2016

Tail non-exchangeability

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ABSTRACT

TAIL NON-EXCHANGEABILITY

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Northern Illinois University, 2016
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Non-exchangeable dependence structures exist in the real world, and we are interested in how to identify the existence of *non-exchangeability* in the joint distributional tails and how to quantify the degree of such *tail non-exchangeability*. The results obtained and the approaches proposed benefit bivariate dependence modeling when dependence patterns in the tails are particularly important, as in the fields of quantitative finance, quantitative risk management, and econometrics. We focus on the bivariate case, and propose to use conditional expectations as the basis quantities. Then, for random variables X and Y, the departure between tail behavior of $E[X|Y > t]$ and $E[Y|X > t]$, or $E[X|Y = t]$ and $E[Y|X = t]$, when t is large, becomes sensible in detecting *tail non-exchangeability*. We use a bivariate copula to model the dependence between X and Y. Various devices of generating non-exchangeable copulas as well as three major tail behaviors for univariate margins are studied, in order to understand the interaction between the departure of those conditional expectations and the non-exchangeable dependence together with various types of margins. Based on the probabilistic properties of the *tail non-exchangeability* structures, we develop graphical approaches and statistical tests for analyzing dataset that may have *non-exchangeability* in the joint tail.

Simulation study and empirical study are then conducted to demonstrate the usefulness of the proposed approaches.

Key words: conditional expectation, copula, tail dependence, tail behavior.

NORTHERN ILLINOIS UNIVERSITY
DE KALB, ILLINOIS

AUGUST 2016

TAIL NON-EXCHANGEABILITY

BY

PARAMAHANSA PRAMANIK

A THESIS SUBMITTED TO THE GRADUATE SCHOOL

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS

FOR THE DEGREE

MASTER OF SCIENCE

DIVISION OF STATISTICS, DEPARTMENT OF MATHEMATICAL SCIENCES

Thesis Director:
Dr. Lei Hua

ACKNOWLEDGEMENTS

I want to thank to my supervisor Professor Dr. Lei Hua, for being patient, helping me during difficulties and supporting me constantly throughout the thesis. Apart from that, I want to express my gratitude to co-supervisor Professor Dr. Alan M. Polansky for giving me valuable suggestions and helping me all the times. I am grateful to the supervisory member Professor Dr. Sanjib Basu for valuable comments and suggestions which led me to improve my thesis. Finally, I want to thank my parents for everything they have done for me.

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CHAPTER 1

INTRODUCTION

The goal of the study is to propose useful measures for quantifying *non-exchangeability* in tails of joint distributions, and to find promising statistics for estimating the measures. The study on tail *non-exchangeability* would be particularly useful for providing more tail-ordered statistical models for modeling *non-exchangeability* structures in the tails, where the statistical inference would be very important for some fields, such as in *risk management, quantitative finance, psychometric, econometric,* and *environmetrics*. In this paper we are using the conditional tail-expectation as the instrument of *exchangeability*. For any two random variables X_1 and X_2 we define conditional tail expectations as $\lim_{t\to\infty} E[X_1|X_2 > t]$ and $\lim_{t\to\infty} E[X_2|X_1 > t]$ respectively and take the *ratio* of them. If the ratio is unity then the system is *exchangeable*, otherwise not. At the beginning of our paper we start deriving the conditions of *non-exchangeability* of a general *non-exchangeable* Copula. Then, we are making an *exchangeable* into a non-exchangeable Copula by using Khoudraji (1996) device. As the integration of the expectation does not have any closed form solution, following Hua and Joe (2014) we use either *Laplace Approximation* or *Watson's Lemma*.

Then we take two special Copulas namely Clayton and Gumbel with *Pareto, Weibull* and *Exponential* margins. The importance of taking these three margins are they represent *power family*, *sub-exponential* and *exponential* families respectively. In the case of Clayton Copula we use *Laplace Approximation* or *Watson's lemma* based on the conditions satisfied by different margins respectively. *Laplace Approximation* and *Watson's Lemma* work only for Khoudraji (1996) *non-exchangeable* Clayton survival Copula with *Pareto* margin. We show that, when the sum of inverse one *non-exchangeability* parameter with the parameter of *pareto* margin is greater than unity, we can use *Laplace Approximation* method, otherwise, we are using *Watson's Lemma*. In the case of *non-exchangeable* survival Clayton Copula with *Weibull* and *Exponential* margins we can *only* use *Watson's Lemma*. At the end of this section with *non-exchangeable* Clayton Copula we plot our simulation results and compare with the existing process. Furthermore, we do some hypothesis testing by using *empirical distribution function* in order to see if *non-exchangeability* occurs.

There are some simple *tail non-exchangeable* copulas, such as , Marshall-Olkin copula, and the generalized Clayton Copula, mentioned in Furman et al. (2015). A bivariate Khoudraji (1996) *non-exchangeable-transformed* Clayton copula is studied in Hofert et al. (2012), which leads to non-exchangeable structures. In what follows, we refer to this copula as KB4 copula. The *Cumulative Distribution Function* (CDF) of a bivariate KB4 copula is

$$C(u, v) = u^{1-\alpha_1} v^{1-\alpha_2} \; (u^{-\alpha_1 \delta} + v^{-\alpha_2 \delta} - 1)^{-1/\delta}, \quad \delta \geq 0, \quad (\alpha_1, \alpha_2) \in \; [0, 1]^2.$$

So, we can study the tail *non-exchangeability* through this specific example. Moreover, when more study can be done for copulas constructed through the *comonotonic* latent variables (see: Hua and Joe (2016)), where a non-exchangeable copula can be easily constructed, and we will need to model the degree of tail *non-exchangeability*.

One reasonable approach is to consider the difference between certain conditional quantities. Namely, we have random variables X_1 and X_2, and without loss of generality, we assume that X_1 and X_2 are identically distributed and non-negative, but not necessarily independent. If there is tail non-exchangeability between X_1 and X_2, then some summary quantities about $X_1|X_2$ and $X_2|X_1$ must be different. The summary quantities can be conditional expectation or conditional quantiles. For example, in Hua and Joe (2014), the forms $\mathrm{E}[X_1|X_2 = t]$ and $\mathrm{E}[X_1|X_2 > t]$ are used to study the strength of tail dependence as $t \to \infty$.

While in Bernard and Czado (2015), conditional quantiles are used to study the strength of tail dependence.

Borrowing the similar ideas from the above two papers, we believe that some metrics such as the following may be useful in quantifying the degree of tail non-exchangeability for a bivariate copula.

1.1 Basic concepts and motivations

In dependence modeling, one often uses copula functions to account for dependence patterns appeared in the tail part of the joint distribution, in particular when these patterns cannot be well modeled by commonly-used multivariate models such as multivariate Normal or Student t distributions. Most of the commonly-used bivariate copulas are of the exchangeable structure, meaning that $C(u,v) \equiv C(v,u)$ for any pairs of $(u,v) \in [0,1]^2$. There is recently research on non-exchangeable structures of bivariate copulas, such as Genest and Nešlehová (2013), where studies have been focused on the overall distributions. As copula modeling often plays an important role in accounting for dependence in the tails, one may be particularly interested in the non-exchangeable structure in the joint tails. Motivated by Hua and Joe (2014), where the tail behavior of $\mathrm{E}[X_1|X_2 > t]$ or $\mathrm{E}[X_1|X_2 = t]$ is studied for capturing the tail dependence strength between the bivariate random vector (X_1, X_2), we can consider the departure between $\mathrm{E}[X_1|X_2 > t]$ and $\mathrm{E}[X_2|X_1 > t]$, or between $\mathrm{E}[X_1|X_2 = t]$ and $\mathrm{E}[X_2|X_1 = t]$, to understand the degree of non-exchangeability appeared in the joint upper tails, as $t \to \infty$. Therefore, by the notion of *tail exchangeability* of a bivariate random vector, we introduce the following definition, which involves, without loss of generality, a pair of identically distributed random variables X_1 and X_2, which are supported on $[0,\infty)$, respectively.

Definition 1. *Let (X_1, X_2) be a bivariate random vector with identically distributed marginals, supported on $[0,\infty)^2$. Then the random vector (X_1, X_2) is said to be tail exchangeable of Type I if the following condition holds:*

$$\textit{Condition I:} \quad \lim_{t\to\infty} \eta_1(t) := \lim_{t\to\infty} \frac{\mathrm{E}[X_1|X_2 > t]}{\mathrm{E}[X_2|X_1 > t]} = 1, \tag{1.1}$$

and is tail exchangeable of Type II if the following condition holds:

$$\textit{Condition II:} \quad \lim_{t\to\infty} \eta_2(t) := \lim_{t\to\infty} \frac{\mathrm{E}[X_1|X_2 = t]}{\mathrm{E}[X_2|X_1 = t]} = 1. \tag{1.2}$$

Note here, we define "tail exchangeability" as a limiting property between two random variables when both of them take large values. When either of the conditions (1.1) and (1.2) does not hold, the random vector is said to be "tail non-exchangeable". The departure of functions $\eta_1(t)$ and $\eta_2(t)$ to 1 as $t \to \infty$ captures the degree of tail non-exchangeability.

Without loss of generality, assume that the (X_1, X_2) has a unique copula $C(\cdot,\cdot)$, of which the survival copula denoted as $\widehat{C}(\cdot,\cdot)$. Therefore the above conditions can be written as:

$$\text{Condition I':} \quad \lim_{t\to\infty} \eta_1(t) = \lim_{t\to\infty} \frac{\int_0^\infty \widehat{C}(\overline{F}(x), \overline{F}(t))dx}{\int_0^\infty \widehat{C}(\overline{F}(t), \overline{F}(x))dx} = 1, \tag{1.3}$$

where F is the cdf of the identical univariate marginals. The second condition is then

$$\text{Condition II':} \quad \lim_{t\to\infty} \eta_2(t) = \lim_{t\to\infty} \frac{\int_0^\infty \widehat{C}_{1|2}(\overline{F}(x)|\overline{F}(t))dx}{\int_0^\infty \widehat{C}_{2|1}(\overline{F}(t)|\overline{F}(x))dx} = 1. \tag{1.4}$$

It is clear that the tail behavior of $\eta_1(t)$ and $\eta_2(t)$ rely on both the copula C and the marginal F. Our goal is to study the effect of different marginals on the degree of tail non-exchangeability.

1.1.1 Notations and Symbols

In this section we discuss different notations and symbols. We are describing them one by one. Firstly, we define distribution functions as $F(x)$, where in the parenthesis we have the argument. In order to define the survival functions we use $\overline{F}(x)$, where in the parenthesis we have the argument. From the traditional literature we know the first order derivative of the distribution function is itself the density function. Hence, in the next sections when we differentiate $F(x)$ with respect to its argument, we write $f(x)$ as the derivative instead of $F'(x)$. In other words. $f(x) = F'(x) = \partial F(x)/\partial x$.

Secondly, throughout our paper we define the survival copula of an ordinary Copula C^* as $\widehat{C}^*(.\ ,.)$. Important thing in this case is that $\widehat{C}^*(.\ ,.)$ is the Copula before non-exchangeable transformation. After Khoudraji (1996) non-exchangeable transformation we have the survival Copula as $\widehat{C}(.\ ,.)$. Here, throughout our paper by Copula we actually mean Survival Copula where it itself is a function of survival functions [i.e. $\overline{F}(x)$]. In order to calculate the first order derivative we further use $\widehat{C}^*_{1|2}(u|v)$ instead of $\partial\widehat{C}^*(u,v)/\partial v$. From the literature we know that, $\widehat{C}^*_{1|2}(u|v)$ is a Cumulative distribution function(cdf) if $u, v \in [0,1]^2$. In this paper we put $u = \overline{F}(x)$ and $v = \overline{F}(y)$ to make them vary in $[0,1]$. At the tail, when we derive conditional expectation by Laplace approximation, we need to calculate second order derivative of our survival Copula. We use the notation $\widehat{C}^*_{1|2,2}(u|v)$. In other words, we define $\widehat{C}^*_{1|2,2}(u|v) = \partial^2\widehat{C}^*(u,v)/\partial v^2 = \partial\widehat{C}^*_{1|2}(u|v)/\partial v$.

In the chapter of Extreme value Copula we use the traditional notation of the Pickands (1981) dependence function as $A(\nu)$. Furthermore, after using non-exchangeable transformation in general extreme value Pickands (1981) dependence function becomes A^*. In the first part of this section we define our own non-exchangeable extreme value Copula. Inside of the parenthesis we define $\nu = \log(\overline{F}(t))/\log(\overline{F}(x)\overline{F}(t))$. According to Durante and Mesiar

(2010) an extreme value survival Copula is non-exchangeable if $A^*(\nu) \neq A^*(1-\nu)$. In our case, we define $A^*_\alpha(\nu) = A^*(1-\nu)$. Thus, using our notation with Durante and Mesiar (2010) we can say that, one survival extreme value Copula is non-exchangeable if $A^*(\nu) \neq A^*_\alpha(\nu)$.

Finally, by $\mathrm{E}[X_1|X_2 > t]$ and $\mathrm{E}[X_1|X_2 = t]$ we define two conditional expectations of X_1 given X_2 when X_2 is greater than and equal to t, respectively. Here, X_1 and X_2 are two random variables. Usually here we take a very large value of t and check the behavior of X_1 at that level.

CHAPTER 2

NON-EXCHANGEABILITY USING KHOUDRAJI DEVICE

In this chapter we will try to get some non-exchangeable structure. By tail non-exchangeability we mean, in a unit square drawn from the origin, if we plot a diagonal line towards north-east, two halves of the square around the point (1,1) are not the same. Throughout this paper we are going to use different kinds of non-exchangeable devices. In this chapter we are using Khoudraji (1996) non-exchangeable transformation (KT transformation) of copula. After doing the transformation, we are trying to find out the conditions under which we can use either Laplace approximation or Watson's lemma. As in this case, the integrations do not have any closed form solutions, we have to use some asymptotic approximations. Finally, we are checking the behavior of the conditional expectations with $t \to \infty$. In order to derive the mathematical properties firstly, we assume general functional forms of survival copula functions. Then, we try to show how our method works with Khoudraji (1996) transformed Copulas. Finally, we are trying to use this same Copula with Weibull and Exponential margins respectively.

