallest, and the evens $o_1 + 1, o_1 + 3, \cdots, o_1 + 9$ on the right, ordered smallest to largest.

We start the algorithm with the first distinct odd, $o_1 = 17$, and consider the list of evens generated by adding odds from the set $\{1, 3, 5, 7, 9\}$.

Each of the following tables shows how the odd parts $o_i$, $i = 1, 2, 3, \ldots$, are recovered.

| $i$ | $e_i$ | $o_1 + 2i - 1$ |
|---|---|---|
| 1 | 20 | |
| | | $\leftarrow 18 = o_1 + 1$ |
| 2 | 16 | |
| | | $20 = o_1 + 3$ |
| 3 | 12 | |
| | | $22 = o_1 + 5$ |
| 4 | 8 | |
| | | $24 = o_1 + 7$ |
| 5 | 6 | |
| | | $26 = o_1 + 9$ |
| 6 | 2 | |
| 7 | 2 | |

| $i$ | $e_i$ | $o_2 + 2i - 1$ |
|---|---|---|
| 1 | 20 | |
| 2 | 18 | |
| | | $14 = o_2 + 3$ |
| 3 | 16 | |
| | | $\leftarrow 16 = o_2 + 5$ |
| 4 | 12 | |
| | | $18 = o_2 + 7$ |
| 5 | 8 | |
| | | $20 = o_2 + 9$ |
| 6 | 6 | |
| 7 | 2 | |
| 8 | 2 | |

| $i$ | $e_i$ | $o_3 + 2i - 1$ |
|---|---|---|
| 1 | 20 | |
| 2 | 18 | |
| 3 | 16 | |
| 4 | 16 | |
| | | $10 = o_3 + 7$ |
| 5 | 12 | |
| | | $\leftarrow 12 = o_3 + 9$ |
| 6 | 8 | |
| 7 | 6 | |
| 8 | 2 | |
| 9 | 2 | |

In the first table, it is clear that 18 was the original even, since this even fits between $e_1$ and $e_2$. Thus we insert $e_2 = 18$ and shift all the evens below down to get $e_3 = 16$, $e_4 = 12$, and so on. This gives the new sequence $e_i$ in the second table.

We continue with the next odd, $o_2 = 11$, and create a list of evens $o_2 + 2i - 1$ from the remaining odds in the set $\{3, 5, 7, 9\}$.

From the second table, it is clear that 14 was not the original even, since 14 does not fit between the second and third even in the list of $e_i$ on the left. Moving to the next even $o_2 + 5 = 16$, it is clear that this was the original even, since 16 fits between $e_3 = 16$ and $e_4 = 12$. Thus, let $e_4 = 16$ and shift the evens less than 16 to get a new list of evens.

Now consider the final odd, $o_3 = 3$, and create a list of evens $o_3 + 2i - 1$ from the remaining odds in the set $\{7, 9\}$.

From the last table, it is clear that 7 was not subtracted, since $10 = o_3 + 7$ does not fit between $e_4 = 16$ and $e_5 = 12$. It is clear that $12 = o_3 + 9$ was the original even, since 12 fits between $e_5 = 12$ and $e_6 = 8$. Thus we insert $e_6 = 12$ and shift all the evens below down to get $e_7 = 8$, $e_8 = 6$, and so on.

Thus, we have the original evens from $\pi_3$, or $20+18+16+16+12+12+8+6+2+2$. The inverse of the remaining steps is routine.

## 3.2 Overpartitions

Motivated by Andrews subtraction of odd parts from even parts in his combinatorial proof of (18), we employ the group $\mathcal{O} = (\mathbb{Z}, +) \times (\mathbb{Z}/2\mathbb{Z}, +)$. In the context of overpartitions, elements $(a, b) \in \mathcal{O}$ are written as $a$, if $b = [0]_2$, and $\overline{a}$, if $b = [1]_2$. Some examples of the group operation are: $3 + \overline{2} = \overline{5}$, $3 + 2 = 5$, $1 - \overline{0} = \overline{1}$, and $\overline{3} + \overline{2} = 5$. This operation is used in this chapter and the next.

The *rank* of an overpartition is defined to be the largest part minus 1 less the number of overlined parts less than the largest part. For example, the rank of $\pi = \overline{5} + 5 + \overline{3} + \overline{2} + 1$ is 2, since the largest part is $\overline{5}$ and there are two overlined parts less than 5.

The following proposition from [7] provides the generating function for overpartitions with restrictions on the number of parts (both overlined and non-overlined), as well as the rank. This proposition gives a way to interpret Fine's function $F(a, b; t)$ after the change of variables $a \mapsto -a$ and $t \mapsto tq$.

**Proposition 26** (Corteel and Lovejoy [7]). Let $\overline{p}(\nu, \nu_o, r, n)$ denote the number of overpartitions of a positive integer $n$ into $\nu$ parts, of which $\nu_o$ are overlined, and rank $r$. Let $\overline{P}$ denote the set of overpartitions counted in $\overline{p}(\nu, \nu_o, r, n)$. Then

$$\sum_{n,\nu_o,r\geq 0} \overline{p}(\nu, \nu_o, r, n) a^{\nu_o} b^r t^\nu q^n = \frac{(-a)_\nu (tq)^\nu}{(bq)_\nu}.$$

*Proof (Corteel and Lovejoy).*

Consider the factor $\frac{(tq)^\nu}{(bq)_\nu}$ on the right side. This is the generating function for partitions $\pi_1$ into $\nu$ parts, where the exponent on $t$ tracks the number of parts and the exponent on $b$ tracks the largest part minus 1. Now, $(-a)_\nu$ generates partitions $\pi_2$ into distinct non-negative parts less than $\nu$. (The statement non-negative is important, since the polynomial $(-a)_\nu$ contains a term with $q^0$, so 0 can occur as a part). Let $\pi_2 = \mu_1 + \mu_2 + \cdots + \mu_j$ where $j \leq \nu - 1$ and $\mu_1 > \mu_2 > \cdots > \mu_j$. Attach the partition $\pi_2$ onto $\pi_1$ using the following algorithm: for each $\mu_i$ starting with the largest, $\mu_1$, add 1 to the first $\mu_i$ parts of $\pi_1$ and over-line the $\mu_i + 1^{\underline{st}}$ part of $\pi_1$. This gives an overpartition $\pi$ that preserves the rank, since each step involves increasing the largest part and overlining another. Reversing this algorithm shows gives a bijection. □

As an example, let $\pi_1 = 9+5+3+3+2+1$ and $\pi_2 = 5+2+0$, where $\pi_1$ is an unrestricted partition and $\pi_2$ is a partition into distinct non-negative parts. We employ the algorithm above by transforming the ordered pair $(\pi_1, \pi_2)$:

$$\begin{aligned}(9+5+3+3+2+1, 5+2+0) &\Longleftrightarrow (10+6+4+4+3+\overline{1}, 2+0) \\ &\Longleftrightarrow (11+7+\overline{4}+4+3+\overline{1}, 0) \\ &\Longleftrightarrow (\overline{11}+7+\overline{4}+4+3+\overline{1}).\end{aligned}$$

Note the rank is 9, since the largest part is 11 and the number of overlined parts less than 11 is two.

We now give an overpartition interpretation to Fine's involution (12), or

$$F(a,b;t) = \frac{1-b}{1-t} F(at/b,t;b).$$

Dividing both sides by $(1-b)$ gives

$$\sum_{n\geq 0} \frac{(a)_n}{(b)_{n+1}} t^n = \sum_{n\geq 0} \frac{(at/b)_n}{(t)_{n+1}} b^n.$$

After a change of variables of $a \mapsto aq$, $b \mapsto bq$, and $t \mapsto tq$, as well as multiplying both sides by $q$ yields

$$\sum_{n\geq 0} \frac{(-aq)_n}{(bq)_{n+1}} t^n q^{n+1} = \sum_{n\geq 0} \frac{(-atq/b)_n}{(tq)_{n+1}} b^n q^{n+1}. \tag{69}$$

We give an interpretation of both sides using the method Corteel and Lovejoy employed for Proposition 26.

**Proposition 27.** Let $\overline{\Lambda}(\nu,\nu_o,r,n)$ denote the number of overpartitions of a positive integer $n$ into $\nu$ parts, of which $\nu_o$ are overlined, and rank $r$, where the first part is not overlined. Then

$$\sum_{\substack{n\geq 1\\ \nu_o,r\geq 0}} \overline{\Lambda}(\nu+1,\nu_o,r,n) a^{\nu_o} b^r t^\nu q^n = \frac{(-aq)_\nu}{(bq)_{\nu+1}} t^\nu q^{\nu+1}. \tag{70}$$

*Proof of Proposition 27.*

Starting with the right side, the factor $t^\nu q^{\nu+1}/(bq)_{\nu+1}$ generates partitions $\pi_1$ into $\nu+1$ parts where the exponent of $b$ counts the largest part minus one and the exponent of $t$ counts the number of parts minus 1. The factor $(-aq)_\nu$ generates partitions $\pi_2$ into distinct parts ranging in size from 1 to $\nu$. Using the algorithm from the proof of Proposition 26, we attach $\pi_2$ to $\pi_1$ to obtain overpartitions into $\nu+1$ parts, $\nu_o$ overlined parts, and rank $r$. The first part is not overlined since $(-aq)_\nu$ generates positive distinct parts. $\square$

Using this combinatorial interpretation, we can apply a change of variables and obtain an interpretation for the right side. Letting $a \mapsto at/b$, $b \mapsto t$, and $t \mapsto b$ in Proposition 27 implies that

$$\begin{aligned}\frac{(-atq/b)_v}{(tq)_{v+1}} b^v q^{v+1} &= \sum_{\substack{m\geq 1\\ k,l\geq 0}} \overline{\Lambda}(v+1,k,l,m)(at/b)^k t^l b^v q^m \\ &= \sum_{\substack{m\geq 1\\ k,l\geq 0}} \overline{\Lambda}(v+1,k,l,m) a^k b^{v-k} t^{l+k} q^m. \qquad (71)\end{aligned}$$

Thus, summing the left side of (70) and the right side of (71) over $\nu \geq 0$, as well as using (69), yields

$$\sum_{\substack{n\geq 1\\ \nu_o,\nu,r\geq 0}} \overline{\Lambda}(\nu+1,\nu_o,r,n) a^{\nu_o} b^r t^\nu q^n = \sum_{\substack{m\geq 1\\ k,l,v\geq 0}} \overline{\Lambda}(v+1,k,l,m) a^k b^{v-k} t^{l+k} q^m.$$

Next make the change of variables on the right side: $k \mapsto \nu_o$, $v-k \mapsto r$, $l+k \mapsto \nu$, and $m \mapsto n$. After this, the inequalities on the right side become: $n \geq 1$, $\nu_o \geq 0$, $\nu - \nu_o \geq 0$, and $r + \nu_o \geq 0$. These reduce to the inequalities on the left side. Equating coefficients on both sides, as well as shifting $\nu \mapsto \nu - 1$ gives the following theorem, which is an overpartition interpretation of the involution (12).

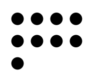
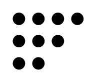
**Theorem 28.** *For $n, \nu \geq 1$ and $\nu_o, r \geq 0$,*

$$\overline{\Lambda}(\nu, \nu_o, r, n) = \overline{\Lambda}(\nu_o + r + 1, \nu_o, \nu - \nu_o - 1, n).$$

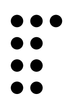
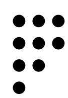
Now consider an example of partitions of 9 into 3 parts, no overlined part, and rank 3. Theorem 28 states that

$$\overline{\Lambda}(3, 0, 3, 9) = \overline{\Lambda}(4, 0, 2, 9).$$

Overpartitions generated by the function on the left side are $4 + 4 + 1$ and $4 + 3 + 2$ which have Ferrers diagrams given below.

Overpartitions generated by the function on the right side are $3+3+2+1$ and $3+2+2+2$ which have Ferrers diagrams given below.

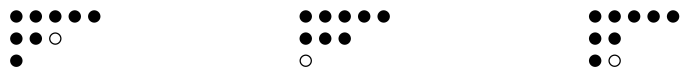



Notice that the conjugate of $4 + 4 + 1$ is $3 + 2 + 2 + 2$ and the conjugate of $4 + 3 + 2$ is $3+3+2+1$. In general, the absence of overlined parts gives a combinatorial understanding in terms of overpartitions for Theorem 28 as conjugation.

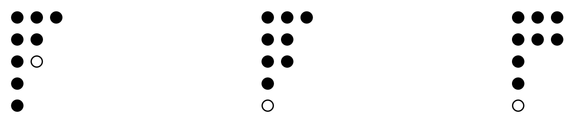
Consider another example of partitions of 9 into 3 parts, 1 overlined part, and rank 3. Theorem 28 states that

$$\overline{\Lambda}(3, 1, 3, 9) = \overline{\Lambda}(5, 1, 1, 9).$$

Overpartitions generated by the function on the left side are $5+\overline{3}+1$, $5+3+\overline{1}$, and $5+\overline{2}+2$, which have Ferrers diagrams given below.



Overpartitions generated by the function on the right side are $3+\overline{2}+2+1+1$, $3+2+2+\overline{1}+1$, and $3+3+\overline{1}+1+1$, which have the following diagrams.



Notice that the conjugate of $5+\overline{3}+1$ is $3+\overline{2}+2+1+1$. Conjugating $5+3+\overline{1}$ gives $\overline{3}+2+2+1+1$. This is similar to the overpartition $3+2+2+\overline{1}+1$ from the right side, however, the largest part cannot be overlined.

We now discuss how Theorem 28 follows from conjugation along with a simple bijection on overpartitions. Let $\Lambda'$ be the set of overpartitions where instead of the largest part being restricted from being overlined, the smallest part has that restriction.

*Combinatorial Proof of Theorem 28.*

We show that conjugation $* : \overline{\Lambda} \mapsto \Lambda'$ is a bijection. Start with an overpartition $\pi$ with $\nu$ parts (the largest of which is $\lambda$), $\nu_o$ overlined parts, and rank $r$ where the largest part is not overlined. The Ferrers diagram for $\pi^*$ has $\lambda$ (or $r+1+\nu_o$) parts with largest part $\nu$. The number of overlined parts remains the same under conjugation. In terms of $\lambda$, the rank for $\pi$ can be expressed as $r = \lambda - \nu_o - 1$. This is possible since the largest part is not overlined. Under conjugation, the new rank becomes $\nu - \nu_o - 1$.

Define the map $f : \Lambda' \mapsto \overline{\Lambda}$ in the following way. If the smallest part is not overlined, then the overpartition maps to the conjugate. If the smallest part is overlined, then the overpartition maps to a overpartition of which the largest part is overlined and then conjugation is applied. In both cases, this gives overpartitions of which the largest part is not overlined. The composition $f \circ *$ is the bijection proving the theorem. $\square$

We now show that Proposition 26 is equivalent to Alladi's [1] identity,

$$\frac{(abq)_n}{(bq)_n} = \sum_{\lambda(\pi) \leq n} (1-a)^{\nu_d(\pi)} b^{\nu(\pi)} q^{\sigma(\pi)}, \tag{72}$$

where $\lambda(\pi)$ denotes the largest part of a partition $\pi$, $\nu(\pi)$ is the number of parts, $\sigma(\pi)$ represents the sum of parts of the partition, and $\nu_d(\pi)$ denotes the number of distinct parts. To see this, map $a \to -a$ in Alladi's identity (72):

$$\sum_{\lambda(\pi) \leq n} (1+a)^{\nu_d(\pi)} b^{\nu(\pi)} q^{\sigma(\pi)} = \frac{(-abq)_n}{(bq)_n} \tag{73}$$

where $\pi$ is a partition, $\sigma(\pi)$ is the sum of the parts of $\pi$, $\nu(\pi)$ is the number of parts, $\nu_d(\pi)$ is the number of different parts, and $\lambda(\pi)$ is the largest part. Alladi's combinatorial proof is based on conjugation, for which the number of different parts are preserved. To see the equivalence, suppose $\nu_o$ parts are overlined from the $\nu_d$ different parts. The number of overpartitions with $\nu_d$ different parts and $\nu_o$ overlined parts is equal to $\binom{\nu_d}{\nu_o}$ times the number of partitions with $\nu_d$ different parts. Multiplying by $a^{\nu_o}$ and summing over $\nu_o$ gives the factor $(1+a)^{\nu_o}$. This shows the equivalence of Alladi's identity (72) and the identity in Proposition 26 of Corteel and Lovejoy.

## 3.3 The Durfee Parameter

Earlier in this chapter, we presented a proof outlined by Andrews of the identity

$$\sum_{n=0}^{\infty} \frac{(a)_n}{(bq)_n} t^n = \sum_{n=0}^{\infty} \frac{(bq/a)_n}{(bq)_n (t)_{n+1}} (-at)^n q^{(n^2-n)/2},$$

which used Cauchy order. Overpartitions can also be used to give a proof where the overlined parts play the role that odd parts played in Andrews' proof. Because of this, the identity can be proved without the dilation $q \mapsto q^2$. In addition, instead of Cauchy order, we define an overpartition parameter called the *Durfee parameter*. We apply this notion to a variety of overpartitions with Durfee rectangles of different sizes. We give a general definition for a Durfee rectangle of size $(aN + b) \times (cN + d)$, where $a, b, c, d \geq 0$. Adapting the definition of Cauchy order, we replace evens with non-overlined parts, odds with overlined parts, and we require a new inequality. Let $h = aN + b$ be the height of the Durfee rectangle, let $s$ be the number of non-overlined parts greater than or equal to $cN + d$, the length of the Durfee rectangle, and let $o$ be the number of overlined parts strictly greater than $(c-1)N + (d-1)$.

**Definition 8** (Durfee Parameter)**.** The *Durfee parameter* $N$ for a partition is the maximal $N$ such that

$$s + o \geq h. \tag{74}$$

For such an $N$ we say the *constrained Durfee rectangle* is of size $(aN + b) \times (cN + d)$.

We now give a combinatorial proof of (18) that uses overpartitions and the Durfee parameter. First, apply the change of variables $a \mapsto -a$ along with the $q$-shift $t \mapsto tq$ to both sides of (18), yields

$$\sum_{n=0}^{\infty} \frac{(-a)_n}{(bq)_n} t^n q^n = \sum_{n=0}^{\infty} \frac{(-bq/a)_n}{(bq)_n (tq)_{n+1}} (at)^n q^{n(n+1)/2}.$$

Rewriting the product on the right side as

$$\begin{aligned}(-bq/a)_n &= (1+bq/a)(1+bq^2/a)\cdots(1+bq^n/a) \\ &= (1+aq^{-1}/b)(1+aq^{-2}/b)\cdot(1+aq^{-n}/b)a^{-n}b^n q^{n(n+1)/2},\end{aligned} \tag{75}$$

gives

$$\sum_{n=0}^{\infty} \frac{(-a)_n}{(bq)_n} t^n q^n = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{n}(1+aq^{-i}/b)}{(bq)_n (tq)_{n+1}} b^n t^n q^{n(n+1)}. \tag{76}$$

Following the proof, an explicit example is presented.

*Overpartition Proof of* (18).

By Proposition 26, the coefficient of $q^n a^{\nu_o} b^r t^\nu$ on the left side of (76) enumerates overpartitions of $n$ into $\nu$ parts, of which $\nu_o$ are overlined, and rank $r$. We show the $N^{\underline{th}}$ term of the sum on the right side of (76),

$$\frac{\prod_{i=1}^{N}(1+aq^{-i}/b)}{(bq)_N (tq)_{N+1}} b^N t^N q^{N(N+1)},$$

enumerates the same overpartitions with Durfee parameter $N$ and constrained Durfee rectangle $N \times (N+1)$. This set of overpartitions is constructed in four steps using Ferrers diagrams.

1. The factor $b^N t^N q^{N(N+1)}$ generates a Durfee rectangle of size $N \times (N+1)$, that is, $N$ parts (enumerated by the exponent of $t$) of size $N+1$, rank $N$ (enumerated by the exponent of $b$), and no overlined parts. Note the Durfee parameter is $N$ since there are $N$ non-overlined parts greater than or equal to $N+1$.

2. Consider $1/(bq)_N$. This is the generating function for partitions into non-overlined parts of size less than or equal to $N$. We conjugate the Ferrers diagrams for these partitions to get diagrams with at most $N$ non-overlined parts. The Ferrers diagrams for these partitions are appended to the right side of the Durfee rectangle obtained from the previous step. This generating function adds to the rank, since the largest part is increased (due to the added Ferrers diagram to the right of the Durfee rectangle); the rank is still enumerated by the exponent of $b$. This term also preserves the Durfee parameter, since there are still $N$ parts greater than or equal to $N+1$.

3. Consider $1/(tq)_{N+1}$. This is the generating function for unrestricted partitions into non-overlined parts of size at most $N+1$. The Ferrers diagrams for these partitions are placed under the partitions generated from the previous step. Each part added under the Durfee rectangle increases the number of parts by 1, which is still counted by the exponent of $t$. Note this preserves the Durfee parameter, since the largest non-overlined part added is of size $N+1$, which increase the value of $n$ and preserves the inequality (74).

4. Consider $\prod_{i=1}^{N}(1+aq^{-i}/b)$. The exponent of $q$ in a typical term from the expansion of this product has the form $-[c_1+c_2+\cdots+c_{\nu_o}]$, where $1 \leq c_1 < c_2 < \cdots < c_{\nu_o} \leq N$. Thus, this factor generates overpartitions into negative overlined parts from the set $\{\overline{1}, \overline{2}, \cdots, \overline{N}\}$. Each overlined part $c_i$, where $1 \leq i \leq \nu_o$, is now subtracted from the $c_i^{th}$ non-overlined part (or the first $N$ parts) from the partition generated by step 3.

The new overlined parts do not change the Durfee parameter of the partition, since the overlined parts replace the non-overlined parts considered in the Durfee parameter from previous steps. (In other words, the number of non-overlined parts previously counted in $s$ is now counted in $o$, preserving the inequality (74).) This factor also decreases the rank (counted by the exponent of $b$) by 1 with the addition of each overlined part (counted by the exponent of $a$).

Thus, we have overpartitions with $\nu$ parts, of which $\nu_o$ are overlined, rank $r$, Durfee parameter $N$, and constrained Durfee rectangle $N \times (N+1)$.

Next we show that every overpartition $\pi$ with Durfee parameter $N$ and constrained Durfee rectangle $N \times (N+1)$ can occur only once. In other words, given an overpartition meeting these requirements, one can recover the terms from the product $\prod_{i=1}^{N}(1 + aq^{-i}/b)$ which formed the partition given in step 4. We first find the Durfee rectangle and parameter giving an upper bound to the size of the overlined parts from the set $\{\overline{1}, \cdots, \overline{N}\}$ which could have been subtracted.

The following algorithm gives the partition of non-overlined parts which was generated by the first three steps. Step (ii) is similar to the one given in the previous combinatorial proof given in Section 3.1.

(i) Start with an overpartition with $\nu_o$ overlined parts, $\nu - \nu_o$ non-overlined parts, Durfee parameter $N$, and constrained Durfee rectangle $N \times (N+1)$. Then separate this overpartition into two tuples, one $(\nu - \nu_o)$-tuple consisting of the non-overlined parts $(n_1, \cdots, n_i, \cdots, n_{\nu-\nu_o})$ and one $\nu_o$-tuple of the overlined parts $(o_1, \cdots, o_j, \cdots, o_{\nu_o})$. Both tuples are ordered largest to smallest.

(ii) Add the overlined numbers $\{\overline{1},\cdots,\overline{N}\}$ in order to the first overlined part, $o_1$, to obtain an increasing sequence of non-overlined parts $o_1+\overline{1}, o_1+\overline{2},\cdots,o_1+\overline{N}$. One of these non-overlined parts must have been the non-overlined part that was transformed into an overlined part in step 4. This non-overlined part is uniquely determined by comparing our increasing non-overlined sequence against the decreasing sequence $n_1,\cdots,n_i,\cdots,n_{\nu-\nu_o}$. In particular, there exists an overlined part $\overline{i}\in\{\overline{1},\overline{2},\cdots,\overline{N}\}$ such that $o_1+\overline{i}$ is between the $i-1^{\underline{st}}$ non-overlined and the $i^{\underline{th}}$ non-overlined part. When this is the case, then shift down all non-overlined parts less than or equal to $n_i$ (in other words, let $n_i\mapsto n_{i+1}\mapsto n_{i+2}\mapsto\cdots$), and set $n_i=o_1+\overline{i}$. Note we now have $\nu-\nu_o+1$ non-overlined parts in our sequence.

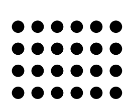
(iii) Repeat this process for all distinct overlined parts $o_2,o_3,\cdots,o_{\nu_o}$.

This algorithm recovers the overlined parts that were subtracted. Thus, finding the Durfee parameter and the constrained Durfee rectangle $N\times(N+1)$ yields the term on the right side of (76) which generates any partition from $\overline{P}$. □

Here is an example of an overpartition generated from the $5^{\underline{th}}$ term of the series:

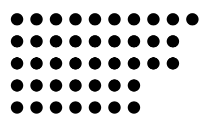
$$\frac{(1+aq^{-1}/b)(1+aq^{-2}/b)\cdots(1+aq^{-5}/b)}{(bq)_5(tq)_6}b^5t^5q^{5\times 6}.$$

We start with the factor $b^5t^5q^{5\times 6}$, which generates a Durfee rectangle of size $5\times 6$ into 5 non-overlined parts of size 6, or $\pi_1=6+6+6+6+6$. The rank of $\pi_1$ is 5 since the largest part is 6 and there are no overlined parts.

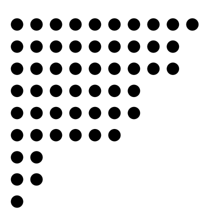


Note this term also gives the initial Durfee parameter of $N = 5$, since there are 5 non-overlined parts of size 6.

The factor $1/(bq)_5$ is the generating function for partitions with non-overlined parts of size at most 5. Conjugation gives partitions with at most 5 non-overlined parts. For example, conjugating the partition $\lambda = 5 + 3 + 3 + 1$ yields $\lambda^* = 4 + 3 + 3 + 1 + 1$. The Ferrers diagram for $\lambda^*$ is placed on the right side of the Durfee rectangle $\pi_1$. In the example, adding $\lambda^*$ to $\pi_1$ yields the partition $\pi_2 = 10 + 9 + 9 + 7 + 7$. The rank of this partition is now 9, since the largest part has increased by 4. The Durfee parameter is still 5, since there are 5 non-overlined parts greater than or equal to 6.

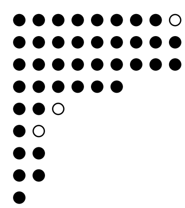


The generating function $1/(tq)_6$ gives partitions into non-overlined parts of size at most 6. An example is $\rho = 6 + 2 + 2 + 1$. The Ferrers diagram for $\rho$ is placed under $\pi_2$, which yields $\pi_3 = 10 + 9 + 9 + 7 + 7 + 6 + 2 + 2 + 1$. The rank of this partition is 9 and the Durfee parameter is 5, since there are 6 non-overlined parts parts greater than or equal to 6 and no overlined parts, giving the maximal integer such that $6 \geq 5$.



Finally, $(1 + aq^{-1}/b)(1 + aq^{-2}/b)(1 + aq^{-3}/b)(1 + aq^{-4}/b)(1 + aq^{-5}/b)$ generates overpartitions into negative distinct overlined parts from the set $\{\overline{1}, \cdots, \overline{5}\}$. Such a partition, for example $-[\overline{1} + \overline{4} + \overline{5}]$, are subtracted from $\pi_3$ to yield a partition with three overlined parts.

Define the sequence $c_i$ by $c_1 = 1$, $c_2 = 4$, and $c_5 = 5$. We always subtract the $i^{\underline{th}}$ overlined part in the list $\overline{1}$ through $\overline{5}$ from the $i^{\underline{th}}$ part of the partition, meaning the largest term that can be subtracted is $\overline{5}$ from the fifth part. Starting with $\overline{1}$ and $c_1 = 1$, we subtract $\overline{1}$ from the first part, or $n_1 = 10$, creating the overlined part $\overline{9}$ in its place. Moving onto $\overline{4}$ with $c_2 = 4$, we subtract 4 from the fourth part. Finally, $\overline{5}$ with $c_3 = 5$, we subtract 5 from the fifth part. After subtraction we have $\pi_4 = \overline{9} + 9 + 9 + \overline{3} + \overline{2} + 6 + 2 + 2 + 1$, which after reordering is $\pi_4 = \overline{9} + 9 + 9 + 6 + \overline{3} + \overline{2} + 2 + 2 + 1$. Note, the Durfee parameter is still 5 since $n$, the number of non-overlined parts from $\pi_3$, is replaced with an equal number of overlined parts, $o$. In other words, $s_{\pi_3} = 6$ and $o_{\pi_3} = 0$ becomes $s_{\pi_4} = 3$ and $o_{\pi_4} = 3$.



Conversely, we now show that given the overpartition $\pi = \overline{9} + 9 + 9 + 6 + \overline{3} + \overline{2} + 2 + 2 + 1$, we can recover which $N^{\underline{th}}$ term of the sum on the right side of (76) which generated this overpartition. This is equivalent to finding the Durfee rectangle and parameter. The Durfee parameter is 5 and the constrained Durfee rectangle has size $5 \times 6$ since there are 3 non-overlined parts larger than 5 and 3 overlined parts. Thus $N = 5$ is maximal giving $6 \geq 5$. This indicates we must use the $5^{\underline{th}}$ summand. This also gives the range of overlined parts that could have been subtracted as $\overline{1}$ to $\overline{5}$.

Now we use the algorithm described at the end of the proof. Start by separating the non-overlined parts $n_1 = 9, n_2 = 9, n_3 = 6, n_4 = 2, n_5 = 2$, and $n_6 = 1$ from the overlined parts $o_1 = \overline{9}, o_2 = \overline{3}$, and $o_3 = \overline{2}$, ordering both largest to smallest.

Start the algorithm with the first overlined part $o_1 = \overline{9}$ and consider the sequence of non-overlined parts $o_1 + \overline{1}, o_1 + \overline{2}, \cdots, o_1 + \overline{5}$. We place this sequence on the right of the non-overlined parts $n_i$.

Each of the following tables shows how the overlined parts $o_i$, $i = 1, 2, 3, \ldots$, are recovered.

| $i$ | $n_i$ | $o_1 + \bar{i}$ |
|---|---|---|
| | | $\leftarrow 10 = o_1 + \overline{1}$ |
| 1 | 9 | |
| | | $11 = o_1 + \overline{2}$ |
| 2 | 9 | |
| | | $12 = o_1 + \overline{3}$ |
| 3 | 6 | |
| | | $13 = o_1 + \overline{4}$ |
| 4 | 2 | |
| | | $14 = o_1 + \overline{5}$ |
| 5 | 2 | |
| 6 | 1 | |

| $i$ | $n_i$ | $o_2 + \bar{i}$ |
|---|---|---|
| 1 | 10 | |
| | | $5 = o_2 + \overline{2}$ |
| 2 | 9 | |
| | | $6 = o_2 + \overline{3}$ |
| 3 | 9 | |
| | | $\leftarrow 7 = o_2 + \overline{4}$ |
| 4 | 6 | |
| | | $8 = o_2 + \overline{5}$ |
| 5 | 2 | |
| 6 | 2 | |
| 7 | 1 | |

| $i$ | $n_i$ | $o_3 + \bar{i}$ |
|---|---|---|
| 1 | 10 | |
| 2 | 9 | |
| 3 | 9 | |
| 4 | 7 | |
| | | $\leftarrow 7 = o_3 + \overline{5}$ |
| 5 | 6 | |
| 6 | 2 | |
| 7 | 2 | |
| 8 | 1 | |

In the first table, it is clear that 10 was the original non-overlined part, since 10 is larger than the first non-overlined part $n_1 = 9$. Thus, we insert 10 into the list of non-overlined parts as the new $n_1$ and shift the other non-overlined parts down, letting $n_2 = 9$, $n_3 = 9$, $n_4 = 6$ and so on. This gives the new sequence $n_i$ in the second table.