2.1 Case I: $\mathrm{E}[X_1|X_2 > t]$

In order to study the conditions for tail non-exchangeability, we study $\mathrm{E}[X_1|X_2 > t]$ and $\mathrm{E}[X_1|X_2 = t]$ first as $t \to \infty$. Here we are assuming $\overline{F}(x), \overline{F}(t)$ are survival functions of distribution functions $F(x)$ and $F(t)$ respectively. Following Khoudraji (1996) device, Genest, Ghoudi and Rivest (1998),Genest, Kojadinovic and Neslehova (2011) and finally, Genest and Nešlehová (2013) define a non-exchangeable copula function;

$$\widehat{C}(\overline{F}(x), \overline{F}(t)) = \overline{F}(x)^{1-\alpha_1}\overline{F}(t)^{1-\alpha_2}\widehat{C}^*(\overline{F}(x)^{\alpha_1}, \overline{F}(t)^{\alpha_2}),\ (\alpha_1, \alpha_2) \in [0,1]^2, \tag{2.1}$$

where $\widehat{C}$ is any copula which is non-exchangeable in nature.

Firstly, we assume $X_1,\ X_2$ have same marginal distribution functions where cumulative distribution function (cdf) F is properly defined and continuous on $[0, \infty)$. Secondly, we also assume that, F and $\widehat{C}$ are absolutely continuous with the density function $f(.,.) = F^{'}(.,.) = \partial F(.,.)/\partial(.,.)$. Thirdly, X_1 has finite mean (i.e $\int_0^\infty \overline{F}(x)dx < \infty$). Now, we are going to apply the method directly.

Now, by using methods in Hua and Joe (2014) we are trying to see the *tail non-exchangeability* of survival copula defined in (2.1) which we are explaining in the next steps. As the conditional expectations do not have any closed form solutions, Hua and Joe (2014) suggests to use either Laplace approximation or Watson's lemma for asymptotic approximation when $t \to \infty$. They used these approximations in *exchangeable* copulas. In our paper, we are using the same method after transforming a copula into a *non-exchangeable* structure.

2.1.1 Theoretical results

In this section, we are constructing the conditional tail expectations by using general form of Survival Copula. We are trying to formulate a slow variation function at the tail. Here tail corresponds to extreme events. In the traditional literature we can find models which do good fit for the entire dataset. Our objective is to see the extreme cases and see the behavior of data there. We also know that conditional expectation is a traditional way of the linear regression analysis. In our case we are also trying to use a conditional expectations to understand the pattern of tail *non-exchangeability*. Our model might not be good for the entire dataset but, it is good for the extreme values. Here we use conditional expectations in order to check tail behavior. We assume the distributions are continuous. In order to calculate, conditional tail expectations we use the definition given by Hua and Joe (2014). As these integrations do not have any closed formed solutions, we have to use either Laplace Approximation or Watson's lemma. We try to find out a slow variation function which is not sensitive to the extreme values. In other words, during simulation if we increase any of the two variables or both of them at a very high level, the conditional expectation does not change that much.

2.1.1.1 General cases

Hua and Joe (2014) use numerical approximation methods to capture the tail order behavior of Gumbel Copula with Pareto, Weibull and Exponential margins respectively. In our paper we go further, starting with Khoudraji (1996) non-exchangeable transformed survival Copulas with these three margins and finally ending up with checking out tail order non-exchangeability. We assume X_1 and X_2 are continuous random variables supported on $[0, \infty)^2$. Firstly, we take any general form of survival Copula. Then we transform this into a non-exchangeable structure by using Khoudraji (1996) device. One important thing to note in this case is we do not include non-exchangeable copula as C^*. Here, we use $\widehat{C}(\overline{F}(x), \overline{F}(t)) = \overline{F}(x)^{1-\alpha_1}\overline{F}(t)^{1-\alpha_2}\widehat{C}^*(\overline{F}(x)^{\alpha_1}, \overline{F}(t)^{\alpha_2})$ as the Khoudraji (1996) non-exchangeable structure; where $(\alpha_1, \alpha_2) \in [0, 1]^2$. Clearly, in this transformation α_1 and α_2 are the non-exchangeable components.

We use some particular form of the conditional expectations. We follow the technique provided by Hua and Joe (2014). In their paper they [i.e. Hua and Joe (2014)] provide some numerical approximation methods to see the behavior of a copula at $t \to \infty$. They use two main methods, namely, Laplace Approximation and Watson's Lemma. In order to use either of them, we need certain conditions. They find out the conditions under which we are able to use either Laplace aproximation or Watson's lemma. In their examples they use Gumbel Copula with Pareto and Weibull margins. In this section we first define general survival Copulas which are exchangeable. Then we use Khoudraji (1996) device to make it non-exchangeable. Then we have the integration of conditional expectation in the form of $\mathrm{E}[X_1|X_2 > t] = \int_0^\infty \overline{F}(x)^{1-\alpha_1}\overline{F}(t)^{1-\alpha_2}\widehat{C}^*(\overline{F}(x)^{\alpha_1}, \overline{F}(t)^{\alpha_2})/\overline{F}(t)\ dx\ \forall\ t$. This integration does not have any closed form solution in general. Following Hua and Joe (2014) we convert the integrand into a multiplicatively separable form of $e^{g(s,T)}$ and $h(s) > 0$. In this case $T = \log \overline{F}(t)$. Thus, $T \to \infty \iff t \to \infty$. Then we use the properties like $g(0,T) = 0$ and $g(\infty, T) = -\infty$ before actually applying either of the above two approximations.

Throughout this section, we are considering Khoudraji-transformed survival Clayton Copula with Pareto margins as an example. We try to show that the theoretical results are still valid when we use the examples. Even the interesting fact is that, only in the case of Clayton Copula with Pareto margin we can use both the Laplace Approximation and Watson's lemma. We found that if $\alpha_1 + 1/\beta > 1$ we can use Laplace Approximation and if $\alpha_1 + 1/\beta \not> 1$ we can use Watson's lemma. Under Laplace approximation we have,

$$\mathrm{E}[X_1|X_2 > t] \sim Te^{Tg(\gamma,T)}\sqrt{\frac{2\pi}{-Tg''(\gamma,T)}}$$

at $t \to \infty$ and $T = -\log \overline{F}(t)$; where $g(s;T) = 1 + s/\beta - T^{-1}[\delta^{-1}\log(e^{\alpha_1\delta sT} + e^{\alpha_2\delta T} - 1) + (1-\alpha_1)sT + (1-\alpha_2)T]$ and $\gamma = (\alpha_2\delta)(\alpha_1\delta(1-\alpha_1))^{-1}$. Furthermore, if we assume the condition $\alpha_1 + \beta^{-1} > 1$ we have, $\gamma \to (\alpha_2)(\alpha_1)^{-1}$ as $\alpha_1 \to 1$. Now on the other hand, we can do Watson's lemma in order to calculate tail order conditional expectations. In this case we have, $\mathrm{E}[X_1|X_2 > t] \sim \left(\beta\left[(2-\alpha_1)+C+D\right]\right)^{-1}$, as $t \to \infty$ around $s \in (0, 0+\epsilon]$ where $C = (\alpha_1 e^{-\alpha_1 sT}\widehat{C}^*_{2|1}(e^{-\alpha_2 T}|e^{-\alpha_1 sT})(\widehat{C}^*(e^{-\alpha_1 sT}, e^{-\alpha_2 T}))^{-1}$ and $D = e^{-sT}f'(F^{-1}(1-e^{-sT}))(f^2(F^{-1}(1-e^{-sT})))^{-1}$. In the next section when we actually do the simulations, Watson's lemma does not give good approximation of this $E[X_1|X_2 > t]$ as $t \to \infty$ but the Laplace Approximation does. That is why we do not show the simulation results for Watson's lemma but we do show for Laplace Approximation.

Proposition 1. *For all* $(\alpha_1, \alpha_2) \in [0,1]^2$*,* $\beta > 1$ *and if* X_1 *and* X_2 *are two random variables which are dependent to each other and if* $g'(0,T) > 0$ *then,*

$$\mathrm{E}[X_1|X_2 > t] \sim -\log(\overline{F}(t))\; e^{-\log(\overline{F}(t))\;\; g(\gamma, -\log[\overline{F}(t)])}\sqrt{\frac{2\pi}{\log[\overline{F}(t)]\;\, g''(\gamma, -\log[\overline{F}(t)])}},$$

at $t \to \infty$*,* $\gamma = \lim_{T\to\infty}\max_s g(s,T)$*,* $T = -\log(\overline{F}(t))$*,* $\alpha_1 \neq \alpha_2$*, and around* $s \in (0, 0+\epsilon]$

Proof. Following the definition of conditional tail expectations provided by Hua and Joe (2014) we know;

$$\begin{aligned}\mathrm{E}[X_1|X_2 > t] &= \int_0^\infty \frac{\widehat{C}(\overline{F}(x), \overline{F}(t))}{\overline{F}(t)}\; dx,\; \forall\, t \\ &= \int_0^\infty \frac{\overline{F}(x)^{1-\alpha_1}\overline{F}(t)^{1-\alpha_2}\widehat{C}^*(\overline{F}(x)^{\alpha_1}, \overline{F}(t)^{\alpha_2})}{\overline{F}(t)}\; dx\; \forall\, t \qquad (2.2)\end{aligned}$$

As $y = -\log \overline{F}(x) \implies dy = -\frac{\frac{\partial \overline{F}(x)}{\partial x}}{\overline{F}(x)} dx \implies \overline{F}(x)dy = -\frac{\partial \overline{F}(x)}{\partial x} dx \implies \overline{F}(x)dy = f(F^{-1}(1-\overline{F}(x)))dx$, after changing of variables we get, $e^{-y}dy = f(F^{-1}(1-e^{-y}))dx \implies e^{-y}[f(F^{-1}(1-e^{-y}))]^{-1}dy = dx$. After putting this condition in (2.2) we get, as $T = -\log(\overline{F}(t))$,

$$\begin{aligned}
\mathrm{E}[X_1|X_2 > t] &= \int_0^\infty e^T e^{-(1-\alpha_1)y} e^{-T(1-\alpha_2)} \widehat{C}^*(e^{-\alpha_1 y}, e^{-\alpha_2 T}) e^{-y} [f(F^{-1}(1-e^{-y}))]^{-1}\, dy \\
&= \int_0^\infty e^{T-y(1-\alpha_1)-T(1-\alpha_2)-y} \widehat{C}^*(e^{-\alpha_1 y}, e^{-\alpha_2 T}) [f(F^{-1}(1-e^{-y}))]^{-1}\, dy \\
&= \int_0^\infty e^{T(1-1+\alpha_2)-y(1-\alpha_1+1)} \widehat{C}^*(e^{-\alpha_1 y}, e^{-\alpha_2 T}) [f(F^{-1}(1-e^{-y}))]^{-1}\, dy \\
&= \int_0^\infty e^{T\alpha_2-y(2-\alpha_1)} \widehat{C}^*(e^{-\alpha_1 y}, e^{-\alpha_2 T}) [f(F^{-1}(1-e^{-y}))]^{-1}\, dy \qquad (2.3)
\end{aligned}$$

Let $y = sT$. If we differentiate this equation with respect to s we get $\frac{dy}{ds} = T \implies dy = Tds$. After putting this condition in (2.3) we get;

$$\begin{aligned}
\mathrm{E}[X_1|X_2 > t] &= T \int_0^\infty e^{T\alpha_2 - sT(2-\alpha_1)} \widehat{C}^*(e^{-\alpha_1 sT}, e^{-\alpha_2 T}) [f(F^{-1}(1-e^{-sT}))]^{-1}\, ds \\
&= T \int_0^\infty e^{T(\alpha_2 - s(2-\alpha_1))} \widehat{C}^*(e^{-\alpha_1 sT}, e^{-\alpha_2 T}) [f(F^{-1}(1-e^{-sT}))]^{-1}\, ds \\
&= T \int_0^\infty e^{T[(\alpha_2 - s(2-\alpha_1)) + \frac{1}{T}\log\{\widehat{C}^*(e^{-\alpha_1 sT}, e^{-\alpha_2 T})[f(F^{-1}(1-e^{-sT}))]^{-1}\}]}\, ds \\
&= Te^{-w} \int_0^\infty e^{T[(\alpha_2 - s(2-\alpha_1)) + \frac{1}{T}\{\log \frac{\widehat{C}^*(e^{-\alpha_1 sT}, e^{-\alpha_2 T})}{f(F^{-1}(1-e^{-sT}))} + w\}]}\, ds \qquad (2.4) \\
&= Te^{-w} \int_0^\infty e^{Tg(s,T)} h(s) ds, \ \textit{here}\ h(s) = 1,\ \forall\ s \in [0, \infty), \qquad (2.5)
\end{aligned}$$

where $w = \log(f(F^{-1}(0)) < \infty$ is a real constant depending on marginal distribution F and $g(s,T) = \alpha_2 - s(2-\alpha_1) + \frac{1}{T}\{\log[\widehat{C}^*(e^{-\alpha_1 sT}, e^{-\alpha_2 T})[f(F^{-1}(1-e^{-sT}))]^{-1}] + w\}$.

From (2.5) we can clearly see that, the integrands can be written as, $e^{Tg(s,T)}h(s)$ where h(s)=1. It is important to say that, $Tg(s,T)$ is the dominant term of the integration. Furthermore, if $\alpha_1 = \alpha_2 = \alpha$ we have exchangeable survival copula. Furthermore, if we consider $\alpha = 1$ we get the same result with Hua and Joe (2014).

As the above integrations do not have any closed form solutions we have to use numerical integration. Hua and Joe (2014) suggested that under these circumstances we can do either Laplace approximation of Watson's lemma. Before going to those numerical approximations directly we have to check certain conditions which we have explained before this section. We assume $(\alpha_1, \alpha_2) \in [0,1]^2$.

Now, from (2.5) we know,

$$\begin{aligned} g(0,T) &= \alpha_2 + \frac{1}{T}[\log[\widehat{C}^*(e^0, e^{-\alpha_2 T})[f(F^{-1}(1-e^0))]^{-1}] + w] \\ &= \alpha_2 + \frac{1}{T}[\log \frac{\widehat{C}^*(1, e^{-\alpha_2 T})}{f(F^{-1}(0))} + w] \end{aligned} \tag{2.6}$$

From (2.6) we can clearly say that, in order to get $g(0,T) = 0$ we need $\frac{1}{T}[\log \frac{\widehat{C}^*(1,e^{-\alpha_2 T})}{f(F^{-1}(0))} + w] = -\alpha_2$ *or* $\log[\widehat{C}^*(1, e^{-\alpha_2 T})[f(F^{-1}(0))]^{-1}] + w = -\alpha_2 T$ which is true as in (2.6) $w = \log(f(F^{-1}(0)))$.

Again, from (2.5) we know,

$$g(s,T) = \alpha_2 - s(2-\alpha_1) + \frac{1}{T}[\log \frac{\widehat{C}^*(e^{-\alpha_1 sT}, e^{-\alpha_2 T})}{f(F^{-1}(1-e^{-sT}))} + w] \tag{2.7}$$

Hence,

$$\lim_{s\to\infty} \log \frac{\widehat{C}^*(e^{-\alpha_1 sT}, e^{-\alpha_2 T})}{f(F^{-1}(1-e^{-sT}))} < \infty$$

where $w = \log(f(F^{-1}(0))) < \infty$.

These are the required conditions of g function before using either Laplace approximation or Watson's lemma or both and these two conditions are satisfied by any copula with *Pareto, Weibull* or *Exponential* margins.

From the previous paragraphs we know, $g(0,T) = 0$ and $g(\infty,T) = -\infty$ as $T \to \infty$. Now, if we want to use Laplace approximation, we have to check if $g^{'}(0,T) > 0$ as $T \to \infty$.

In this case the asymptotic rate might be $T^{1/2}e^{T\gamma}h(s_0(T);T)[-g''(s_0(T);T)]^{1/2}$. Again, here also $\gamma = \lim_{T\to\infty}\max_s g(s,T)$ and $-g''(s_0(T);T) > 0$ exist and continuous as $T \to \infty$. From (2.4) we know, $g(s,T) = \alpha_2 - s(2-\alpha_1) + \frac{1}{T}[\log \frac{\widehat{C}^*(e^{-\alpha_1 sT}, e^{-\alpha_2 T})}{f(F^{-1}(1-e^{-sT}))} + w]$ [in case $\mathrm{E}[X_1|X_2 > t]$]. If we differentiate this function with respect to s, we get,

$$
\begin{aligned}
g'(s,T) &= -(2-\alpha_1) - \frac{\alpha_1 e^{-\alpha_1 sT}\widehat{C}^*_{2|1}(e^{-\alpha_2 T}|e^{-\alpha_1 sT})}{\widehat{C}^*(e^{-\alpha_1 sT}, e^{-\alpha_2 T})} \\
&\quad - \frac{e^{-sT} f'(F^{-1}(1-e^{-sT}))[f(F^{-1}(1-e^{-sT}))]^{-1}}{f(F^{-1}(1-e^{-sT}))} \\
&= -(2-\alpha_1) - \frac{\alpha_1 e^{-\alpha_1 sT}\widehat{C}^*_{2|1}(e^{-\alpha_2 T}|e^{-\alpha_1 sT})}{\widehat{C}^*(e^{-\alpha_1 sT}, e^{-\alpha_2 T})} \\
&\quad - \frac{e^{-sT} f'(F^{-1}(1-e^{-sT}))}{f^2(F^{-1}(1-e^{-sT}))} \qquad (2.8)
\end{aligned}
$$

Now in order to satisfy the condition of Laplace approximation we should have;

$$
\begin{aligned}
\lim_{T\to\infty} g'(0,T) &= (\alpha_1 - 2) - \lim_{T\to\infty} \frac{\alpha_1 \widehat{C}^*_{2|1}(e^{-\alpha_1 sT}|1)}{e^{-\alpha_2 T}} \qquad (2.9) \\
&\quad - \frac{f'(F^{-1}(0))}{f^2(F^{-1}(0))} > 0 \qquad (2.10)
\end{aligned}
$$

In (2.8) if we do the first order condition, we are able to get α_1. If all the above conditions are satisfied and if $h(s) = 1$ then, by Laplace approximation we get,

$$
\begin{aligned}
\mathrm{E}[X_1|X_2 > t] &= T\int_0^\infty e^{Tg(s,T)}h(s)ds \\
&\sim T\int_0^\infty exp\{Tg(\gamma,T) + \frac{1}{2}(s-\gamma)^2 g''(\gamma,T)\}ds \\
&\sim Te^{Tg(\gamma,T)}\sqrt{\frac{2\pi}{-Tg''(\gamma,T)}}. \qquad (2.11)
\end{aligned}
$$

In order to satisfy the final expression $g''(\gamma,T)$ must exist and $-g''(\gamma,T) > 0$. This completes the proof. □

Remark 1. By doing easy calculations for the case of $\widehat{C}^*$ [where $\widehat{C}^*$ is a Clayton Copula with Pareto margins] we find, $w = \log\beta$. Apart from that after calculation we get $\frac{1}{T}\log\widehat{C}^*(e^{-\alpha_1 sT}, e^{-\alpha_2 T}) = \frac{1}{T}\log e^{-\alpha_2 T} = -\alpha_2$. Thus, $g(0,T) = 0$ and $g(\infty,T) = -\infty$.

We know, marginal cumulative distribution function (CDF) of Pareto distribution is $F(x) = 1-(1+x)^{-\beta}$. Thus, the density function is $f(x) = F'(x) = \beta(1+x)^{-(1+\beta)}$ and $f'(x) = -\beta(1+\beta)(1+x)^{-(2+\beta)}$, $\forall\ \beta > 1$. Thus, the extreme right hand side of (2.9) becomes, $-\lim_{T\to\infty} f'(F^{-1}(0))f^{-2}(F^{-1}(0)) = -\lim_{T\to\infty} -\beta(1+\beta)\beta^{-2}$. Thus, $-\lim_{T\to\infty} f'(F^{-1}(0))f^{-2}(F^{-1}(0)) = 1+\beta^{-1} > 1$. Now, let us solve the second term of the right hand side of equation (2.9). At first let us write the original survival Clayton copula before Khoudraji (1996) transformation. The numerator part is just the first order derivative of Clayton Copula with respect to its first argument and the denominator term is just the copula itself.

We know, Clayton Copula is $\widehat{C}^*(u,v) = (u^{-\delta}+v^{-\delta}-1)^{-1/\delta}$. Hence,

$$\begin{aligned}\widehat{C}^*_{2|1}(v|u) &= -\frac{1}{\delta}(u^{-\delta}+v^{-\delta}-1)^{-(1+1/\delta)}(-\delta)u^{-(1+\delta)}\\ &= u^{-(1+\delta)}(u^{-\delta}+v^{-\delta}-1)^{-(1+1/\delta)},\ \forall\ \delta > 0 \end{aligned}\tag{2.12}$$

From (2.9) we know that the middle term is $-\lim_{T\to\infty}\frac{\alpha_1\widehat{C}^*_{2|1}(e^{-\alpha_2 T}|1)}{\widehat{C}(1,e^{-\alpha_2 T})}$, that is,

$$\begin{aligned}-\lim_{T\to\infty}\frac{\alpha_1\widehat{C}^*_{2|1}(e^{-\alpha_2 T}|1)}{\widehat{C}^*(1,e^{-\alpha_2 T})} &= -\lim_{T\to\infty}\frac{\alpha_1 e^{-\alpha_2\delta T(1+1/\delta)}}{e^{\alpha_2\delta T}}\\ &= -\lim_{T\to\infty}\alpha_1\ e^{-\alpha_2\delta T} = 0,\end{aligned}\tag{2.13}$$

From our above discussion it is clear that we can use Laplace Approximation under Clayton copula with Pareto margins when $\lim_{T\to\infty} g'(0,T) > 0$. After using conditions (2.12),(2.13) we get $-(2-\alpha_1)+1+1/\beta > 0$ or, $1+\alpha_1+\beta^{-1} > 2 \implies \alpha_1+\beta^{-1} > 1$ otherwise we have to use Watson's lemma. In the next section we'll see that we need exactly the same condition before applying Laplace approximation and we will see in the cases of Weibull and exponential margins, $\lim_{T\to\infty} g'(0,T) < 0$.

Remark 2. From the above example it is clear that, when $\alpha_1 + \beta^{-1} > 1$ we can use Laplace Approximation. Furthermore, under this case, $\gamma \to \ \alpha_2\ (\alpha_1)^{-1}$ as $T \to \infty$. On the other hand, in general $\gamma \to \ \alpha_2 \delta \left\{\alpha_1 \delta + (1-\alpha_1)\right\}^{-1}$, as $T \to \infty$, $\forall\ (\alpha_1, \alpha_2) \in [0,1]^2$, and $\delta > 0$.

Proposition 2. *Suppose X_1 and X_2 are two dependent random variables and $\int_0^\infty e^{Tg(s,T)}\, ds < \infty$. For all $(\alpha_1, \alpha_2) \in [0,1]^2$ and if $g'(0,T) \not> 0$ Watson's lemma gives,*

$$\mathrm{E}[X_1|X_2 > t] \sim \frac{1}{[(2-\alpha_1) + D_1 + D_2]},\ \textit{as } t \to \infty,$$

where $D_1 = \frac{\alpha_1 \widehat{C}^*_{2|1}(e^{-\alpha_2 T}|1)}{\widehat{C}^*(1, e^{-\alpha_2 T})}$, $D_2 = \frac{f'(F^{-1}(0))}{f^2(F^{-1}(0))}$ *and* $T = -\log\ \overline{F}(t)$.

Proof. Let us assume $\int_0^\infty e^{Tg(s,T)} < \infty$. From our definition we know that, $T = -\log\ \overline{F}(t)$. This implies, $T \to \infty \iff t \to \infty$. Thus, if we are able to prove,

$$\mathrm{E}[X_1|X_2 > t] \sim \frac{1}{[(2-\alpha_1) + D_1 + D_2]},$$

as $T \to \infty$; where

$$D_1 = \frac{\alpha_1 e^{-\alpha_1 sT} \widehat{C}^*_{2|1}(e^{-\alpha_2 T}|e^{-\alpha_1 sT})}{\widehat{C}^*(e^{-\alpha_1 sT}, e^{-\alpha_2 T})}$$

and

$$D_2 = \frac{e^{-sT} f'(F^{-1}(1-e^{-sT}))}{f^2(F^{-1}(1-e^{-sT}))}.$$

Furthermore, as $g'(0,T) \leq 0$, then We have to use Watson's lemma as $g'(s;T) < 0$. From our previous results , that is enough to prove the proposition. We know, $g(0,T) = 0$ and $g(\infty, T) = -\infty$.

Now we can use Watson's lemma. Here we are not using the Watson's lemma directly. We are using a version of *Watson's* Lemma. In order to do the approximation of this integration we are using Theorem 36 of Breitung (1994) [p. 48]. As $g(s,T)$ is a real valued function on the semi-infinite interval $[0,\infty)$ and in an interval $(0, 0+\epsilon]$ with $\epsilon > 0$, this function is continuously differentiable and

$$\sup_{0+\epsilon \le s \le \infty} g(s,T) \le g(0,T) - \psi, \tag{2.14}$$

with $\psi > 0$.

Now for $g'(s,T)$ we have $g'(s,T) < 0$ as $\alpha_1 + \beta^{-1} \le 1$ and $s \to 0$. We can also write

$$g'(s,T) = -as^{r-1} + o(s^{r-1}) \ \forall r > 0.$$

Now if we assume $r = 1$ then

$$g'(s,T) = -a = -\left[(2-\alpha_1) + \frac{\alpha_1 e^{-\alpha_1 sT}\widehat{C}^*_{2|1}(e^{-\alpha_2 T}|e^{-\alpha_1 sT})}{\widehat{C}^*(e^{-\alpha_1 sT}, e^{-\alpha_2 T})} + \frac{e^{-sT}f'(F^{-1}(1-e^{-sT}))}{f^2(F^{-1}(1-e^{-sT}))}\right].$$

Watson's lemma requires,

$$\lim_{s^+ \to 0} g'(s,T) = -[(2-\alpha_1) + \frac{\alpha_1 e^{-\alpha_1 sT}\widehat{C}^*_{2|1}(e^{-\alpha_2 T}|e^{-\alpha_1 sT})}{\widehat{C}^*(e^{-\alpha_1 sT}, e^{-\alpha_2 T})} + \frac{e^{-sT}f'(F^{-1}(1-e^{-sT}))}{f^2(F^{-1}(1-e^{-sT}))}],$$

which is a constant if

$$\frac{e^{-sT}f'(F^{-1}(1-e^{-sT}))}{f^2(F^{-1}(1-e^{-sT}))}$$

goes to a constant as $s^+ \to 0$. Thus,

$$-a = -[(2-\alpha_1) + \frac{\alpha_1 \widehat{C}^*_{2|1}(e^{-\alpha_2 T}|1)}{\widehat{C}^*(1, e^{-\alpha_2 T})} + \frac{f'(F^{-1}(0))}{f^2(F^{-1}(0))}]$$

or, $a = (2-\alpha_1) + \frac{\alpha_1 \widehat{C}^*_{2|1}(e^{-\alpha_2 T}|1)}{\widehat{C}^*(1,e^{-\alpha_2 T})} + \frac{f'(F^{-1}(0))}{f^2(F^{-1}(0))} = 2 + \frac{f'(F^{-1}(0))}{f^2(F^{-1}(0))} > 0.$

Now, let us assume there is another real and continuous function $h(s) \in [0,\infty)$ such that,

$$h(s) = bs^{m-1} + o(s^{m-1})$$

with $m > 0$. More specifically in our case we have, $h(s) = 1$ in our case. Thus,

$$bs^{m-1} + o(s^{m-1}) = 1 \implies b = 1 \tag{2.15}$$

when $m = 1$. we are assuming $\int_0^\infty e^{g(s,T)} ds < \infty$ then by Watson's lemma we can write the approximated value of the integral $I(T) = \int_0^\infty e^{Tg(s,T)} ds$ with $T \geq 1$ are all finite and the asymptotic approximation is;

$$I(T) \sim \left(\frac{1}{(2-\alpha_1) + D_1 + D_2}\right)\left(T^{-1}\right) e^{Tg(0,T)} \ as, \quad T \to \infty$$

Thus,

$$\mathrm{E}[X_1|X_2 > t] \sim \frac{1}{[(2-\alpha_1) + D_1 + D_2]} \tag{2.16}$$

as $t \to \infty$, $\forall\ (\alpha_1, \alpha_2) \in [0,1]^2$, $\beta > 1$ and where $D_1 = \frac{\alpha_1 \widehat{C}^*_{2|1}(e^{-\alpha_2 T}|1)}{\widehat{C}^*(1,e^{-\alpha_2 T})} = \alpha_1$ and $D_2 = \frac{f'(F^{-1}(0))}{f^2(F^{-1}(0))}$. This completes the proof. □

Example 1. From the above proposition we can write, $a = (2-\alpha_1) + \frac{\alpha_1 \widehat{C}^*_{2|1}(e^{-\alpha_2 T}|1)}{\widehat{C}^*(1,e^{-\alpha_2 T})} + \frac{f'(F^{-1}(0))}{f^2(F^{-1}(0))} > 0$. In our case, the marginal cumulative distribution function (CDF) of Pareto distribution is $F(x) = 1-(1+x)^{-\beta}$. Thus, the density function is $f(x) = F'(x) = \beta(1+x)^{-(1+\beta)}$ and $f'(x) = -\beta(1+\beta)(1+x)^{-(2+\beta)}$, $\forall\ \beta > 1$. Thus, $\lim_{T\to\infty} \frac{f'(F^{-1}(0))}{f^2(F^{-1}(0))} =$

$\lim_{T\to\infty} \frac{-\beta(1+\beta)}{\beta^2}$ for $s \in (0, 0+\epsilon]$. Hence, in the case of Pareto margins we have $\lim_{T\to\infty} \frac{f'(F^{-1}(0))}{f^2(F^{-1}(0))} = -\left[1+\frac{1}{\beta}\right]$ around $s = 0$ [i.e. $s \in (0, 0+\epsilon]$].