Continue with the next overlined part, $o_2 = \overline{3}$, and create a list of non-overlined parts $o_2 + \bar{i}$. Note that the set of distinct overlined parts that could have been subtracted to give a non-overlined part is now $\{\overline{2}, \overline{3}, \overline{4}, \overline{5}\}$. From the second table, it is clear that $o_2 + \overline{2} = 5$ was not the original non-overlined part, since 5 does not fit between the first and second non-overlined part in the list of $n_i$. Moving to the next non-overlined part, $o_2 + \overline{3} = 6$, it is clear that this was also not the original non-overlined part, since 6 does not fit between $n_2 = 9$ and $n_3 = 9$. It is clear that $o_2 + \overline{4} = 7$ was the original non-overlined part, since 7 fits between $n_3 = 9$ and $n_4 = 6$. Thus, let $n_4 = 7$ and shift down all other non-overlined parts as $n_5 = 6$, $n_6 = 2$, etc.

Finally, consider the final overlined part, $o_3 = \overline{2}$, and create a list of non-overlined parts $o_3 + \overline{i}$. Note that $\overline{5}$ is the only remaining overlined part that could have been subtracted. From the last table, it is clear that $o_3 + \overline{5} = 7$ was the original non-overlined part, since 7 fits between $n_4 = 7$ and $n_5 = 6$. Thus insert $n_5 = 7$ and shift all other non-overlined parts down to get the original list of non-overlined parts $10 + 9 + 9 + 7 + 7 + 6 + 2 + 2 + 1$ (which is the same as $\pi_3$ above).

(18) now has two combinatorial proofs. The first, given by Andrews, used the concept of Cauchy order after a dilation is applied to the identity. The method used here employs overpartitions, avoids using dilation, and replaced Cauchy order with Durfee parameter. This method can be applied to other identities proved in Chapter 2, as we demonstrate in the last section of this chapter.

## 3.4 Overpartition Proofs of New Identities

In this section, we apply the techniques of Andrews as well as the concept of overpartitions from Corteel and Lovejoy to give combinatorial proofs of some of the identities presented in Chapter 2. We use the concept of the Durfee parameter and the constrained Durfee rectangle in later proofs.

### 3.4.1 Combinatorial Proof of Theorem 11

We begin this section with a combinatorial proof of Theorem 11 which relies only on methods from Corteel and Lovejoy. First make the change of variables $a \mapsto -a$ and $t \mapsto tq$ to obtain:

$$F(-a, b; tq) = \sum_{n=0}^{\infty} \frac{(-a)_{2n}(-atq/b)_n}{(bq)_{2n}(tq)_{n+1}}(1 + atq^{3n+1})b^n t^{2n} q^{2n(n+1)} \qquad (77)$$
$$+ \sum_{n=0}^{\infty} \frac{(-a)_{2n+1}(-atq/b)_{n+1}}{(bq)_{2n+1}(tq)_{n+1}} b^{n+1} t^{2n+1} q^{(2n+1)(n+2)}.$$

Since we have shown the involution is equivalent to conjugation, the proof of Corollary 12 results from conjugating the Ferrers diagrams given throughout the proof of Theorem 11.

*Combinatorial Proof of Theorem 11.*

By Proposition 26, the coefficient of $a^{\nu_o} b^r t^\nu q^n$ on the left side of (77) enumerates overpartitions of $n$ into $\nu$ parts, of which $\nu_o$ are overlined, and rank $r$.

Consider the $N^{\underline{th}}$ term of the first sum on the right side of (77):

$$\frac{(-a)_{2N}(-atq/b)_N}{(bq)_{2N}(tq)_{N+1}}(1 + atq^{3N+1})b^N t^{2N} q^{2N(N+1)}.$$

We show this term enumerates overpartitions of $n$ with $\nu$ parts, $\nu_o$ overlined parts, rank $r$, and Durfee rectangle of size $2N \times (N + 1)$ such that $N$ gives the largest rectangle contained in the Ferrers diagrams of these overpartitions. Note this enumeration gives a refinement of the left side. This set of overpartitions is constructed in four steps using Ferrers diagrams.

1. The factor $b^N t^{2N} q^{2N(N+1)}$ generates a Durfee rectangle of size $2N \times (N+1)$, which represents $2N$ parts (counted by the exponent of $t$) of size $N+1$. Note the rank is $N$, which is counted by the exponent of $b$.

2. Consider $(-a)_{2N}/(bq)_{2N}$. This is the generating function for overpartitions into $2N$ non-negative parts. The factor $1/(bq)_{2N}$ generates partition $\lambda$ into parts of size at most $2N$. Conjugate these partitions and append the Ferrers diagrams to the right side of the Durfee rectangle from the previous step. This yields partitions into $2N$ parts of size at least $N+1$ where the rank is still enumerated by the exponent of $b$ since the largest part was increased. The factor $(-a)_{2N}$ generates partitions $\mu$ into distinct parts of size ranging from 0 to $2N-1$. We apply these partitions to the Durfee rectangle using the algorithm described in Proposition 26. This yields overpartitions into $2N$ parts of size at least $N+1$. The number of overlined parts of these overpartitions adds to the number of overlined parts from Durfee rectangle, which is counted by the exponent of $a$ (which was 0 at the end of the previous step), and similar for the rank with the exponent of $b$.

3. Consider $(-atq/b)_N/(tq)_{N+1}$. This is the generating function for overpartitions into parts of size at most $N+1$. The Ferrers diagrams for such overpartitions are placed under the diagrams from the previous step. Note, if $N+1$ occurs, it cannot be an overlined part. This factor creates overlined parts and for each added overlined part, the rank decreases by 1, as does the exponent of $b$.

4. Consider $(1+atq^{3N+1})$, which we write as $(1+atq^{2N+(N+1)})$. This factor allows for a possible overlined part of size $N+1$ to be placed in the Ferrers diagram between the Durfee rectangle and above the partition generated from the second step. If this part

occurs, increase the first $2N$ parts by 1. (Note that the largest $\eta$ such that the Durfee rectangle of size $2\eta \times (\eta+1)$ will fit in the Ferrers diagram is still $\eta = N$).

Now we consider the $N^{\underline{th}}$ term of the second sum on the right side of (77):

$$\frac{(-a)_{2N+1}(-atq/b)_{N+1}}{(bq)_{2N+1}(tq)_{N+1}} b^{N+1} t^{2N+1} q^{(2N+1)(N+2)}.$$

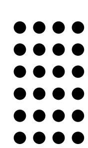
The interpretation for this sum is similar to the first sum, except now the Durfee rectangle is of size $(2N+1) \times (N+2)$, the overpartition added on to the right has at most $2N+1$ non-negative parts, and the overpartition added below is into parts of size at most $N+1$. Thus we have an overpartition with $\nu$ parts, of which $\nu_o$ are overlined, rank $r$, and Durfee rectangle $(2N+1) \times (N+2)$.

The second summand covers the case where there is a dot in the $N+2^{\underline{nd}}$ column and $2N+1^{\underline{st}}$ row, which is not covered in the first summand. Together, these form overpartitions of $n$ into $\nu$ parts, $\nu_o$ overlined parts, and rank $r$. □

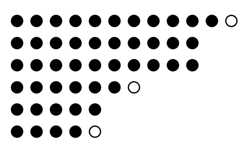
We now consider an example where $N = 3$; the $3^{\underline{rd}}$ term of the first sum on the right side of (77) is:

$$\frac{(-a)_6(-atq/b)_3}{(bq)_6(tq)_4}(1 + atq^{10}) b^3 t^6 q^{6\times 4}.$$

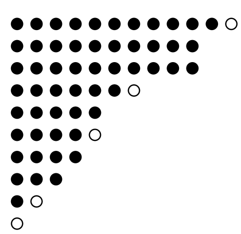
The factor $b^3 t^6 q^{6\times 4}$ generates a Durfee rectangle of size $6 \times 4$ and rank 3, giving $\pi_1 = 4+4+4+4+4+4$ with Ferrers diagram below.



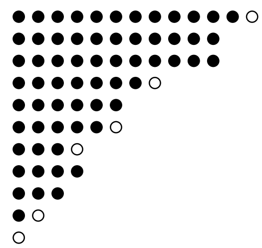
We give an example of an overpartition into at most 6 parts arising from this factor $(-a)_6/(bq)_6$. First consider $1/(bq)_6$, which generates partitions into parts of size at most 6. These partitions are conjugated and placed to the right of the Durfee rectangle. For example, $\lambda = 6+4+4+3+1+1$ is conjugated to give $\lambda^* = 6+4+4+3+1+1$. Then $\lambda^*$ placed to the right of the $\pi_1$ to get $\pi_2 = 10+8+8+7+5+5$. The numerator $(-a)_6$ generates partitions into distinct non-negative parts of size at most 5. An example is $\mu = 5+3+0$. Following the algorithm from Proposition 26, take the largest part of $\mu$, or $\mu_1$, and add 1 to the first $\mu_1$ parts of $\pi_2$, overlining the $\mu_1 + 1^{\underline{st}}$ part of $\pi_2$. Continue this algorithm for all parts of $\mu$. Using $\pi_2$ and $\mu$, we have an overpartition into at most 6 parts, or $\pi_3 = \overline{12}+10+10+\overline{8}+6+\overline{5}$. $\pi_3$ now has 3 overlined parts and rank 9.



The factor $(-atq/b)_3/(tq)_4$ generates overpartitions into parts of size at most 4. An example is $\rho = 4+3+\overline{2}+\overline{1}$. Placing the Ferrers diagram for $\rho$ under the diagram for $\pi_3$ gives $\pi_4 = \overline{12}+10+10+\overline{8}+6+\overline{5}+4+3+\overline{2}+\overline{1}$. Now $\pi_4$ has 10 parts, 5 overlined parts, and rank 7. The addition of the two overlined parts decreases the rank by 2.



The factor $(1+atq^{10}) = (1+atq^{4+6})$ allows for a possible overlined part of size 4. Since there is not an overlined part of size 4, an overlined 4 is added and the first 6 parts are increased by 1. In our example, we now have $\pi_5 = \overline{13} + 11 + 11 + \overline{9} + 7 + \overline{6} + \overline{4} + 4 + 3 + \overline{2} + \overline{1}$. Thus an overpartition with 11 parts, 5 overlined parts, and rank 5 is obtained.



In Chapter 4, we consider a more general identity derived by iterating $(\alpha\beta)^k\tau^l$.

### 3.4.2 Combinatorial Proof of Theorem 9

We now give a combinatorial proof of Theorem 9. After the change of variables $a \mapsto -aq$ and the $q$-shift $t \mapsto tq$ as well as writing the product $(-bq/a)_n$ as in (75), the identity takes the form

$$\sum_{n=0}^{\infty} \frac{(-a)_n}{(bq)_n} t^n q^n = \sum_{n=0}^{\infty} \frac{(-a)_n \prod_{i=1}^{n}(1 + aq^{-i}/b)}{(tq)_{n+1}(bq)_{2n}} (1 + atq^{2n+1})(bt^2)^n q^{2n(n+1)} + \sum_{n=0}^{\infty} \frac{(-a)_{n+1} \prod_{i=1}^{n}(1 + aq^{-i}/b)}{(tq)_{n+1}(bq)_{2n+1}} b^{n+1} t^{2n+1} q^{(2n+1)(n+2)}.$$

Break up the factor $(1 + atq^{2n+1})$ in the first sum. The identity is proved in the following form:

$$\sum_{n=0}^{\infty} \frac{(-a)_n}{(bq)_n} t^n q^n = \sum_{n=0}^{\infty} \frac{(-a)_n \prod_{i=1}^{n}(1 + aq^{-i}/b)}{(tq)_{n+1}(bq)_{2n}} (bt^2)^n q^{2n(n+1)} \tag{78}$$

$$+\sum_{n=0}^{\infty}\frac{(-a)_n\prod_{i=1}^{n}(1+aq^{-i}/b)}{(tq)_{n+1}(bq)_{2n}}ab^nt^{2n+1}q^{(2n+1)(n+1)+n}$$
$$+\sum_{n=0}^{\infty}\frac{(-a)_{n+1}\prod_{i=1}^{n}(1+aq^{-i}/b)}{(tq)_{n+1}(bq)_{2n+1}}b^{n+1}t^{2n+1}q^{(2n+1)(n+2)}.$$

*Combinatorial Proof of Theorem 9.*

By Proposition 26, the coefficient of $q^n a^{\nu_o} b^r t^\nu$ on the left side of (78) enumerates overpartitions of $n$ into $\nu$ parts, of which $\nu_o$ are overlined, and rank $r$.

Consider the $N^{\underline{th}}$ term of the first sum on the right side of (78):

$$\frac{(-a)_N\prod_{i=1}^{N}(1+aq^{-i}/b)}{(tq)_{N+1}(bq)_{2N}}(bt^2)^Nq^{2N(N+1)}.$$

We show this term enumerates overpartitions of $n$ with $\nu$ parts, $\nu_o$ overlined parts, rank $r$, Durfee parameter $N$, and constrained Durfee rectangle $2N\times(N+1)$. Note this enumeration gives a refinement of the left side of (78). This set of overpartitions is constructed in four steps using Ferrers diagrams.

1. The factor $b^Nt^{2N}q^{2N(N+1)}$ generates a Durfee rectangle of size $2N\times(N+1)$, which represents $2N$ non-overlined parts (enumerated by the exponent of $t$) of size $N+1$, rank $N$ (enumerated by the exponent of $b$), and no overlined parts. Note the Durfee parameter is $N$ since there are $2N$ non-overlined parts greater than or equal to $N+1$.

2. Consider $(-a)_N/(bq)_{2N}$. The factor $1/(bq)_{2N}$ generates partitions $\lambda$ into non-overlined parts less than or equal to $2N$, which are conjugated to give partitions into at most $2N$ parts. We append the Ferrers diagrams of these partitions to the right side of the Durfee rectangle from the previous step, obtaining a partition into $2N$ non-overlined

parts of size greater than or equal to $N+1$. Next, multiply by the generating function $(-a)_N$, which generates distinct non-negative partitions $\mu$ with parts strictly less than $N$. Note the multiplication above uses the same algorithm discussed in the proof of Proposition 26. This generates overpartitions where, at most, the first $N$ parts of the Ferrers diagrams are overlined, leaving at least the remaining $N$ parts non-overlined. This is because the largest part from $(-a)_N$ is at most $N-1$. (Again, the overlined part $\overline{0}$ can occur corresponding to the factor $(1+a)$ in the product $(-a)_N$.)
The Durfee parameter is preserved, since the factor $(-a)_N$ replaces the non-overlined parts counted from the factor $1/(bq)_{2N}$. Thus the inequality (74) is preserved, since the sum on the left side of the inequality does not change.

3. Consider $1/(tq)_{N+1}$. This is the generating function for partitions into non-overlined parts of size at most $N+1$. Note the exponent of $t$ enumerates the number of parts. The Ferrers diagrams for these partitions are placed under the diagrams from step 2. This step preserves the Durfee parameter, since the parts added are of size at most $N+1$, thus preserving the inequality (74).

We now have partitions into at least $2N$ parts greater than or equal to $N+1$, where only the first $N$ parts can be overlined.

4. Consider $\prod_{i=1}^{N}(1+aq^{-i}/b)$. The exponent of $q$ in a typical term from the expansion of this product has the form $-[c_1+c_2+\cdots+c_k]$, where $1 \leq c_1 < c_2 < \cdots < c_k \leq N$ and $k \leq \nu_o$ (since the first $N$ parts could have been overlined in step 2). Thus, this factor generates overpartitions into negative overlined parts from the set $\{\overline{1}, \overline{2}, \cdots, \overline{N}\}$. Each overlined part $c_i$, where $1 \leq i \leq \nu_o$, is now subtracted from the $c_i + N^{\underline{th}}$ non-overlined part from the Ferrers diagram generated by step 2. Thus the subtraction may apply to the $N+1^{\underline{st}}$ through the $2N^{\underline{th}}$ part. The new overlined parts do not change the Durfee

parameter of the partition, since the overlined parts replace the non-overlined parts considered in the Durfee parameter from previous steps. (In other words, quantity previously counted in $s$ is now counted in $o$, preserving the desired inequality.) This factor also decreases the rank (the exponent of $b$) by one with the addition of each overlined part (the exponent of $a$).

Now consider the $N^{\underline{th}}$ term of the second sum on the right side of (78):

$$\frac{(-a)_N \prod_{i=1}^{N}(1+aq^{-i}/b)}{(tq)_{N+1}(bq)_{2N}} ab^N t^{2N+1} q^{(2N+1)(N+1)+N}.$$

Much like the first sum, this term generates the same overpartitions as above and follows the same the four steps illustrated above, with the insertion of a new step between the third and fourth. The factor $atq^{2N+1}$ generates an overlined part of size $N+1$ and increases the first $N$ parts by 1. The increase of the first $N$ parts also increases the Durfee rectangle to $(2N+1)\times(N+1)$. The Durfee parameter is still $N$, since the size of the Durfee rectangle increased. After this step is applied, use the same subtraction argument outlined in step 4 in the first sum.

Now consider the $N^{\underline{th}}$ term of the third sum on the right side of (78):

$$\frac{(-a)_{N+1} \prod_{i=1}^{N}(1+aq^{-i}/b)}{(tq)_{N+1}(bq)_{2N+1}} b^{N+1} t^{2N+1} q^{(2N+1)(N+2)}.$$

The interpretation for this sum is similar to the first sum except now the Durfee rectangle is of size $(2N+1)\times(N+2)$, the partition added to the right has at most $2N+1$ parts, the partition added below has part of size at most $N+1$, and the subtraction argument is applied to the $N+2^{\underline{nd}}$ through $2N+1^{\underline{st}}$ parts. Thus, we have an overpartition with $\nu$ parts, of which $\nu_o$ are overlined, rank $r$, Durfee rectangle $(2N+1)\times(N+2)$, and Durfee parameter $N$.

Next it is shown that every overpartition in the set $\overline{P}$ with Durfee parameter $N$ and constrained Durfee rectangle $2N \times (N+1)$ occurs only once. Note the argument for the constrained Durfee rectangle $(2N+1)\times(N+2)$ is similar. Given an overpartition enumerated by $\overline{p}(\nu, \nu_0, r, n)$, we show we can recover the factors from $\prod_{i=1}^{N}(1+aq^{-i}/b)$ which formed the overpartition described in step 3. In other words, we must first find the constrained Durfee rectangle and parameter. The Durfee parameter gives the upper bound to the overlined parts that were subtracted, which ranges from $\overline{1}$ to $\overline{N}$.

The following algorithm gives the overpartition which was generated by the first three steps.

(i) Start with an overpartition with $\nu_o$ overlined parts, $\nu - \nu_o$ non-overlined parts, Durfee parameter $N$, and constrained Durfee rectangle $2N \times (N+1)$. Note that the first $N$ parts could be overlined, and of those parts, $k$ are overlined and $l$ are non-overlined. Separate this partition into two tuples, one $(\nu-\nu_o)$-tuple consisting of the non-overlined parts $(n_1, \cdots, n_i, \cdots, n_{\nu-\nu_o})$, and one $\nu_o$-tuple of the distinct overlined parts $(o_1, \cdots, o_j, \cdots, o_{\nu_o})$. Here, both tuples are ordered largest to smallest.

(ii) Add the overlined numbers $\{\overline{1}, \cdots, \overline{N}\}$ in order to the first overlined part which is not in the first $N$ parts, $o_{k+1}$, to obtain an increasing sequence of non-overlined parts $o_{k+1}+\overline{1}, o_{k+1}+\overline{2}, \cdots, o_{k+1}+\overline{N}$. One of these non-overlined parts must have been the non-overlined part that was transformed into an overlined part in step 4. This non-overlined part is uniquely determined by comparing our increasing non-overlined sequence against the decreasing sequence $n_1, \cdots, n_i, \cdots, n_{\nu-\nu_o}$. In particular, there exists an overlined part $\overline{i} \in \{\overline{1}, \overline{2}, \cdots, \overline{N}\}$ such that $o_{k+1}+\overline{i}$ is between the $l+i-1^{\underline{st}}$

non-overlined and the $l + i^{\underline{th}}$ non-overlined part. When this is the case, shift down all non-overlined parts less than or equal to $n_{l+i}$ (in other words, let $n_{l+i} \mapsto n_{l+i+1} \mapsto n_{l+i+2} \mapsto \cdots$), and let $o_{k+1} + \bar{i}$ be the $l + i^{\underline{th}}$ non-overlined part. Note we now have $\nu - k + 1$ non-overlined parts in our sequence. (This step differs from the proof of the previous, since subtraction was applied to the last $N$ parts of the constrained Durfee rectangle.)

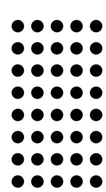
(iii) Repeat this process for all distinct overlined parts $o_{k+2}, o_{k+3}, \cdots, o_{\nu_o}$.

This algorithm recovers the overlined parts that were subtracted as well as the original overpartition generated by the first three steps. Thus we have the existence of a generating function for any overpartition enumerated by $\overline{p}(\nu, \nu_o, r, n)$ with Durfee parameter $N$ and constrained Durfee rectangle $2N \times (N + 1)$. □

Here is an example of an overpartition generated from the $4^{\underline{th}}$ term of the first sum:

$$\frac{(-a)_4(1 + aq^{-1}/b)(1 + aq^{-2}/b)(1 + aq^{-3}/b)(1 + aq^{-4}/b)}{(tq)_5(bq)_8}(bt^2)^4 q^{8\times 5}.$$

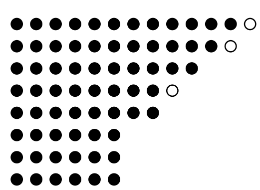
The factor $b^4t^8q^{8\times 5}$ generates a Durfee rectangle with 8 non-overlined parts of size 5, giving $\pi_1 = 5+5+5+5+5+5+5+5$. Note this factor also gives the initial Durfee parameter of $N = 4$, since there are 8 parts of size 5.



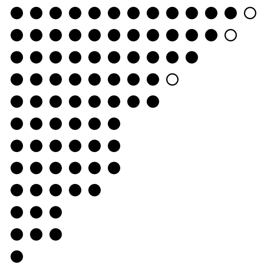
The generating function $(-a)_4/(bq)_8$ can be interpreted using the algorithm stated in the proof of Proposition 26. Take the parts generated by $1/(bq)_8$, or partitions with parts less than or equal to 8 and conjugate to get partitions into at most 8 parts. For example, conjugating $\lambda = 8+5+5+4+2+2$ gives the partition $\lambda^* = 6+6+4+4+3+1+1+1$. Appending the Ferrers diagram of $\lambda^*$ to $\pi_1$ above yields $\pi_2 = 11+11+9+9+8+6+6+6$. Now consider $(-a)_4$, which generates partitions into distinct non-negative parts less than or equal to 3. For example, $\mu = 3+1+0$. We follow the algorithm outlined in the proof by taking the largest part of $\mu$, say $\mu_1$, and adding 1 to the first $\mu_1$ parts of $\pi_2$, overlining the $\mu_1 + 1^{\underline{st}}$ part of $\pi_2$. Repeat this for all parts of $\mu$ from largest to smallest. In our example,

$$(11+11+9+9+8+6+6+6, 3+1+0) = (\overline{13}+\overline{12}+10+\overline{9}+8+6+6+6).$$

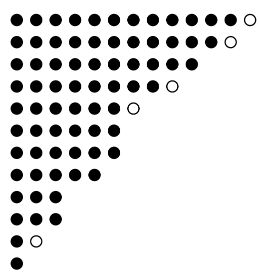
Thus, $\pi_3 = \overline{13}+\overline{12}+10+\overline{9}+8+6+6+6$. Note that only the first four parts have the possibility of being overlined. This preserves the Durfee parameter, since there are still 8 non-overlined parts greater than or equal to 5, giving $N = 4$ as maximal in the inequality (74).



The factor $1/(tq)_5$ generates partitions with parts of size at most 5, for example $\rho = 5+3+3+1$. The Ferrers diagram for this partition is placed under the diagram for $\pi_3$. Placing $\rho$ under $\pi_3$ yields $\pi_4 = \overline{13}+\overline{12}+10+\overline{9}+8+6+6+6+5+3+3+1$. Note this preserves the Durfee parameter since the non-overlined parts added are less than or equal to 5 and do not increase the value of $s$ in the inequality (74).



The factor $(1+aq^{-1}/b)(1+aq^{-2}/b)(1+aq^{-3}/b)(1+aq^{-4}/b)$ creates additional overlined parts by a similar subtraction argument as the previous example. Say we want to subtract $\overline{1}$ and $\overline{4}$. Then, subtract $\overline{1}$ from the fifth part and $\overline{4}$ from the eighth part following the algorithm described in the proof. This generates the sequence $c_i$ as $c_1 = 1$ and $c_2 = 4$. In our example, we have $\pi_5 = \overline{13}+\overline{12}+10+\overline{9}+\overline{7}+6+6+\overline{2}+5+3+3+1$, or with reordering, $\pi_5 = \overline{13}+\overline{12}+10+\overline{9}+\overline{7}+6+6+5+3+3+\overline{2}+1$. This preserves the Durfee parameter since non-overlined parts counted in $s$ are replaced with overlined parts counted in $o$, preserving the inequality (74).



We now show that given the overpartition $\pi = \overline{13}+\overline{12}+10+\overline{9}+\overline{7}+6+6+5+3+3+\overline{2}+1$, we can recover which $N^{\underline{th}}$ term of the sum on the right side of (78) generated this overpartition. This is equivalent to finding the Durfee parameter and constrained Durfee rectangle. The constrained Durfee rectangle has size $8 \times 5$ and the Durfee parameter is 4 since there are 4 non-overlined parts greater than or equal to 5 and 5 overlined parts. Thus $N = 4$ is maximal giving $9 \geq 4$. This indicates we must use the $4^{\underline{th}}$ summand and also gives the set of overlined parts that could have been subtracted: $\{\overline{1}, \overline{2}, \overline{3}, \overline{4}\}$.

Now use the algorithm described at the end of the proof. We begin by noting that the first four parts have the possibility of being overlined. Then separate the non-overlined parts $n_1 = 10, n_2 = 6, n_3 = 6, n_4 = 5, n_5 = 3, n_6 = 3$ and $n_7 = 1$ from the overlined parts $o_1 = \overline{13}, o_2 = \overline{12}, o_3 = \overline{9}, o_4 = \overline{7}$, and $o_5 = \overline{2}$, ordering both largest to smallest. Here, we have $o_1, o_2$, and $o_3$ as overlined parts from before step 4. Thus, we do not consider those parts in the algorithm. Also, $n_1$ is within the first four parts, thus $k = 3$ and $l = 1$.

Each of the following tables shows how the overlined parts $o_i$, $i = 1, 2, \ldots$ are recovered.

| $i$ | $n_i$ | $o_4 + \overline{i}$ |
|---|---|---|
| 1 | 10 | |
| | | $\leftarrow 8 = o_4 + \overline{1}$ |
| 2 | 6 | |
| | | $9 = o_4 + \overline{2}$ |
| 3 | 6 | |
| | | $10 = o_4 + \overline{3}$ |
| 4 | 5 | |
| | | $11 = o_4 + \overline{4}$ |
| 5 | 3 | |
| 6 | 3 | |
| 7 | 1 | |

| $i$ | $n_i$ | $o_5 + \overline{i}$ |
|---|---|---|
| 1 | 10 | |
| 2 | 8 | |
| | | $4 = o_5 + \overline{2}$ |
| 3 | 6 | |
| | | $5 = o_4 + \overline{3}$ |
| 4 | 6 | |
| | | $\leftarrow 6 = o_4 + \overline{4}$ |
| 5 | 5 | |
| 6 | 3 | |
| 7 | 3 | |
| 8 | 1 | |

We start the algorithm with the first overlined part that was not in the first four parts, $o_4 = \overline{7}$, and consider the sequence of non-overlined parts $o_4 + \overline{1}, o_4 + \overline{2}, o_4 + \overline{3}, o_4 + \overline{4}$. We place this sequence on the right of the non-overlined parts $n_i$.

From the table on the left, it is clear that 8 was the original non-overlined part, since 8 fits between $n_l = n_1 = 10$ and $n_{l+1} = n_2 = 6$. Thus we insert 8 into the list of non-overlined parts as the new $n_2$ and shift the other non-overlined parts down, letting $n_3 = 6$, $n_4 = 6$, $n_5 = 5$ and so on.

We continue with the last overlined part, $o_5 = \overline{2}$, and create a list of non-overlined parts $o_5 + \overline{i}$. Note that the set of distinct overlined parts that could have been subtracted to give a non-overlined part is now $\{\overline{2}, \overline{3}, \overline{4}\}$.

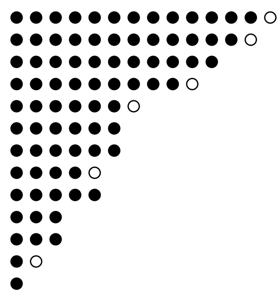
From the table on the right, it is clear that $o_5 + \overline{2} = 4$ was not the original non-overlined part, since 4 does not fit between the second and third non-overlined part in the list of $n_i$. Moving to the next non-overlined part $o_5 + \overline{3} = 5$, it is clear that this was also not the original non-overlined part, since 5 does not fit between $n_3 = 6$ and $n_4 = 6$. It is clear that $o_5 + \overline{4} = 6$ was the original non-overlined part, since 6 fits between $n_4 = 6$ and $n_5 = 5$. Thus, let $n_5 = 6$ and shift down all other non-overlined parts as $n_6 = 5$, $n_7 = 3$, etc. Thus we have recovered the overpartition which was generated from the third step as $\pi_4$, or $\overline{13} + \overline{12} + 10 + \overline{9} + 8 + 6 + 6 + 6 + 5 + 3 + 3 + 1$.

As another example, consider the $4^{\underline{th}}$ term of the second sum:

$$\frac{(-a)_4(1 + aq^{-1}/b)(1 + aq^{-2}/b)(1 + aq^{-3}/b)(1 + aq^{-4}/b)}{(tq)_5(bq)_8} ab^4t^9q^{(8\times5)+9}.$$

We use the overpartition $\pi_4 = \overline{13} + \overline{12} + 10 + \overline{9} + 8 + 6 + 6 + 6 + 5 + 3 + 3 + 1$ generated in step 3 in the previous example to illustrate the impact of the factor $atq^9$ in this overpartition. This factor creates an overlined part $\overline{5}$ and increases the first four parts by 1, giving $\overline{14} + \overline{13} + 11 + \overline{10} + 8 + 6 + 6 + 6 + \overline{5} + 5 + 3 + 3 + 1$. Note, inserting this part and increasing the first four parts of this overpartition alters the constrained Durfee rectangle to be $9 \times 5$ (or $(2N + 1) \times (N + 1)$), since there are 4 non-overlined parts larger than 5 and there are 6 overlined parts, giving the Durfee parameter $N = 4$ as the maximal for the inequality (74).

Now, we apply the same subtraction argument from step 4 in the first sum and subtract $\overline{1}$ and $\overline{4}$, yielding $\overline{14}+\overline{13}+11+\overline{10}+\overline{7}+6+6+\overline{2}+\overline{5}+5+3+3+1$, which after reordering gives $\overline{14}+\overline{13}+11+\overline{10}+\overline{7}+6+6+\overline{5}+5+3+3+\overline{2}+1$



### 3.4.3 Combinatorial Proof of Theorem 14

The final combinatorial proof given in this chapter is of Theorem 14:

$$F(a,b;t)=\sum_{n=0}^{\infty}\frac{(a)_n(at/b)_n(bq/a)_n}{(bq)_{2n}(t)_{2n+1}}(1-atq^{3n})(-abt^2)^n q^{\frac{n(7n-1)}{2}} + bt\sum_{n=0}^{\infty}\frac{(a)_{n+1}(at/b)_{n+1}(bq/a)_n}{(bq)_{2n+1}(t)_{2n+2}}(-abt^2)^n q^{\frac{7n^2+7n+2}{2}}.$$

Apply the same change of variables as before, use the factor $(1+atq^{3n+1})$ to split the first sum into two sums, and rewrite $(-bq/a)_n$ as in (75) to obtain

$$\sum_{n=0}^{\infty}\frac{(-a)_n}{(bq)_n}t^nq^n = \sum_{n=0}^{\infty}\frac{(-a)_n(-atq/b)_n\prod_{i=1}^{n}(1+aq^{-i}/b)}{(bq)_{2n}(tq)_{2n+1}}b^{2n}t^{2n}q^{2n(2n+1)} + \sum_{n=0}^{\infty}\frac{(-a)_n(-atq/b)_n\prod_{i=1}^{n}(1+aq^{-i}/b)}{(bq)_{2n}(tq)_{2n+1}}ab^{2n}t^{2n+1}q^{(2n+1)^2+n} + \sum_{n=0}^{\infty}\frac{(-a)_{n+1}(-atq/b)_{n+1}\prod_{i=1}^{n}(1+aq^{-i}/b)}{(bq)_{2n+1}(tq)_{2n+2}}b^{2n+1}t^{2n+1}q^{(2n+1)(2n+2)}. \tag{79}$$

The proof of this particular identity involves aspects from all our previous combinatorial proofs. Because of this, please see previous proofs for full details of similar steps.

*Combinatorial Proof of Theorem 14.*

By Proposition 26, the coefficient of $q^n a^{\nu_o} b^r t^\nu$ on the left side of (79) enumerates overpartitions of $n$ into $\nu$ parts, of which $\nu_o$ are overlined, and rank $r$.

Consider the $N^{\underline{th}}$ term of the first sum on the right side of (79):

$$\frac{(-a)_N(-atq/b)_N \prod_{i=1}^{N}(1+aq^{-i}/b)}{(bq)_{2N}(tq)_{2N+1}} b^{2N}t^{2N}q^{2N(2N+1)}.$$

We show this term enumerates overpartitions of $n$ into $\nu$ parts, of which $\nu_o$ are overlined, rank $r$, Durfee parameter $N$, and constrained Durfee rectangle $2N \times (2N+1)$. Note this enumeration gives a refinement of the left side. This set of overpartitions is constructed in four steps using Ferrers diagrams.

1. The factor $b^{2N}t^{2N}q^{2N(2N+1)}$ generates a Durfee rectangle of size $2N \times (2N+1)$ with $2N$ parts of size $2N+1$, giving a Durfee parameter of $N$.

2. The factor $(-a)_N/(bq)_{2N}$ generates overpartitions using the same method as in the proof of Theorem 9. The overpartitions generated have at most $2N$ parts where only the first $N$ are overlined and are attached to the right side of the Durfee rectangle from step 1.

3. Use step 3 from the proof of Theorem 9 for $(-atq/b)_N/(tq)_{2N+1}$ to give the interpretation as overpartitions into non-overlined parts of size at most $2N+1$ and overlined parts of size at most $N$. Note the Durfee parameter is $N$, since there are at least $N$

non-overlined parts greater than or equal to $2N+1$, and there are at most $N$ overlined parts strictly larger than $N$, giving maximal $N$ in the inequality (74).

4. This proof is similar to the combinatorial proof of Theorem 9 since the factor $\prod_{i=1}^{N}(1+aq^{-i}/b)$ applies the subtraction of overlined parts from the set $\{\overline{1},\overline{2},\cdots,\overline{N}\}$ to the $N+1^{\underline{st}}$ through $2N^{\underline{th}}$ parts. This factor generates overlined parts greater than or equal to $N+1$. The sequence $c_i$ from the proof of Theorem 9 can be used here with the same subtraction argument. Note this does not alter the Durfee parameter $N$.

The factor $atq^{3N+1}$ in the $N^{\underline{th}}$ term of the second sum ,

$$\frac{(-a)_N(-atq/b)_N(1+aq^{-1}/b)(1+aq^{-2}/b)\cdots(1+aq^{-N}/b)}{(bq)_{2N}(tq)_{2N+1}}ab^{2N}t^{2N+1}q^{2N(2N+1)+3N+1},$$

generates the same overpartitions as the first sum and follows the same four steps given above with the insertion of a new step between steps 3 and 4. The factor $atq^{3N+1}$ creates an overlined part of size $N+1$ and increases the first $2N$ parts by 1. Note, this step must occur before subtraction, since this gives a Durfee parameter $N$ with a new constrained Durfee rectangle of size $(2N+1)\times(2N+1)$.

Now consider the $N^{\underline{th}}$ term of the third sum on the right side of (79):

$$\frac{(-a)_{N+1}(-atq/b)_{N+1}(1+aq^{-1}/b)\cdots(1+aq^{-N}/b)}{(bq)_{2N+1}(tq)_{2N+2}}b^{2N+1}t^{2N+1}q^{(2N+1)(2N+2)}.$$

The interpretation for this sum is similar to the first sum except now the constrained Durfee rectangle is of size $(2N+1)\times(2N+2)$, the overpartitions added to the right has at most $2N+1$ parts where the first $N+1$ are overlined, the overpartitions placed below has overlined parts as large as $N+1$ and non-overlined parts as large as $2N+2$, and the subtraction argument is applied through the $N+2^{\underline{nd}}$ through the $2N+1^{\underline{st}}$ parts. Much like the first sum, the Durfee parameter is preserved throughout each step. Thus we have an overpartition with $\nu$ parts, of which $\nu_o$ are overlined, rank $r$, Durfee parameter $N$, and constrained Durfee rectangle $(2N+1)\times(2N+2)$.

Next it is shown that every overpartition in the set $\overline{P}$ with Durfee parameter $N$ and constrained Durfee rectangle $2N\times(2N+1)$ occurs only once. Note the argument for constrained Durfee rectangle $(2N+1)\times(2N+2)$ is the similar. Given an overpartition enumerated by $\overline{p}(\nu,\nu_0,r,n)$, we show we can recover the factors from $\prod_{i=1}^{N}(1+aq^{-i}/b)$ which formed the overpartition described in step 3. In other words, we must first find the Durfee parameter and constrained Durfee rectangle. After finding the Durfee parameter, we have the upper bound of overlined parts that were subtracted to give overlined parts, which would be from the set $\{\overline{1},\cdots,\overline{N}\}$.

The following algorithm gives the overpartition which was generated by the first three steps.

(i) Start with an overpartition with $\nu_o$ overlined parts, $\nu-\nu_o$ non-overlined parts, Durfee parameter $N$, and constrained Durfee rectangle $2N\times(2N+1)$. Let $k_1$ denote the number of overlined parts in the first $N$ parts and let $l$ be the number of non-overlined parts in the first $N$ parts. Also note that parts of size less than $N$ can also be overlined; let $k_2$ be the number of overlined parts of size less than $N$. Separate this partition into two tuples, one $(\nu-\nu_o)$-tuple consisting of the non-overlined parts $(n_1,\cdots,n_i,\cdots,n_{\nu-\nu_o})$ and one $\nu_o$-tuple of the distinct overlined parts $(o_1,\cdots,o_j,\cdots,o_{\nu_o})$. Here, both tuples are ordered largest to smallest.

(ii) Add the overlined numbers $\{\overline{1},\cdots,\overline{N}\}$ in order to the first overlined part which is not in the first $N$ parts, $o_{k_1+1}$, to obtain an increasing sequence of non-overlined parts $o_{k_1+1}+\overline{1}, o_{k_1+1}+\overline{2},\cdots,o_{k_1+1}+\overline{N}$. One of these non-overlined parts must have been the non-overlined part that was transformed into an overlined part in step 4. This non-overlined part is uniquely determined by comparing our increasing non-overlined sequence against the decreasing sequence $n_1, \cdots, n_i, \cdots, n_{\nu-\nu_o}$. In particular, there exists an overlined part $\bar{i} \in \{\overline{1}, \overline{2}, \cdots, \overline{N}\}$ such that $o_{k_1+1} + \bar{i}$ is between the $l+i-1^{\underline{st}}$ non-overlined and the $l+i^{\underline{th}}$ non-overlined part. When this is the case, then shift down all non-overlined parts less than or equal to $n_{l+i}$ (in other words, let $n_{l+i} \mapsto n_{l+i+1} \mapsto n_{l+i+2} \mapsto \cdots$) and set $n_{l+i} = o_{k_1+1} + \bar{i}$. Note we now have $\nu - k_1 + 1$ non-overlined parts in our sequence. (This step differs from the proof of the previous, since subtraction was applied to the last $N$ parts of the constrained Durfee rectangle.)

(iii) Repeat this process for all distinct overlined parts $o_{k_1+2}, o_{k_1+3}, \cdots, o_{\nu_o-k_2}$.

This algorithm recovers the overlined parts that were subtracted as well as the original overpartition generated by the first three steps. (Note the algorithm given above is similar to the one from the proof of Theorem 9. The difference between the two is the existence of overlined parts of size less than $N$.) Thus we have the existence of a generating function for any overpartition in the set $\overline{P}$ with Durfee parameter $N$ and constrained Durfee rectangle $2N \times (2N+1)$. □

Here is an example of an overpartition generated from the $4^{\underline{th}}$ term the first sum:

$$\frac{(-a)_4(-atq/b)_4(1+aq^{-1}/b)(1+aq^{-2}/b)\cdots(1+aq^{-4}/b)}{(bq)_8(tq)_9} b^8 t^8 q^{8\times 9}.$$

The factor $b^8t^8q^{8\times 9}$ generates a Durfee rectangle with 8 parts of size 9, or $\pi_1 = 9+9+9+9+9+9+9+9$. Note the Durfee parameter is 4 since there are 8 non-overlined parts greater than or equal to 9.

The factor $1/(bq)_8$ generates partitions into parts of size at most 8. We conjugate these partitions and place them on the right side of $\pi_1$. For example, $\lambda = 7+5+4+1$ conjugated gives $\lambda^* = 4+3+3+3+2+1+1$. Placing the Ferrers diagram for $\lambda^*$ on the right side of the diagram for $\pi_1$ yields the partition $\pi_2 = 13+12+12+12+11+10+10+9$. The factor $(-a)_4$ generates partitions into non-negative distinct parts of size at most 3, for example $\mu = 3+0$. This partition is added to the partition $\pi_2$ using the algorithm given in the proof of Proposition 26, which yields $\pi_3 = \overline{14}+13+13+\overline{12}+11+10+10+9$. Note the Durfee parameter is still 4, since before this operation we had $o_{\pi_2} = 0$ and $s_{\pi_2} = 8$, and after we have $o_{\pi_3} = 2$ and $s_{\pi_3} = 6$, which preserves the inequality (74).

The factor $(-atq/b)_4/(tq)_9$ generates overpartitions into non-overlined parts as large as 9 and overlined parts as large as 4. For example, $\rho = 9+8+5+\overline{4}+2+\overline{2}+1$. The Ferrers diagram for $\rho$ is placed under the diagram for $\pi_3$, giving $\pi_4 = \overline{14}+13+13+\overline{12}+11+10+10+9+9+8+5+\overline{4}+2+\overline{2}+1$. The Durfee parameter remains 4, since there are 7 non-overlined parts greater than or equal to 9, and 2 overlined parts strictly larger than 4, making 9 maximal such that the inequality (74) is preserved.

The factor $(1+aq^{-1}/b)(1+aq^{-2}/b)(1+aq^{-3}/b)(1+aq^{-4}/b)$ generates overpartitions into at most 4 parts of size 4. These parts are subtracted from the partition $\pi_4$ to create overlined parts. Subtraction is applied to the fifth part through the eighth part in the Ferrers diagram for $\pi_4$. For example, if we subtract $\overline{4}, \overline{2}$, and $\overline{1}$, we subtract $\overline{1}$ from the fifth part, $\overline{2}$ from the sixth part, and $\overline{4}$ from the eighth part to get a new overpartition $\pi_5 = \overline{14}+13+13+\overline{12}+\overline{10}+\overline{8}+10+\overline{5}+9+8+5+\overline{4}+2+\overline{2}+1$, or after rearranging, $\pi_5 = \overline{14}+13+13+\overline{12}+\overline{10}+10+9+\overline{8}+8+\overline{5}+5+\overline{4}+2+\overline{2}+1$. Note that the smallest part that can occur is $9-\overline{4} = \overline{5}$, which is from the width of the Durfee rectangle. This factor does not create overlined parts which conflict with the parts generated by $(-atq/b)_4/(tq)_9$ since the largest overlined part from the latter generating function is $\overline{4}$. This preserves the Durfee parameter since overlined parts added are strictly greater than $\overline{4}$, meeting the threshold for overlined parts counted.

Note, the algorithm given in the example of the proof of Theorem 9 is very similar to the one used in this case.

Now we consider the same example from the $4^{\underline{th}}$ term of the second sum:

$$\frac{(-a)_4(-atq/b)_4(1+aq^{-1}/b)(1+aq^{-2}/b)\cdots(1+aq^{-4}/b)}{(bq)_8(tq)_9}ab^8t^9q^{(8\times 9)+13}.$$

We use the example of $\pi_4$ given above. The term $atq^{13}$ generates an overlined part of size 5, and if it occurs, increase the first 8 parts by one to give $\overline{15}+14+14+\overline{13}+12+11+11+10+9+8+5+\overline{5}+\overline{4}+2+\overline{2}+1$. This factor alters the Durfee rectangle to be $9\times 9$ with Durfee parameter 4, since there are 4 non-overlined parts greater than or equal to 9 and 5 overlined parts strictly larger than 5, making $N=4$ maximal such that $4+5\geq 9$. Now apply the same subtracted terms as in the fourth step from the proof of the first sum by subtracting $\overline{1}$ from the fifth part, $\overline{2}$ from the sixth part, and $\overline{4}$ from the eighth part. This gives $\overline{15}+14+14+\overline{13}+\overline{11}+\overline{9}+11+\overline{6}+9+8+5+\overline{5}+\overline{4}+2+\overline{2}+1$, or after rearranging, $\overline{15}+14+14+\overline{13}+\overline{11}+11+\overline{9}+9+8+\overline{6}+5+\overline{5}+\overline{4}+2+\overline{2}+1$. Note that after this operation of adding 1 to the first 8 parts of $\pi_4$, the smallest overlined part that can occur is $\overline{6}$, since the length of the Durfee rectangle is increased. Thus must occur before subtraction to yield distinct overlined parts.

# CHAPTER 4

# COMBINATORIAL PROOFS OF GENERALIZED IDENTITIES

In this chapter we give two general theorems: Theorem 29, and Theorem 32. The first theorem generalizes the Rogers-Fine identity. We prove Theorem 29 algebraically using the extension of Fine's method; a combinatorial proof is also given in the chapter. Two special cases, Corollaries 30 and 31 extend Theorem 11 and Corollary 12, respectively. Both of these corollaries also extend the Rogers-Fine identity.

The second general theorem, Theorem 32, extends (18), as well as Corollary 10, proved in Chapter 2. We prove this identity algebraically using the extensions of Fine's method and also give a combinatorial proof using the concept of Durfee parameter from Chapter 3.

## 4.1 General Identities and Overpartitions

The signal theorem of this section is the following two parameter generalization of the Rogers-Fine identity. The identities are subject to cases of $a$, $b$, and $t$ for which the series on both sides converge. Restrictions on continuous variables can be loosened in some cases. For example, $|b| < 1$ (or similarly for the variable $t$) can be lifted under analytic continuation if the coefficient of the quadratic exponent of $q$ is non-zero.

**Theorem 29.** *Iterating* $\sigma = (\alpha\beta)^k\tau^l$ *with* $k + l > 0$ *and* $|b|, |t| < 1$ *gives*

$$F(a,b;t) = \sum_{i=0}^{k-1}\sum_{n=0}^{\infty} \frac{(a)_{kn+i}(at/b)_{ln}}{(bq)_{kn+i}(t)_{ln}} b^{ln}t^{kn+i}q^{ln(kn+i)}$$
$$+ \sum_{j=0}^{l-1}\sum_{n=0}^{\infty} \frac{(a)_{k(n+1)}(at/b)_{ln+j}}{(bq)_{k(n+1)}(t)_{ln+j+1}}(1 - bq^{k(n+1)})b^{ln+j}t^{k(n+1)}q^{k(n+1)(ln+j)}.$$

The special cases $k = 1$ and $l = 1$ give the following corollaries.

**Corollary 30.** *Iterating* $\sigma = \alpha^k\beta^k\tau$ *with* $k > 0$ *and* $|b|, |t| < 1$ *gives*

$$F(a,b;t) = \sum_{n=0}^{\infty} \frac{(a)_{kn}(at/b)_n}{(bq)_{kn}(t)_{n+1}}(1 - atq^{(k+1)n})b^n t^{kn} q^{kn^2}$$
$$+ \sum_{i=1}^{k-1}\sum_{n=0}^{\infty} \frac{(a)_{kn+i}(at/b)_{n+1}}{(bq)_{kn+i}(t)_{n+1}} b^{n+1}t^{kn+i}q^{(n+1)(kn+i)}.$$

**Corollary 31.** *Iterating* $\sigma = \alpha\beta\tau^l$ *with* $l > 0$ *and* $|b|, |t| < 1$ *gives*

$$F(a,b;t) = \sum_{n=0}^{\infty} \frac{(a)_n(at/b)_{ln}}{(bq)_n(t)_{ln}} b^{ln}t^n q^{ln^2}$$
$$+ \sum_{i=0}^{l-1}\sum_{n=0}^{\infty} \frac{(a)_{n+1}(at/b)_{ln+i}}{(bq)_n(t)_{ln+i+1}} b^{ln+i}t^{n+1}q^{(ln+i)(n+1)}.$$

Corollaries 30 and 31 extend Theorem 11 and Corollary 12 from Chapter 2, as well as the Rogers-Fine identity (20). Note that the right side in Corollary 30 reduces to the case of $l = 1$ in Theorem 29 after regrouping terms.

We begin with an algebraic proof of Theorem 29.

*Proof of Theorem 29.*

Employ (52), (53), (59), and (60) (given here again for convenience):

$$\begin{aligned}
A_{(\alpha\beta)^k} &= \frac{(a)_k}{(bq)_k} t^k, \\
B_{(\alpha\beta)^k} &= \sum_{i=0}^{k-1} \frac{(a)_i}{(bq)_i} t^i, \\
A_{\tau^l} &= \frac{(at/b)_l}{(t)_l} b^l, \\
B_{\tau^l} &= (1-b) \sum_{j=0}^{l-1} \frac{(at/b)_j}{(t)_{j+1}} b^j.
\end{aligned}$$

From (29), (33), and the above identities,

$$\begin{aligned}
A_{((\alpha\beta)^k\tau^l)^n} &= \prod_{i=0}^{n-1} ((\alpha\beta)^k\tau^l)^i A_{(\alpha\beta)^k\tau^l} \\
&= \prod_{i=0}^{n-1} ((\alpha\beta)^k\tau^l)^i \left(A_{(\alpha\beta)^k} (\alpha\beta)^k A_{\tau^l}\right) \\
&= \prod_{i=0}^{n-1} ((\alpha\beta)^k\tau^l)^i \left(\frac{(a)_k}{(bq)_k} t^k (\alpha\beta)^k \frac{(at/b)_l}{(t)_l} b^l\right) \\
&= \prod_{i=0}^{n-1} ((\alpha\beta)^k\tau^l)^i \left(\frac{(a)_k}{(bq)_k} t^k \cdot \frac{(at/b)_l}{(t)_l} b^l q^{kl}\right) \\
&= \prod_{i=0}^{n-1} ((\alpha\beta)^k\tau^l)^i \left(\frac{(a)_k (at/b)_l}{(bq)_k (t)_l} b^l t^k q^{kl}\right) \\
&= \prod_{i=0}^{n-1} \frac{(aq^{ki})_k (atq^{li}/b)_l}{(bq^{1+ki})_k (tq^{li})_l} b^l t^k q^{kl(2i+1)} \\
&= \frac{(a)_{kn} (at/b)_{ln}}{(bq)_{kn} (t)_{ln}} b^{ln} t^{kn} q^{kln^2}.
\end{aligned}$$

Using (30),

$$\begin{aligned}
B_{(\alpha\beta)^k\tau^l} &= B_{(\alpha\beta)^k} + A_{(\alpha\beta)^k}(\alpha\beta)^k B_{\tau^l} \\
&= \sum_{i=0}^{k-1} \frac{(a)_i}{(bq)_i} t^i + \frac{(a)_k}{(bq)_k} t^k (\alpha\beta)^k \left( (1-b) \sum_{j=0}^{l-1} \frac{(at/b)_j}{(t)_{j+1}} b^j \right) \\
&= \sum_{i=0}^{k-1} \frac{(a)_i}{(bq)_i} t^i + \frac{(a)_k}{(bq)_k} t^k (1-bq^k) \sum_{j=0}^{l-1} \frac{(at/b)_j}{(t)_{j+1}} b^j q^{jk} \\
&= \sum_{i=0}^{k-1} \frac{(a)_i}{(bq)_i} t^i + (1-bq^k) \sum_{j=0}^{l-1} \frac{(a)_k (at/b)_j}{(bq)_k (t)_{j+1}} b^j t^k q^{jk}.
\end{aligned}$$

Thus by (34),

$$\begin{aligned}
B_{((\alpha\beta)^k\tau^l)^n} &= \sum_{s=0}^{n-1} A_{((\alpha\beta)^k\tau^l)^s} ((\alpha\beta)^k\tau^l)^s B_{(\alpha\beta)^k\tau^l} \\
&= \sum_{s=0}^{n-1} \frac{(a)_{ks}(at/b)_{ls}}{(bq)_{ks}(t)_{ls}} b^{ls} t^{ks} q^{kls^2} \cdot ((\alpha\beta)^k\tau^l)^s \left( \sum_{i=0}^{k-1} \frac{(a)_i}{(bq)_i} t^i \right. \\
&\left. + (1-bq^k) \sum_{j=0}^{l-1} \frac{(a)_k (at/b)_j}{(bq)_k (t)_{j+1}} b^j t^k q^{jk} \right) \\
&= \sum_{s=0}^{n-1} \frac{(a)_{ks}(at/b)_{ls}}{(bq)_{ks}(t)_{ls}} b^{ls} t^{ks} q^{kls^2} \left( \sum_{i=0}^{k-1} \frac{(aq^{ks})_i}{(bq^{1+ks})_i} t^i q^{ils} \right. \\
&\left. + (1-bq^{k(s+1)}) \sum_{j=0}^{l-1} \frac{(aq^{ks})_k (atq^{ls}/b)_j}{(bq^{1+ks})_k (tq^{ls})_{j+1}} b^j t^k q^{jk(1+s)+kls} \right) \\
&= \sum_{i=0}^{k-1} \sum_{s=0}^{n-1} \frac{(a)_{ks+i}(at/b)_{ls}}{(bq)_{ks+i}(t)_{ls}} b^{ls} t^{ks+i} q^{ls(ks+i)} \\
&+ \sum_{j=0}^{l-1} \sum_{s=0}^{n-1} \frac{(a)_{k(s+1)}(at/b)_{ls+j}}{(bq)_{k(s+1)}(t)_{ls+j+1}} (1-bq^{k(s+1)}) b^{ls+j} t^{k(s+1)} q^{k(s+1)(ls+j)}.
\end{aligned}$$

Hence,

$$
\begin{aligned}
F(a,b;t) &= B_{\sigma^n} + A_{\sigma^n}\sigma^n F(a,b;t) \\
&= \sum_{i=0}^{k-1}\sum_{s=0}^{n-1} \frac{(a)_{ks+i}(at/b)_{ls}}{(bq)_{ks+i}(t)_{ls}} b^{ls}t^{ks+i}q^{ls(ks+i)} \\
&+ \sum_{j=0}^{l-1}\sum_{s=0}^{n-1} \frac{(a)_{k(s+1)}(at/b)_{ls+j}}{(bq)_{k(s+1)}(t)_{ls+j+1}}(1-bq^{k(s+1)})b^{ls+j}t^{k(s+1)}q^{k(s+1)(ls+j)} \\
&+ \frac{(a)_{kn}(at/b)_{ln}}{(bq)_{kn}(t)_{ln}} b^{ln}t^{kn}q^{kln^2}F(aq^{kn},bq^{kn};tq^{ln}).
\end{aligned}
$$

Letting $n \to \infty$ and re-indexing gives Theorem 29. □

### 4.1.1 Combinatorial Proof of Theorem 29

Theorem 29 can be proved combinatorially using the same methods of proof from Theorem 11 given in Chapter 3. Apply the change of variables $a \mapsto -a$ and $t \mapsto tq$ to obtain:

$$
\begin{aligned}
F(-a,b;tq) &= \sum_{i=0}^{k-1}\sum_{n=0}^{\infty} \frac{(-a)_{kn+i}(-atq/b)_{ln}}{(bq)_{kn+i}(tq)_{ln}} b^{ln}t^{kn+i}q^{(ln+1)(kn+i)} \\
&+ \sum_{j=0}^{l-1}\sum_{n=0}^{\infty} \frac{(-a)_{k(n+1)}(-atq/b)_{ln+j}}{(bq)_{k(n+1)}(tq)_{ln+j+1}}(1-bq^{k(n+1)})b^{ln+j}t^{k(n+1)}q^{k(n+1)(ln+j+1)}.
\end{aligned}
\tag{80}
$$

*Proof of Theorem 29.*

By Proposition 26, the coefficient of $a^{\nu_o}b^r t^\nu q^n$ on the left side of (80) enumerates overpartitions of $n$ into $\nu$ parts, $\nu_o$ overlined parts, and rank $r$.

Fix $i$ and consider the $N^{\underline{th}}$ term of the first sums on the right side of (80):

$$\frac{(-a)_{kN+i}(-atq/b)_{lN}}{(bq)_{kN+i}(tq)_{lN}} b^{lN} t^{kN+i} q^{(lN+1)(kN+i)}.$$

We show this term generates overpartitions of $n$ with $\nu$ parts, of which $\nu_o$ are overlined, rank $r$, and Durfee rectangle $(kN+i) \times (lN+1)$. Note this enumeration gives a refinement of the left side. This set of overpartitions is constructed in three steps using Ferrers diagrams.

1. The factor $b^{lN}t^{kN+i}q^{(lN+1)(kN+i)}$ generates a Durfee rectangle of size $(kN+i)\times(lN+1)$, which represents $kN+i$ parts (counted by the exponent of $t$) of size $lN+1$. The rank is $lN$ (counted by the exponent of $b$) and there are no overlined parts.

2. Consider $(-a)_{kN+i}/(bq)_{kN+i}$. This is the generating function for overpartitions into $kN+i$ non-negative parts. The factor $1/(bq)_{kN+i}$ generates partitions into parts of size at most $kN+i$. The Ferrers diagram for these partitions are conjugated and attached to the right side of the Durfee rectangle generated in the first step. This yields partitions into $kN+i$ parts of size at least $lN+1$. The factor $(-a)_{kN+i}$ generates partitions into distinct parts of size ranging from 0 to $kN+i$. Following the algorithm described in Proposition 26, we apply this partition with distinct parts to the Durfee rectangle to generate an overpartition $kN+i$ non-negative parts of size at least $lN+1$. The number of overlined parts is counted by the exponent of $a$, the number of parts is still counted by the exponent of $t$, and the rank is counted by the exponent of $b$. Note the rank can change depending on the number of parts generated by the factor $1/(bq)_{kN+i}$ before conjugation and also depending on the number of overlined parts added by the factor $(-a)_{kN+i}$.

3. Consider $(-atq/b)_{lN}/(tq)_{lN}$. This factor generates overpartitions into parts of size at most $lN$. The Ferrers diagrams for these overpartitions are placed under the diagrams from the previous step. This factor creates overlined parts and for each added overlined part, the rank decrease by 1, as does the exponent of $b$.

Fix $j$ and consider the $N^{\underline{th}}$ term and $j^{\underline{th}}$ term of second sums on the right side of (80):

$$\frac{(-a)_{k(N+1)}(-atq/b)_{lN+j}}{(bq)_{k(N+1)}(tq)_{lN+j+1}}(1-bq^{k(N+1)})b^{lN+j}t^{k(N+1)}q^{k(N+1)(lN+j+1)}.$$

We distribute the factor $(1-bq^{k(N+1)})$ and consider two terms. The interpretation for this first term,

$$\frac{(-a)_{k(N+1)}(-atq/b)_{lN+j}}{(bq)_{k(N+1)}(tq)_{lN+j+1}}b^{lN+j}t^{k(N+1)}q^{k(N+1)(lN+j+1)},$$

is similar to the first sums, except now the Durfee rectangle is of size $k(N+1)\times(lN+j+1)$, the overpartition added to the right has at most $k(N+1)$ non-negative parts, and the overpartition added below has parts of size at most $lN+j+1$, where the largest overlined part that can occur is of size $lN+j$; in other words, the largest part is not overlined. Thus we have shown this term generate overpartitions of $n$ with $\nu$ parts, $\nu_o$ overlined parts, rank $r$, and Durfee rectangle of size $(kN+k)\times(lN+j+1)$.

The interpretation for this first term,

$$\frac{(-a)_{k(N+1)}(-atq/b)_{lN+j}}{(bq)_{k(N+1)}(tq)_{lN+j+1}}b^{lN+j+1}t^{k(N+1)}q^{(kN+k+1)(lN+j+1)},$$

is similar to the first sums, except now the Durfee rectangle is of size $(kN+k+1)\times(lN+j+1)$, the overpartition added to the right has at most $k(N+1)$ non-negative parts, and the overpartition added below has parts of size at most $lN+j+1$, where the largest overlined part that can occur is of size $lN+j$; in other words, the largest part is not overlined. Thus we have shown this term generate overpartitions of $n$ with $\nu$ parts, $\nu_o$ overlined parts, rank $r$, and Durfee rectangle of size $(kN+k+1)\times(lN+j+1)$.