In the case of Clayton copula we know that,

$$\widehat{C}^*_{2|1}(e^{-\alpha_2 T}|e^{-\alpha_1 sT}) = \alpha_1 e^{sT(1+\alpha_1\delta)}(e^{\alpha_1\delta sT} + e^{\alpha_2\delta T} - 1)^{-(1+\frac{1}{\delta})}$$

We know, $g(s,T) = (2-\alpha_1) + \frac{\alpha_1 e^{-\alpha_1 sT}\widehat{C}^*_{2|1}(e^{-\alpha_2 T}|e^{-\alpha_1 sT})}{\widehat{C}^*(e^{-\alpha_1 sT}, e^{-\alpha_2 T})} + \frac{e^{-sT} f'(F^{-1}(1-e^{-sT}))}{f^2(F^{-1}(1-e^{-sT}))}$. We have to calculate the value of $g(s,T)$ around $s = 0$ and as $T \to \infty$ [i.e. a]. We have;

$$\begin{aligned} g(s,T) =& (2-\alpha_1) + \frac{\alpha_1 e^{-\alpha_1 sT}\widehat{C}^*_{2|1}(e^{-\alpha_2 T}|e^{-\alpha_1 sT})}{\widehat{C}^*(e^{-\alpha_1 sT}, e^{-\alpha_2 T})} + \frac{e^{-sT} f'(F^{-1}(1-e^{-sT}))}{f^2(F^{-1}(1-e^{-sT}))} \\ =& (2-\alpha_1) + \frac{\alpha_1 e^{-\alpha_1 sT}\alpha_1 e^{sT(1+\alpha_1\delta)}(e^{\alpha_1\delta sT} + e^{\alpha_2\delta T} - 1)^{-(1+\frac{1}{\delta})}}{(e^{\alpha_1\delta sT} + e^{\alpha_2\delta T} - 1)^{-\frac{1}{\delta}}} \\ &+ \frac{e^{-sT} f'(F^{-1}(1-e^{-sT}))}{f^2(F^{-1}(1-e^{-sT}))} \end{aligned} \tag{2.17}$$

$$\begin{aligned} a =& (2-\alpha_1) - \left[1+\frac{1}{\beta}\right],\ \textit{at } T \to \infty \textit{ and } s \in (0, 0+\epsilon] \\ =& 1 - \alpha_1 - \frac{1}{\beta},\ \textit{at } T \to \infty \textit{ and } s \in (0, 0+\epsilon]. \end{aligned} \tag{2.18}$$

In order to satisfy $a > 0$ we need $\alpha_1 + \beta^{-1} \leq 1$. This condition is true. Thus, by using Watson's lemma we have,

$$\mathrm{E}[X_1|X_2 > t] \sim \frac{1}{\left(1 - \alpha_1 - \frac{1}{\beta}\right)},\ \textit{at } t \to \infty \textit{ and } s \in (0, 0+\epsilon]. \tag{2.19}$$

2.1.2 Examples with Clayton Copula with Three Margins

In our case, we are using a particular form of Archimedean copula named Clayton Copula. We know by definition this Copula looks like $\widehat{C}^*(u,v) = (u^{-\delta} + v^{-\delta} - 1)^{-\frac{1}{\delta}}$, where $(u,v) \in [0,1]^2$ and $\delta \geq 0$. Then we use Khoudraji non-exchangeable device to transform it into Khoudraji (1996) transformed Clayton Copula (i.e., KB4). Furthermore, when we discuss the *non-exchangeability* at the tail, we take three distributions as margins. They are Pareto, Weibull and Exponential margins respectively, which represent three main patterns of tail behavior of univariate margins, based on extreme value theory. Our objective in this paper is to study the effects from both marginals and dependence on *tail non-exchangeability*.

A bivariate Khoudraji-transformed Clayton copula has been studied in Hofert and Vrins (2013), which takes care about the non-exchangeable structure. In our study we name this as KB4 copula. The Cumulative Distribution Function (CDF) of bivariate KB4 copula can be written as;

$$\widehat{C}(u,v) = (u^{-\alpha_1\delta} + v^{-\alpha_2\delta} - 1)^{-1/\delta} u^{1-\alpha_1} v^{1-\alpha_2}, \quad \delta \geq 0, \quad (\alpha_1, \alpha_2) \in [0,1]^2, \tag{2.20}$$

where the original survival Clayton Copula before Khoudraji transformation is $\widehat{C}^*(u,v) = (u^{-\delta} + v^{-\delta} - 1)^{-1/\delta}$. At first, we are interested in checking the behavior of our measure of tail *non-exchangeability* represented by;

$$\eta_1(t) = \frac{\mathrm{E}[X_1|X_2 > t]}{\mathrm{E}[X_2|X_1 > t]},$$

where a general tail behavior of $E[X_1|X_2 > t]$ as $t \to \infty$ has been studied earlier in the thesis.

2.1.2.1 Pareto Margins

Firstly, consider using Pareto distribution as univariate marginals, with CDF $F(x) = 1-(1+x)^{-\beta}$, $\beta > 1$, for both. This marginal is of power law, and is of relatively heavier tail. We are taking this margin with KT transformed Clayton Copula. In economics, actuarial science, geophysics etc. we use this distribution a lot. In some discipline we call it as Bradford distribution. As this distribution is heavily skewed to the right, it would take very large values. It was developed by *Vilfredo Pareto* in order to develop the distribution of wealth of class-based society in his famous article *The New Theories of Economics* in *The Journal of Political Economy* [i.e.,Pareto (1897)]. He found out that 20% of the citizen holds 80% of the wealth of the society [in Pareto (1897)]. In that way, he found out the unequal allocation of resources. This implies a less proportion of people owns a huge number of wealth. This is the case of extreme conditions. Thus, we need to use extreme value theory. This distribution is very popular in insurance and finance because, there are risks involved in this sector. There is a big probability to lose wealth and it is very hard to predict this by traditional asymptotic theory. By using Pareto distribution with Khoudraji (1996) transformed Clayton Copula we actually can predict the extreme events.

In our model, we try to calculate a stable conditional expectation at extreme values. As Pareto distribution is heavily skewed to right, with non-exchangeable Clayton Copula we use this margin. In order to find the g function we directly use KT-transformed Clayton Copula with Pareto margins, unlike in the previous section where in the example, we put the forms with Clayton Copula instead of general Copula functions with some margins. We show that, the conditions and the results in the previous section and this section are the same. This is good for us as the results in this section are consistent with the theory we developed already. Here we use both Laplace Approximation and Watson's lemma to get the form of tail order non-exchangeable form. After doing simulation studies we also find that, Laplace Approximation is better method than Watson's lemma in this case. By following propositions and lemmas we are calculating the form tail-order conditional expectations.

Lemma 3. *Khoudraji transformed Clayton Copula with Pareto margins the integrand function g satisfies $g(0,T)=0$ and $g(\infty,T)=-\infty$ as $T\to\infty$.*

Proof. Now we consider a simple case with random variables X_1, X_2 follow Pareto distribution with identical *Cumulative Distribution Function* (CDF) $F(x) = 1-(1+x)^{-\beta}, \beta > 1$. Let $\widehat{C}$ be the Khoudraji (1996)-transformed Clayton survival Copula of (X_1, X_2).

$$F(x) = 1-(1+x)^{-\beta},\ \beta>1$$
$$Thus,\ \overline{F}(x) = 1-F(x) = 1-1+(1+x)^{-\beta}\ = (1+x)^{-\beta}, \tag{2.21}$$

where $F(x)$ and $\overline{F}(x)$ represent distribution and survival functions respectively.

Following Hua and Joe (2014) we will transform the survival function $\overline{F}(t)\to\ e^{-T}$ $or, T = -\log\overline{F}(t) = \beta\log(1+t)$ we have, letting $T=\beta\log(1+t)$

$$\mathrm{E}[X_1|X_2>t] = \frac{T}{\beta}\int_0^\infty e^{Tg(s,T)}ds,\ \forall\ T \tag{2.22}$$

where, $g(s,T) = 1+\frac{s}{\beta}+\frac{1}{T}\log\widehat{C}(e^{-sT},e^{-T})$. Clearly, $t\to\infty \iff T\to\infty$. Now,

$$\begin{aligned}\widehat{C}(e^{-sT},e^{-T}) &= (e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1)^{-1/\delta}e^{-(1-\alpha_1)sT}e^{-(1-\alpha_2)T}\\ &= (e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1)^{-1/\delta}e^{-sT+\alpha_1 sT-T+\alpha_2 T}\end{aligned} \tag{2.23}$$

Hence,

$$\begin{aligned}
&\frac{1}{T}\log\widehat{C}(e^{-sT},e^{-T}) = -\frac{1}{T}[1/\delta\log(e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1)+sT-\alpha_1 sT+T-\alpha_2 T]\\
\implies &\frac{1}{T}\log\widehat{C}(e^{-sT},e^{-T}) = -\frac{1}{T}[1/\delta\log(e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1)+(1-\alpha_1)sT+(1-\alpha_2)T]
\end{aligned} \tag{2.24}$$

In order to do either Laplace Approximation or Watson's Lemma we need to check if (i) $g(0;T)=0$ and (ii) $g(\infty;T)=-\infty$ are true. Lets check the first case; (i) $\implies$

$$\begin{aligned}
g(0;T) &= 1-\frac{1}{T}[\frac{1}{\delta}\log(1+e^{\alpha_2\delta T}-1)+(1-\alpha_2)T], as\ e^0=1\\
&= 1-\frac{1}{T}[\frac{1}{\delta}\log e^{\alpha_2\delta T}+(1-\alpha_2)T]\\
&= 1-\frac{1}{T}[\alpha_2 T+T-\alpha_2 T] = 1-1 = 0,
\end{aligned} \tag{2.25}$$

as $T\to\infty$. From the above calculation we conclude that, condition (i) holds. In order to prove (ii) let use write g(s;T) one more time. We know from the previous calculations,

$$\begin{aligned}
&g(s;T) = 1+\frac{s}{\beta}-\frac{1}{T}[\frac{1}{\delta}\log(e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1)+(1-\alpha_1)sT+(1-\alpha_2)T]\\
\implies &g(\infty;T) = 1+\infty-\frac{1}{T}[\frac{1}{\delta}\log(e^{\infty}+e^{\alpha_2\delta T}-1)+\infty+(1-\alpha_2)T]\\
\implies &g(\infty;T) = -\infty,
\end{aligned} \tag{2.26}$$

as $T\to\infty$. From (2.25) and (2.26) we get our desirable result. □

Lemma 4. *$g'(0,T)>0$ if $\alpha_1+\beta^{-1}>1$ and $\alpha_2>\alpha_1$, for all $(\alpha_1,\alpha_2)\in[0,1]^2$ and $\beta>1$.*

Proof. In our case we know the g function as,

$$g(s;T) = 1 + \frac{s}{\beta} - \frac{1}{T}[\frac{1}{\delta}\log(e^{\alpha_1\delta sT} + e^{\alpha_2\delta T} - 1)$$
$$+ (1-\alpha_1)sT + (1-\alpha_2)T] \quad (2.27)$$
$$Thus,\ g'(s;T) = \frac{1}{\beta} - \frac{1}{T}[\frac{\alpha_1\delta T e^{\alpha_1\delta sT}}{\delta(e^{\alpha_1\delta sT} + e^{\alpha_2\delta T} - 1)} + (1-\alpha_1)T]$$
$$= \frac{1}{\beta} - \frac{1}{T}[\frac{\alpha_1\delta T}{\delta[1 + e^{\alpha_2\delta T - \alpha_1\delta sT} - e^{-\alpha_1\delta sT}]} + (1-\alpha_1)T]$$
$$= \frac{1}{\beta} - \frac{\alpha_1}{1 + e^{\alpha_2\delta T - \alpha_1\delta sT} - e^{-\alpha_1\delta sT}} - (1-\alpha_1) \quad (2.28)$$
$$\Longrightarrow \lim_{T\to\infty} g'(0,T) = \frac{1}{\beta} + \alpha_1 - 1\ ,\ if\ \alpha_2 > \alpha_1 \quad (2.29)$$

where $\alpha_1 \in [0,1]$ and $\beta^{-1} \in [0,1)$. [as by *Pareto Margin* we know $\beta > 1$] In order to get $g'(0,T) > 0$ we need $\beta^{-1} + \alpha_1 - 1 > 0$ or, $\beta^{-1} + \alpha_1 > 1$. If we compare this result with the general case, we can see that, both the results are exactly the same. □

Remark 3. Now we can use Laplace approximation. In this case asymptotic rate might be;

$$T^{1/2}e^{T\gamma}h(s_0(T))[-g''(s_0(T);T)]^{-1/2}$$

where $h(s_0(T)) = 1$ and $\gamma = \lim_{T\to\infty} max_s g(s;T)$, $s_0(T) = argmax_s g(s,T)$.