The first sum allows the height of the Durfee rectangle to vary while holding the length fixed. The change in the height depends on which term of the $k-1$ sums is chosen. The second sum fixes the height of the Durfee rectangle and allows the length to vary depending which term of the $l-1$ sums is chosen. Together, these form overpartitions of $n$ into $\nu$ parts, $\nu_o$ overlined parts, and rank $r$. □

Note that the case of $k=0$ and $l=1$ in Theorem 29, we recover the $q$-shifted version of Theorem 28.

In the next section, we consider generalizations from other Rogers-Fine like identities for which combinatorial proofs require the Durfee parameter.

## 4.2 A Generalization of Corollary 10

In Chapter 3 we gave combinatorial proofs of several identities from Chapter 2 using the Durfee parameter. In this section, we prove Theorem 32, which a generalization of Corollary 10 and (18) given by Fine. Note that Theorem 32 is equivalent to $\sigma = \alpha^{k-1}\beta^k\tau$ under the involution (12). Thus the case when $k \neq 2$ yields a new theorem.

We begin by giving an algebraic proof of Theorem 32 and conclude with the combinatorial proof. Note that identity (18) of Fine is a special case of Jackson's identity ($III.4$) of Appendix $III$ from [10]. Theorem 32 is not a special case of Jackson's identity, since Theorem 32 expresses $F$ as the sum of $k$ basic hypergeometric series.

**Theorem 32.** *Iterating $\sigma = \beta\tau^k$ for $k \geq 1$ gives*

$$\begin{aligned} F(a,b;t) = &\sum_{n=0}^{\infty} \frac{(bq/a)_n (at/b)_{(k-1)n}}{(bq)_n (t)_{kn+1}} (-ab^{k-1}t)^n q^{(2k-1)\frac{n(n-1)}{2}+(k-1)n} \\ &+ \sum_{i=0}^{k-2}\sum_{n=0}^{\infty} \frac{(bq/a)_{n+1}(at/b)_{(k-1)n+i}}{(bq)_n (t)_{kn+i+2}} (-at)^{n+1} b^{(k-1)n+i} q^{(2k-1)\frac{n(n+1)}{2}+i(n+1)}. \end{aligned}$$

Under the involution (12), the conjugate identity is given below.

**Corollary 33.** *Iterating $\sigma = \alpha^{k-1}\beta^k\tau$ with $k \geq 1$ gives*

$$\begin{aligned} F(a,b;t) = &\sum_{n=0}^{\infty} \frac{(bq/a)_n (a)_{(k-1)n}}{(t)_{n+1} (bq)_{kn}} (-at^k)^n q^{(2k-1)\frac{n(n-1)}{2}+(k-1)n} \\ &+ \sum_{i=0}^{k-2}\sum_{n=0}^{\infty} \frac{(bq/a)_{n+1}(a)_{(k-1)n+i}}{(t)_{n+1}(bq)_{nk+i+1}} (-a)^{n+1} t^{kn+i+1} q^{(2k-1)\frac{n(n+1)}{2}+i(n+1)}. \end{aligned}$$

*Proof of Theorem 32.*

Using (52) and (54) as well as (29) and (33) with $k \geq 1$,

$$\begin{aligned} A_{(\beta\tau^k)^n} &= \prod_{i=0}^{n-1} (\beta\tau^k)^i A_{\beta\tau^k} \\ &= \prod_{i=0}^{n-1} (\beta\tau^k)^i \left(A_{\beta\tau}(\beta\tau)A_{\tau^{k-1}}\right) \\ &= \prod_{i=0}^{n-1} (\beta\tau^k)^i \left(-\frac{(1-bq/a)}{(1-bq)(1-t)} at \cdot (\beta\tau)\frac{(at/b)_{k-1}}{(t)_{k-1}} b^{k-1}\right) \\ &= \prod_{i=0}^{n-1} (\beta\tau^k)^i \left(-\frac{(1-bq/a)}{(1-bq)(1-t)} at \cdot \frac{(at/b)_{k-1}}{(tq)_{k-1}} b^{k-1} q^{k-1}\right) \\ &= \prod_{i=0}^{n-1} (\beta\tau^k)^i \left(-\frac{(1-bq/a)(at/b)_{k-1}}{(1-bq)(t)_k} ab^{k-1}q^{k-1}t\right) \end{aligned}$$

$$= \prod_{i=0}^{n-1} -\frac{(1-bq^{i+1}/a)(atq^{(k-1)i}/b)_{k-1}}{(1-bq^{i+1})(tq^{ki})_k} ab^{k-1}tq^{(2k-1)i+k-1}$$
$$= \frac{(bq/a)_n(at/b)_{(k-1)n}}{(bq)_n(t)_{kn}}(-ab^{k-1}t)^n q^{(2k-1)\frac{n(n-1)}{2}+(k-1)n}.$$

Next, using (59), (61), and the formula (30) with $k \geq 1$,

$$\begin{aligned}
B_{\beta\tau^k} &= B_{\beta\tau} + A_{\beta\tau}(\beta\tau)B_{\tau^{k-1}} \\
&= \frac{1}{1-t} - \frac{1-bq/a}{(1-bq)(1-t)}at\cdot(\beta\tau)\left((1-b)\sum_{i=0}^{k-2}\frac{(at/b)_i}{(t)_{i+1}}b^i\right) \\
&= \frac{1}{1-t} - \frac{1-bq/a}{(1-bq)(1-t)}at\cdot(1-bq)\sum_{i=0}^{k-2}\frac{(at/b)_i}{(tq)_{i+1}}b^iq^i \\
&= \frac{1}{1-t} - (1-bq/a)at\sum_{i=0}^{k-2}\frac{(at/b)_i}{(t)_{i+2}}b^iq^i.
\end{aligned}$$

Employing (34) yields

$$\begin{aligned}
B_{(\beta\tau^k)^n} &= \sum_{j=0}^{n-1} A_{(\beta\tau^k)^j}(\beta\tau^k)^j B_{\beta\tau^k} \\
&= \sum_{j=0}^{n-1}\frac{(bq/a)_j(at/b)_{(k-1)j}}{(bq)_j(t)_{kj}}(-ab^{k-1}t)^j q^{(2k-1)\frac{j(j-1)}{2}+(k-1)j} \\
&\cdot(\beta\tau^k)^j\left(\frac{1}{1-t} - (1-bq/a)at\sum_{i=0}^{k-2}\frac{(at/b)_i}{(t)_{i+2}}b^iq^i\right) \\
&= \sum_{j=0}^{n-1}\frac{(bq/a)_j(at/b)_{(k-1)j}}{(bq)_j(t)_{kj}}(-ab^{k-1}t)^j q^{(2k-1)\frac{j(j-1)}{2}+(k-1)j} \\
&\cdot\left(\frac{1}{1-tq^{kj}} - (1-bq^{j+1}/a)atq^{kj}\sum_{i=0}^{k-2}\frac{(atq^{(k-1)j}/b)_i}{(tq^{kj})_{i+2}}b^iq^{i+ij}\right)
\end{aligned}$$

$$= \sum_{j=0}^{n-1} \frac{(bq/a)_j(at/b)_{(k-1)j}}{(bq)_j(t)_{kj+1}}(-ab^{k-1}t)^j q^{(2k-1)\frac{j(j-1)}{2}+(k-1)j}$$

$$+ \sum_{i=0}^{k-2}\sum_{j=0}^{n-1} \frac{(bq/a)_{j+1}(at/b)_{(k-1)j+i}}{(bq)_j(t)_{kj+i+2}}(-at)^{j+1}b^{(k-1)j+i}q^{(2k-1)\frac{j(j+1)}{2}+i(j+1)}.$$

Hence,

$$F(a,b;t) = \sum_{j=0}^{n-1} \frac{(bq/a)_j(at/b)_{(k-1)j}}{(bq)_j(t)_{kj+1}}(-ab^{k-1}t)^j q^{(2k-1)\frac{j(j-1)}{2}+(k-1)j}$$

$$+ \sum_{i=0}^{k-2}\sum_{j=0}^{n-1} \frac{(bq/a)_{j+1}(at/b)_{(k-1)j+i}}{(bq)_j(t)_{kj+i+2}}(-at)^{j+1}b^{(k-1)j+i}q^{(2k-1)\frac{j(j+1)}{2}+i(j+1)}$$

$$+ \frac{(bq/a)_n(at/b)_{(k-1)n}}{(bq)_n(t)_{kn}}(-1)^n a^n b^{(k-1)n} t^n q^{(2k-1)\frac{n(n-1)}{2}+(k-1)j} F(a, bq^n; tq^{kn}).$$

Letting $n \to \infty$ and re-indexing gives Theorem 32. □

### 4.2.1 Combinatorial Proof of Theorem 32

Theorem 32 can be proved combinatorially using the same methods of proof from Theorems 9 and 14 given in Chapter 3. Apply the change of variables $a \mapsto -a$ and $t \mapsto tq$ to obtain:

$$F(-a,b;tq) = \sum_{n=0}^{\infty} \frac{(-bq/a)_n(-atq/b)_{(k-1)n}}{(bq)_n(tq)_{kn+1}}(ab^{k-1}t)^n q^{(2k-1)\frac{n(n-1)}{2}+kn} \tag{81}$$

$$+ \sum_{i=0}^{k-2}\sum_{n=0}^{\infty} \frac{(-bq/a)_{n+1}(-atq/b)_{(k-1)n+i}}{(bq)_n(tq)_{kn+i+2}}(at)^{n+1}b^{(k-1)n+i}q^{(2k-1)\frac{n(n+1)}{2}+(i+1)(n+1)}.$$

Rewriting the $q$-product $(-bq/a)_n$ as in (75), we prove the identity in the following form:

$$F(-a,b;tq) = \sum_{n=0}^{\infty} \frac{(1+\frac{a}{b}q^{-1})\cdots(1+\frac{a}{b}q^{-n})(-atq/b)_{(k-1)n}}{(bq)_n(tq)_{kn+1}} b^{kn}t^n q^{n(kn+1)} \tag{82}$$
$$+ \sum_{i=0}^{k-2}\sum_{n=0}^{\infty} \frac{(1+\frac{a}{b}q^{-1})\cdots(1+\frac{a}{b}q^{-n-1})(-atq/b)_{(k-1)n+i}}{(bq)_n(tq)_{kn+i+2}} b^{kn+i+1}t^{n+1}q^{(kn+i+2)(n+1)}.$$

The proof of Theorem 32 uses (82) and involves Durfee rectangles of sizes $n \times (kn+1)$ and $(n+1)\times(kn+i+2)$.

*Combinatorial Proof of Theorem 32.*

By (26), the coefficient of $q^n a^{\nu_o} b^r t^\nu$ on left side of (82) enumerates overpartitions of $n$ into $\nu$ parts, $\nu_o$ overlined parts, and rank $r$.

Consider the $N^{\underline{th}}$ term of the first sum on the right side of (82):

$$\frac{\prod_{j=1}^{N}(1+aq^{-j}/b)(-atq/b)_{(k-1)N}}{(bq)_N(tq)_{kN+1}} b^{kN}t^N q^{N(kN+1)}.$$

We show this term enumerates overpartitions of $n$ with $\nu$ parts, $\nu_o$ overlined parts, rank $r$, Durfee parameter $N$, and constrained Durfee rectangle $N\times(kN+1)$. Note this enumeration gives a refinement of the left side of (82). This set of overpartitions is constructed in four steps using Ferrers diagrams.

1. The factor $b^{kN}t^N q^{N(kN+1)}$ generates a Durfee rectangle of size $N\times(kN+1)$, that is, $N$ non-overlined parts (enumerated by the exponent of $t$) of size $kN+1$ and rank $kN$ (enumerated by the exponent of $b$). Note the Durfee parameter is $N$ since there are $N$ non-overlined parts equal to $kN+1$.

2. Consider $1/(bq)_N$. This is the generating function for partitions into non-overlined parts of size at most $N$. We conjugate the Ferrers diagrams for these partitions to give at most $N$ parts of unrestricted size. The resulting Ferrers diagram is placed to the right of the Durfee rectangle constructed in the first step. Note this preserves the Durfee parameter as $N$, since there are still $N$ non-overlined parts greater than or equal to $kN+1$.

3. Consider $(-atq/b)_{(k-1)N}/(tq)_{kN+1}$. This is the generating function for overpartitions into non-overlined parts of size at most $kN+1$ and overlined parts of size at most $(k-1)N$. The Ferrers diagrams for these overpartitions are placed under the diagrams from the previous step. Note this preserves the Durfee parameter since the overlined parts added are less than $(k-1)N$ and, thus, are not counted in the value of $o$ in the inequality (74).

We now have overpartitions into at least $N$ parts greater than or equal to $kN+1$ and other non-overlined parts of size at most $kN+1$ as well as other overlined parts of size at most $(k-1)N$.

4. Consider $\prod_{i=1}^{N}(1+aq^{-i}/b)$. The exponent of $q$ in a typical term from the expansion of this product has the form $-[c_1+c_2+\cdots+c_{\nu_o}]$, where $1 \le c_1 < c_2 < \cdots < c_{\nu_o} \le N$. Thus this factor generates overpartitions into negative overlined parts from the set $\{\overline{1}, \overline{2}, \cdots, \overline{N}\}$. Each overlined part $c_i$, where $1 \le i \le \nu_o$, is now subtracted from the $c_i^{th}$ non-overlined parts (or the first $N$ parts). The new overlined parts do not change the Durfee parameter since the smallest overlined part that can be generated is $(k-1)N+1$, thus, the value of $s$ decrease (due to the replacement of non-overlined parts with overlined part) and the value of $o$ increases at the same rate. With the left side of (74) preserved, this operation preserves the Durfee parameter. This term also

decreases the rank (the exponent of $b$) by one with the addition of each overlined part (the exponent of $a$).

Next, we show that every overpartition in the set $\overline{P}$ with Durfee parameter $N$ and constrained Durfee rectangle $N \times (kN+1)$ can occur only once. In other words, given an overpartition meeting these qualifications, we show we can recover the terms from the product $\prod_{i=1}^{N}(1+aq^{-i}/b)$ which formed the overpartition from step 4. We first find the Durfee parameter and constrained Durfee rectangle, which gives an upper bound for the size of the overlined parts from the set $\{\overline{1},\cdots,\overline{N}\}$ which could have been subtracted. The following algorithm recovers the overpartition which was generated by the first three steps.

(i) Start with an overpartition with $\nu_o$ overlined parts, $\nu-\nu_o$ non-overlined parts, Durfee parameter $N$, and constrained Durfee rectangle $N\times(kN+1)$. Say there are $k$ overlined parts strictly less than $k(N-1)$. Then, separate the partition into two tuples, one $(\nu-\nu_o)$-tuple consisting of the non-overlined parts $(n_1,\cdots,n_i,\cdots n_{\nu-\nu_o})$ and a $\nu_o$-tuple of the distinct overlined parts $(o_1,\cdots,o_j,\cdots,o_{\nu_o})$. Here, both tuples are ordered largest to smallest.

(ii) Add the overlined numbers $\{\overline{1},\cdots,\overline{N}\}$ in order to the first overlined part, $o_1$, to obtain an increasing sequence of non-overlined parts $o_1+\overline{1},o_1+\overline{2},\cdots,o_1+\overline{N}$. One of these non-overlined terms must have been the non-overlined term that was transformed into an overlined term in step 4. This non-overlined term is uniquely determined by comparing our increasing non-overlined sequence against the decreasing sequence $n_1,\cdots,n_i,\cdots n_{\nu-\nu_o}$. In particular, there exists an overlined part $\overline{i}\in\{\overline{1},\overline{2},\cdots,\overline{N}\}$ such that $o_1+\overline{i}$ is between the $i-1^{\underline{st}}$ non-overlined and the $i^{\underline{th}}$ non-overlined part. When this is the case, shift down all non-overlined parts less than or equal to $n_i$ (in other words, let $n_i \mapsto n_{i+1} \mapsto n_{i+2} \mapsto \cdots$), and let $o_1 + \bar{i}$ be the $i^{\underline{th}}$ non-overlined part. Note, we now have $\nu - \nu_o + 1$ non-overlined terms in our sequence.

(iii) Repeat this process for all distinct overlined parts $o_2, o_3, \cdots, o_{\nu_o - k}$, noting that the remaining $k$ overlined parts were generated before step 4 above and are thus not considered in this algorithm.

Now we consider the $N^{\underline{th}}$ summand of the $k-1$ other sums

$$\frac{\prod_{j=1}^{N+1}(1+aq^{-j}/b)(-atq/b)_{(k-1)N+i}}{(bq)_N(tq)_{kN+i+2}} b^{kN+i+1} t^{N+1} q^{(kN+i+2)(N+1)}$$

with $i \neq 0$. The interpretation for this sum is similar to the first. Now the Durfee rectangle is of size $(N+1) \times (kN+i+2)$, the partition added to the right has at most $N$ parts of unrestricted size, and the overpartition placed below has non-overlined parts of size at most $kN+i+2$ and overlined parts of size at most $(k-1)N+i$. Finally, apply the same subtraction argument as the first sum, only with a larger set $\{\overline{1}, \cdots, \overline{N}, \overline{N+1}\}$. Furthermore, the Durfee parameter is preserved throughout for the same reasons as in the first term above.

Thus, the right side generates overpartitions of $n$ into $\nu$ parts, $\nu_o$ overlined parts, and rank $r$. $\square$

Note that the involution applied to Theorem 32 gives proof to Corollary 33.

# CHAPTER 5

# GENERAL FORMULAS AND APPLICATIONS

In this chapter we apply Fine's method of iteration to the general case $\sigma = \alpha^{n_1}\beta^{n_2}\tau^{n_3}$. To accomplish this, put $n = 3$ in (32) to obtain,

$$B_{\sigma_1\sigma_2\sigma_3} = B_{\sigma_1} + A_{\sigma_1}\sigma_1 B_{\sigma_2} + A_{\sigma_1\sigma_2}\sigma_1\sigma_2 B_{\sigma_3},$$

for choices of $\sigma_1$, $\sigma_2$, and $\sigma_3$. However, letting $\sigma_1 = \alpha^{n_1}$, $\sigma_2 = \beta^{n_2}$, and $\sigma_3 = \tau^{n_3}$ does not yield a general identity in an appealing form. From the Multinomial Theorem, there are $\binom{n_1+n_2+n_3}{n_1,\, n_2,\, n_3}$ words over the alphabet $\{\alpha, \beta, \tau\}$ corresponding to $\sigma$. Moreover, each of these orderings can be grouped into products of elements of the $\sigma$-column of Table 2.1 in many ways. Depending on how this is done, different identities can arise. The following cases lead to conducive expressions for a general $\sigma$. Case $I$ is determined by the inequality $n_1 \geq n_2$, where $\sigma_1 = (\alpha\beta)^k$, $\sigma_2 = \alpha^l$, and $\sigma_3 = \tau^m$, with $k, l, m \geq 0$. Case $II$ is split into two sub-cases. Case $II(a)$ is determined by the inequalities $n_1 \leq n_2$ and $n_2 \leq n_1 + n_3$, where $\sigma_1 = (\alpha\beta)^k$, $\sigma_2 = (\beta\tau)^l$, and $\sigma_3 = \tau^m$, with $k, l, m \geq 0$. Finally, Case $II(b)$ is determined by the inequality $n_2 \geq n_1 + n_3$, where $\sigma_1 = (\alpha\beta)^k$, $\sigma_2 = (\beta\tau)^l$, and $\sigma_3 = \beta^m$, with $k, l, m \geq 0$.

Applying our extension of Fine's method in these cases gives the theorem below. As discussed in Chapter 4, the identities are subject to cases of $a$, $b$, and $t$ for which the series on both sides converge. Restrictions on continuous variables can be loosened in some cases. Note that the case of $l = 0$ in (83) in Theorem 34 recovers Theorem 29.

**Theorem 34.** *Iterating $\sigma = (\alpha\beta)^k \alpha^l \tau^m$ with $k+m>0$, $l \geq 0$, and $|b|, |t| < 1$ gives*

$$F(a,b;t) = \sum_{i=0}^{k-1} t^i \sum_{n=0}^{\infty} \frac{(a)_{(k+l)n+i}(at/b)_{(l+m)n}}{(bq)_{kn+i}(a/b)_{ln}(t)_{mn}} b^{mn} t^{kn} q^{mn(kn+i)} \tag{83}$$
$$- \frac{at^k}{b} \sum_{i=0}^{l-1} q^i \sum_{n=0}^{\infty} \frac{(a)_{(k+l)n+k+i}(at/b)_{(l+m)n+i}}{(bq)_{k(n+1)}(aq/b)_{ln+i+1}(t)_{mn}} (1-bq^{k(n+1)}) b^{mn} t^{kn} q^{kmn(n+1)+ln}$$
$$+ t^k \sum_{i=0}^{m-1} b^i \sum_{n=0}^{\infty} \frac{(a)_{(k+l)(n+1)}(at/b)_{(l+m)n+l+i}}{(bq)_{k(n+1)}(a/b)_{l(n+1)}(t)_{mn+i+1}} (1-bq^{k(n+1)}) b^{mn} t^{kn} q^{k(mn+i)(n+1)}.$$

*Iterating $\sigma = (\alpha\beta)^k (\beta\tau)^l \tau^m$ with $k+l>0$, $m \geq 0$, and $|b|, |t| < 1$ gives*

$$F(a,b;t) = \sum_{i=0}^{k-1} t^i \sum_{n=0}^{\infty} \frac{(a)_{kn+i}(bq/a)_{ln}(at/b)_{mn}}{(bq)_{(k+l)n+i}(t)_{(l+m)n}} \tag{84}$$
$$\cdot (-a)^{ln} b^{mn} t^{(k+l)n} q^{in(l+m)+\binom{ln}{2}+kln^2+mn^2(k+l)}$$
$$+ t^k \sum_{i=0}^{l-1} (-a)^i t^i q^{\binom{i}{2}+ik} \sum_{n=0}^{\infty} \frac{(a)_{k(n+1)}(bq/a)_{ln+i}(at/b)_{mn}}{(bq)_{(k+l)n+k+i}(t)_{(l+m)n+i+1}}$$
$$\cdot (-a)^{ln} b^{mn} t^{(k+l)n} q^{ikn+(k+i)(l+m)n+\binom{ln}{2}+kln^2+mn^2(k+l)}$$
$$+ (-a)^l t^{k+l} q^{\binom{l}{2}+lk} \sum_{i=0}^{m-1} b^i q^{i(k+l)} \sum_{n=0}^{\infty} \frac{(a)_{k(n+1)}(bq/a)_{l(n+1)}(at/b)_{mn+i}}{(bq)_{(k+l)(n+1)}(t)_{(l+m)n+l+i+1}} (1-bq^{(k+l)(n+1)})$$
$$\cdot (-a)^{ln} b^{mn} t^{(k+l)n} q^{lkn+n(k+l)(i+l+m)+\binom{ln}{2}+kln^2+mn^2(k+l)}.$$

*Iterating $\sigma = (\alpha\beta)^k (\beta\tau)^l \beta^m$ with $k+l>0$, $m \geq 0$, and $|t| < 1$ gives*

$$F(a,b;t) = \sum_{i=0}^{k-1} t^i \sum_{n=0}^{\infty} \frac{(a)_{kn+i}(bq/a)_{(l+m)n}}{(bq)_{(k+l+m)n+i}(t)_{ln}(bq/at)_{mn}} (-a)^{ln} t^{(k+l)n} q^{iln+\binom{ln}{2}+kln^2} \tag{85}$$
$$+ t^k \sum_{i=0}^{l-1} (-a)^i t^i q^{\binom{i}{2}+ik} \sum_{n=0}^{\infty} \frac{(a)_{kn+k}(bq/a)_{(l+m)n+i}}{(bq)_{(k+l+m)n+k+i}(t)_{ln+i+1}(bq/at)_{mn}}$$
$$\cdot t^{(k+l)n} (-a)^{ln} q^{kni+ln(k+i)+\binom{ln}{2}+kln^2}$$

$$+ b(-a)^{l-1} t^{k+l-1} q^{1+\binom{l}{2}+lk} \sum_{i=0}^{m-1} q^i \sum_{n=0}^{\infty} \frac{(a)_{kn+k}(bq/a)_{(l+m)n+l+i}}{(bq)_{(k+l+m)n+k+l+i}(t)_{ln+l}(bq/at)_{mn+i+1}}$$
$$\cdot\, t^{(k+l)n}(-a)^{ln} q^{(k+l)ln+lkn+mn+\binom{ln}{2}+kln^2}.$$

Applying conjugation to (83), (84), and (85) gives the following corollary.

**Corollary 35.** *An iteration of* $\sigma = \alpha^{l+m}\beta^m\tau^k$ *with* $k+m>0$*,* $l \geq 0$*, and* $|b|, |t| < 1$ *gives*

$$F(a,b;t) = (1-b)\sum_{i=0}^{k-1} b^i \sum_{n=0}^{\infty} \frac{(at/b)_{(k+l)n+i}(a)_{(l+m)n}}{(t)_{kn+i+1}(a/b)_{ln}(b)_{mn}} t^{mn} b^{kn} q^{mn(kn+i)} \tag{86}$$
$$- a(1-b)b^{k-1}\sum_{i=0}^{l-1} q^i \sum_{n=0}^{\infty} \frac{(at/b)_{(k+l)n+k+i}(a)_{(l+m)n+i}}{(t)_{k(n+1)+1}(a/b)_{ln+i+1}(b)_{mn}}(1-tq^{k(n+1)}) t^{mn} b^{kn} q^{kmn(n+1)+ln}$$
$$+ b^k \sum_{i=0}^{m-1} t^i \sum_{n=0}^{\infty} \frac{(at/b)_{(k+l)(n+1)}(a)_{(l+m)n+l+i}}{(t)_{k(n+1)+1}(a/b)_{l(n+1)}(bq)_{mn+i}}(1-tq^{k(n+1)}) t^{mn} b^{kn} q^{k(mn+i)(n+1)}.$$

*An iteration of* $\sigma = \alpha^m\beta^{l+m}\tau^{k+l}$ *with* $k+l>0$*,* $m \geq 0$*, and* $|b|, |t| < 1$ *gives*

$$F(a,b;t) = \sum_{i=0}^{k-1} b^i \sum_{n=0}^{\infty} \frac{(at/b)_{kn+i}(bq/a)_{ln}(a)_{mn}}{(t)_{(k+l)n+i+1}(bq)_{(l+m)n-1}} \tag{87}$$
$$\cdot\, (-a)^{ln} t^{(l+m)n} b^{kn} q^{in(l+m)+\binom{ln}{2}+kln^2+mn^2(k+l)}$$
$$+ b^k \sum_{i=0}^{l-1} (-a)^i t^i q^{\binom{i}{2}+ik} \sum_{n=0}^{\infty} \frac{(at/b)_{k(n+1)}(bq/a)_{ln+i}(a)_{mn}}{(t)_{(k+l)n+k+i+1}(bq)_{(l+m)n+i}}$$
$$\cdot\, (-a)^{ln} t^{(m+l)n} b^{kn} q^{ikn+(k+i)(l+m)n+\binom{ln}{2}+kln^2+mn^2(k+l)}$$
$$+ (-a)^l b^k t^l q^{\binom{l}{2}+lk} \sum_{i=0}^{m-1} t^i q^{i(k+l)} \sum_{n=0}^{\infty} \frac{(at/b)_{k(n+1)}(bq/a)_{l(n+1)}(a)_{mn+i}}{(t)_{(k+l)(n+1)+1}(bq)_{(l+m)n+l+i}}(1-bq^{k+l+(k+l)n})$$
$$\cdot\, (-a)^{ln} t^{(m+l)n} b^{kn} q^{lkn+n(k+l)(i+l+m)+\binom{ln}{2}+kln^2+mn^2(k+l)}.$$

*An iteration of* $\sigma = \alpha^{-m}\beta^l\tau^{k+l+m}$ *with with* $k+l>0$, $m \geq 0$, *and* $|b|<1$ *gives*

$$F(a,b;t) = (1-b)\sum_{i=0}^{k-1} b^i \sum_{n=0}^{\infty} \frac{(at/b)_{kn+i}(bq/a)_{(l+m)n}}{(t)_{(k+l+m)n+i+1}(b)_{ln}(q/a)_{mn}}(-at)^{ln}b^{kn}q^{iln+\binom{ln}{2}+kln^2} \tag{88}$$
$$+ b^k \sum_{i=0}^{l-1}(-a)^i q^{\binom{i}{2}+ik}\sum_{n=0}^{\infty}\frac{(at/b)_{kn+k}(bq/a)_{(l+m)n+i}}{(t)_{(k+l+m)n+k+i+1}(bq)_{ln+i}(q/a)_{mn}}$$
$$\cdot b^{kn}(-at)^{ln}q^{kni+ln(k+i)+\binom{ln}{2}+kln^2}$$
$$+ (1-b)(-a)^{l-1}b^k q^{1+\binom{l}{2}+lk}\sum_{i=0}^{m-1} q^i \sum_{n=0}^{\infty}\frac{(at/b)_{kn+k}(bq/a)_{(l+m)n+l+i}}{(t)_{(k+l+m)n+k+l+i+1}(b)_{ln+l}(q/a)_{mn+i+1}}$$
$$\cdot b^{kn}(-at)^{ln}q^{(k+l)ln+lkn+mn+\binom{ln}{2}+kln^2}.$$

Note that in Case $I$, conjugation preserves the condition $n_1 \geq n_2$. Case $II(a)$ is also preserved under conjugation with $n_1 \leq n_2$ and $n_2 \leq n_1+n_3$. Case $II(b)$ is transformed to $n_2 \leq n_1 + n_3$ with $n_1 \leq 0$.

Numerical checks for Theorem 34 and Corollary 35 are provided in the appendix in Section A.5.

Letting $k \to \infty$ in (83), (84), and (85) recovers (1). Similarly, sending $k \to \infty$ in (86), (87), and (88) recovers the involution (12). Sending $l \to \infty$ in (83) recovers (15). Sending $l \to \infty$ in (84) and (85) with $|at|<1$ recovers (18). Sending $m \to \infty$ in (83) gives a result which follows from (92) and (95) below with $m=0$. Similarly, sending $m \to \infty$ in (84) with $|b|<1$ gives a result which follows from (93) and (96) below with $m=0$. Finally, sending $m \to \infty$ in (85) with $|b|<1$ gives a result which follows from (94) and (97) below with $m=0$. In fact, sending $k$, $l$, or $m \to \infty$ in (83), (84), or (85) can be explained by considering $\sigma$. For example, (83) is obtained by iterating $\sigma = (\alpha\beta)^k\alpha^l\tau^m$. Letting $k \to \infty$ in this iteration overpowers the $\alpha^l$ and $\tau^m$ factors, yielding $(\alpha\beta)^\infty$, which recovers $F$.

The next section considers applications of Theorem 34 and Corollary 35. The proofs are given in the following section.

## 5.1 Applications of Theorem 34 and Corollary 35

In this section we present several applications of Theorem 34 and Corollary 35. We begin with the divisor function which has generating function $\frac{q}{1-q}F(q,q;q)$ (as in (2)). We show that a special case of Theorem 34 gives

$$\begin{aligned}\sum_{n\geq 1} d(n)q^n &= \sum_{n=0}^{\infty} \frac{1}{(1-q^{n+1})\cdots(1-q^{3n+1})} \cdot \frac{q^{(2n+1)^2}}{(1-q^{2n+1})} \\ &- \sum_{n=0}^{\infty} \frac{1}{(1-q^{n+1})\cdots(1-q^{3n+2})} \cdot \frac{q^{(4n+3)(n+1)}}{(1-q^{2n+2})} \\ &+ \sum_{n=0}^{\infty} \frac{1}{(1-q^{n+1})\cdots(1-q^{3n+3})} q^{(2n+3)(2n+2)}.\end{aligned}$$

The right hand side generates three different types of partitions for which interpretations are provided in this section. Such an interpretation yields Theorem 36.

Finally, sending $t \to 1^-$ in (85), letting $a = -q$, and $b = 0$ yields the following:

$$\begin{aligned}\sum_{n\geq 0} p(n|\text{distinct parts})q^n &= \sum_{n=0}^{\infty} \frac{(-q)_n}{(q)_{2n-1}} q^{n(4n+1)} + \sum_{n=0}^{\infty} \frac{(-q)_{n+1}}{(q)_{2n}} q^{n(4n+3)} \\ &+ \sum_{n=0}^{\infty} \frac{(-q)_{n+1}}{(q)_{2n+1}} q^{(n+1)(4n+2)}.\end{aligned}$$

Giving an overpartition interpretation to the right side yields Theorem 39.

### 5.1.1 The Divisor Function

In Fine [9], Sections 12 and 14 of Chapter 1 considers applications of his identities to obtain transformations for the generating function of $d(n)$, the divisor function. In [15], Patkowski gives a combinatorial interpretation to Fine's identity (12.45). Patkowski expresses the divisor function as the difference of two functions of sums over partitions of a certain type. Here we express the divisor function as a simple linear combination of three partition functions. For this application, we use $\frac{q}{1-q}F(q,q;q)$ given in (2), or

$$\sum_{n\geq 1} d(n)q^n = \sum_{n\geq 1} \frac{q^n}{1-q^n} = \frac{q}{1-q}F(q,q;q).$$

From Theorem 34, consider the identity generated by iterating $\sigma = (\alpha\beta)^k(\beta\tau)^l\tau^m$, or (84), with $a=b=t=q$. Then, using (2) yields

$$\begin{aligned}
\sum_{n\geq 1} d(n)q^n &= \frac{q}{1-q}F(q,q;q) \qquad (89)\\
&= \sum_{i=0}^{k-1}\sum_{n=0}^{\infty} \frac{(q)_{kn+i}(q)_{ln}(q)_{mn}}{(q)_{(k+l)n+i+1}(q)_{(l+m)n}}(-1)^{ln}q^{1+i+(2l+m+k)n+in(l+m)+\binom{ln}{2}+kln^2+mn^2(k+l)}\\
&+ \sum_{i=0}^{l-1}\sum_{n=0}^{\infty} \frac{(q)_{k(n+1)}(q)_{ln+i}(q)_{mn}}{(q)_{(k+l)n+k+i+1}(q)_{(l+m)n+i+1}}(-1)^{ln+i}\\
&\quad\cdot q^{1+k+(2l+m+k)n+\binom{i}{2}+i(k+2+kn)+(k+i)(l+m)n+\binom{ln}{2}+kln^2+mn^2(k+l)}\\
&+ \sum_{i=0}^{m-1}\sum_{n=0}^{\infty} \frac{(q)_{k(n+1)}(q)_{l(n+1)}(q)_{mn+i}}{(q)_{(k+l)(n+1)+1}(q)_{(l+m)n+l+i+1}}(1-q^{(k+l)(n+1)+1})(-1)^{l(n+1)}\\
&\quad\cdot q^{1+2l+k+\binom{l}{2}+lk(n+1)+(2l+m+k)n+i(1+k+l)+n(k+l)(i+l+m)+\binom{ln}{2}+kln^2+mn^2(k+l)}.
\end{aligned}$$

Letting $k = 0$ with $l > 0$, followed by cancellation, yields

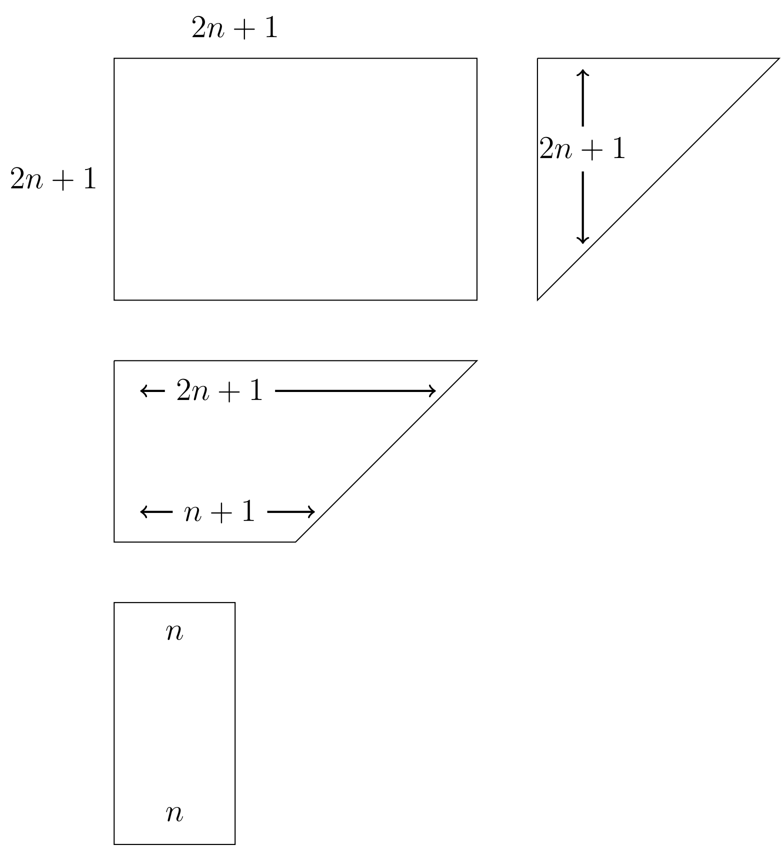
$$\sum_{n\geq 1} d(n)q^n = \sum_{n=0}^{\infty} \frac{1}{(1-q^{n+1})\cdots(1-q^{3n+1})} \cdot \frac{q^{(2n+1)^2}}{(1-q^{2n+1})} - \sum_{n=0}^{\infty} \frac{1}{(1-q^{n+1})\cdots(1-q^{3n+2})} \cdot \frac{q^{(4n+3)(n+1)}}{(1-q^{2n+2})} + \sum_{n=0}^{\infty} \frac{1}{(1-q^{n+1})\cdots(1-q^{3n+3})} q^{(2n+3)(2n+2)}. \quad (90)$$

We now provide partition interpretations to each sum on the right side of (90) above.

Consider the $i^{\underline{th}}$ term of the first sum on the right of (90),

$$\frac{q^{(2i+1)^2}}{(1-q^{i+1})\cdots(1-q^{2i+1})(1-q^{2i+1})\cdots(1-q^{3i+1})}.$$

The factor $q^{(2i+1)^2}$ generates a Durfee square with side length $2i+1$ as a partition into $2i+1$ parts of size $2i+1$. The factor $1/((1-q^{i+1})\cdots(1-q^{2i+1}))$ generates partitions into parts of size ranging from $i+1$ to $2i+1$. These partitions are conjugated and then placed to the right of the Durfee square. Thus, we now have a partition into $2i+1$ parts greater than or equal to $2i+1$, where at least the first $i+1$ parts are the same size. The factor $1/((1-q^{2i+1})\cdots(1-q^{3i+1}))$ generates partitions into parts ranging in size from $2i+1$ to $3i+1$. We transformation these partitions by removing $i$ from each part, leaving two sets of partitions: the first set is into parts of size ranging from $i+1$ to $2i+1$, and the second set is into parts of size $i$. These two partitions are stacked vertically from the largest part to smallest and are placed under the Durfee square. Note that the number of parts of size $i$ are equal to the number of parts where the size ranges from $i+1$ to $2i+1$. *Let $R_1(n)$ be the set of partitions of $n$ for which there exists an $i$ such that the first $2i+1$ parts are greater than or equal to $2i+1$ and the remaining parts ($2i+2^{\underline{nd}}$ part to the last part) are less than* *or equal to* $2i+1$*. In addition, from the* $2i+2^{\underline{nd}}$ *part on, there are an equal number of parts of size equal to* $i$ *as there are the parts less than or equal to* $2i+1$ *and greater than or equal to* $i+1$*. If the largest part is larger than* $2i+1$*, then at least the first* $i+1$ *parts must be the same size. Let* $\rho_1(n)$ *denote the number of integer partitions of* $n$ *from the set* $R_1(n)$*.* The image below represents the construction of this partition interpretation. We give the image for the first construction and give verbal descriptions for all others.




Now consider the $i^{\underline{th}}$ term of the second sum on the right side of (90),

$$\frac{q^{(4i+3)(i+1)}}{(1-q^{i+1})\cdots(1-q^{2i+2})(1-q^{2i+2})\cdots(1-q^{3i+2})}.$$

The factor $q^{(4i+3)(i+1)}$ generates a Durfee rectangle which represents a partition with $i+1$ parts of size $4i+3$. The factor $1/((1-q^{2i+2})\cdots(1-q^{3i+2}))$ generates partitions into parts of size $2i+2$ to $3i+2$. We place these partitions under the Durfee rectangle and obtain partitions into $i+1$ parts of size $4i+3$ and the remaining parts greater than or equal to $2i+2$ and less than or equal to $3i+2$. The factor $1/((1-q^{i+1})\cdots(1-q^{2i+2}))$ generates partitions into parts of size $i+1$ to $2i+2$. We transform these partitions into two partitions; the first containing only the parts of size $2i+2$ and the second containing parts of size ranging from $2i+1$ to $i+1$. Transform the partition of parts of size equal to $2i+2$ by splitting into two partitions of size $i+1$ and stacking vertically. This gives a partition into an even number of parts of size $i+1$. Conjugate this partition to give $i+1$ even parts and place on the right of the Durfee rectangle. Place the partitions with parts of size $2i+1$ to $i+1$ below the partitions with parts ranging in size $2i+2$ to $3i+2$. *Let $R_2(n)$ be the set of partitions of $n$ for which there exists an $i$ such that there are $i+1$ parts of size greater than or equal to $4i+3$ and the remaining parts are of size less than or equal to $3i+2$ and greater than or equal to $i+1$. In addition, note the first $i+1$ parts are odd and of the same size. Let $\rho_2(n)$ denote the number of integer partitions of $n$ from the set $R_2(n)$.*

Finally, consider the $i^{\underline{th}}$ term of the third sum on the right side of (90),

$$\frac{q^{(2i+3)(2i+2)}}{(1-q^{i+1})\cdots(1-q^{3i+3})}q^{(2i+3)(2i+2)}.$$

The factor $q^{(2i+3)(2i+2)}$ generates a Durfee rectangle which represents a partition into $2i+2$ parts of size $2i+3$. The factor $1/((1-q^{i+1})\cdots(1-q^{2i+2}))$ generates partitions into parts of size $i+1$ to $2i+2$. These partitions are conjugated to give partitions into at most $2i+2$ parts and are placed on the right of the Durfee rectangle. Thus we have partitions in $2i+2$ parts of size at least $2i+3$ where if the largest part is larger than $2i+3$, then at least the first $i+1$ parts are the same size. The factor $1/((1-q^{2i+3})\cdots(1-q^{3i+3}))$ generates partitions into parts of size $2i+3$ to $3i+3$. We transform these partitions by removing $i+1$ from all parts to create two sets of partitions: one with parts of size exactly $i+1$, and the other with parts of size ranging from $i+2$ to $2i+2$. Note, the number of parts of size equal to $i+1$ is equal to the number of parts with size between $i+2$ and $2i+2$. These partitions are stacked largest to smallest and are placed under the Durfee rectangle. *Let $R_3(n)$ be the set of partitions of $n$ for which there exists an $i$ such that there are $2i+2$ parts of size greater than or equal to $2i+3$ and the remaining parts less than or equal to $2i+2$ and greater than or equal to $i+1$. In addition, there are the same number of parts of size equal to $i+1$ as there are parts of size between $i+2$ and $2i+2$. If the at least the first $i+1$ parts are larger than $2i+3$, then they must be of equal size. Let $\rho_3(n)$ denote the number of integer partitions of $n$ from the set $R_3(n)$.*

Thus,

$$\sum_{n\geq 1} d(n)q^n = \sum_{n\geq 1}(\rho_1(n) - \rho_2(n) + \rho_3(n))q^n.$$

Equating coefficients on both sides gives the following theorem.

**Theorem 36.** *For $n \geq 1$,*

$$d(n) = \rho_1(n) - \rho_2(n) + \rho_3(n).$$

This gives an interpretation to the divisor function as the sum of three different partition generating functions. We have the immediate corollary.

**Corollary 37.** *$n$ is prime if and only if $\rho_1(n) + \rho_3(n) = 2 + \rho_2(n)$.*

As an example of Theorem 36, consider $n = 7$ noting that $d(7) = 2$. To start, $\rho_1(7) = 7$; the elements of $R_1(7)$ are listed below.

$$1+1+1+1+1+1+1$$

$$2+1+1+1+1+1$$

$$3+1+1+1+1$$

$$4+1+1+1$$

$$5+1+1$$

$$6+1$$

$$7$$

Next, $\rho_2(7) = 6$; the elements of $R_2(7)$ are listed below.

$$7$$

$$3+1+1+1+1$$

$$3+2+1+1$$

$$3+2+2$$

$$5+2$$

$$5+1+1$$

Finally, $\rho_3(7) = 1$. Below is the element of $R_3(7)$.

$$4+3$$

Thus we have,

$$\rho_1(7) - \rho_2(7) + \rho_3(7) = 7 - 6 + 1 = 2 = d(7).$$

As another example, consider the partitions of 9 with $d(9) = 3$. For $R_1(9)$, we have the following partitions listed below.

$$9$$

$$8 + 1$$

$$7 + 1 + 1$$

$$6 + 1 + 1 + 1$$

$$5 + 1 + 1 + 1 + 1$$

$$4 + 1 + 1 + 1 + 1 + 1$$

$$3 + 1 + 1 + 1 + 1 + 1 + 1$$

$$2 + 1 + 1 + 1 + 1 + 1 + 1 + 1$$

$$1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1$$

$$3 + 3$$

This yields $\rho_1(9) = 10$.

$R_2(9)$ contains the partitions listed below.

$$9$$

$$7 + 2 + 1$$

$$7 + 1 + 1 + 1$$

$$5 + 2 + 2$$

$$5 + 2 + 1 + 1$$

$$5 + 1 + 1 + 1 + 1$$

$$3 + 2 + 2 + 2$$

$$3 + 2 + 2 + 1 + 1$$

$$3 + 2 + 1 + 1 + 1 + 1$$

$$3 + 1 + 1 + 1 + 1 + 1 + 1$$

This gives $\rho_2(9) = 10$.

Finally, $R_3(9)$ contains the following partitions listed below.

$$5 + 4$$

$$6 + 3$$

$$3 + 3 + 2 + 1$$

This gives $\rho_3(9) = 3$. Thus we have,

$$\rho_1(9) - \rho_2(9) + \rho_3(9) = 10 - 10 + 3 = 3 = d(9).$$

There may exist other expressions of $d(n)$ by rearranging the Ferrers diagrams and moving away from the idea of Durfee rectangles. Also note that Theorem 36 is given from one choice of $k$, $l$, and $m$ from (89). The case $k=1$, $l=2$, and $m=0$ yields a similar identity:

$$\begin{aligned}\sum_{n\geq 1} d(n)q^n &= \sum_{n=0}^{\infty} \frac{1}{(1-q^{n+1})\cdots(1-q^{3n+1})} q^{(2n+1)^2} \\ &+ \sum_{n=0}^{\infty} \frac{1}{(1-q^{n+2})\cdots(1-q^{3n+2})} \cdot \frac{q^{(2n+1)(2n+2)}}{(1-q^{2n+1})} \\ &- \sum_{n=0}^{\infty} \frac{1}{(1-q^{n+2})\cdots(1-q^{3n+3})} \cdot \frac{q^{(4n+5)(n+1)}}{(1-q^{2n+2})}.\end{aligned}$$

**Theorem 38.** *For $n \geq 1$,*

$$d(n) = \pi_1(n) + \pi_2(n) - \pi_3(n).$$

In this theorem, let $P_1(n)$ be the set of partitions of $n$ for which there exists an $i$ such that the first $2i+1$ parts are greater than or equal to $2i+1$ and the remaining parts (from the $2i+2^{\underline{nd}}$ part to the last part) are less than or equal to $2i+1$ and greater than or equal to $i$. In addition, from the $2i+2^{\underline{nd}}$ part on, there are an equal number of parts of size equal to $i$ as there are the parts less than or equal to $2i+1$ and greater than or equal to $i+2$. If the largest part is larger than $2i+1$, then at least the first $i+1$ parts must be the same size. Let $\pi_1(n)$ denote the number of integer partitions of $n$ from the set $P_1(n)$.

Let $P_2(n)$ be the set of partitions of $n$ for which there exists an $i$ such that the first $2i+1$ parts are greater than or equal to $2i+2$ and the remaining parts (from the $2i+2^{\underline{nd}}$ part to the last part) are less than or equal to $2i+2$ and greater than or equal to $i$. In addition, from the $2i+2^{\underline{nd}}$ part on, there are an equal number of parts of size equal to $i$ as there are the parts less than or equal to $2i+2$ and greater than or equal to $i+1$. If the largest part is larger than $2i+2$, then at least the first $i+2$ parts must be the same size. Let $\pi_2(n)$ denote the number of integer partitions of $n$ from the set $P_2(n)$.

Finally, let $P_3(n)$ be the set of partitions of $n$ for which there exists an $i$ such that there are $i+1$ parts of size greater than or equal to $4i+5$ and the remaining parts are of size less than or equal to $3i+3$ and greater than or equal to $i+2$. In addition, note the first $i+1$ parts are odd and of the same size. Let $\pi_3(n)$ denote the number of integer partitions of $n$ from the set $P_3(n)$.

Thus we have an interpretation for Theorem 38 as the sum of three partition functions.

The case $k=m=0$ and $l=2$ in (89) reduces to (12.42) in [9].

### 5.1.2 Partitions into Distinct Parts

After iterating $\sigma = (\alpha\beta)^k(\beta\tau)^l\beta^m$ with $k, l, m \geq 0$, consider the case where $t \to 1$:

$$\begin{aligned}
\frac{(a)_\infty}{(bq)_\infty} &= \sum_{i=0}^{k-1}\sum_{n=0}^{\infty} \frac{(a)_{kn+i}(bq/a)_{(l+m)n}}{(bq)_{(k+l+m)n+i}(q)_{ln-1}(bq/a)_{mn}}(-a)^{ln}q^{iln+\binom{ln}{2}+kln^2} \\
&\quad + \sum_{i=0}^{l-1}(-a)^i q^{\binom{i}{2}+ik}\sum_{n=0}^{\infty}\frac{(a)_{kn+k}(bq/a)_{(l+m)n+i}}{(bq)_{(k+l+m)n+k+i}(q)_{ln+i}(bq/a)_{mn}} \\
&\qquad \cdot (-a)^{ln}q^{kni+ln(k+i)+\binom{ln}{2}+kln^2} \\
&\quad + b(-a)^{l-1}q^{1+\binom{l}{2}+lk}\sum_{i=0}^{m-1}q^i\sum_{n=0}^{\infty}\frac{(a)_{kn+k}(bq/a)_{(l+m)n+l+i}}{(bq)_{(k+l+m)n+k+l+i}(q)_{ln+l-1}(bq/a)_{mn+i+1}} \\
&\qquad \cdot (-a)^{ln}q^{(k+l)ln+lkn+mn+\binom{ln}{2}+kln^2}.
\end{aligned}$$

Then letting $a = -q$, $b = 0$, $k = 1$, and $l = 2$ yields

$$(-q)_\infty = \sum_{n=0}^{\infty} \frac{(-q)_n}{(q)_{2n-1}} q^{n(4n+1)} + \sum_{n=0}^{\infty} \frac{(-q)_{n+1}}{(q)_{2n}} q^{n(4n+3)} + \sum_{n=0}^{\infty} \frac{(-q)_{n+1}}{(q)_{2n+1}} q^{(n+1)(4n+2)}.$$

From Theorem 2, the left hand side is the generate partitions into distinct parts, or

$$(-q)_\infty = \sum_{n\geq 0} p(n|\text{distinct parts}) q^n.$$

Thus, we give a partition interpretation for the right side of

$$\sum_{n\geq 0} p(n|\text{distinct parts}) q^n = \sum_{n=0}^{\infty} \frac{(-q)_n}{(q)_{2n-1}} q^{n(4n+1)} + \sum_{n=0}^{\infty} \frac{(-q)_{n+1}}{(q)_{2n}} q^{n(4n+3)} + \sum_{n=0}^{\infty} \frac{(-q)_{n+1}}{(q)_{2n+1}} q^{(n+1)(4n+2)}. \tag{91}$$

Thus, we have the following theorem.

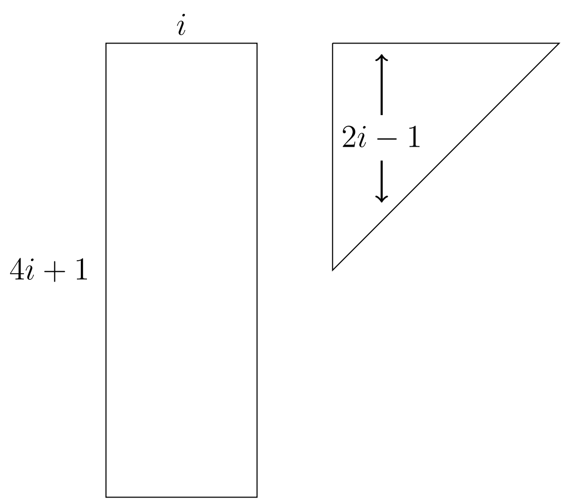
**Theorem 39.** *For $n \geq 0$,*

$$p(n|\textit{distinct parts}) = a_1(n) + a_2(n) + a_3(n).$$

Consider the $i^{\underline{th}}$ term of the first sum on the right side of (91),

$$\frac{(-q)_i}{(q)_{2i-1}} q^{i(4i+1)}.$$

The factor $q^{i(4i+1)}$ generates a Durfee rectangle as a partition into $4i+1$ parts of size $i$. The factor $1/(q)_{2i-1}$ generates partitions into $2i-1$ parts. We conjugate these partitions and place them on the right side of the Durfee rectangle, which gives partitions into $4i+1$ parts where the first $2i-1$ parts are greater than or equal to $i$ and the remaining parts are of size equal to $i$. The factor $(-q)_i$ generates partitions into distinct parts of size ranging from 1 to $i$. These partitions are appended to the partitions into $4i+1$ parts to give overlined parts. (Note, this statement used Proposition 26). Thus, we have a partition into $4i+1$ parts where the first $2i-1$ parts are greater than or equal to $i$ and where the first $i$ parts can be overlined. The remaining parts are of size exactly equal to $i$. *Let $A_1(n)$ be the set of overpartitions of $n$ for which there exists an $i$ such that there are $4i+1$ parts, where the first $2i-1$ parts are greater than or equal to $i$, the first $i$ parts can be overlined, and the remaining $2i+2$ parts are of size $i$. Let $a_1(n)$ denote the number of overpartitions of $n$ from the set $A_1(n)$.* The image below represents the construction of this partition interpretation. We give the image for the first construction and give verbal descriptions for all others.




Consider the $i^{\underline{th}}$ term of the second sum on the right,

$$\frac{(-q)_{i+1}}{(q)_{2i}} q^{i(4i+3)}.$$

The factor $q^{i(4i+3)}$ generates a Durfee rectangle as a partition into $4i+3$ parts of size $i$. The factor $1/(q)_{2i}$ generates partitions into $2i$ parts. We conjugate these partitions and place them on the right side of the Durfee rectangle, which gives partitions into $4i+3$ parts where the first $2i$ parts are greater than or equal to $i$ and the remaining parts are of size equal to $i$. The factor $(-q)_{i+1}$ generates partitions into distinct parts of size ranging from 1 to $i+1$. These partitions are appended to the partitions into $4i+3$ parts to give overlined parts. Thus, we have a partition into $4i+3$ parts where the first $2i$ parts are greater than or equal to $i$ and the first $i+1$ parts are overlined. The remaining parts are of size exactly equal to $i$. *Let $A_2(n)$ be the set of overpartitions of $n$ for which there exists an $i$ such that there are $4i+3$ parts, where the first $2i$ parts are greater than or equal to $i$, the first $i+1$ parts can be overlined, and the remaining $2i+3$ parts are of size exactly $i$. Let $a_2(n)$ denote the number of overpartitions of $n$ from the set $A_2(n)$.*

Consider the $i^{\underline{th}}$ term of the first sum on the right,

$$\frac{(-q)_{i+1}}{(q)_{2i+1}} q^{(i+1)(4i+2)}.$$

The factor $q^{(i+1)(4i+2)}$ generates a Durfee rectangle as a partition into $4i+2$ parts of size $i+1$. The factor $1/(q)_{2i+1}$ generates partitions into $2i+1$ parts. We conjugate these partitions and place them on the right side of the Durfee rectangle, which gives partitions into $4i+2$ parts where the first $2i+1$ parts are greater than or equal to $i+1$ and the remaining parts are of size equal to $i$. The factor $(-q)_{i+1}$ generates partitions into distinct parts of size ranging from 1 to $i+1$. These partitions are appended to the partitions into $4i+2$ parts to give overlined parts. Thus, we have a partition into $4i+2$ parts where the first $2i+1$ parts are greater than or equal to $i$ and the first $i+1$ parts are overlined. The remaining parts are of size exactly equal to $i+1$. *Let $A_3(n)$ be the set of overpartitions of $n$ for which there exists an $i$ such that there are $4i+2$ parts, where the first $2i+1$ parts are greater than or equal to*

*$i+1$, the first $i+1$ parts can be overlined, and the remaining $2i+1$ parts are of size exactly $i+1$. Let $a_3(n)$ denote the number of overpartitions of $n$ from the set $A_3(n)$.*

As an example of Theorem 39, we consider $n = 9$, noting that $D(9) = 8$.

$$9$$

$$8+1$$

$$7+2$$

$$6+3$$

$$6+2+1$$

$$5+4$$

$$5+3+1$$

$$4+3+2$$

We now consider the number of partitions in each set $A_1(9)$, $A_2(9)$, and $A_3(9)$.

Note $a_1(9) = 2$; the elements of $A_1(9)$ are listed below.

$$5+1+1+1+1$$

$$\overline{5}+1+1+1+1$$

Next, $a_2(9) = 4$ with the elements of $A_2(9)$ listed below.

$$3 + 1 + 1 + 1 + 1 + 1 + 1$$

$$\overline{3} + 1 + 1 + 1 + 1 + 1 + 1$$

$$2 + 2 + 1 + 1 + 1 + 1 + 1$$

$$\overline{2} + 2 + 1 + 1 + 1 + 1 + 1$$

Finally, $a_3(9) = 2$ with elements of $A_3(9)$ listed below.

$$8 + 1$$

$$\overline{8} + 1$$

Thus, we have,

$$a_1(9) + a_2(9) + a_3(9) = 2 + 4 + 2 = 8 = p(9|\text{distinct parts}).$$

## 5.2 Proof of Theorem 34

In this proof we employ Lemmas 7 and 8 from Chapter 2. This proof has the same structure as previous algebraic proofs using Fine's method. We employ (34) with $n = 3$ to obtain the first lemma. Since Corollary 35 results from applying conjugation to Theorem 34, it suffices to prove Theorem 34.

**Lemma 40.** *For $k, l, m \geq 0$,*

$$B_{(\alpha\beta)^k\alpha^l\tau^m} = \sum_{i=0}^{k-1} \frac{(a)_i}{(bq)_i} t^i - \frac{a(1-bq^k)}{b} \sum_{i=0}^{l-1} \frac{(a)_{k+i}(at/b)_i}{(bq)_k (a/b)_{i+1}} t^k q^i + (1-bq^k) \sum_{i=0}^{m-1} \frac{(a)_{k+l}(at/b)_{l+i}}{(bq)_k (a/b)_l (t)_{i+1}} b^i t^k q^{ik}, \tag{92}$$

$$B_{(\alpha\beta)^k(\beta\tau)^l\tau^m} = \sum_{i=0}^{k-1} \frac{(a)_i}{(bq)_i} t^i + \sum_{i=0}^{l-1} \frac{(a)_k (bq/a)_i}{(bq)_{k+i}(t)_{i+1}} (-a)^i t^{k+i} q^{\binom{i}{2}+ik} + (1-bq^{k+l}) \sum_{i=0}^{m-1} \frac{(a)_k (bq/a)_l (at/b)_i}{(bq)_{k+l}(t)_{l+i+1}} (-a)^l b^i t^{k+l} q^{i(k+l)+\binom{l}{2}+lk}, \tag{93}$$

*and*

$$B_{(\alpha\beta)^k(\beta\tau)^l\beta^m} = \sum_{i=0}^{k-1} \frac{(a)_i}{(bq)_i} t^i + \sum_{i=0}^{l-1} \frac{(a)_k (bq/a)_i}{(bq)_{k+i}(t)_{i+1}} (-a)^i t^{k+i} q^{\binom{i}{2}+ik} - \frac{b}{at} \sum_{i=0}^{m-1} \frac{(a)_k (bq/a)_{l+i}}{(bq)_{k+l+i}(t)_l (bq/at)_{i+1}} t^{k+l} (-a)^l q^{i+1+\binom{l}{2}+lk}. \tag{94}$$

*Proof of Lemma 40.*

$$\begin{aligned}
B_{(\alpha\beta)^k\alpha^l\tau^m} &= B_{(\alpha\beta)^k} + A_{(\alpha\beta)^k}(\alpha\beta)^k B_{\alpha^l} + A_{(\alpha\beta)^k\alpha^l}(\alpha\beta)^k \alpha^l B_{\tau^m} \\
&= \sum_{i=0}^{k-1} \frac{(a)_i}{(bq)_i} t^i + \frac{(a)_k}{(bq)_k} t^k (\alpha\beta)^k \left( -\frac{a(1-b)}{b} \sum_{i=0}^{l-1} \frac{(a)_i (at/b)_i}{(a/b)_{i+1}} q^i \right) \\
&+ \frac{(a)_{k+l}(at/b)_l}{(bq)_k (a/b)_l} t^k (\alpha\beta)^k \alpha^l \left( (1-b) \sum_{i=0}^{m-1} \frac{(at/b)_i}{(t)_{i+1}} b^i \right) \\
&= \sum_{i=0}^{k-1} \frac{(a)_i}{(bq)_i} t^i + \frac{(a)_k}{(bq)_k} t^k \cdot - \frac{aq^k(1-bq^k)}{bq^k} \sum_{i=0}^{l-1} \frac{(aq^k)_i (at/b)_i}{(a/b)_{i+1}} q^i
\end{aligned}$$

$$+\frac{(a)_{k+l}(at/b)_l}{(bq)_k(a/b)_l}t^k\cdot(1-bq^k)\sum_{i=0}^{m-1}\frac{(atq^l/b)_i}{(t)_{i+1}}b^iq^{ik}.$$

Simplification yields (92).