Proposition 5. *Under Clayton Copula with Pareto marginal,* $\gamma \to \alpha_2\ (\alpha_1)^{-1}$ *as* $t \to \infty$*, which is the ratio of two non-exchangeable parameters.*

Proof. In our case as we are taking pareto marginals and KB4 copula our g function becomes $g(s,T) = 1+s\beta^{-1}-T^{-1}[\delta^{-1}\log(e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1)+(1-\alpha_1)sT+(1-\alpha_2)T]$, which is continuous everywhere through the real line. Before going to condition (ii) let us first calculate the exact form of $s_0(T) = argmax_s\ g(s,T)$. From (2.27) we know,

$$g'(s;T) = \frac{1}{\beta} - \frac{1}{T}[\frac{\alpha_1 \delta T}{\delta(1 + e^{\alpha_2 \delta T - \alpha_1 \delta s T} - e^{-\alpha_1 \delta s T})} + (1-\alpha_1)T]$$
$$= \frac{1}{\beta} - \frac{\alpha_1}{1 + e^{\alpha_2 \delta T - \alpha_1 \delta s T} - e^{-\alpha_1 \delta s T}} - (1-\alpha_1) \tag{2.30}$$

After putting the first order condition we get,

$$\frac{1}{\beta} - \frac{\alpha_1}{1 + e^{\alpha_2 \delta T - \alpha_1 \delta s T} - e^{-\alpha_1 \delta s T}} - (1-\alpha_1) = 0$$
$$\frac{\alpha_1}{1 + e^{-\alpha_1 \delta s T}(e^{\alpha_2 \delta T} - 1)} = \frac{1}{\beta} - (1-\alpha_1)$$
$$e^{-\alpha_1 \delta s T}(e^{\alpha_2 \delta T} - 1) = \frac{\alpha_1 \beta}{1 - \beta(1-\alpha_1)} - 1$$
$$e^{-\alpha_1 \delta s T} = \frac{1}{e^{\alpha_2 \delta T} - 1}[\frac{\alpha_1 \beta}{1 - \beta(1-\alpha_1)} - 1]$$
$$e^{\alpha_1 \delta s T} = \frac{(e^{\alpha_2 \delta T} - 1)[1 - \beta(1-\alpha_1)]}{\alpha_1 \beta - (1 - \beta(1-\alpha_1))}$$
$$\alpha_1 \delta s T = \log \frac{(e^{\alpha_2 \delta T} - 1)[1 - \beta(1-\alpha_1)]}{\alpha_1 \beta - (1 - \beta(1-\alpha_1))}$$
$$s_0(T) = \frac{1}{\alpha_1 \delta T} \log \frac{(e^{\alpha_2 \delta T} - 1)[1 - \beta(1-\alpha_1)]}{\alpha_1 \beta - (1 - \beta(1-\alpha_1))} \tag{2.31}$$

From (2.31) we get the exact expression of $s_0(T)$. Now we have to check the behavior of this above function when $T \to \infty$. By our definition we know,$\gamma = \lim_{T\to\infty} max_s g(s;T)$. Using the expression of $s_0(T)$ in (2.31) and taking the limit of T in both the sides we get;

$$
\begin{aligned}
\gamma &= \lim_{T\to\infty} s_0(T) \\
&= \lim_{T\to\infty}[\frac{1}{\alpha_1\delta T}\log\frac{(e^{\alpha_2\delta T}-1)[1-\beta(1-\alpha_1)]}{\alpha_1\beta-(1-\beta(1-\alpha_1))}] \\
&= \lim_{T\to\infty}\frac{\log(e^{\alpha_2\delta T}-1)}{\alpha_1\delta T} + \lim_{T\to\infty}\frac{\log(1-\beta(1-\alpha_1))}{\alpha_1\delta T} \\
&\qquad - \lim_{T\to\infty}\frac{\alpha_1\beta-(1-\beta(1-\alpha_1))}{\alpha_1\delta T} \\
&= \lim_{T\to\infty}\frac{\log(e^{\alpha_2\delta T}-1)}{\alpha_1\delta T}[\ \textit{two right hand side terms go to } 0 \textit{ as } T\to\infty\ ] \\
&= \lim_{T\to\infty}\frac{\alpha_2\delta e^{\alpha_2\delta T}}{\alpha_1\delta(e^{\alpha_2\delta T}-1)},\ [\textit{by } L'Hospital\ Rule] \\
&= \lim_{T\to\infty}\frac{\alpha_2}{\alpha_1(1-e^{-\alpha_2\delta T})},\ [\textit{dividing by } \delta e^{\alpha_2\delta T}\ \textit{from numeritor and denominator}] \\
&= \frac{\alpha_2}{\alpha_1},\ \forall\ (\alpha_1,\alpha_2)\in\ [0,1]^2
\end{aligned} \tag{2.32}
$$

This completes the proof. □

Remark 4. If we compare this result with using the theoretical results in the previous section, we have to approach in the following way: In the case of Clayton copula we know that,

$$
\widehat{C}^*_{2|1}(e^{-\alpha_2 T}|e^{-\alpha_1 sT}) = \alpha_1 e^{sT(1+\alpha_1\delta)}(e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1)^{-(1+\frac{1}{\delta})}
$$

From our previous result we know that, $-g'(s,T) = (2-\alpha_1) + \frac{\alpha_1 e^{-\alpha_1 sT}\widehat{C}^*_{2|1}(e^{-\alpha_2 T}|e^{-\alpha_1 sT})}{\widehat{C}^*(e^{-\alpha_1 sT}, e^{-\alpha_2 T})} + \frac{e^{-sT}f'(F^{-1}(1-e^{-sT}))}{f^2(F^{-1}(1-e^{-sT}))}$. If we calculate the first order differentiation, we have to use $g'(s,T) = 0$. Thus, we have;

$$
\begin{aligned}
(2-\alpha_1) + \frac{\alpha_1 e^{-\alpha_1 sT}\widehat{C}^*_{2|1}(e^{-\alpha_2 T}|e^{-\alpha_1 sT})}{\widehat{C}^*(e^{-\alpha_1 sT}, e^{-\alpha_2 T})} + \frac{e^{-sT}f'(F^{-1}(1-e^{-sT}))}{f^2(F^{-1}(1-e^{-sT}))} = 0 \\
(2-\alpha_1) + \frac{\alpha_1 e^{-\alpha_1 sT}\widehat{C}^*_{2|1}(e^{-\alpha_2 T}|e^{-\alpha_1 sT})}{\widehat{C}^*(e^{-\alpha_1 sT}, e^{-\alpha_2 T})} = 0, \ as\ T \to \infty \\
e^{-\alpha_1 \delta sT} = \frac{1}{e^{\alpha_2 \delta T} - 1}\left[\frac{\alpha_1^2 e^{(1-\alpha_1)sT}}{\alpha_1 - 2} - 1\right] \qquad (2.33)
\end{aligned}
$$

If we solve further the above equation we get,

$$
\begin{aligned}
e^{-\alpha_1 \delta sT} &= \frac{1}{e^{\alpha_2 \delta T} - 1}\left[\frac{\alpha_1^2 e^{(1-\alpha_1)sT}}{\alpha_1 - 2} - 1\right] \\
e^{\alpha_1 \delta sT} &= \frac{(\alpha_1 - 2)(e^{\alpha_2 \delta T} - 1)}{\alpha_1^2 e^{(1-\alpha_1)sT} - \alpha_1 + 2} \\
\alpha_1 \delta sT &= \log(e^{\alpha_2 \delta T} - 1) + \log(\alpha_1 - 2) - \log[\alpha_1^2 e^{(1-\alpha_1)sT} - \alpha_1 + 2] \\
s &= \frac{\log(e^{\alpha_2 \delta T} - 1)}{\alpha_1 \delta T} + \frac{\log(\alpha_1 - 2)}{\alpha_1 \delta T} - \frac{\log[\alpha_1^2 e^{(1-\alpha_1)sT} - \alpha_1 + 2]}{\alpha_1 \delta T} \\
&= \frac{\alpha_2 e^{\alpha_2 \delta T}}{\alpha_1(e^{\alpha_2 \delta T} - 1)} - \frac{\alpha_1(1-\alpha_1)se^{(1-\alpha_1)sT}}{\delta[\alpha_1^2 e^{(1-\alpha_1)sT} + 2 - \alpha_1]}, \ by\ L'Hospital\ Rule \\
&= \frac{\alpha_2}{\alpha_1} - \frac{\alpha_1 s(1-\alpha_1)}{\delta[\alpha_1^2 + (2-\alpha_1)e^{-(1-\alpha_1)sT}]}, \ as\ T \to \infty \\
&= \frac{\alpha_2}{\alpha_1} - \frac{(1-\alpha_1)s}{\alpha_1 \delta}, as\ T \to \infty \\
\gamma &= \frac{\alpha_2 \delta}{\alpha_1 \delta + (1-\alpha_1)}, \ as\ T \to \infty \qquad (2.34)
\end{aligned}
$$

From the earlier example we know that, for Clayton Copula with Pareto margin we can use Laplace Approximation if $\alpha_1 + \beta^{-1} > 1$. By our assumption we also know that, $\beta > 1$. Thus, $\beta^{-1} \in [0,1)$. In order to fulfill both the conditions we need α_1 is very close to 1. Then the expression $\gamma \to \alpha_2 (\alpha_1)^{-1}$ as $\alpha_1 \to 1$. In the previous proposition we see the expression is exactly same. Thus, $\gamma \to \frac{\alpha_2}{\alpha_1}$ as $T \to \infty$ and $\alpha_1 \to 1$.

Proposition 6. *Second order sufficient condition of the integrand g of Khoudraji transformed Clayton Copula satisfies.*

Proof. From the previous section we know that, $g'(s;T) = \beta^{-1} - T^{-1}[(\alpha_1 \delta T e^{\alpha_1 \delta s T})\{\delta(e^{\alpha_1 \delta s T} + e^{\alpha_2 \delta T} - 1)\}^{-1} + (1-\alpha_1)T]$. In order to satisfy Laplace Approximation we need negative second order value of the integrand. Here we prove that when $\alpha_2 = \alpha_1 s$ holds, the second order condition for maximization occurs.

$$
\begin{aligned}
As,\ g'(s;T) &= \frac{1}{\beta} - \frac{1}{T}\left[\frac{\alpha_1 \delta T e^{\alpha_1 \delta s T}}{\delta(e^{\alpha_1 \delta s T} + e^{\alpha_2 \delta T} - 1)} + (1-\alpha_1)T\right] \\
g''(s;T) &= \frac{\partial g'(s;T)}{\partial s} \\
&= -\left[\frac{\alpha_1^2 \delta T e^{\delta s T}(e^{\alpha_1 \delta s T} + e^{\alpha_2 \delta T} - 1)}{(e^{\alpha_1 \delta s T} + e^{\alpha_2 \delta T} - 1)^2} - \frac{\alpha_1 e^{\delta s T}(\alpha_1 \delta T e^{\alpha_1 \delta s T})}{(e^{\alpha_1 \delta s T} + e^{\alpha_2 \delta T} - 1)^2}\right] \\
\implies -g''(s;T) &= \left[\frac{\alpha_1^2 \delta T e^{\delta s T}(e^{\alpha_1 \delta s T} + e^{\alpha_2 \delta T} - 1)}{(e^{\alpha_1 \delta s T} + e^{\alpha_2 \delta T} - 1)^2} - \frac{\alpha_1 e^{\delta s T}(\alpha_1 \delta T e^{\alpha_1 \delta s T})}{(e^{\alpha_1 \delta s T} + e^{\alpha_2 \delta T} - 1)^2}\right] \\
&= \left[\frac{\alpha_1^2 \delta T e^{\delta s T}}{e^{\alpha_1 \delta s T} + e^{\alpha_2 \delta T} - 1} - \frac{\alpha_1 e^{\delta s T}(\alpha_1 \delta T e^{\alpha_1 \delta s T})}{(e^{\alpha_1 \delta s T} + e^{\alpha_2 \delta T} - 1)^2}\right] \\
&= \frac{\alpha_1^2 \delta T e^{\delta s T}}{e^{\alpha_1 \delta s T} + e^{\alpha_2 \delta T} - 1}\left(1 - \frac{e^{\alpha_1 \delta s T}}{e^{\alpha_1 \delta s T} + e^{\alpha_2 \delta T} - 1}\right) \qquad (2.35)
\end{aligned}
$$

Taking limit of T in both sides of the (2.35) we get,

$$
\lim_{T\to\infty} -g''(s;T) = \lim_{T\to\infty} \frac{\alpha_1^2 \delta T e^{\delta s T}}{e^{\alpha_1 \delta s T} + e^{\alpha_2 \delta T} - 1}\left(1 - \frac{e^{\alpha_1 \delta s T}}{e^{\alpha_1 \delta s T} + e^{\alpha_2 \delta T} - 1}\right) \qquad (2.36)
$$

In order to make $\lim_{T\to\infty} -g''(s;T) > 0$ we need $(1 - e^{\alpha_1 \delta s T}(e^{\alpha_1 \delta s T} + e^{\alpha_2 \delta T} - 1)^{-1}) > 0$ as $\lim_{T\to\infty} \alpha_1^2 \delta T e^{\delta s T}(e^{\alpha_1 \delta s T} + e^{\alpha_2 \delta T} - 1)^{-1} > 0$ (because the denominator is greater than 0) and

$0 < \alpha_1 < 1$ and $\delta \geq 0$. Now from the above condition we get, $e^{\alpha_1 \delta s T} + e^{\alpha_2 \delta T} - 1 - e^{\alpha_1 \delta s T} > 0$ which further implies

$$e^{\alpha_2 \delta T} > 1$$

This is possible only when $s \in [0, \infty)$, $0 < \alpha_1 < 1$ and $\delta \geq 0$.