Similarly,

$$\begin{aligned}
B_{(\alpha\beta)^k(\beta\tau)^l\tau^m} &= B_{(\alpha\beta)^k}+A_{(\alpha\beta)^k}(\alpha\beta)^kB_{(\beta\tau)^l}+A_{(\alpha\beta)^k\cdot(\beta\tau)^l}(\alpha\beta)^k\cdot(\beta\tau)^lB_{\tau^m}\\
&=\sum_{i=0}^{k-1}\frac{(a)_i}{(bq)_i}t^i+\frac{(a)_k}{(bq)_k}t^k\cdot(\alpha\beta)^k\left(\sum_{i=0}^{l-1}\frac{(bq/a)_i}{(bq)_i(t)_{i+1}}(-at)^iq^{\binom{i}{2}}\right)\\
&+\frac{(a)_k(bq/a)_l}{(bq)_{k+l}(t)_l}t^{k+l}(-a)^lq^{\binom{l}{2}+lk}(\alpha\beta)^k\cdot(\beta\tau)^l\left((1-b)\sum_{i=0}^{m-1}\frac{(at/b)_i}{(t)_{i+1}}b^i\right)\\
&=\sum_{i=0}^{k-1}\frac{(a)_i}{(bq)_i}t^i+\frac{(a)_k}{(bq)_k}t^k\cdot\sum_{i=0}^{l-1}\frac{(bq/a)_i}{(bq^{1+k})_i(t)_{i+1}}(-at)^iq^{\binom{i}{2}+ik}\\
&+\frac{(a)_k(bq/a)_l}{(bq)_{k+l}(t)_l}t^{k+l}(-a)^lq^{\binom{l}{2}+lk}\cdot(1-bq^{k+l})\sum_{i=0}^{m-1}\frac{(at/b)_i}{(tq^l)_{i+1}}b^iq^{i(k+l)}.
\end{aligned}$$

Simplification yields (93).

Finally,

$$\begin{aligned}
B_{(\alpha\beta)^k(\beta\tau)^l\beta^m} &= B_{(\alpha\beta)^k}+A_{(\alpha\beta)^k}(\alpha\beta)^kB_{(\beta\tau)^l}+A_{(\alpha\beta)^k\cdot(\beta\tau)^l}(\alpha\beta)^k\cdot(\beta\tau)^lB_{\beta^m}\\
&=\sum_{i=0}^{k-1}\frac{(a)_i}{(bq)_i}t^i+\frac{(a)_k}{(bq)_k}t^k\cdot(\alpha\beta)^k\left(\sum_{i=0}^{l-1}\frac{(bq/a)_i}{(bq)_i(t)_{i+1}}(-at)^iq^{\binom{i}{2}}\right)\\
&+\frac{(a)_k(bq/a)_l}{(bq)_{k+l}(t)_l}t^{k+l}(-a)^lq^{\binom{l}{2}+lk}(\alpha\beta)^k\cdot(\beta\tau)^l\left(-\frac{b}{at}\sum_{i=0}^{m-1}\frac{(bq/a)_i}{(bq)_i(bq/at)_{i+1}}q^{i+1}\right)\\
&=\sum_{i=0}^{k-1}\frac{(a)_i}{(bq)_i}t^i+\frac{(a)_k}{(bq)_k}t^k\cdot\sum_{i=0}^{l-1}\frac{(bq/a)_i}{(bq^{1+k})_i(t)_{i+1}}(-at)^iq^{\binom{i}{2}+ik}
\end{aligned}$$

$$+\frac{(a)_k(bq/a)_l}{(bq)_{k+l}(t)_l}t^{k+l}(-a)^l q^{\binom{l}{2}+lk}\cdot -\frac{b}{at}\sum_{i=0}^{m-1}\frac{(bq^{1+l}/a)_i}{(bq^{k+l})_i(bq/at)_{i+1}}q^{i+1}.$$

Simplification yields (94).

□

The following two lemmas easily gives Theorem 34.

**Lemma 41.** *For $k, l, m \geq 0$, we have*

$$A_{((\alpha\beta)^k\alpha^l\tau^m)^n}=\frac{(a)_{(k+l)n}(at/b)_{(l+m)n}}{(bq)_{kn}(a/b)_{ln}(t)_{mn}}b^{mn}t^{kn}q^{kmn^2}, \tag{95}$$

$$A_{((\alpha\beta)^k(\beta\tau)^l\tau^m)^n}=\frac{(a)_{kn}(bq/a)_{ln}(at/b)_{mn}}{(bq)_{(k+l)n}(t)_{(l+m)n}}(-a)^{ln}b^{mn}t^{(k+l)n}q^{\binom{ln}{2}+kln^2+mn^2(k+l)}, \tag{96}$$

$$A_{((\alpha\beta)^k(\beta\tau)^l\beta^m)^n}=\frac{(a)_{kn}(bq/a)_{(l+m)n}}{(bq)_{(k+l+m)n}(t)_{ln}(bq/at)_{mn}}t^{(k+l)n}(-a)^{ln}q^{\binom{ln}{2}+kln^2}. \tag{97}$$

*Proof of Lemma 41.*

For (95),

$$\begin{aligned}A_{((\alpha\beta)^k\alpha^l\tau^m)^n}&=A_{(\alpha\beta)^{kn}\alpha^{ln}\tau^{mn}}\\&=\left(A_{(\alpha\beta)^{kn}}(\alpha\beta)^{kn}A_{\alpha^{ln}}\right)(\alpha\beta)^{kn}\alpha^{ln}A_{\tau^{mn}}\end{aligned}$$

$$
\begin{aligned}
&= \frac{(a)_{kn}}{(bq)_{kn}} t^{kn} (\alpha\beta)^{kn} \left( \frac{(a)_{ln}(at/b)_{ln}}{(a/b)_{ln}} \right) (\alpha\beta)^{kn} \alpha^{ln} \left( \frac{(at/b)_{mn}}{(t)_{mn}} b^{mn} \right) \\
&= \frac{(a)_{kn}}{(bq)_{kn}} t^{kn} \cdot \frac{(aq^{kn})_{ln}(at/b)_{ln}}{(a/b)_{ln}} \cdot \frac{(atq^{ln}/b)_{mn}}{(t)_{mn}} b^{mn} q^{kmn^2} \\
&= \frac{(a)_{(k+l)n}(at/b)_{(l+m)n}}{(bq)_{kn}(a/b)_{ln}(t)_{mn}} b^{mn} t^{kn} q^{kmn^2}.
\end{aligned}
$$

For (96),

$$
\begin{aligned}
A_{((\alpha\beta)^k(\beta\tau)^l\tau^m)^n} &= A_{(\alpha\beta)^{kn}}(\alpha\beta)^{kn} A_{(\beta\tau)^{ln}}(\alpha\beta)^{kn}(\beta\tau)^{ln} A_{\tau^m} \\
&= \frac{(a)_{kn}}{(bq)_{kn}} t^{kn} \cdot (\alpha\beta)^{kn} \left( \frac{(bq/a)_{ln}}{(bq)_{ln}(t)_{ln}} (-at)^{ln} q^{\binom{ln}{2}} \right) \\
&\cdot (\alpha\beta)^{kn}(\beta\tau)^{ln} \left( \frac{(at/b)_{mn}}{(t)_{mn}} b^{mn} \right) \\
&= \frac{(a)_{kn}}{(bq)_{kn}} t^{kn} \cdot \frac{(bq/a)_{ln}}{(bq^{1+kn})_{ln}(t)_{ln}} (-at)^{ln} q^{\binom{ln}{2}+kln^2} \cdot \frac{(at/b)_{mn}}{(tq^{ln})_{mn}} b^{mn} q^{mn^2(k+l)} \\
&= \frac{(a)_{kn}(bq/a)_{ln}(at/b)_{mn}}{(bq)_{(k+l)n}(t)_{(l+m)n}} (-a)^{ln} b^{mn} t^{(k+l)n} q^{\binom{ln}{2}+kln^2+mn^2(k+l)}.
\end{aligned}
$$

For (97),

$$
\begin{aligned}
A_{((\alpha\beta)^k(\beta\tau)^l\beta^m)^n} &= A_{(\alpha\beta)^{kn}}(\alpha\beta)^{kn} A_{(\beta\tau)^{ln}}(\alpha\beta)^{kn}(\beta\tau)^{ln} A_{\beta^m} \\
&= \frac{(a)_{kn}}{(bq)_{kn}} t^{kn} \cdot (\alpha\beta)^{kn} \left( \frac{(bq/a)_{ln}}{(bq)_{ln}(t)_{ln}} (-at)^{ln} q^{\binom{ln}{2}} \right) \\
&\cdot (\alpha\beta)^{kn}(\beta\tau)^{ln} \left( \frac{(bq/a)_{mn}}{(bq)mn(bq/at)_{mn}} \right) \\
&= \frac{(a)_{kn}}{(bq)_{kn}} t^{kn} \cdot \frac{(bq/a)_{ln}}{(bq^{1+kn})_{ln}(t)_{ln}} (-at)^{ln} q^{\binom{ln}{2}+kln^2} \\
&\cdot \frac{(bq^{1+ln}/a)_{mn}}{(bq^{1+(k+l)n})_{mn}(bq/at)_{mn}} \\
&= \frac{(a)_{kn}(bq/a)_{(l+m)n}}{(bq)_{(k+l+m)n}(t)_{ln}(bq/at)_{mn}} t^{(k+l)n}(-a)^{ln} q^{\binom{ln}{2}+kln^2}.
\end{aligned}
$$

□

**Lemma 42.** *For $k, l, m > 0$, we have*

$$\begin{aligned}
B_{((\alpha\beta)^k\alpha^l\tau^m)^n} &= \sum_{i=0}^{k-1}\sum_{s=0}^{n-1}\frac{(a)_{(k+l)s+i}(at/b)_{(l+m)s}}{(bq)_{ks+i}(a/b)_{ls}(t)_{ms}}b^{ms}t^{ks+i}q^{ms(ks+i)} \\
&- \frac{a}{b}\sum_{i=0}^{l-1}\sum_{s=0}^{n-1}\frac{(a)_{(k+l)s+k+i}(at/b)_{(l+m)s+i}}{(bq)_{k(s+1)}(a/b)_{ls+i+1}(t)_{ms}}(1-bq^{k(n+1)})b^{ms}t^{k(s+1)}q^{i+kms(s+1)+ls} \\
&+ \sum_{i=0}^{m-1}\sum_{s=0}^{n-1}\frac{(a)_{(k+l)(s+1)}(at/b)_{(l+m)s+l+i}}{(bq)_{k(s+1)}(a/b)_{l(s+1)}(t)_{ms+i+1}}(1-bq^{k(n+1)})b^{ms+i}t^{k(s+1)}q^{k(ms+i)(s+1)},
\end{aligned} \tag{98}$$

$$\begin{aligned}
B_{((\alpha\beta)^k(\beta\tau)^l\tau^m)^n} &= \sum_{i=0}^{k-1}\sum_{s=0}^{n-1}\frac{(a)_{ks+i}(bq/a)_{ls}(at/b)_{ms}}{(bq)_{(k+l)s+i}(t)_{(l+m)s}} \\
&\quad \cdot (-a)^{ls}b^{ms}t^{(k+l)s+i}q^{is(l+m)+\binom{ls}{2}+kls^2+ms^2(k+l)} \\
&+ \sum_{i=0}^{l-1}\sum_{s=0}^{n-1}\frac{(a)_{k(s+1)}(bq/a)_{ls+i}(at/b)_{ms}}{(bq)_{(k+l)s+k+i}(t)_{(l+m)s+i+1}} \\
&\quad \cdot (-a)^{ls+i}b^{ms}t^{(k+l)s+k+i}q^{\binom{i}{2}+ik(s+1)+(k+i)(l+m)s+\binom{ls}{2}+kls^2+ms^2(k+l)} \\
&+ \sum_{i=0}^{m-1}\sum_{s=0}^{n-1}\frac{(a)_{k(s+1)}(bq/a)_{l(s+1)}(at/b)_{ms+i}}{(bq)_{(k+l)(s+1)}(t)_{(l+m)s+l+i+1}}(1-bq^{k+l+(k+l)s}) \\
&\quad \cdot (-a)^{l(s+1)}b^{ms+i}t^{(k+l)(s+1)} \\
&\quad \cdot q^{i(k+l)+\binom{l}{2}+lk(s+1)+(k+l)s(i+l+m)+\binom{ls}{2}+kls^2+ms^2(k+l)},
\end{aligned} \tag{99}$$

*and*

$$\begin{aligned}
B_{((\alpha\beta)^k(\beta\tau)^l\beta^m)^n} &= \sum_{i=0}^{k-1}\sum_{s=0}^{n-1}\frac{(a)_{ks+i}(bq/a)_{(l+m)s}}{(bq)_{(k+l+m)s+i}(t)_{ls}(bq/at)_{ms}}(-a)^{ls}t^{(k+l)s+i}q^{ils+ls(ls-1)/2+kls^2} \\
&+ \sum_{i=0}^{l-1}\sum_{s=0}^{n-1}\frac{(a)_{ks+k}(bq/a)_{(l+m)s+i}}{(bq)_{(k+l+m)s+k+i}(t)_{ls+i+1}(bq/at)_{ms}} \\
&\quad \cdot t^{(k+l)s+k+i}(-a)^{ls+i}q^{\binom{i}{2}+ik+ksi+ls(k+i)+\binom{ls}{2}+kls^2}
\end{aligned} \tag{100}$$

$$- \frac{b}{at} \sum_{i=0}^{m-1} \sum_{s=0}^{n-1} \frac{(a)_{ks+k}(bq/a)_{(l+m)s+l+i}}{(bq)_{(k+l+m)s+k+l+i}(t)_{ls+l}(bq/at)_{ms+i+1}}$$
$$\cdot t^{(k+l)s+k+l}(-a)^{ls+l} q^{i+1+\binom{l}{2}+lk+(k+l)ls+lks+ms+ls\binom{ls}{2}+kls^2}.$$

*Proof of Lemma 42.*

For (98), use (92) and (95) in Lemma 41 to obtain

$$\begin{aligned}
B_{((\alpha\beta)^k\alpha^l\tau^m)^n} &= \sum_{s=0}^{n-1} A_{((\alpha\beta)^k\alpha^l\tau^m)^s}((\alpha\beta)^k\alpha^l\tau^m)^s B_{(\alpha\beta)^k\alpha^l\tau^m} \\
&= \sum_{s=0}^{n-1} \frac{(a)_{(k+l)s}(at/b)_{(l+m)s}}{(bq)_{ks}(a/b)_{ls}(t)_{ms}} b^{ms}t^{ks}q^{kms^2} \cdot \\
&\quad ((\alpha\beta)^k\alpha^l\tau^m)^s \left( \sum_{i=0}^{k-1} \frac{(a)_i}{(bq)_i} t^i - \frac{a(1-bq^k)}{b} \sum_{i=0}^{l-1} \frac{(a)_{k+i}(at/b)_i}{(bq)_k(a/b)_{i+1}} t^k q^i \right. \\
&\quad \left. +(1-bq^k) \sum_{i=0}^{m-1} \frac{(a)_{k+l}(at/b)_{l+i}}{(bq)_k(a/b)_l(t)_{i+1}} b^i t^k q^{ik} \right).
\end{aligned}$$

Distributing the operator $((\alpha\beta)^k\alpha^l\tau^m)^s = \alpha^{(k+l)s}\beta^{ks}\tau^{ms}$ yields

$$\begin{aligned}
B_{((\alpha\beta)^k\alpha^l\tau^m)^n} &= \sum_{s=0}^{n-1} \frac{(a)_{(k+l)s}(at/b)_{(l+m)s}}{(bq)_{ks}(a/b)_{ls}(t)_{ms}} b^{ms}t^{ks}q^{kms^2} \cdot \\
&\quad \left( \sum_{i=0}^{k-1} \frac{(aq^{(k+l)s})_i}{(bq^{1+ks})_i} t^i q^{msi} \right. \\
&\quad + - \frac{aq^{(k+l)s}(1-bq^{k(s+1)})}{bq^{ks}} \sum_{i=0}^{l-1} \frac{(aq^{(k+l)s})_{k+i}(atq^{(l+m)s}/b)_i}{(bq^{1+ks})_k(aq^{ls}/b)_{i+1}} t^k q^{i+kms} \\
&\quad \left. +(1-bq^{k(s+1)}) \sum_{i=0}^{m-1} \frac{(aq^{(k+l)s})_{k+l}(atq^{(l+m)s}/b)_{l+i}}{(bq^{1+ks})_k(aq^{ls}/b)_l(tq^{ms})_{i+1}} b^i t^k q^{ik+iks+kms} \right).
\end{aligned}$$

Distributing the product and sum to all other sums gives

$$\begin{aligned}
B_{((\alpha\beta)^k\alpha^l\tau^m)^n} &= \sum_{s=0}^{n-1}\sum_{i=0}^{k-1} \frac{(a)_{(k+l)s+i}(at/b)_{(l+m)s}}{(bq)_{ks+i}(a/b)_{ls}(t)_{ms}} b^{ms}t^{ks+i}q^{ms(ks+i)} \\
&- \frac{a}{b}\sum_{s=0}^{n-1}\sum_{i=0}^{l-1} \frac{(a)_{(k+l)s+k+i}(at/b)_{(l+m)s+i}}{(bq)_{k(s+1)}(a/b)_{ls+i+1}(t)_{ms}}(1-bq^{k(s+1)})b^{ms}t^{k(s+1)}q^{i+kms(s+1)+ls} \\
&+ \sum_{s=0}^{n-1}\sum_{i=0}^{m-1} \frac{(a)_{(k+l)(s+1)}(at/b)_{(l+m)s+l+i}}{(bq)_{k(s+1)}(a/b)_{l(s+1)}(t)_{ms+i+1}}(1-bq^{k(s+1)})b^{ms+i}t^{k(s+1)}q^{k(ms+i)(s+1)}.
\end{aligned}$$

Finally, switching the order of summation yields

$$\begin{aligned}
B_{((\alpha\beta)^k\alpha^l\tau^m)^n} &= \sum_{i=0}^{k-1}\sum_{s=0}^{n-1} \frac{(a)_{(k+l)s+i}(at/b)_{(l+m)s}}{(bq)_{ks+i}(a/b)_{ls}(t)_{ms}} b^{ms}t^{ks+i}q^{ms(ks+i)} \\
&- \frac{a}{b}\sum_{i=0}^{l-1}\sum_{s=0}^{n-1} \frac{(a)_{(k+l)s+k+i}(at/b)_{(l+m)s+i}}{(bq)_{k(s+1)}(a/b)_{ls+i+1}(t)_{ms}}(1-bq^{k(s+1)})b^{ms}t^{k(s+1)}q^{i+kms(s+1)+ls} \\
&+ \sum_{i=0}^{m-1}\sum_{s=0}^{n-1} \frac{(a)_{(k+l)(s+1)}(at/b)_{(l+m)s+l+i}}{(bq)_{k(s+1)}(a/b)_{l(s+1)}(t)_{ms+i+1}}(1-bq^{k(s+1)})b^{ms+i}t^{k(s+1)}q^{k(ms+i)(s+1)}.
\end{aligned}$$

For (99), use (93) and (96) in Lemma 41 to obtain

$$\begin{aligned}
B_{((\alpha\beta)^k(\beta\tau)^l\tau^m)^n} &= \sum_{s=0}^{n-1} A_{((\alpha\beta)^k(\beta\tau)^l\tau^m)^s}((\alpha\beta)^k(\beta\tau)^l\tau^m)^s B_{(\alpha\beta)^k(\beta\tau)^l\tau^m} \\
&= \sum_{s=0}^{n-1} \frac{(a)_{ks}(bq/a)_{ls}(at/b)_{ms}}{(bq)_{(k+l)s}(t)_{(l+m)s}}(-a)^{ls}b^{ms}t^{(k+l)s}q^{\binom{ls}{2}+kls^2+ms^2(k+l)} \\
&\cdot((\alpha\beta)^k(\beta\tau)^l\tau^m)^s \left(\sum_{i=0}^{k-1}\frac{(a)_i}{(bq)_i}t^i\right. \\
&+ \sum_{i=0}^{l-1}\frac{(a)_k(bq/a)_i}{(bq)_{k+i}(t)_{i+1}}(-a)^i t^{k+i}q^{\binom{i}{2}+ik} \\
&\left.+(1-bq^{k+l})\sum_{i=0}^{m-1}\frac{(a)_k(bq/a)_l(at/b)_i}{(bq)_{k+l}(t)_{l+i+1}}(-a)^l b^i t^{k+l} q^{i(k+l)+\binom{l}{2}+lk}\right).
\end{aligned}$$

Distributing the operator $((\alpha\beta)^k(\beta\tau)^l\tau^m)^s = \alpha^{ks}\beta^{(k+l)s}\tau^{(l+m)s}$ yields

$$\begin{aligned}B_{((\alpha\beta)^k(\beta\tau)^l\tau^m)^n} = \sum_{s=0}^{n-1} & \frac{(a)_{ks}(bq/a)_{ls}(at/b)_{ms}}{(bq)_{(k+l)s}(t)_{(l+m)s}}(-a)^{ls}b^{ms}t^{(k+l)s}q^{\binom{ls}{2}+kls^2+ms^2(k+l)} \\ & \cdot\left(\sum_{i=0}^{k-1}\frac{(aq^{ks})_i}{(bq^{1+(k+l)s})_i}t^i q^{is(l+m)}\right. \\ & +\sum_{i=0}^{l-1}\frac{(aq^{ks})_k(bq^{1+ls}/a)_i}{(bq^{1+(k+l)s})_{k+i}(tq^{(l+m)s})_{i+1}}(-a)^i t^{k+i}q^{\binom{i}{2}+ik+iks+(k+i)(l+m)s} \\ & +(1-bq^{k+l+(k+l)s})\sum_{i=0}^{m-1}\frac{(aq^{ks})_k(bq^{1+ls}/a)_l(atq^{ms}/b)_i}{(bq^{1+(k+l)s})_{k+l}(tq^{(l+m)s})_{l+i+1}} \\ & \qquad\left.\cdot(-a)^l b^i t^{k+l}q^{i(k+l)+\binom{l}{2}+lk+lks+i(k+l)s+(k+l)(l+m)s}\right).\end{aligned}$$

Distributing the product and the sum to all other sums gives

$$\begin{aligned}B_{((\alpha\beta)^k(\beta\tau)^l\tau^m)^n} = & \sum_{s=0}^{n-1}\sum_{i=0}^{k-1}\frac{(a)_{ks+i}(bq/a)_{ls}(at/b)_{ms}}{(bq)_{(k+l)s+i}(t)_{(l+m)s}}(-a)^{ls}b^{ms}t^{(k+l)s+i} \\ & \qquad\cdot q^{is(l+m)+\binom{ls}{2}+kls^2+ms^2(k+l)} \\ & +\sum_{s=0}^{n-1}\sum_{i=0}^{l-1}\frac{(a)_{k(s+1)}(bq/a)_{ls+i}(at/b)_{ms}}{(bq)_{(k+l)s+k+i}(t)_{(l+m)s+i+1}} \\ & \qquad\cdot(-a)^{ls+i}b^{ms}t^{(k+l)s+k+i}q^{\binom{i}{2}+ik(s+1)+(k+i)(l+m)s+\binom{ls}{2}+kls^2+ms^2(k+l)} \\ & +\sum_{s=0}^{n-1}\sum_{i=0}^{m-1}\frac{(a)_{k(s+1)}(bq/a)_{l(s+1)}(at/b)_{ms+i}}{(bq)_{(k+l)(s+1)}(t)_{(l+m)s+l+i+1}}(1-bq^{(k+l)(s+1)}) \\ & \qquad\cdot(-a)^{l(s+1)}b^{ms+i}t^{(k+l)(s+1)} \\ & \qquad\cdot q^{i(k+l)+\binom{l}{2}+lk(s+1)+(k+l)s(i+l+m)+\binom{ls}{2}+kls^2+ms^2(k+l)}.\end{aligned}$$

Finally, switching the order of summation yields

$$B_{((\alpha\beta)^k(\beta\tau)^l\tau^m)^n} = \sum_{i=0}^{k-1}\sum_{s=0}^{n-1}\frac{(a)_{ks+i}(bq/a)_{ls}(at/b)_{ms}}{(bq)_{(k+l)s+i}(t)_{(l+m)s}}(-a)^{ls}b^{ms}t^{(k+l)s+i}q^{is(l+m)+\binom{ls}{2}+kls^2+ms^2(k+l)}$$

$$+\sum_{i=0}^{l-1}\sum_{s=0}^{n-1}\frac{(a)_{k(s+1)}(bq/a)_{ls+i}(at/b)_{ms}}{(bq)_{(k+l)s+k+i}(t)_{(l+m)s+i+1}}$$
$$\cdot(-a)^{ls+i}b^{ms}t^{(k+l)s+k+i}q^{\binom{i}{2}+ik(s+1)+(k+i)(l+m)s+\binom{ls}{2}+kls^2+ms^2(k+l)}$$
$$+\sum_{i=0}^{m-1}\sum_{s=0}^{n-1}\frac{(a)_{k(s+1)}(bq/a)_{l(s+1)}(at/b)_{ms+i}}{(bq)_{(k+l)(s+1)}(t)_{(l+m)s+l+i+1}}(1-bq^{(k+l)(s+1)})$$
$$\cdot(-a)^{l(s+1)}b^{ms+i}t^{(k+l)(s+1)}$$
$$\cdot q^{i(k+l)+\binom{l}{2}+lk(s+1)+(k+l)s(i+l+m)+\binom{ls}{2}+kls^2+ms^2(k+l)}.$$

For (100), use (94) and (97) in Lemma 41 to obtain

$$\begin{aligned}B_{((\alpha\beta)^k(\beta\tau)^l\beta^m)^n} &= \sum_{s=0}^{n-1}A_{((\alpha\beta)^k(\beta\tau)^l\beta^m)^s}((\alpha\beta)^k(\beta\tau)^l\beta^m)^sB_{(\alpha\beta)^k(\beta\tau)^l\beta^m}\\
&=\sum_{s=0}^{n-1}\frac{(a)_{ks}(bq/a)_{(l+m)s}}{(bq)_{(k+l+m)s}(t)_{ls}(bq/at)_{ms}}t^{(k+l)s}(-a)^{ls}q^{\binom{ls}{2}+kls^2}\\
&\cdot((\alpha\beta)^k(\beta\tau)^l\beta^m)^s\left(\sum_{i=0}^{k-1}\frac{(a)_i}{(bq)_i}t^i\right.\\
&+\sum_{i=0}^{l-1}\frac{(a)_k(bq/a)_i}{(bq)_{k+i}(t)_{i+1}}(-a)^it^{k+i}q^{\binom{i}{2}+ik}\\
&\left.-\frac{b}{at}\sum_{i=0}^{m-1}\frac{(a)_k(bq/a)_{l+i}}{(bq)_{k+l+i}(t)_l(bq/at)_{i+1}}t^{k+l}(-a)^lq^{i+1+\binom{l}{2}+lk}\right).\end{aligned}$$

Distributing the operator $((\alpha\beta)^k(\beta\tau)^l\beta^m)^s=\alpha^{ks}\beta^{(k+l+m)s}\tau^{ls}$ yields

$$\begin{aligned}B_{((\alpha\beta)^k(\beta\tau)^l\beta^m)^n} &= \sum_{s=0}^{n-1}\frac{(a)_{ks}(bq/a)_{(l+m)s}}{(bq)_{(k+l+m)s}(t)_{ls}(bq/at)_{ms}}t^{(k+l)s}(-a)^{ls}q^{ls(ls-1)/2+kls^2}\\
&\cdot\left(\sum_{i=0}^{k-1}\frac{(aq^{ks})_i}{(bq^{1+(k+l+m)s})_i}t^iq^{ils}\right.\end{aligned}$$

$$+\sum_{i=0}^{l-1}\frac{(aq^{ks})_k(bq^{1+(l+m)s}/a)_i}{(bq^{1+(k+l+m)s})_{k+i}(tq^{ls})_{i+1}}(-a)^i t^{k+i} q^{\binom{i}{2}+ik+ksi+ls(k+i)}$$

$$-\frac{bq^{ms}}{at}\sum_{i=0}^{m-1}\frac{(aq^{ks})_k(bq^{1+(l+m)s}/a)_{l+i}}{(bq^{1+(k+l+m)s})_{k+l+i}(tq^{ls})_l(bq^{1+ms}/at)_{i+1}}$$

$$\cdot t^{k+l}(-a)^l q^{i+1+\binom{l}{2}+lk+(k+l)ls+lks}\Bigg).$$

Distributing the product and the sum to all other sums gives

$$\begin{aligned}B_{((\alpha\beta)^k(\beta\tau)^l\beta^m)^n} &= \sum_{s=0}^{n-1}\sum_{i=0}^{k-1}\frac{(a)_{ks+i}(bq/a)_{(l+m)s}}{(bq)_{(k+l+m)s+i}(t)_{ls}(bq/at)_{ms}}(-a)^{ls}t^{(k+l)s+i}q^{ils+\binom{ls}{2}+kls^2}\\ &+\sum_{s=0}^{n-1}\sum_{i=0}^{l-1}\frac{(a)_{ks+k}(bq/a)_{(l+m)s+i}}{(bq)_{(k+l+m)s+k+i}(t)_{ls+i+1}(bq/at)_{ms}}\\ &\qquad\cdot t^{(k+l)s+k+i}(-a)^{ls+i}q^{\binom{i}{2}+ik+ksi+ls(k+i)+\binom{ls}{2}+kls^2}\\ &-\frac{b}{at}\sum_{s=0}^{n-1}\sum_{i=0}^{m-1}\frac{(a)_{ks+k}(bq/a)_{(l+m)s+l+i}}{(bq)_{(k+l+m)s+k+l+i}(t)_{ls+l}(bq/at)_{ms+i+1}}\\ &\qquad\cdot t^{(k+l)s+k+l}(-a)^{ls+l}q^{i+1+\binom{l}{2}+lk+(k+l)ls+lks+ms+\binom{ls}{2}+kls^2}.\end{aligned}$$

Finally, switching the order of summation yields

$$\begin{aligned}B_{((\alpha\beta)^k(\beta\tau)^l\beta^m)^n} &= \sum_{i=0}^{k-1}\sum_{s=0}^{n-1}\frac{(a)_{ks+i}(bq/a)_{(l+m)s}}{(bq)_{(k+l+m)s+i}(t)_{ls}(bq/at)_{ms}}(-a)^{ls}t^{(k+l)s+i}q^{ils+\binom{ls}{2}+kls^2}\\ &+\sum_{i=0}^{l-1}\sum_{s=0}^{n-1}\frac{(a)_{ks+k}(bq/a)_{(l+m)s+i}}{(bq)_{(k+l+m)s+k+i}(t)_{ls+i+1}(bq/at)_{ms}}\\ &\qquad\cdot t^{(k+l)s+k+i}(-a)^{ls+i}q^{\binom{i}{2}+ik+ksi+ls(k+i)+\binom{ls}{2}+kls^2}\\ &-\frac{b}{at}\sum_{i=0}^{m-1}\sum_{s=0}^{n-1}\frac{(a)_{ks+k}(bq/a)_{(l+m)s+l+i}}{(bq)_{(k+l+m)s+k+l+i}(t)_{ls+l}(bq/at)_{ms+i+1}}\\ &\qquad\cdot t^{(k+l)s+k+l}(-a)^{ls+l}q^{i+1+\binom{l}{2}+lk+(k+l)ls+lks+ms+\binom{ls}{2}+kls^2}.\end{aligned}$$

□

Using the lemmas, a proof of Theorem 83 is now forthcoming.

*Proof of Theorem 83.*

Using (95) and (98), gives

$$\begin{aligned} F(a,b;t) &= \sum_{i=0}^{k-1}\sum_{s=0}^{n-1} \frac{(a)_{(k+l)s+i}(at/b)_{(l+m)s}}{(bq)_{ks+i}(a/b)_{ls}(t)_{ms}} b^{ms}t^{ks+i}q^{ms(ks+i)} \\ &- \frac{a}{b}\sum_{i=0}^{l-1}\sum_{s=0}^{n-1} \frac{(a)_{(k+l)s+k+i}(at/b)_{(l+m)s+i}}{(bq)_{k(s+1)}(a/b)_{ls+i+1}(t)_{ms}}(1-bq^{k(s+1)})b^{ms}t^{k(s+1)}q^{i+kms(s+1)+ls} \\ &+ \sum_{i=0}^{m-1}\sum_{s=0}^{n-1} \frac{(a)_{(k+l)(s+1)}(at/b)_{(l+m)s+l+i}}{(bq)_{k(s+1)}(a/b)_{l(s+1)}(t)_{ms+i+1}}(1-bq^{k(s+1)})b^{ms+i}t^{k(s+1)}q^{k(ms+i)(s+1)} \\ &+ \frac{(a)_{(k+l)n}(at/b)_{(l+m)n}}{(bq)_{kn}(a/b)_{ln}(t)_{mn}} b^{mn}t^{kn}q^{kmn^2}\cdot F(aq^{(k+l)n}, bq^{kn}; tq^{mn}). \end{aligned} \tag{101}$$

Using (96) and (99), gives

$$\begin{aligned} F(a,b;t) &= \sum_{i=0}^{k-1}\sum_{s=0}^{n-1} \frac{(a)_{ks+i}(bq/a)_{ls}(at/b)_{ms}}{(bq)_{(k+l)s+i}(t)_{(l+m)s}}(-a)^{ls}b^{ms}t^{(k+l)s+i}q^{is(l+m)+\binom{ls}{2}+kls^2+ms^2(k+l)} \\ &+ \sum_{i=0}^{l-1}\sum_{s=0}^{n-1} \frac{(a)_{k(s+1)}(bq/a)_{ls+i}(at/b)_{ms}}{(bq)_{(k+l)s+k+i}(t)_{(l+m)s+i+1}} \\ &\qquad \cdot (-a)^{ls+i}b^{ms}t^{(k+l)s+k+i}q^{\binom{i}{2}+ik(s+1)+(k+i)(l+m)s+\binom{ls}{2}+kls^2+ms^2(k+l)} \\ &+ \sum_{i=0}^{m-1}\sum_{s=0}^{n-1} \frac{(a)_{k(s+1)}(bq/a)_{l(s+1)}(at/b)_{ms+i}}{(bq)_{(k+l)(s+1)}(t)_{(l+m)s+l+i+1}}(1-bq^{k+l+(k+l)s}) \\ &\qquad \cdot (-a)^{l(s+1)}b^{ms+i}t^{(k+l)(s+1)}q^{i(k+l)+\binom{l}{2}+lk(s+1)+(k+l)s(i+l+m)+\binom{ls}{2}+kls^2+ms^2(k+l)} \\ &+ \frac{(a)_{kn}(bq/a)_{ln}(at/b)_{mn}}{(bq)_{(k+l)n}(t)_{(l+m)n}}(-a)^{ln}b^{mn}t^{(k+l)n}q^{\binom{ln}{2}+kln^2+mn^2(k+l)} \\ &\qquad \cdot F(aq^{kn}, bq^{(k+l)n}; tq^{(l+m)n}). \end{aligned} \tag{102}$$

Using (97) and (100), gives

$$F(a,b;t) = \sum_{i=0}^{k-1}\sum_{s=0}^{n-1} \frac{(a)_{ks+i}(bq/a)_{(l+m)s}}{(bq)_{(k+l+m)s+i}(t)_{ls}(bq/at)_{ms}}(-a)^{ls}t^{(k+l)s+i}q^{ils+\binom{ls}{2}+kls^2} \tag{103}$$

$$+\sum_{i=0}^{l-1}\sum_{s=0}^{n-1} \frac{(a)_{ks+k}(bq/a)_{(l+m)s+i}}{(bq)_{(k+l+m)s+k+i}(t)_{ls+i+1}(bq/at)_{ms}}$$

$$\cdot\, t^{(k+l)s+k+i}(-a)^{ls+i}q^{\binom{i}{2}+ik+ksi+ls(k+i)+\binom{ls}{2}+kls^2}$$

$$-\frac{b}{at}\sum_{i=0}^{m-1}\sum_{s=0}^{n-1} \frac{(a)_{ks+k}(bq/a)_{(l+m)s+l+i}}{(bq)_{(k+l+m)s+k+l+i}(t)_{ls+l}(bq/at)_{ms+i+1}}$$

$$\cdot\, t^{(k+l)s+k+l}(-a)^{ls+l}q^{i+1+\binom{l}{2}+lk+(k+l)ls+lks+ms+\binom{ls}{2}+kls^2}$$

$$+\frac{(a)_{kn}(bq/a)_{(l+m)n}}{(bq)_{(k+l+m)n}(t)_{ln}(bq/at)_{mn}}t^{(k+l)n}(-a)^{ln}q^{\binom{ln}{2}+kln^2}\cdot F(aq^{kn}, bq^{(k+l+m)n}; tq^{ln}).$$

Let $n\to\infty$ in the (101), (102), and (103) above and re-index to get Theorem 34.

□

*Proof of Corollary 35.*

Apply the involution (12) to (83), (84), and (85) in Theorem 34. After a change of variables as $a' = at/b$, $b' = t$, and $t' = b$, as well as noting $at/b \mapsto a'$, then drop the primes to obtain (86), (87), and (88).

□

# APPENDIX

# ADDITIONAL RESOURCES

## A.1 List of Fine's Iterative Transformations

In the following list of Fine's identities from [9] we use a different definition for $F(a,b;t)$; his $F(a,b;t)$ being our $F(aq,b;t)$.

Fine's iteration $a \mapsto aq$, (15), (11.1) in [9]:

$$F(a,b;t) = -\frac{a(1-b)}{b}\sum_{n\geq 0}\frac{(a)_n(at/b)_n}{(a/b)_{n+1}}q^n + \frac{(a)_\infty(at/b)_\infty}{(a/b)_\infty}\sum_{n\geq 0}\frac{t^n}{(bq)_n}.$$

The conjugate identity is

$$F(a,b;t) = -\frac{a(1-b)}{b}\sum_{n\geq 0}\frac{(at/b)_n(a)_n}{(a/b)_{n+1}}q^n + \frac{(1-b)(at/b)_\infty(a)_\infty}{(a/b)_\infty}\sum_{n\geq 0}\frac{b^n}{(t)_{n+1}}.$$

Fine's iteration $b \mapsto bq$ for $|t| < 1$, (5), (7.1) in [9]:

$$F(a,b;t) = -\frac{b}{at}\sum_{n\geq 0}\frac{(bq/a)_n}{(bq)_n(bq/at)_{n+1}}q^{n+1} + \frac{(bq/a)_\infty}{(bq)_\infty(bq/at)_\infty}\sum_{n\geq 0}(a)_n t^n.$$

The conjugate identity is, for $|b| < 1$,

$$F(a,b;t) = -\frac{(1-b)}{a}\sum_{n\geq 0}\frac{(bq/a)_n}{(t)_{n+1}(q/a)_{n+1}}q^{n+1} + \frac{(1-b)(bq/a)_\infty}{(t)_\infty(q/a)_\infty}\sum_{n\geq 0}(at/b)_n b^n.$$

Fine's iteration $t \mapsto tq$ for $|b| < 1$, (11), (6.3) in [9]:

$$F(a,b;t) = (1-b)\sum_{n=0}^{\infty}\frac{(at/b)_n}{(t)_{n+1}}b^n.$$

The conjugate identity is (1).

Fine's iteration $(b,t) \mapsto (bq,tq)$, (18), (12.1) in [9]:

$$(1-t)F(a,b;t) = \sum_{n\geq 0} \frac{(bq/a)_n}{(bq)_n(tq)_n}(-at)^n q^{(n^2-n)/2}.$$

This identity is invariant under conjugation.

Fine's iteration $(a,t) \mapsto (aq,tq)$ for $|b|<1$, (4), (13.1) in [9]:

$$\frac{1-t}{1-b}F(a,b;t) = \sum_{n\geq 0} \frac{(aq)_n(at/b)_{2n}}{(tq)_n(aq/b)_n}b^n - aq\sum_{n\geq 0} \frac{(aq)_n(at/b)_{2n+1}}{(tq)_n(aq/b)_{n+1}}(bq)^n.$$

The conjugate identity for $|t|<1$ is

$$F(a,b;t) = \sum_{n\geq 0} \frac{(atq/b)_n(a)_{2n}}{(bq)_n(aq/b)_n}t^n - atq/b\sum_{n\geq 0} \frac{(atq/b)_n(a)_{2n+1}}{(bq)_n(aq/b)_{n+1}}(tq)^n$$

The Rogers-Fine identity is derived from the iteration $(a,b,t) \mapsto (aq,bq,tq)$, (20), (14.1) in [9]:

$$(1-t)F(a,b;t) = \sum_{n\geq 0} \frac{(a)_n(at/b)_n}{(bq)_n(tq)_n}(1-atq^{2n})(bt)^n q^{n^2}.$$

This identity is invariant under conjugation.

## A.2 List of New Identities

Theorem 9:

$$F(a,b;t)=\sum_{n=0}^{\infty}\frac{(a)_n(bq/a)_n}{(t)_{n+1}(bq)_{2n}}(1-atq^{2n})(-at^2)^n q^{\frac{n(3n-1)}{2}}$$
$$+bt\sum_{n=0}^{\infty}\frac{(a)_{n+1}(bq/a)_n}{(t)_{n+1}(bq)_{2n+1}}(-at^2)^n q^{\frac{(3n+2)(n+1)}{2}}.$$

Corollary 10:

$$F(a,b;t)=\sum_{n=0}^{\infty}\frac{(at/b)_n(bq/a)_n}{(t)_{2n+1}(bq)_n}(-abt)^n q^{\frac{n(3n-1)}{2}}$$
$$+\frac{1}{b}\sum_{n=0}^{\infty}\frac{(at/b)_n(bq/a)_{n+1}}{(t)_{2n+2}(bq)_n}(-abt)^{n+1} q^{3\binom{n+1}{2}}.$$

Theorem 11:

$$F(a,b;t)=\sum_{n=0}^{\infty}\frac{(a)_{2n}(at/b)_n}{(bq)_{2n}(t)_{n+1}}(1-atq^{3n})(bt^2)^n q^{2n^2}$$
$$+\frac{1}{t}\sum_{n=0}^{\infty}\frac{(a)_{2n+1}(at/b)_{n+1}}{(bq)_{2n+1}(t)_{n+1}}(bt^2)^{n+1} q^{(2n+1)(n+1)}.$$

Corollary 12:

$$F(a,b;t)=\sum_{n=0}^{\infty}\frac{(a)_n(at/b)_{2n}}{(bq)_n(t)_{2n+1}}(1-atq^{3n})(b^2t)^n q^{2n^2}$$
$$+bt\sum_{n=0}^{\infty}\frac{(a)_{n+1}(at/b)_{2n+1}}{(bq)_n(t)_{2n+2}}(b^2t)^n q^{(n+1)(2n+1)}.$$

Theorem 13:

$$F(a,b;t)=\sum_{n=0}^{\infty}\frac{(a)_{2n}(at/b)_{2n}}{(bq)_n(t)_{n+1}(a/b)_n}(1-atq^{3n})(bt)^n q^{n^2}$$
$$-\frac{a}{b}\sum_{n=0}^{\infty}\frac{(a)_{2n+1}(at/b)_{2n+1}}{(bq)_n(t)_{n+1}(a/b)_{n+1}}(bt)^{n+1}q^{n^2+3n+1}.$$

Theorem 14:

$$F(a,b;t)=\sum_{n=0}^{\infty}\frac{(a)_n(at/b)_n(bq/a)_n}{(bq)_{2n}(t)_{2n+1}}(1-atq^{3n})(-abt^2)^n q^{\frac{n(7n-1)}{2}}$$
$$+btq\sum_{n=0}^{\infty}\frac{(a)_{n+1}(at/b)_{n+1}(bq/a)_n}{(bq)_{2n+1}(t)_{2n+2}}(-abt^2)^n q^{\frac{n(7n+1)}{2}}.$$

Theorem 15:

$$F(a,b;t)=\sum_{n=0}^{\infty}\frac{(a)_{2n}(at/b)_{3n}}{(t)_{2n+1}(a/b)_n(bq)_n}(1-atq^{4n})(b^2t)^n q^{2n^2}$$
$$+\frac{1}{b}\sum_{n=0}^{\infty}\frac{(a)_{2n+1}(at/b)_{3n+1}}{(t)_{2n+2}(a/b)_n(bq)_n}(b^2t)^{n+1}q^{(n+1)(2n+1)}$$
$$-\frac{a}{b}\sum_{n=0}^{\infty}\frac{(a)_{2n+1}(at/b)_{3n+2}}{(t)_{2n+2}(a/b)_{n+1}(bq)_n}(b^2t)^{n+1}q^{(2n+1)(n+2)}.$$

Corollary 16:

$$F(a,b;t)=\sum_{n=0}^{\infty}\frac{(at/b)_{2n}(a)_{3n}}{(bq)_{2n}(a/b)_n(t)_{n+1}}(1-atq^{4n})(bt^2)^n q^{2n^2}$$
$$+\frac{1}{t}\sum_{n=0}^{\infty}\frac{(at/b)_{2n+1}(a)_{3n+1}}{(bq)_{2n+1}(a/b)_n(t)_{n+1}}(bt^2)^{n+1}q^{(n+1)(2n+1)}$$
$$-\frac{a}{b}\sum_{n=0}^{\infty}\frac{(at/b)_{2n+1}(a)_{3n+2}}{(bq)_{2n+1}(a/b)_{n+1}(t)_{n+1}}(bt^2)^{n+1}q^{(2n+1)(n+2)}.$$

Theorem 17 for $|b| < 1$:

$$F(a,b;t) = (1-b)\sum_{n=0}^{\infty}\frac{(a)_{2n}(at/b)_{3n}}{(a/b)_{2n}(t)_{n+1}}b^n - a(1-b)\sum_{n=0}^{\infty}\frac{(a)_{2n}(at/b)_{3n+1}}{(a/b)_{2n+1}(t)_{n+1}}(bq^2)^n$$
$$- a(1-b)\sum_{n=0}^{\infty}\frac{(a)_{2n+1}(at/b)_{3n+2}}{(a/b)_{2n+2}(t)_{n+1}}b^n q^{2n+1}.$$

Corollary 18 for $|t| < 1$:

$$F(a,b;t) = \sum_{n=0}^{\infty}\frac{(at/b)_{2n}(a)_{3n}}{(a/b)_{2n}(bq)_n}t^n - at/b\sum_{n=0}^{\infty}\frac{(at/b)_{2n}(a)_{3n+1}}{(a/b)_{2n+1}(bq)_n}(tq^2)^n$$
$$- at/b\sum_{n=0}^{\infty}\frac{(at/b)_{2n+1}(a)_{3n+2}}{(a/b)_{2n+2}(bq)_n}t^n q^{2n+1}.$$

Theorem 19 for $|b| < 1$:

$$F(a,b;t) = (1-b)\sum_{n=0}^{\infty}\frac{(a)_n(at/b)_{3n}}{(t)_{2n+1}(a/b)_n}b^{2n} - a(1-b)\sum_{n=0}^{\infty}\frac{(a)_n(at/b)_{3n+1}}{(t)_{2n+1}(a/b)_{n+1}}(b^2q)^n$$
$$+ (1-b)\sum_{n=0}^{\infty}\frac{(a)_{n+1}(at/b)_{3n+2}}{(t)_{2n+2}(a/b)_{n+1}}b^{2n+1}.$$

Corollary 20 for $|t| < 1$:

$$F(a,b;t) = \sum_{n=0}^{\infty}\frac{(at/b)_n(a)_{3n}}{(bq)_{2n}(a/b)_n}t^{2n} - at/b\sum_{n=0}^{\infty}\frac{(at/b)_n(a)_{3n+1}}{(bq)_{2n}(a/b)_{n+1}}(t^2q)^n$$
$$+ \sum_{n=0}^{\infty}\frac{(at/b)_{n+1}(a)_{3n+2}}{(bq)_{2n+1}(a/b)_{n+1}}t^{2n+1}.$$

Theorem 21 for $|t| < 1$:

$$F(a,b;t) = \sum_{n=0}^{\infty} \frac{(a)_n (bq/a)_n}{(bq)_{2n}(bq/at)_n} t^n - \frac{b}{at} \sum_{n=0}^{\infty} \frac{(a)_{n+1}(bq/a)_n}{(bq)_{2n+1}(bq/at)_{n+1}} (tq)^{n+1}.$$

Corollary 22 for $|b| < 1$:

$$F(a,b;t) = (1-b) \sum_{n=0}^{\infty} \frac{(at/b)_n (bq/a)_n}{(t)_{2n+1}(q/a)_n} b^n - \frac{1-b}{a} \sum_{n=0}^{\infty} \frac{(at/b)_{n+1}(bq/a)_n}{(t)_{2n}(q/a)_{n+1}} (tq)^{n+1}.$$

Theorem 23:

$$\begin{aligned} F(a,b;t) &= \sum_{n=0}^{\infty} \frac{(bq/a)_{2n}}{(bq)_{2n}(bq/at)_n (t)_{n+1}} (-at)^n q^{\binom{n}{2}} \\ &\quad + b \sum_{n=0}^{\infty} \frac{(bq/a)_{2n+1}}{(bq)_{2n+1}(bq/at)_{n+1}(t)_{n+1}} (-at)^n q^{\binom{n+2}{2}}. \end{aligned}$$

Corollary 24:

$$\begin{aligned} F(a,b;t) &= \sum_{n=0}^{\infty} \frac{(bq/a)_{2n}}{(t)_{2n+1}(q/a)_n (bq)_n} (-at)^n q^{\binom{n}{2}} \\ &\quad + b \sum_{n=0}^{\infty} \frac{(bq/a)_{2n+1}}{(t)_{2n+2}(q/a)_{n+1}(bq)_n} (-at)^n q^{\binom{n+2}{2}}. \end{aligned}$$

Theorem 29 with $k + l > 0$ and $|b|, |t| < 1$:

$$\begin{aligned} F(a,b;t) &= \sum_{i=0}^{k-1} \sum_{n=0}^{\infty} \frac{(a)_{kn+i}(at/b)_{ln}}{(bq)_{kn+i}(t)_{ln}} b^{ln} t^{kn+i} q^{ln(kn+i)} \\ &\quad + \sum_{j=0}^{l-1} \sum_{n=0}^{\infty} \frac{(a)_{k(n+1)}(at/b)_{ln+j}}{(bq)_{k(n+1)-1}(t)_{ln+j+1}} b^{ln+j} t^{k(n+1)} q^{k(n+1)(ln+j)}. \end{aligned}$$

Corollary 30 with $k > 0$ and $|b|, |t| < 1$:

$$F(a,b;t) = \sum_{n=0}^{\infty} \frac{(a)_{kn}(at/b)_n}{(bq)_{kn}(t)_{n+1}}(1 - atq^{(k+1)n})b^n t^{kn} q^{kn^2} + \sum_{i=1}^{k-1}\sum_{n=0}^{\infty} \frac{(a)_{kn+i}(at/b)_{n+1}}{(bq)_{kn+i}(t)_{n+1}} b^{n+1} t^{kn+i} q^{(n+1)(kn+i)}.$$

Corollary 31 with $l > 0$ and $|b|, |t| < 1$:

$$F(a,b;t) = \sum_{n=0}^{\infty} \frac{(a)_n(at/b)_{ln}}{(bq)_n(t)_{ln}} b^{ln} t^n q^{ln^2} + \sum_{i=0}^{l-1}\sum_{n=0}^{\infty} \frac{(a)_{n+1}(at/b)_{ln+i}}{(bq)_n(t)_{ln+i+1}} b^{ln+i} t^{n+1} q^{(ln+i)(n+1)}.$$

Theorem 32 for $k \geq 1$:

$$F(a,b;t) = \sum_{n=0}^{\infty} \frac{(bq/a)_n(at/b)_{(k-1)n}}{(bq)_n(t)_{kn+1}}(-ab^{k-1}t)^n q^{(2k-1)\frac{n(n-1)}{2}+(k-1)n} + \sum_{i=0}^{k-2}\sum_{n=0}^{\infty} \frac{(bq/a)_{n+1}(at/b)_{(k-1)n+i}}{(bq)_n(t)_{kn+i+2}}(-at)^{n+1} b^{(k-1)n+i} q^{(2k-1)\frac{n(n+1)}{2}+i(n+1)}.$$

Corollary 33 for $k \geq 1$:

$$F(a,b;t) = \sum_{n=0}^{\infty} \frac{(bq/a)_n(a)_{(k-1)n}}{(t)_{n+1}(bq)_{kn}}(-at^k)^n q^{(2k-1)\frac{n(n-1)}{2}+(k-1)n} + \sum_{i=0}^{k-2}\sum_{n=0}^{\infty} \frac{(bq/a)_{n+1}(a)_{(k-1)n+i}}{(t)_{n+1}(bq)_{nk+i+1}}(-a)^{n+1} t^{kn+i+1} q^{(2k-1)\frac{n(n+1)}{2}+i(n+1)}.$$

Theorem 34:

(83) valid for $k+m>0$, $l\geq 0$, and $|b|,|t|<1$:

$$F(a,b;t)=\sum_{i=0}^{k-1}t^i\sum_{n=0}^{\infty}\frac{(a)_{(k+l)n+i}(at/b)_{(l+m)n}}{(bq)_{kn+i}(a/b)_{ln}(t)_{mn}}b^{mn}t^{kn}q^{mn(kn+i)}$$
$$-\frac{at^k}{b}\sum_{i=0}^{l-1}q^i\sum_{n=0}^{\infty}\frac{(a)_{(k+l)n+k+i}(at/b)_{(l+m)n+i}}{(bq)_{k(n+1)-1}(aq/b)_{ln+i+1}(t)_{mn}}b^{mn}t^{kn}q^{kmn(n+1)+ln}$$
$$+t^k\sum_{i=0}^{m-1}b^i\sum_{n=0}^{\infty}\frac{(a)_{(k+l)(n+1)}(at/b)_{(l+m)n+l+i}}{(bq)_{k(n+1)-1}(a/b)_{l(n+1)}(t)_{mn+i+1}}b^{mn}t^{kn}q^{k(mn+i)(n+1)}.$$

(84) valid for $k+l>0$, $m\geq 0$, and $|b|,|t|<1$:

$$F(a,b;t)=\sum_{i=0}^{k-1}t^i\sum_{n=0}^{\infty}\frac{(a)_{kn+i}(bq/a)_{ln}(at/b)_{mn}}{(bq)_{(k+l)n+i}(t)_{(l+m)n}}$$
$$\cdot(-a)^{ln}b^{mn}t^{(k+l)n}q^{in(l+m)+\binom{ln}{2}+kln^2+mn^2(k+l)}$$
$$+t^k\sum_{i=0}^{l-1}(-a)^it^iq^{\binom{i}{2}+ik}\sum_{n=0}^{\infty}\frac{(a)_{k(n+1)}(bq/a)_{ln+i}(at/b)_{mn}}{(bq)_{(k+l)n+k+i}(t)_{(l+m)n+i+1}}$$
$$\cdot(-a)^{ln}b^{mn}t^{(k+l)n}q^{ikn+(k+i)(l+m)n+\binom{ln}{2}+kln^2+mn^2(k+l)}$$
$$+(-a)^lt^{k+l}q^{\binom{l}{2}+lk}\sum_{i=0}^{m-1}b^iq^{i(k+l)}\sum_{n=0}^{\infty}\frac{(a)_{k(n+1)}(bq/a)_{l(n+1)}(at/b)_{mn+i}}{(bq)_{(k+l)(n+1)}(t)_{(l+m)n+l+i+1}}(1-bq^{(k+l)(n+1)})$$
$$\cdot(-a)^{ln}b^{mn}t^{(k+l)n}q^{lkn+n(k+l)(i+l+m)+\binom{ln}{2}+kln^2+mn^2(k+l)}.$$

(85) valid for $k+l>0$, $m\geq 0$, and $|t|<1$:

$$F(a,b;t)=\sum_{i=0}^{k-1}t^i\sum_{n=0}^{\infty}\frac{(a)_{kn+i}(bq/a)_{(l+m)n}}{(bq)_{(k+l+m)n+i}(t)_{ln}(bq/at)_{mn}}(-a)^{ln}t^{(k+l)n}q^{iln+\binom{ln}{2}+kln^2}$$

$$+ t^k \sum_{i=0}^{l-1} (-a)^i t^i q^{\binom{i}{2}+ik} \sum_{n=0}^{\infty} \frac{(a)_{kn+k}(bq/a)_{(l+m)n+i}}{(bq)_{(k+l+m)n+k+i}(t)_{ln+i+1}(bq/at)_{mn}}$$
$$\cdot t^{(k+l)n}(-a)^{ln} q^{kni+ln(k+i)+\binom{ln}{2}+kln^2}$$
$$+ b(-a)^{l-1} t^{k+l-1} q^{1+\binom{l}{2}+lk} \sum_{i=0}^{m-1} q^i \sum_{n=0}^{\infty} \frac{(a)_{kn+k}(bq/a)_{(l+m)n+l+i}}{(bq)_{(k+l+m)n+k+l+i}(t)_{ln+l}(bq/at)_{mn+i+1}}$$
$$\cdot t^{(k+l)n}(-a)^{ln} q^{(k+l)ln+lkn+mn+\binom{ln}{2}+kln^2}.$$

Corollary 35:

(86) valid for $k+m>0$, $l \geq 0$, and $|b|, |t| < 1$:

$$F(a,b;t) = (1-b) \sum_{i=0}^{k-1} b^i \sum_{n=0}^{\infty} \frac{(at/b)_{(k+l)n+i}(a)_{(l+m)n}}{(t)_{kn+i+1}(a/b)_{ln}(b)_{mn}} t^{mn} b^{kn} q^{mn(kn+i)}$$
$$- a(1-b)b^{k-1} \sum_{i=0}^{l-1} q^i \sum_{n=0}^{\infty} \frac{(at/b)_{(k+l)n+k+i}(a)_{(l+m)n+i}}{(t)_{k(n+1)}(a/b)_{ln+i+1}(b)_{mn}} t^{mn} b^{kn} q^{kmn(n+1)+ln}$$
$$+ b^k \sum_{i=0}^{m-1} t^i \sum_{n=0}^{\infty} \frac{(at/b)_{(k+l)(n+1)}(a)_{(l+m)n+l+i}}{(t)_{k(n+1)}(a/b)_{l(n+1)}(bq)_{mn+i}} t^{mn} b^{kn} q^{k(mn+i)(n+1)}.$$

(87) valid for $k+l>0$, $m \geq 0$, and $|b|, |t| < 1$:

$$F(a,b;t) = \sum_{i=0}^{k-1} b^i \sum_{n=0}^{\infty} \frac{(at/b)_{kn+i}(bq/a)_{ln}(a)_{mn}}{(t)_{(k+l)n+i+1}(bq)_{(l+m)n-1}}$$
$$\cdot (-a)^{ln} t^{(l+m)n} b^{kn} q^{in(l+m)+\binom{ln}{2}+kln^2+mn^2(k+l)}$$
$$+ b^k \sum_{i=0}^{l-1} (-a)^i t^i q^{\binom{i}{2}+ik} \sum_{n=0}^{\infty} \frac{(at/b)_{k(n+1)}(bq/a)_{ln+i}(a)_{mn}}{(t)_{(k+l)n+k+i+1}(bq)_{(l+m)n+i}}$$
$$\cdot (-a)^{ln} t^{(m+l)n} b^{kn} q^{ikn+(k+i)(l+m)n+\binom{ln}{2}+kln^2+mn^2(k+l)}$$

$$+ (-a)^l b^k t^l q^{\binom{l}{2}+lk} \sum_{i=0}^{m-1} t^i q^{i(k+l)} \sum_{n=0}^{\infty} \frac{(at/b)_{k(n+1)}(bq/a)_{l(n+1)}(a)_{mn+i}}{(t)_{(k+l)(n+1)+1}(bq)_{(l+m)n+l+i}} (1 - bq^{k+l+(k+l)n})$$

$$\cdot (-a)^{ln} t^{(m+l)n} b^{kn} q^{lkn+n(k+l)(i+l+m)+\binom{ln}{2}+kln^2+mn^2(k+l)}.$$

(88) valid for $k + l > 0$, $m \geq 0$, and $|b| < 1$:

$$F(a, b; t) = (1 - b) \sum_{i=0}^{k-1} b^i \sum_{n=0}^{\infty} \frac{(at/b)_{kn+i}(bq/a)_{(l+m)n}}{(t)_{(k+l+m)n+i+1}(b)_{ln}(q/a)_{mn}} (-at)^{ln} b^{kn} q^{iln+\binom{ln}{2}+kln^2}$$

$$+ b^k \sum_{i=0}^{l-1} (-a)^i q^{\binom{i}{2}+ik} \sum_{n=0}^{\infty} \frac{(at/b)_{kn+k}(bq/a)_{(l+m)n+i}}{(t)_{(k+l+m)n+k+i+1}(bq)_{ln+i}(q/a)_{mn}}$$

$$\cdot b^{kn} (-at)^{ln} q^{kni+ln(k+i)+\binom{ln}{2}+kln^2}$$

$$+ (1 - b)(-a)^{l-1} b^k q^{1+\binom{l}{2}+lk} \sum_{i=0}^{m-1} q^i \sum_{n=0}^{\infty} \frac{(at/b)_{kn+k}(bq/a)_{(l+m)n+l+i}}{(t)_{(k+l+m)n+k+l+i+1}(b)_{ln+l}(q/a)_{mn+i+1}}$$

$$\cdot b^{kn} (-at)^{ln} q^{(k+l)ln+lkn+mn+\binom{ln}{2}+kln^2}.$$

## A.3 Derivations of Seeds

Previously, we have derived the identities for $\sigma = \alpha\beta$, $\sigma = \tau$, and $\sigma = \beta\tau$. We continue on with $\sigma = \alpha\beta\tau$. Putting together the identity $\sigma = \alpha\beta$ and then applying $\sigma = \tau$, we can get $\sigma = \alpha\beta\tau$ as

$$\begin{aligned} F(a,b;t) &= 1 + \frac{1-a}{1-bq} t F(aq,bq;t) \\ &= 1 + \frac{1-a}{1-bq} t \left( \frac{1-bq}{1-t} + \frac{bq-atq}{1-t} bqF(aq,bq;tq) \right) \\ &= 1 + \frac{1-a}{1-t} t + \frac{(1-a)(1-at/b)}{(1-t)(1-bq)} btqF(aq,bq;tq) \\ &= \frac{1-at}{1-t} + \frac{(1-a)(1-at/b)}{(1-t)(1-bq)} btqF(aq,bq;tq) \end{aligned}$$

which gives (48) and (49) in Table 2.1. For example, $\sigma = \beta$, we can use $\tau$ and $\beta\tau$. We will use another form of $\sigma = \beta\tau$ for this derivation, namely solving for $F(a, bq; tq)$ to get

$$F(a,bq;tq) = \frac{1-bq}{at(1-bq/a)} - \frac{(1-bq)(1-t)}{at(1-bq/a)} F(a,b;t).$$

Using this along with a $\beta$ shift of $\sigma = \tau$, we have

$$\begin{aligned} F(a,bq;t) &= \frac{1-bq}{1-t} + \frac{bq-at}{1-t} F(a,b;tq) \\ &= \frac{1-bq}{1-t} + \frac{bq-at}{1-t} \cdot \left( \frac{1-bq}{at(1-bq/a)} - \frac{(1-bq)(1-t)}{at(1-bq/a)} F(a,b;t) \right) \\ &= \frac{1-bq}{1-t} + \frac{(1-bq)(bq-at)}{at(1-bq/a)(1-t)} - \frac{(1-bq)(bq-at)}{at(1-bq/a)} F(a,b;t) \\ &= \frac{1-bq}{1-t} + \frac{(1-bq)(1-bq/at)(-at)}{at(1-bq/a)(1-t)} - \frac{(1-bq)(1-bq/at)(-at)}{at(1-bq/a)} F(a,b;t) \\ &= \frac{1-bq}{1-t} - \frac{(1-bq)(1-bq/at)}{(1-bq/a)(1-t)} + \frac{(1-bq)(1-bq/at)}{1-bq/a} F(a,b;t). \end{aligned}$$

Solving for $F(a,b;t)$ gives

$$\begin{aligned} F(a,b;t) &= \frac{1-bq/a}{(1-bq/at)(1-bq)}\left(F(a,bq;t) - \frac{1-bq}{1-t} + \frac{(1-bq/at)(1-bq)}{(1-bq/a)(1-t)}\right) \\ &= \frac{1-bq/a}{(1-bq/at)(1-bq)}F(a,bq;t) - \frac{1-bq/a}{(1-t)(1-bq/at)} + \frac{1}{1-t} \\ &= \frac{1-bq/a}{(1-bq/at)(1-bq)}F(a,bq;t) - \frac{bq}{(1-bq/at)at}. \end{aligned}$$

This gives (38) and (39) in Table 2.1.