Let us discuss the property of $g''(s,T)$ in further details. From (2.36) we know that,

$$\begin{aligned}
\lim_{T\to\infty} -g''(s;T) &= \lim_{T\to\infty} \frac{\alpha_1^2 \delta T e^{\alpha_1 \delta s T}}{e^{\alpha_1 \delta s T} + e^{\alpha_2 \delta T} - 1} \left(1 - \frac{e^{\alpha_1 \delta s T}}{e^{\alpha_1 \delta s T} + e^{\alpha_2 \delta T} - 1}\right) \\
&= \lim_{T\to\infty} \frac{\alpha_1^2 \delta T e^{\alpha_1 \delta s T}}{e^{\alpha_1 \delta s T} + e^{\alpha_2 \delta T} - 1} \left(1 - \frac{e^{(\alpha_1 \delta s - \alpha_2 \delta)T}}{e^{(\alpha_1 \delta s - \alpha_2 \delta)T} + 1 - e^{-\alpha_2 \delta T}}\right) \\
&= \lim_{T\to\infty} \frac{\alpha_1^2 \delta T}{1 + e^{(\alpha_2 \delta - \alpha_1 \delta s)T} - e^{-\alpha_1 \delta s T}} \left(1 - \frac{e^{(\alpha_1 \delta s - \alpha_2 \delta)T}}{e^{(\alpha_1 \delta s - \alpha_2 \delta)T} + 1 - e^{-\alpha_2 \delta T}}\right) \\
&= \lim_{T\to\infty} \frac{\alpha_1^2 \delta T}{2} \left(1 - \frac{1}{2}\right) \\
&= \lim_{T\to\infty} \frac{\alpha_1^2 \delta T}{4} > 0, \ \textit{as } \alpha_1 = \alpha_2, \ \textit{based on previous proposition}
\end{aligned} \tag{2.37}$$

This completes the proof. □

Combining the above conditions of function g we get, $g(0;T) = 0$ and $g(\infty, T) = -\infty$. Thus, $g(s;T)$ is strictly increasing for $s \in (0, \gamma]$ and is strictly decreasing for $s \in [\gamma; \infty)$. Now at $t \to \infty$ we can write,

$$\mathrm{E}[X_1|X_2 > t] \sim \frac{T}{\beta} e^{Tg(s_0(T);T)} \sqrt{\frac{2\pi}{-Tg''(s_0(T);T)}} \sim \frac{1}{\beta} e^{Tg(s_0(T);T)} \sqrt{\frac{2\pi T}{-g''(s_0(T);T)}} \tag{2.38}$$

Again in our case

$$g(s;T) = 1 + s_0(T)\beta^{-1} - T^{-1}[\delta^{-1} \log(e^{\alpha_1 \delta s_0(T)\ T} + e^{\alpha_2 \delta T} - 1) + (1-\alpha_1)s_0(T)\ T + (1-\alpha_2)T],$$

$$-g''(s_0(T);T) = (\alpha_1^2 \delta T e^{\alpha_1 \delta s_0(T)\ T})(e^{\alpha_1 \delta s_0(T)\ T} + e^{\alpha_2 \delta T} - 1)^{-1}(1 - e^{\alpha_1 \delta s T}(e^{\alpha_1 \delta s T} + e^{\alpha_2 \delta T} - 1)^{-1}),$$

$$s_0(T) = (\alpha_1 \delta T)^{-1} \log \left[(e^{\alpha_2 \delta T} - 1)[1 - \beta(1-\alpha_1)](\alpha_1 \beta - (1 - \beta(1-\alpha_1)))^{-1} \right]$$

and $T = \beta \ \log(1+t)$.

Proposition 7. *For all $(\alpha_1, \alpha_2) \in [0,1]^2$ and very large value of T, the integrand function g varies slowly and takes the value;*

$$g_a(s,T) = \begin{cases} 1 + (1-\alpha_1)s & \text{if } \alpha_2 > \alpha_1; \\ 1 + s - \alpha_2 & \text{if } \alpha_2 < \alpha_1; \\ (1 - \frac{\alpha_2}{2}) + (1 - \frac{\alpha_1}{2})s & \text{if } \alpha_2 = \alpha_1, \end{cases}$$

where $g_a(s,T) = T^{-1}[\delta^{-1} \log(e^{\alpha_1 \delta s T} + e^{\alpha_2 \delta T} - 1) + (1-\alpha_1)sT + (1-\alpha_2)T]$.

Proof. Please see the appendix. □

Again, as we have calculated the optimal s, then we can say that the limit goes to a constant number. Thus, the conditional expectation at $T \to \infty$ becomes,

$$\mathrm{E}[X_1 | X_2 > t] \sim \frac{1}{\beta} e^{\beta \log(1+t) g(\alpha_1, \alpha_2, \delta)} \sqrt{\frac{2\pi\beta \log(1+t)}{-g''(\alpha_1, \alpha_2, \delta, \beta \log(1+t))}}, \tag{2.39}$$

as $T = \beta \log(1+t)$.

Now after using conditions(2.32), (2.37) and (A.2) we get,

$$\begin{aligned} \mathrm{E}[X_1 | X_2 > t] &\sim \frac{1}{\beta} e^{\beta \log(1+t) g(\alpha_1, \alpha_2, \delta)} \sqrt{\frac{8\pi\beta \log(1+t)}{\alpha_1^2 \delta \beta \log(1+t)}} \\ &\sim \frac{1}{\beta} (1+t)^{\beta[1 + \frac{\alpha_2}{\alpha_1 \beta} - F]} \sqrt{\frac{8\pi}{\alpha_1^2 \delta}}, \ \forall \ (\alpha_1, \alpha_2) \in [0,1]^2, \ \beta > 1 \textit{ and } \delta > 0 \end{aligned} \tag{2.40}$$

where F is a constant where $F = (1 - \frac{\alpha_2}{2}) + (1 - \frac{\alpha_1}{2})\frac{\alpha_2}{\alpha_1}$.

Corollary 8. *Suppose X_1 and X_2 are two dependent random variables. For all $(\alpha_1, \alpha_2) \in [0,1]^2$, $\delta \geq 0$, $\beta > 1$ and as $t \to \infty$, we have;*

$$\mathrm{E}[X_2|X_1 > t] \sim \frac{1}{\beta}(1+t)^{\beta\left[1+\frac{\alpha_2}{\alpha_1\beta}-\left\{(1-\frac{\alpha_1}{2})+(1-\frac{\alpha_2}{2})\frac{\alpha_2}{\alpha_1}\right\}\right]}\sqrt{\frac{8\pi}{\alpha_2^2\delta}} \tag{2.41}$$

In order to measure the *non-exchangeability* we have to take the ratio of two conditional expectations [i.e $\mathrm{E}[X_1|X_2 > t]$ $\mathrm{E}[X_2|X_1 > t]^{-1}$]. Using conditions (2.40) and (2.41) we get;

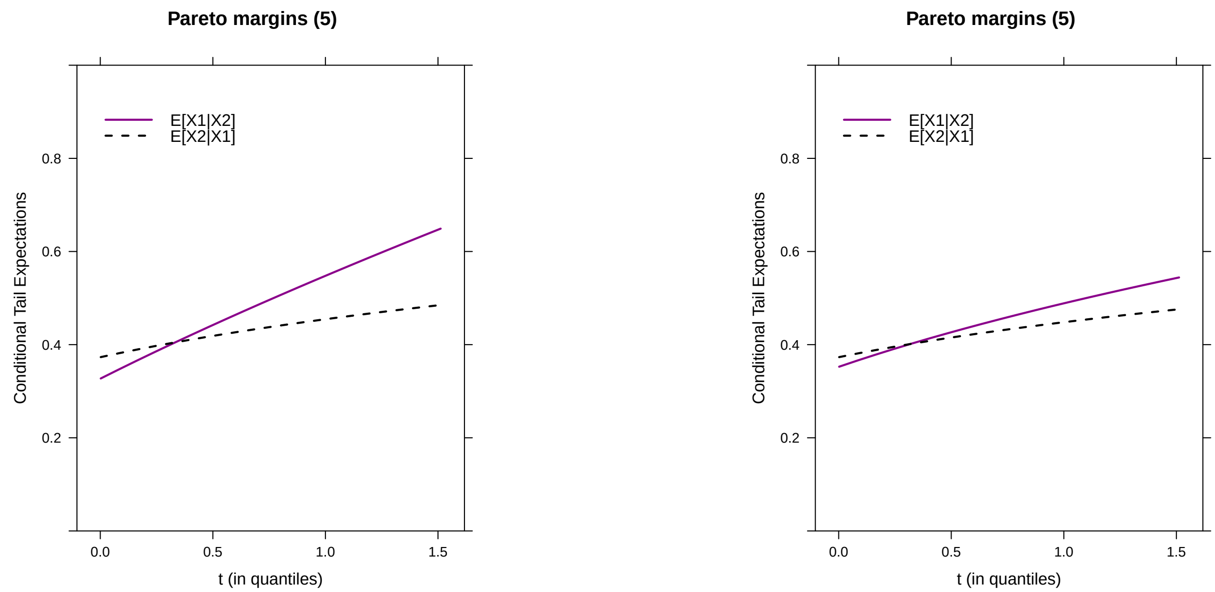
$$\frac{\mathrm{E}[X_1|X_2 > t]}{\mathrm{E}[X_2|X_1 > t]} \sim \frac{(1+t)^{\beta\left[1+\frac{\alpha_2}{\alpha_1\beta}-(1-\frac{\alpha_2}{2})-(1-\frac{\alpha_1}{2})\frac{\alpha_2}{\alpha_1}\right]}}{(1+t)^{\beta\left[1+\frac{\alpha_1}{\alpha_2\beta}-(1-\frac{\alpha_1}{2})-(1-\frac{\alpha_2}{2})\frac{\alpha_1}{\alpha_2}\right]}}$$

$$\forall\ (\alpha_1, \alpha_2) \in [0,1]^2,\ \beta > 1,\ \delta > 0,\ \text{and}\ s_0 \in [0, \infty) \tag{2.42}$$

From (2.42) it is clear that the ratio of two conditional expectations depends on some constant power of $(1+t)$. In order to satisfy Laplace approximation we need just only two conditions, which are $\beta^{-1} + \alpha_1 > 1$ and $\beta^{-1} + \alpha_2 > 1$. By the framework we also know that, $(\alpha_1, \alpha_2) \in [0,1]^2$, $\beta > 1$, $\delta > 0$ and $s_0 \in [0, \infty)$.

Remark 5. In (2.42) we can see that, if $\alpha_1 = \alpha_2$, the ratio the two conditional expectations becomes 1. Intuitively, $\alpha_1 = \alpha_2$ means the KB4 copula is *exchangeable.* Then, there is no difference between two conditional expectations.

In Figure 2.1a and Figure 2.1b we try to do the simulation using Laplace approximation. In Figure 2.1a we take α_1 and α_2 0.97 and 0.85 respectively. In Figure 2.1b we reduce α_1 such that , its value comes closer to α_2. We do this because if α_1 and α_2 are very close to each other, we get *exchangeability* as the result of symmetric copulas. In these two panels we assume $\beta = 5$ and $\delta = 10$ throughout this simulation. The values of α_1 and α_2 are very high. The main reason is that, if we take lower vales, the distance between two conditional



(a) $\alpha_1 = 0.97$, $\alpha_2 = 0.85$, $\delta = 10$ (b) $\alpha_1 = 0.90$, $\alpha_2 = 0.85$, $\delta = 10$

Figure 2.1: Comparison of $\mathrm{E}[X_1|X_2 > t]$ and $\mathrm{E}[X_2|X_1 > t]$ when α_1 and α_2 are different, using Laplace Approximation

expectations are so big that we cannot find any pattern. Apart from that these Laplace approximation simulations look similar to the simulations when we use Pareto margins and use the definitions of conditional expectations.

Proposition 9. *Suppose X_1 and X_2 are two dependent random variables. If $\alpha_1 + \beta^{-1} < 1$ and $\int_0^\infty e^{g(s,T)} < \infty$, then the integrand function g slowly converges to a constant as $t \to \infty$; where $(\alpha_1, \alpha_2) \in [0,1]^2$, $\delta > 0$, $\beta > 1$ and $s \in [0, 0+\epsilon)$. In this case conditional tail expectation converges to*

$$\mathrm{E}[X_1|X_2 > t] \sim \frac{1}{\beta}\left(\frac{1}{1-\frac{1}{\beta}-\alpha_1}\right),$$

as $t \to \infty$.