Using $\sigma = \beta$, we can obtain the identity for $\sigma = \alpha$. Similar to the derivation for $\beta$, we will use a shifted version of $\beta$ and $\alpha\beta$ after solving for $F(aq,bq;t)$,

$$F(aq,bq;t) = \frac{1-bq}{(1-a)t}F(a,b;t) - \frac{1-bq}{(1-a)t}.$$

Thus,

$$\begin{aligned} F(aq,b;t) &= -\frac{b}{(1-b/at)at} + \frac{1-b/a}{(1-bq)(1-b/at)}F(aq,bq;t) \\ &= -\frac{b}{(1-b/at)at} + \frac{1-b/a}{(1-bq)(1-b/at)}\left(\frac{1-bq}{(1-a)t}F(a,b;t) - \frac{1-bq}{(1-a)t}\right) \\ &= -\frac{b}{(1-b/at)at} + \frac{1-b/a}{(1-a)(1-b/at)t}F(a,b;t) - \frac{1-b/a}{(1-a)(1-b/at)t}. \end{aligned}$$

Solving for $F(a,b;t)$ gives

$$\begin{aligned} F(a,b;t) &= \frac{(1-a)(1-b/at)t}{1-b/a}\left(F(aq,b;t) + \frac{b}{(1-b/at)at} + \frac{1-b/a}{(1-a)(1-b/at)t}\right) \\ &= \frac{(1-a)(1-b/at)t}{(1-b/a)}F(aq,b;t) + \frac{(1-a)b}{(1-b/a)a} + 1 \\ &= \frac{(1-a)(1-at/b)}{1-a/b}F(aq,b;t) - \frac{(1-b)a}{(1-a/b)b}. \end{aligned}$$

This gives (36) and (37) in Table 2.1.

Lastly, we derive the identity $\sigma = \alpha\tau$ by using a shifted version of $\alpha$ and also $\tau$ after solving for $F(a, b; tq)$,

$$F(a,b;tq) = \frac{1-t}{(1-at/b)b}F(a,b;t) - \frac{1-b}{(1-at/b)b}.$$

Thus,

$$\begin{aligned}
F(a,b;tq) &= -\frac{(1-b)a}{(1-a/b)b} + \frac{(1-a)(1-atq/b)}{(1-a/b)}F(aq,b;tq)\\
\frac{1-t}{(1-at/b)b}F(a,b;t) - \frac{1-b}{(1-at/b)b} &= -\frac{(1-b)a}{(1-a/b)b} + \frac{(1-a)(1-atq/b)}{(1-a/b)}F(aq,b;tq)\\
F(a,b;t) &= \frac{(1-at/b)b}{1-t}\left(\frac{1-b}{(1-at/b)b} - \frac{(1-b)a}{(1-a/b)b}\right.\\
&\quad \left.+\frac{(1-a)(1-atq/b)}{(1-a/b)}F(aq,b;tq)\right)\\
F(a,b;t) &= \frac{1-b}{1-t} - \frac{(1-b)(1-at/b)a}{(1-a/b)(1-t)}\\
&\quad + \frac{(1-a)(1-at/b)(1-atq/b)b}{(1-t)(1-a/b)}F(aq,b;tq)
\end{aligned}$$

This gives (46) and (47) in Table 2.1.

## A.4 Proof of $\mathrm{n}^{\underline{\mathrm{th}}}$ Iteration of Seeds

*Proof.* For (51) and (58),

$$\begin{aligned}
A_{\beta^n} &= \prod_{i=0}^{n-1}\beta^i A_\beta = \prod_{i=0}^{n-1}\beta^i\left(\frac{(1-bq/a)}{(1-bq)(1-bq/at)}\right)\\
&= \prod_{i=0}^{n-1}\frac{(1-bq^{i+1}/a)}{(1-bq^{i+1})(1-bq^{i+1}/at)} = \frac{(bq/a)_n}{(bq)_n(bq/at)_n};
\end{aligned}$$

$$
\begin{aligned}
B_{\beta^n} &= \sum_{i=0}^{n-1} A_{\beta^i}\beta^i B_\beta = \sum_{i=0}^{n-1} \frac{(bq/a)_i}{(bq)_i(bq/at)_i}\beta^i \left(\frac{-bq}{(1-bq/at)at}\right) \\
&= \sum_{i=0}^{n-1} \frac{(bq/a)_i}{(bq)_i(bq/at)_i} \cdot \frac{-bq^{i+1}}{(1-bq^{i+1}/at)at} = -\frac{b}{at}\sum_{i=0}^{n-1} \frac{(bq/a)_i}{(bq)_i(bq/at)_{i+1}} q^{i+1}.
\end{aligned}
$$

For (52) and (59),

$$
\begin{aligned}
A_{\tau^n} &= \prod_{i=0}^{n-1} \tau^i A_\tau = \prod_{i=0}^{n-1} \tau^i \left(\frac{(1-at/b)}{(1-t)} b\right) \\
&= \prod_{i=0}^{n-1} \frac{(1-atq^i/b)}{(1-tq^i)} b = \frac{(at/b)_n}{(t)_n} b^n;
\end{aligned}
$$

$$
\begin{aligned}
B_{\tau^n} &= \sum_{i=0}^{n-1} A_{\tau^i}\tau^i B_\tau = \sum_{i=0}^{n-1} \frac{(at/b)_i}{(t)_i} b^i \tau^i \left(\frac{1-b}{1-t}\right) \\
&= \sum_{i=0}^{n-1} \frac{(at/b)_i}{(t)_i} b^i \cdot \frac{1-b}{1-tq^i} = (1-b)\sum_{i=0}^{n-1} \frac{(at/b)_i}{(t)_{i+1}} b^i.
\end{aligned}
$$

For (53) and (60),

$$
A_{(\alpha\beta)^n} = \prod_{i=0}^{n-1} (\alpha\beta)^i A_{\alpha\beta} = \prod_{i=0}^{n-1} (\alpha\beta)^i \left(\frac{(1-a)}{(1-bq)} t\right) = \prod_{i=0}^{n-1} \frac{(1-aq^i)}{(1-bq^{i+1})} t = \frac{(a)_n}{(bq)_n} t^n;
$$

$$
B_{(\alpha\beta)^n} = \sum_{i=0}^{n-1} A_{(\alpha\beta)^i} (\alpha\beta)^i B_{(\alpha\beta)} = \sum_{i=0}^{n-1} \frac{(a)_i}{(bq)_i} t^i (\alpha\beta)^i (1) = \sum_{i=0}^{n-1} \frac{(a)_i}{(bq)_i} t^i.
$$

For (54) and (61),

$$\begin{aligned}
A_{(\beta\tau)^n} &= \prod_{i=0}^{n-1} (\beta\tau)^i A_{\beta\tau} = \prod_{i=0}^{n-1} (\beta\tau)^i \left( -\frac{(1-bq/a)}{(1-bq)(1-t)} ta \right) \\
&= \prod_{i=0}^{n-1} -\frac{(1-bq^{i+1}/a)}{(1-bq^{i+1})(1-tq^i)} atq^i = \frac{(bq/a)_n}{(bq)_n (t)_n} (-at)^n q^{n(n-1)/2};
\end{aligned}$$

$$\begin{aligned}
B_{(\beta\tau)^n} &= \sum_{i=0}^{n-1} A_{(\beta\tau)^i} (\beta\tau)^i B_{(\beta\tau)} = \sum_{i=0}^{n-1} \frac{(bq/a)_i}{(bq)_i (t)_i} (at)^i q^{i(i-1)/2} (\beta\tau)^i \left( \frac{1}{1-t} \right) \\
&= \sum_{i=0}^{n-1} \frac{(bq/a)_i}{(bq)_i (t)_{i+1}} (at)^i q^{i(i-1)/2}.
\end{aligned}$$

For (55) and (62),

$$\begin{aligned}
A_{(\alpha\tau)^n} &= \prod_{i=0}^{n-1} (\alpha\tau)^i A_{\alpha\tau} = \prod_{i=0}^{n-1} (\alpha\tau)^i \left( \frac{(1-a)(1-at/b)(1-atq/b)}{(1-t)(1-a/b)} b \right) \\
&= \prod_{i=0}^{n-1} \frac{(1-aq^i)(1-atq^{2i}/b)(1-atq^{2i+1}/b)}{(1-tq^i)(1-aq^i/b)} b \\
&= \frac{(a)_n (at/b)_{2n}}{(t)_n (a/b)_n} b^n;
\end{aligned}$$

$$\begin{aligned}
B_{(\alpha\tau)^n} &= \sum_{i=0}^{n-1} A_{(\alpha\tau)^i} (\alpha\tau)^i B_{(\alpha\tau)} \\
&= \sum_{i=0}^{n-1} \frac{(a)_i (at/b)_{2i}}{(t)_i (a/b)_i} b^i (\alpha\tau)^i \left( \frac{1-b}{1-t} - \frac{(1-b)(1-at/b)a}{(1-t)(1-a/b)} \right) \\
&= \sum_{i=0}^{n-1} \frac{(a)_i (at/b)_{2i}}{(t)_i (a/b)_i} b^i \cdot \frac{1-b}{1-tq^i} - \sum_{i=0}^{n-1} \frac{(a)_i (at/b)_{2i}}{(t)_i (a/b)_i} b^i \cdot \frac{(1-b)(1-atq^{2i}/b)a}{(1-tq^i)(1-aq^i/b)} \\
&= (1-b) \sum_{i=0}^{n-1} \frac{(a)_i (at/b)_{2i}}{(t)_{i+1} (a/b)_i} b^i - a(1-b) \sum_{i=0}^{n-1} \frac{(a)_i (at/b)_{2i+1}}{(t)_{i+1} (a/b)_{i+1}} (bq)^i.
\end{aligned}$$

Lastly, for (56) and (63),

$$A_{(\alpha\beta\tau)^n} = \prod_{i=0}^{n-1}(\alpha\beta\tau)^i A_{\alpha\beta\tau} = \prod_{i=0}^{n-1}(\alpha\beta\tau)^i \left(\frac{(1-a)(1-at/b)}{(1-bq)(1-t)}btq\right)$$
$$= \prod_{i=0}^{n-1}\frac{(1-aq^i)(1-atq^i/b)}{(1-bq^{i+1})(1-tq^i)}btq^{2i+1} = \frac{(a)_n(at/b)_n}{(bq)_n(t)_n}(bt)^n q^{n^2};$$

$$B_{(\alpha\beta\tau)^n} = \sum_{i=0}^{n-1} A_{(\alpha\beta\tau)^i}(\alpha\beta\tau)^i B_{(\alpha\beta\tau)} = \sum_{i=0}^{n-1}\frac{(a)_i(at/b)_i}{(bq)_i(t)_i}(bt)^i q^{i^2}(\alpha\beta\tau)^i\left(\frac{1-at}{1-t}\right)$$
$$= \sum_{i=0}^{n-1}\frac{(a)_i(at/b)_i}{(bq)_i(t)_i}(bt)^i q^{i^2}\cdot\frac{1-atq^{2i}}{1-tq^i} = \sum_{i=0}^{n-1}\frac{(a)_i(at/b)_i}{(bq)_i(t)_{i+1}}(bt)^i q^{i^2}(1-atq^{2i}).$$

□

## A.5 Numerical Checks Via Maxima

We use the following definition for q-products in our Maxima code,

```
qprod(a,n,q):=bfloat(block( if n=0 then return(1)
else return(product(1-a*q^i,i,0,n-1))));
```

along with the function,

```
G(a,b,t,q):=bfloat((1/(1-t))*sum((qprod(b*q/a,i,q)/
(qprod(b*q,i,q)*qprod(t*q,i,q)))
*(-a*t)^(i)*q^(i*(i-1)*0.5),i,0,10));
```

to represent $F(a, b; t)$. We choose this representation since convergence is much faster. Along with this, we must specify values of $a$, $b$, $t$, $q$ (noting that $|q| < 1$) and $n$, and for the general identity, $k$, $l$, and $m$.

```
a:bfloat(0.356);
q:bfloat(0.025);
b:bfloat(0.033);
t:bfloat(0.847);
n:bfloat(10);
```

The first check is of identities presented in Theorem 34, starting with (83):

```
bfloat(sum(sum(qprod(a,(k+l)*n+i,q)*qprod(a*t/b,(l+m)*n,q)
/(qprod(b*q,k*n+i,q)*qprod(a/b,l*n,q)*qprod(t,m*n,q))*b^(m*n)
*t^(k*n+i)*q^(m*n*(k*n+i)),n,0,10),i,0,k-1)
-(a/b)*sum(sum((1-b*q^(k*n+k))*qprod(a,(k+l)*n+k+i,q)
*qprod(a*t/b,(l+m)*n+i,q)/(qprod(b*q,k*(n+1),q)*qprod(a/b,l*n+i+1,q)
*qprod(t,m*n,q))*b^(m*n)*t^(k*n+k)
*q^(i+k*m*n*(n+1)+l*n),n,0,10),i,0,l-1)
+sum(sum((1-b*q^(k*n+k))*qprod(a,(k+l)*(n+1),q)*qprod(a*t/b,(l+m)*n+l+i,q)
/(qprod(b*q,k*(n+1),q)*qprod(a/b,l*(n+1),q)*qprod(t,m*n+i+1,q))*b^(m*n+i)
*t^(k*(n+1))*q^(k*(m*n+i)*(n+1)),n,0,10),i,0,m-1) -G(a,b,t,q));
```

$$-9.6281200236379792425670740077[45 digits]9261566831050935210232881 7b-94$$

For (84):

```
bfloat(sum(sum(qprod(a,k*n+i,q)*qprod(b*q/a,l*n,q)*qprod(a*t/b,m*n,q)
/(qprod(b*q,(k+l)*n+i,q)*qprod(t,(l+m)*n,q))*(-a)^(l*n)*b^(m*n)
*t^((k+l)*n+i)*q^(i*n*(l+m)+((l*n-1)*l*n/2)
+k*l*n^2+m*n^2*(k+l)),n,0,10),i,0,k-1)
+sum(sum(qprod(a,k*(n+1),q)*qprod(b*q/a,l*n+i,q)*qprod(a*t/b,m*n,q)
/(qprod(b*q,(k+l)*n+k+i,q)*qprod(t,(l+m)*n+i+1,q))*(-a)^(l*n+i)*b^(m*n)
*t^((k+l)*n+k+i)*q^((i*(i-1)/2)+i*k*(n+1)+(k+i)*(l+m)*n+((l*n-
1)*l*n/2)+k*l*n^2+m*n^2*(k+l)),n,0,10),i,0,l-1)
+sum(sum(qprod(a,k*n+k,q)*qprod(b*q/a,l*n+l,q)*qprod(a*t/b,m*n+i,q)
/(qprod(b*q,(k+l)*(n+1),q)*qprod(t,(l+m)*n+l+i+1,q))
*(-1*a)^(l*n+l)*b^(m*n+i)*t^((k+l)*(n+1))*q^(i*(k+l)+(l*(l-
1)/2)+l*k*(n+1)+n*(k+l)*(i+l+m)
+(l*n*(l*n-1)/2)+k*l*n^2+m*n^2*(k+l))*(1-b*q^((k+l)*(n+1))),n,0,10),i,0,m-1)
-G(a,b,t,q));
```

$$-9.6281200236379792425670740077[45 digits]92615668310509352102328817b-94$$

For (85):

```
bfloat(sum(sum(qprod(a,k*n+i,q)*qprod(b*q/a,(l+m)*n,q)
/(qprod(b*q,(k+l+m)*n+i,q)*qprod(t,l*n,q)*qprod(b*q/(a*t),m*n,q))
*(-a)^(l*n)*t^((k+l)*n+i)*q^(i*n*l+((l*n-1)*l*n/2)+k*l*n^2),n,0,10),i,0,k-1)
+sum(sum(qprod(a,k*(n+1),q)*qprod(b*q/a,(l+m)*n+i,q)
/(qprod(b*q,(k+l+m)*n+k+i,q)*qprod(t,l*n+i+1,q)*qprod(b*q/(a*t),m*n,q))
*(-a)^(l*n+i)*t^((k+l)*n+k+i)*q^((i*(i-1)/2)+i*k*(n+1)+(k+i)*l*n+((l*n-
1)*l*n/2)+k*l*n^2),n,0,10),i,0,l-1)
-(b/(a*t))*sum(sum(qprod(a,k*n+k,q)*qprod(b*q/a,(l+m)*n+l+i,q)
/(qprod(b*q,(k+l+m)*n+k+l+i,q)*qprod(t,l*n+l,q)*qprod(b*q/(a*t),m*n+i+1,q))
*(-1*a)^(l*n+l)*t^((k+l)*(n+1))*q^(i+1+(l*(l-1)/2)+l*k+(k+l)*l*n+l*k*n+m*n
+(l*n*(l*n-1)/2)+k*l*n^2),n,0,10),i,0,m-1) -G(a,b,t,q));
```

$$-9.6281200236379792425670740077[45 digits]92615668310509352102328817b-94$$

The following is a check of the identity obtained by iterating $\sigma = \alpha^2\beta$ with is conjugate to the identity generated by iterating $\sigma = \alpha\tau$.

```
bfloat(sum(qprod(a,2*i,q)*qprod(a*t/b,i,q)/(qprod(b*q,i,q)*qprod(a/b,i,q))*t^i,i,0,n-1)
-(a*t/b)*sum(qprod(a,2*i+1,q)*qprod(a*t/b,i,q)/(qprod(b*q,i,q)*qprod(a/b,i+1,q))
*t^i*q^i,i,0,n-1)+(qprod(a,2*n,q)*qprod(a*t/b,n,q)/(qprod(b*q,n,q)*qprod(a/b,n,q)))
*t^n*G(a*q^(2*n),b*q^n,t,q)-G(a,b,t,q));
```

$$-9.6281200236379792425670740077[45 digits]92615668310509352102328817b-94$$

The following is a check of the identity presented in Theorem 21.

```
bfloat(sum(qprod(a,i,q)*qprod(b*q/a,i,q)/(qprod(b*q,2*i,q)
*qprod(b*q/(a*t),i,q))*t^i,i,0,n-1)
-(b/(a*t))*sum(qprod(a,i+1,q)*qprod(b*q/a,i,q)/(qprod(b*q,2*i+1,q)
*qprod(b*q/(a*t),i+1,q))*t^(i+1)*q^(i+1),i,0,n-1)
+(qprod(a,n,q)*qprod(b*q/a,n,q)/(qprod(b*q,2*n,q)
*qprod(b*q/(a*t),n,q)))*t^n*G(a*q^n,b*q^(2*n),t,q) - G(a,b,t,q));
```

$$-9.62811773766319667801710957761[45 digits]27343854834118001206285096b-94$$

The following is a check of the identity presented in Corollary 10.

```
bfloat(sum(qprod(b*q/a,i,q)*qprod(a*t/b,i,q)/(qprod(b*q,i,q)*qprod(t,2*i+1,q))
*(-a*b*t)^i*q^((3*i-1)*i/2),i,0,n-1)-a*t*sum(qprod(b*q/a,i+1,q)*qprod(a*t/b,i,q)
/(qprod(b*q,i,q)*qprod(t,2*i+2,q))*(-a*b*t)^i*q^((i+1)*3*i/2),i,0,n-1)
+(qprod(b*q/a,n,q)*qprod(a*t/b,n,q)/(qprod(b*q,n,q)*qprod(t,2*n,q)))
*(-a*b*t)^n*q^((3*n-1)*n/2)*G(a,b*q^n,t*q^(2*n),q)-G(a,b,t,q));
```

$$-9.62812002363797924256707400077[45 digits]92615668310509352102328817b-94$$

The following is a check of the identity presented in Theorem 23.

```
bfloat(sum(qprod(b*q/a,2*i,q)/(qprod(b*q,2*i,q)*qprod(t,i+1,q)*qprod(b*q/(a*t),i,q))
(-a*t)^i*q^(i*(i-1)/2),i,0,n-1) +b*sum(qprod(b*q/a,2*i+1,q)/(qprod(b*q,2*i+1,q)
*qprod(t,i+1,q)
*qprod(b*q/(a*t),i+1,q))*(-a*t)^i*q^((i+2)*(i+1)/2),i,0,n-1)
+(qprod(b*q/a,2*n,q)/(qprod(b*q,2*n,q)*qprod(t,n,q)*qprod(b*q/(a*t),n,q)))
*(-a*t)^n*q^(n*(n-1)/2)*G(a,b*q^(2*n),t*q^n,q)-G(a,b,t,q));
```

$$-9.62811888065058796029209179 19[45 digits]599797615723136766543069 57b-94$$

The following is a check of the identity presented in Theorem 17.

```
bfloat((1-b)*sum(qprod(a,2*i,q)*qprod(a*t/b,3*i,q)/(qprod(t,i+1,q)
*qprod(a/b,2*i,q))*b^i,i,0,n-1)-(1-b)*a*sum(qprod(a,2*i,q)*qprod(a*t/b,3*i+1,q)
/(qprod(t,i+1,q)*qprod(a/b,2*i+1,q))*b^i*q^(2*i),i,0,n-1)
-(1-b)*a*sum(qprod(a,2*i+1,q)*qprod(a*t/b,3*i+2,q)/(qprod(t,i+1,q)
*qprod(a/b,2*i+2,q))*b^i*q^(2*i+1),i,0,n-1)+(qprod(a,2*n,q)
*qprod(a*t/b,3*n,q)/(qprod(t,n,q)*qprod(a/b,2*n,q)))*b^n
*G(a*q^(2*n),b,t*q^n,q)-G(a,b,t,q));
```

$$-9.62812002363797924256707400 77[45 digits]926156683105093521023288 17b-94$$

The following is a check of the identity presented in Theorem 19.

```
bfloat((1-b)*sum(qprod(a,i,q)*qprod(a*t/b,3*i,q)
/(qprod(t,2*i+1,q)*qprod(a/b,i,q))*b^(2*i),i,0,n-1)
-(1-b)*a*sum(qprod(a,i,q)*qprod(a*t/b,3*i+1,q)
/(qprod(t,2*i+1,q)*qprod(a/b,i+1,q))*b^(2*i)*q^(i),i,0,n-1)
+(1-b)*sum(qprod(a,i+1,q)*qprod(a*t/b,3*i+2,q)
/(qprod(t,2*i+2,q)*qprod(a/b,i+1,q))*b^(2*i+1),i,0,n-1)
+(qprod(a,n,q)*qprod(a*t/b,3*n,q)/(qprod(t,2*n,q)*qprod(a/b,n,q)))*b^(2*n)
*G(a*q^(n),b,t*q^(2*n),q)-G(a,b,t,q));
```

$$-9.628120023637979242567074007 7[45 digits]926156683105093521023288 17b-94$$

The following is a check of the identity presented in Theorem 12.

```
bfloat(sum(qprod(a,i,q)*qprod(a*t/b,2*i,q)/(qprod(b*q,i,q)*qprod(t,2*i+1,q))
*b^(2*i)*t^i*q^(2*i^2)*(1-a*t*q^(3*i)),i,0,n-1)
+sum(qprod(a,i+1,q)*qprod(a*t/b,2*i+1,q)
/(qprod(b*q,i,q)*qprod(t,2*i+2,q))*b^(2*i+1)*t^(i+1)*q^((2*i+1)*(i+1)),i,0,n-1)
+(qprod(a,n,q)*qprod(a*t/b,2*n,q)/(qprod(b*q,n,q)*qprod(t,2*n,q)))
*b^(2*n)*t^n*q^(2*n^2)*G(a*q^n,b*q^n,t*q^(2*n),q) -G(a,b,t,q));
```

$$-9.628121166625370524842056223 5[45 digits]252515750487050275503506 77b-94$$

The following is a check of the identity obtained by Theorem 9.

```
bfloat(sum(qprod(a,i,q)*qprod(b*q/a,i,q)/(qprod(b*q,2*i,q)*qprod(t,i+1,q))*(-a*t^2)^i
*q^((3*i-1)*i/2)*(1-a*t*q^(2*i)),i,0,n-1)+b*t*sum(qprod(a,i+1,q)*qprod(b*q/a,i,q)
/(qprod(b*q,2*i+1,q)*qprod(t,i+1,q))*(-a*t^2)^i*q^((3*i+2)*(i+1)/2),i,0,n-1)
+(qprod(a,n,q)*qprod(b*q/a,n,q)/(qprod(b*q,2*n,q)*qprod(t,n,q)))
*(-a*t^2)^n*q^((3*n-1)*n/2)*G(a*q^n,b*q^(2*n),t*q^(n),q)-G(a,b,t,q));
```

$$-9.628121166625370524842056223 5[45 digits]2525157504870502755 0350677b-94$$

The following is a check of the identity presented in Theorem 13.

```
bfloat(sum(qprod(a,2*i,q)*qprod(a*t/b,2*i,q)/(qprod(b*q,i,q)*qprod(t,i+1,q)*qprod(a/b,i,q))
*b^(i)*t^i*q^(i^2)*(1-a*t*q^(3*i)),i,0,n-1)-a*t*sum(qprod(a,2*i+1,q)
*qprod(a*t/b,2*i+1,q)/(qprod(b*q,i,q)*qprod(t,i+1,q)*qprod(a/b,i+1,q))*b^(i)
*t^(i)*q^(i^2+3*i+1),i,0,n-1)+(qprod(a,2*n,q)*qprod(a*t/b,2*n,q)/(qprod(b*q,n,q)
*qprod(t,n,q)*qprod(a/b,n,q)))*b^(n)*t^n*q^(n^2)
*G(a*q^(2*n),b*q^n,t*q^(n),q)-G(a,b,t,q));
```

$$-9.628120023637979242567074007 7[45 digits]9261566831050935210 23288817b-94$$

The following is a check of the identity presented in Theorem 11.

```
bfloat(sum(qprod(a,2*i,q)*qprod(a*t/b,i,q)/(qprod(b*q,2*i,q)*qprod(t,i+1,q))*(b*t^2)^i
*q^(2*i^2)*(1-a*t*q^(3*i)),i,0,n-1)+sum(qprod(a,2*i+1,q)*qprod(a*t/b,i+1,q)
/(qprod(b*q,2*i+1,q)*qprod(t,i+1,q))*b^(i+1)*t^(2*i+1)*q^((2*i+1)*(i+1)),i,0,n-1)
+(qprod(a,2*n,q)*qprod(a*t/b,n,q)/(qprod(b*q,2*n,q)*qprod(t,n,q)))*b^n*t^(2*n)
*q^(2*n^2)*G(a*q^(2*n),b*q^(2*n),t*q^n,q)-G(a,b,t,q));
```

$$-9.62812230961276180711703843 93[45digits]57887481786900702998372537b-94$$

The following is a check of the identity presented in Theorem 14.

```
bfloat(sum(qprod(a,i,q)*qprod(a*t/b,i,q)*qprod(b*q/a,i,q)
/(qprod(b*q,2*i,q)*qprod(t,2*i+1,q))
*(-a*b*t^2)^i*q^(i*(7*i-1)/2)*(1-a*t*q^(3*i)),i,0,n-1)+b*t*sum(qprod(a,i+1,q)
*qprod(a*t/b,i+1,q)*qprod(b*q/a,i,q)/(qprod(b*q,2*i+1,q)*qprod(t,2*i+2,q))
*(-a*b*t^2)^i*q^((7*i^2+7*i+2)/2),i,0,n-1)+(qprod(a,n,q)*qprod(a*t/b,n,q)
*qprod(b*q/a,n,q)/(qprod(b*q,2*n,q)*qprod(t,2*n,q)))*(-a*b*t^2)^n
*q^(n*(7*n-1)/2)*G(a*q^n,b*q^(2*n),t*q^(2*n),q)-G(a,b,t,q));
```

$$-9.628120023637979242567074 0077[45digits]92615668310509352102328817b-94$$

The following is a check of the identity presented in Theorem 15.

```
bfloat(sum(qprod(a,2*i,q)*qprod(a*t/b,3*i,q)/(qprod(b*q,i,q)*qprod(t,2*i+1,q)
*qprod(a/b,i,q))*(b^2*t)^i*q^(2*i^2)*(1-a*t*q^(4*i)),i,0,n-1)
+sum(qprod(a,2*i+1,q)*qprod(a*t/b,3*i+1,q)/(qprod(b*q,i,q)*qprod(t,2*i+2,q)
*qprod(a/b,i,q))*b^(2*i+1)*t^(i+1)*q^((2*i+1)*(i+1)),i,0,n-1)
-a*sum(qprod(a,2*i+1,q)*qprod(a*t/b,3*i+2,q)/(qprod(b*q,i,q)*qprod(t,2*i+2,q)
*qprod(a/b,i+1,q))*b^(2*i+1)*t^(i+1)*q^((2*i+1)*(i+2)),i,0,n-1)+(qprod(a,2*n,q)
*qprod(a*t/b,3*n,q)/(qprod(b*q,n,q)*qprod(t,2*n,q)*qprod(a/b,n,q)))*(b^2*t)^n
*q^(2*n^2)*G(a*q^(2*n),b*q^n,t*q^(2*n),q)-G(a,b,t,q));
```

$$-9.62812002363797924256707400 77[45digits]926156683105093521023288 17b-94$$

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