Proof. Now let us consider the other case when Laplace approximation does not work. We have the use Watson's lemma if $g'(s;T) < 0$. From our previous results we know, $g(0,T) = 0$ and $g(\infty, T) = -\infty$. Again we know, $g(s;T) = 1 + s\beta^{-1} - T^{-1}[\delta^{-1}\log(e^{\alpha_1\delta sT} + e^{\alpha_2\delta T} - 1) + (1-\alpha_1)sT + (1-\alpha_2)T]$. Thus,

$$g(s;T) = 1 + \frac{s}{\beta} - \frac{1}{T}\left[\frac{1}{\delta}\log(e^{\alpha_1\delta sT} + e^{\alpha_2\delta T} - 1) + (1-\alpha_1)sT + (1-\alpha_2)T\right]$$

$$\implies g'(s;T) = \frac{1}{\beta} - \frac{1}{T}\left[\frac{\alpha_1\delta T e^{\alpha_1\delta sT}}{\delta(e^{\alpha_1\delta sT} + e^{\alpha_2\delta T} - 1)} + (1-\alpha_1)T\right] \tag{2.43}$$

Suppose we consider, like in the previous case $\beta^{-1} + \alpha_1 - 1 > 0$ does not hold anymore as $T \to \infty$. Then we no longer can use Laplace approximation. We have to check if $g'(s,T)$ is decreasing in s. In this situation if we divide the numerator and denominator of the second term of the right hand side of (2.43) by $e^{\alpha_1 sT}$, we get,

$$\begin{aligned} g'(s;T) &= \frac{1}{\beta} - \frac{\alpha_1}{1 + e^{(\alpha_2-\alpha_1 s)\delta T} - e^{-\alpha_1\delta sT}} - (1-\alpha_1),\ \textit{with}\ \alpha_2 > \alpha_1 \\ \implies \lim_{s\to 0^+,\ T\to\infty} g'(s,T) &= \frac{1}{\beta} - \lim_{s\to 0^+,\ T\to\infty} \frac{\alpha_1}{1 + e^{(\alpha_2-\alpha_1 s)\delta T} - e^{-\alpha_1\delta sT}} - (1-\alpha_1) \\ &= \frac{1}{\beta} + \alpha_1 - 1 \end{aligned} \tag{2.44}$$

In order to show $g'(s;T) < 0$ we have to assume $\beta^{-1} + \alpha_1 \not> 1$ where $\beta > 1$ *and* $\alpha_1 \in [0,1]$. Now we can use Watson's lemma. Here we are not using the Watson's lemma directly. We are using a version of Watson's lemma. In order to do the approximation of this integration we are using theorem 36 of Breitung (1994) [p. 48]. As $g(s,T)$ is a real function on the semi-infinite interval $[0,\infty)$ and in $(0, 0+\epsilon]$ with $\epsilon > 0$, this function is continuously differentiable and

$$\sup_{0+\epsilon\le s\le\infty} g(s,T) \le g(0,T) - \psi, \tag{2.45}$$

with $\psi > 0$.

Now for $g'(s,T)$ we have $g'(s,T) < 0$ as $\beta^{-1} + \alpha_1 < 1$ and $s \to \infty$. We can also write

$$g'(s,T) = -as^{r-1} + o(s^{r-1}) \;\forall\; r > 0.$$

Now if we assume $r = 1$ then $g'(s,T) = -a$. From our previous results we know that, $\lim_{s\to 0^+,\ T\to\infty} g'(s,T) = \beta^{-1} - 1 + \alpha_1$, which is a constant. Thus, $-a = \beta^{-1} + \alpha_1 - 1$ or, $a = 1 - \beta^{-1} - \alpha_1 > 0$. Let us assume there is another real and continuous function $h(s,T) \in [0,\infty)$ such that,

$$h(s,T) = bs^{m-1} + o(s^{m-1})$$

with $m > 0$. More specifically we assume $h(s,T) = 1$ in our case. Thus,

$$bs^{m-1} + o(s^{m-1}) = 1 \implies b = 1 \tag{2.46}$$

when $m = 1$.

Finally, as we are assuming $\int_0^\infty e^{g(s,T)} ds < \infty$. We are trying to find out the condition under which the above integration is finite. Now, after putting the value of $g(s,T)$ in the integrand, we have;

$$\begin{aligned}
\int_0^\infty e^{g(s,T)}\, ds &= \int_0^\infty e^{1+\frac{s}{\beta}-\frac{1}{T}\left[\frac{1}{\delta}\log(e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1)+(1-\alpha_1)sT+(1-\alpha_2)T\right]}\, ds \\
&= \int_0^\infty e^{1+\frac{s}{\beta}-\frac{1}{\delta T}\log(e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1)-(1-\alpha_1)s-(1-\alpha_2)}\, ds \\
&= e^{\alpha_2} \int_0^\infty e^{\frac{s}{\beta}-\frac{1}{\delta T}\log(e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1)-(1-\alpha_1)s}\, ds \\
&= \vartheta \int_0^\infty e^{\frac{s}{\beta}-\frac{1}{\delta T}\log(e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1)-(1-\alpha_1)s}\, ds,\ where\ \vartheta = e^{\alpha_2} < \infty
\end{aligned} \tag{2.47}$$

If we are able to show the integration in (2.47) is finite, then we are able to use *Watson's* lemma. When T is very large [i.e., $T \to \infty$],

$$\begin{aligned}
&\int_0^{\infty} e^{\frac{s}{\beta}-\frac{1}{\delta T}\log(e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1)-(1-\alpha_1)s}\ ds \\
&\leq \int_0^{\frac{\alpha_2}{\alpha_1}} e^{\frac{s}{\beta}-\frac{\alpha_1\delta sT}{\delta T}-(1-\alpha_1)s}\ ds + \int_{\frac{\alpha_2}{\alpha_1}}^{\infty} e^{\frac{s}{\beta}-\frac{\alpha_2\delta T}{\delta T}-(1-\alpha_1)s}\ ds\ \ as\ T \to \infty \\
&= \int_0^{\frac{\alpha_2}{\alpha_1}} e^{\frac{s}{\beta}-\alpha_1 s-(1-\alpha_1)s}\ ds + \int_{\frac{\alpha_2}{\alpha_1}}^{\infty} e^{\frac{s}{\beta}-\frac{\alpha_2}{\delta}-(1-\alpha_1)s}\ ds \\
&= \int_0^{\frac{\alpha_2}{\alpha_1}} e^{\left[\frac{1}{\beta}-\alpha_1-(1-\alpha_1)\right]s}\ ds + e^{-\alpha_2}\ \int_{\frac{\alpha_2}{\alpha_1}}^{\infty} e^{\left[\frac{1}{\beta}-(1-\alpha_1)\right]s}\ ds \\
&= \frac{e^{\left[\frac{1}{\beta}-1\right]\frac{\alpha_2}{\alpha_1}}}{\frac{1}{\beta}-1} + e^{-\alpha_2}\ \frac{e^{\left[\frac{1}{\beta}-(1-\alpha_1)\right]\ s}}{\frac{1}{\beta}-(1-\alpha_1)}\Bigg|_{\frac{\alpha_2}{\alpha_1}}^{\infty}
\end{aligned} \tag{2.48}$$

It is clear that in (2.48) the first term is always finite . The second term in (2.48) is finite if $\alpha_1+\beta^{-1}<1$ which is obvious as $\beta>1$. Thus, we can use *Watson's Lemma* all the time in this case.

Now, by Watson's lemma we can write the approximated value of the integral $I(T) = \int_0^{\infty} e^{Tg(s,T)}ds$ with $T \geq 1$ are all finite and the asymptotic approximation is;

$$I(T) \sim \left(\frac{1}{1-\frac{1}{\beta}-\alpha_1}\right)\left(T^{-1}\right)e^{Tg(0,T)}\ as,\quad T \to \infty$$

$$\implies I(T) \sim \left(\frac{1}{1-\frac{1}{\beta}-\alpha_1}\right)\left(T^{-1}\right)e^{0}\ as,\ g(0,T)=0,\ T \to \infty \tag{2.49}$$

$$\implies I(T) \sim \left(\frac{1}{1-\frac{1}{\beta}-\alpha_1}\right)\left(T^{-1}\right)\ as,\ g(0,T)=0,\ T \to \infty \tag{2.50}$$

Hence, from (2.50) we know that,

$$\mathrm{E}[X_1|X_2 > t] \sim \left(\frac{1}{1-\frac{1}{\beta}-\alpha_1}\right)\left(\beta^{-1}\right) \; as, \; g(0,T)=0, \; T \to \infty$$
$$\sim \frac{1}{\beta}\left(\frac{1}{1-\frac{1}{\beta}-\alpha_1}\right) \; as \; t \to \infty \tag{2.51}$$

This completes the proof. □

Corollary 10. *Suppose X_1 and X_2 are two random variables. For all $(\alpha_1, \alpha_2) \in [0,1]^2$, $\delta > 0$ and $\beta > 1$ we have;*

$$\mathrm{E}[X_2|X_1 > t] \sim \frac{1}{\beta}\left(\frac{1}{1-\frac{1}{\beta}-\alpha_2}\right), \tag{2.52}$$

as $t \to \infty$.

Again in this case in order to measure non-exchangeability we have to take the absolute difference between two conditional expectations defined in (2.51) and (2.52) respectively.Using conditions (2.51) and (2.52) we get;

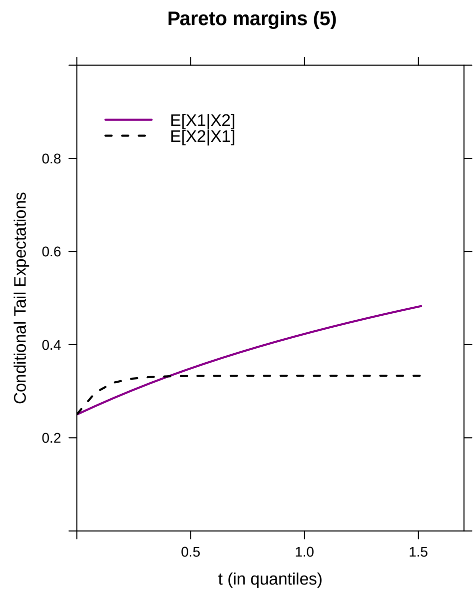
$$\frac{\mathrm{E}[X_1|X_2 > t]}{\mathrm{E}[X_2|X_1 > t]} \sim \left(\frac{1-\frac{1}{\beta}-\alpha_2}{1-\frac{1}{\beta}-\alpha_1}\right) \tag{2.53}$$
$$\forall \; (\alpha_1, \alpha_2) \in [0,1]^2, \; and \; \beta > 1$$

From (2.53) it is clear that the ratio of two conditional expectations goes to 1 as $t \to \infty$ if $\alpha_1 = \alpha_2$. If we carefully look at (2.51) and (2.52), both of the conditional expectations equal $O(\log(1+t)^{-1})$. If we do simulations we always get horizontal lines of $E[X_1|X_2 > t]$ and $E[X_2|X_1 > t]$ respectively.

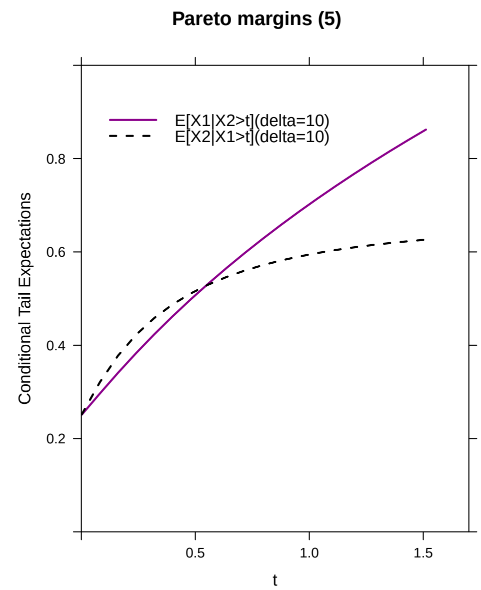
Remark 6. In order to satisfy Watson's lemma, we need just only two conditions, which are $\beta^{-1} + \alpha_1 < 1$ and $\beta^{-1} + \alpha_2 < 1$. By the framework we also know that, $(\alpha_1, \alpha_2) \in [0,1]^2,\ \beta > 1,\ \delta > 0$, and $s \in [0, 0 + \epsilon)$.

Remark 7. In (2.53) we observe that, if $\alpha_1 = \alpha_2$, the ratio of two conditional expectations becomes 1. Intuitively, $\alpha_1 = \alpha_2$ means the KB4 copula becomes exchangeable. Therefore, there is no difference between two conditional expectations.

Now let us see if our mathematical results are consistent with the graphs of simulations. We choose $\beta = 5,\ \alpha_1 = 0.85$ and $\delta = 10$ first and then vary α_2 in order to see if non-ex changeability holds. We set $\alpha_2 = 0.2$ first and then set it to 0.5. The main objective is to check how different values of α_2 makes the conditional expectations non-exchangeable. In the following plots on the vertical axis we plot two conditional tail expectations namely $\mathrm{E}[X_1|X_2 > t]$ and $\mathrm{E}[X_2|X_1 > t]$ and, on the horizontal axis we plot t. Here when we define t-axis , we mean t in quantiles. Thus, in Figure 2.2a and 2.2b when we see t-value is 1.5, it almost covers 99th percentile of all t. At this value t is really very large. As $\alpha_1 = 0.85$, if we set α_2 from 0.2 to 0.5 , the distance between α_1 and α_2 is getting reduced. In Figure 2.2a and Figure 2.2b the dotted lines represent $\mathrm{E}[X_2|X_1 > t]$ and the solid line represents $\mathrm{E}[X_1|X_2 > t]$. In Figure 2.2a we have α_2=0.2. At this level $\mathrm{E}[X_2|X_1 > t]$ line was increasing at around 0.25 then, it becomes almost parallel to the horizontal axis. Even the solid line did not increase that much compared with Figure 2.2a. That is why the difference between two conditional expectations is smaller when $\alpha_2 = 0.2$. On the other hand, we see there is bigger difference between these two conditional tail expectations in Figure 2.2b when $\alpha_2 = 0.5$. That means, if α_2 is relatively bigger, we can have more tail non-ex changeability. Again, we have to be more careful in order to choose α_2. We should not choose α_2 in such a way that its value does not get very close to α_1. Then, we probably cannot see *non-exchangeability.*



(a) $\alpha_1 = 0.85$, $\alpha_2 = 0.2$, $\delta = 10$



(b) $\alpha_1 = 0.85$, $\alpha_2 = 0.5$, $\delta = 10$

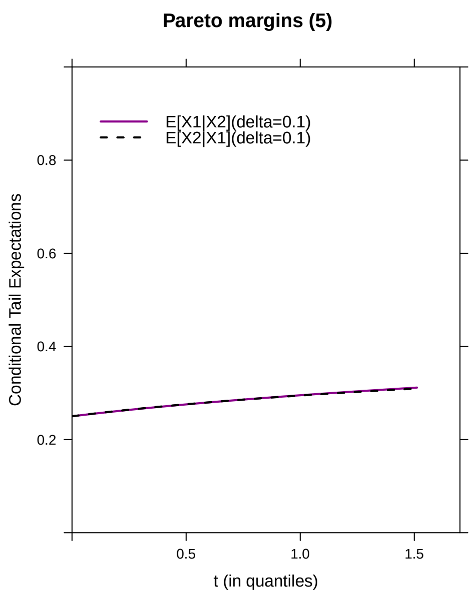
Figure 2.2: Comparison of $\mathrm{E}[X_1|X_2 > t]$ and $\mathrm{E}[X_2|X_1 > t]$ when α_1 and α_2 are different

Now, in Figures 2.3a, 2.3b, 2.3c and 2.3d we are only varying the value of δ. In our case, we fix the other parameters respectively at $\beta = 5$, $\alpha_1 = 0.85$, and $\alpha_2 = 0.5$. Again, like in the previous cases, the dotted line represents $\mathrm{E}[X_2|X_1]$ and, the solid line represents $\mathrm{E}[X_1|X_2]$. In Figure 2.3a and Figure 2.3b we are varying δ between 0 and 1. In panel 2.3a δ is very low (i.e. 0.1). Here we can not distinguish the two lines representing two conditional expectations. From (2.42) we know that, both the terms have $\sqrt{\delta}$ in themselves. As already δ is very small and it is multiplied by square of another fraction α_1 , the denominator is going to be very low. Thus, when δ is low, it should not have any impact of any of the conditional expectation curves. We can see this in Figure 2.3a. Now, we increase the value of δ a little bit to 0.5. We can see there is some gap between two conditional expectations at the tail (i.e. at 99th percentile). That means, if we increase the value of δ a little bit, we get tail *non-exchangeability* at certain level even the variation is between 0 and 1.

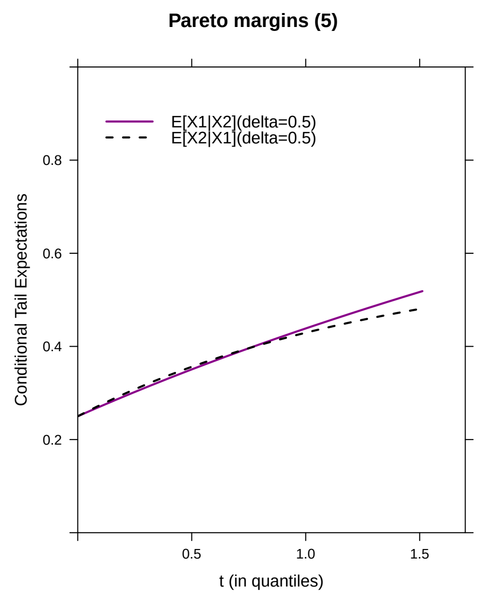
In Figures 2.3c and 2.3d we are taking the value of δ greater than 1. In panel 2.3c we are taking $\delta = 10$ and in panel 2.3d $\delta = 100$. In this way, we are able to increase the value of δ significantly. From these two panels it is clear that, due to increase in δ from 10 to 100 we do not find any significant difference in two conditional tail expectations. On the other hand, if we compare these differences with Figure 2.3a and Figure 2.3b, they are relatively higher when $\delta > 1$. As we increase δ after 1 the difference between two conditional tail expectations increases but not like when we increase δ in [0,1]. In (2.42) if we put 100 instead of 10 in δ, the difference does not change much because, it is in the denominator with the root form. Thus, the pictures obtained in Figures 2.3a, 2.3b, 2.3c and 2.3d are consistent with the theoretical results.

2.1.2.2 Weibull Margins

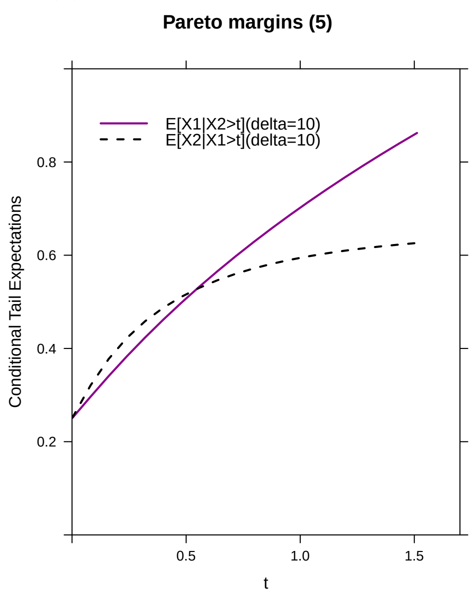
Now, in this section we are considering Weibull distribution. The reason behind to choose this distribution is because this is sub-exponential distribution. The advantage to use this distribution we can use this with a very small size of sample. In our paper, we are trying to develop a model on the extreme value theory. Here we are interested at the extreme; in other words, if we have a density function, we must consider the tails of that distribution. In extreme value theory sometimes getting more data at the tail is very expensive. Suppose, let us consider the case of the incidence of tornado in a certain region in the United States. We know there are some Southern parts in the United States where there is a higher probability of tornado than other parts. There are certain categories of tornadoes based on the severity of damage. Traditionally we know that, tornado at EF-5 level is the most dangerous. Under the extreme value theory we try to find out the probability of occurrence this kind of tornado.



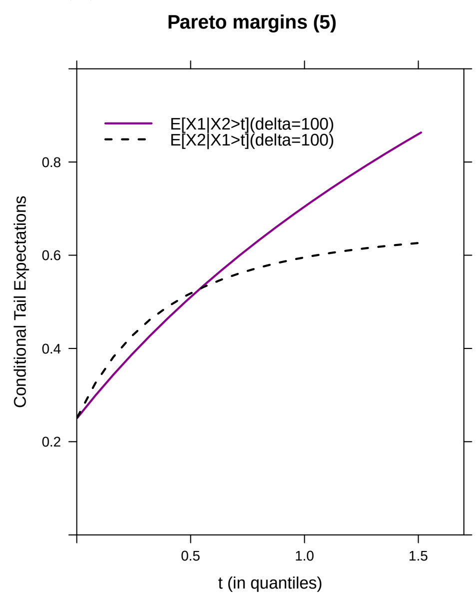
(a) $\alpha_1 = 0.85$, $\alpha_2 = 0.5$, $\delta = 0.1$



(b) $\alpha_1 = 0.85$, $\alpha_2 = 0.5$, $\delta = 0.5$



(c) $\alpha_1 = 0.85$, $\alpha_2 = 0.5$, $\delta = 10$



(d) $\alpha_1 = 0.85$, $\alpha_2 = 0.5$, $\delta = 100$

Figure 2.3: Comparison of $\mathrm{E}[X_1|X_2 > t]$ and $\mathrm{E}[X_2|X_1 > t]$ of Survival Copula with Pareto Margins

Obviously, these EF-5 categories are extremely rare. Hence, the availability of sample of this data is very small. Getting a big data set is very expensive here.

Another example might be the case of economics and finance. Suppose one person is interested to invest some money in the stock market. Stock market is extremely volatile. When an investor invests money in that system, he calculates expected returns based on probability of winning and losing some money there. Before, investing money the investor need to pay some price as a cost. If we deduct this cost from the expected return, we get expected profit of that investor by investing money in that market. Let us discuss a simple model. Then the scenario will be more clear. Suppose, an investor has only two possibilities, win some money (say, W) and lose money (say, L). As there are only two possibilities, the probability of win is equal to the probability of loss which is $\frac{1}{2}$. Thus, expected return of that investor from the stock market is $\mathrm{E}(R) = \frac{1}{2}W + \frac{1}{2}(-L)$. This kind of expectations are called Von Neuman Morgenstern type return. Let us also suppose, that investor needs to pay c before putting money in the stock market. Thus, the expected profit of that investor from the stock market is $\mathrm{E}(R) - c$. Further, let us assume, the extreme cases in this environment as one person completely loses his money. If we try to figure out the probability of loss in this case, the availability of this kind of data set is going to be very small. If we try to find more incidence of this data, it is expensive. Here we have to use small data set to predict the incidence of this kind of loss of the investor. Under this case, Weibull distribution is very helpful. Apart from that, this distribution has very good graphical plot. Hence, it is easier to understand the behavior at the extreme by just seeing the plots.

Now, in this section, we consider X_1 and X_2 follow Weibull distribution with identical cumulative distribution functions $F(x) = 1 - e^{-x^{\gamma}}$, $\forall x, \gamma > 0$. Now the survival function should be $\overline{F}(x) = 1 - F(x) = e^{-x^{\gamma}}$ $\forall$ $x, \gamma > 0$. The reason behind to take the positive value of γ is to show some significance at the tail. We use our KB4 copula with this margin. As

Weibull distribution does not need many observations, we think the conditional expectations based on this are going to be a good fit at the tail. Again, as these conditional expectations do not have a proper closed form solutions, we need to see whether we can use Laplace Approximation or Watson's lemma. Before going directly to these processes we transform the integrand function into $e^{g(s,T)}$ and check whether if $g(0, T) = 0$ and $g(\infty, T) = -\infty$ hold. Throughout this paper we assume s as a proxy of X_1. Then we find out that in this case we cannot do Laplace Approximation. Only Watson's lemma does work. Even here we use Theorem 36 of Breitung (1994) [p. 48] in order to get the exact value of conditional expectations when $t \to \infty$. We get, $\mathrm{E}[X_1|X_2 > t] \sim \gamma^{-1}\Gamma(\frac{1}{\gamma})(1-\alpha_1)^{-1}$ as $t \to \infty$ *and* $T = t^\gamma$, which is constant. In a similar way by, using Watson's lemma we get, $\mathrm{E}[X_2|X_1 > t \sim \gamma^{-1}\Gamma(\frac{1}{\gamma})(1-\alpha_2)^{-1}$ as $t \to \infty$. As both of them are constants, their ratio is going to be unity. Finally, when we do the simulation studies our simulated pictures become similar to the original simulations done by software.

Proposition 11. *Integrand of the conditional expectations are multiplicative separative of two monotonic functions.*

Proof. In this example we consider two dependent random variables X_1 and X_2 which follow Weibull distribution with identical cumulative distribution functions $F(x) = 1 - e^{-x^\gamma},\ \forall x, \gamma > 0$. Now the survival function should be $\overline{F}(x) = 1 - F(x) = e^{-x^\gamma}\ \forall x, \gamma > 0$. Following Hua and Joe (2014) we will transform the survival function $\overline{F}(t) \to\ e^{-T}\ or, T = -\log \overline{F}(t) = t^\gamma,\ y = -log\overline{F}(x) = x^\gamma \implies x = y^{\frac{1}{\gamma}}$. Differentiating totally both sides of the previous equation we get; $dx = \frac{1}{\gamma} y^{\frac{1}{\gamma}-1} dy$.

Now, by using the method provided by Hua and Joe (2014) we get;

$$
\begin{aligned}
\mathrm{E}(X_1|X_2>t) &= \int_0^\infty e^T \widehat{C}(e^{-y}, e^{-T}) y^{\frac{1}{\gamma}-1}\gamma^{-1}dy \\
&= \gamma^{-1}T^{\frac{1}{\gamma}} \int_0^\infty e^T \widehat{C}(e^{-sT}, e^{-T}) s^{\frac{1}{\gamma}-1} ds, \; by\ using\ y = sT \\
&= \gamma^{-1}T^{\frac{1}{\gamma}} \int_0^\infty e^{T[1+\frac{1}{T}\log \widehat{C}(e^{-sT}, e^{-T})]} s^{\frac{1}{\gamma}-1} ds \\
&= \gamma^{-1}T^{\frac{1}{\gamma}} \int_0^\infty e^{Tg(s;T)} h(s) ds, \; \forall s \in [0,\infty)
\end{aligned} \tag{2.54}
$$

where, $g(s;T) = 1 + T^{-1}\log \widehat{C}(e^{-sT}, e^{-T})$ and , $h(s) = s^{\frac{1}{\gamma}-1} > 0$. □

Remark 8. Clearly, from the above two equations we can say that, the behavior of the conditional expectation depends on $g(s;T)$ function.

In order to check if Laplace approximation or Watson's lemma is valid, we need to check two conditions first, (i) $g(0;T) = 0$ and, (ii) $g(\infty;T) = -\infty$ first. After using KB4 copula in the g function above we have,

$$
g(s,T) = 1 - \frac{1}{T}[\frac{1}{\delta}\log(e^{\alpha_1\delta sT} + e^{\alpha_2\delta T} - 1) + (1-\alpha_1)sT + (1-\alpha_2)T] \tag{2.55}
$$

Thus,

$$
g(0;T) = 1 - \frac{1}{T}[\alpha_2 T + T - \alpha_2 T] = 1 - 1 = 0 \tag{2.56}
$$

and,

$$
\begin{aligned}
& g(\infty;T) = 1 - \frac{1}{T}[\frac{1}{\delta}\log(e^{\infty} + e^{\alpha_2\delta T} - 1) + \infty + (1-\alpha_2)T] \\
\implies & g(\infty;T) = -\infty, \; as\ T \to \infty.
\end{aligned} \tag{2.57}
$$

From the above discussion we can see that in the case of KB4 Copula with Weibull margins we can have the g function which satisfies $g(0,T)=0$ and $g(\infty,T)=-\infty$. Now in order to calculate the conditional tail expectations we need to check if $g'(0,T)>0$ as $T\to\infty$.

Claim 12. *Khoudraji transformed Clayton Copula does not have any solution of conditional expectations at the tail by Laplace Approximation.*

Proof. Please see in the Appendix. □

Proposition 13. *Suppose X_1 and X_2 are two dependent random variables. If $\gamma>1$, conditional tail expectation of KB4 Copula with Weibull margin goes to some constant ,*

$$\mathrm{E}[X_1|X_2>t]\sim\gamma^{-1}\Gamma\left(\frac{1}{\gamma}\right)\frac{1}{1-\alpha_1}$$

as $t\to\infty$.

Proof. In order to satisfy Watson's lemma we have to check if $g(s;T)$ is decreasing in s or if $g'(s;T)<0$.We know from above;

$$\begin{aligned}
g'(s;T) &= -\frac{1}{T}[\frac{\alpha_1\delta Te^{\alpha_1\delta sT}}{\delta(e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1)}+(1-\alpha_1)T]\\
&= -[\frac{\alpha_1 e^{\alpha_1\delta sT}}{(e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1)}+(1-\alpha_1)]\\
&= -[\frac{\alpha_1}{(e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1)e^{-\alpha_1\delta sT}}+(1-\alpha_1)]\\
&= -[\frac{\alpha_1}{e^{0}+e^{(\alpha_2-\alpha_1 s)\delta T}-e^{-\alpha_1\delta sT}}+(1-\alpha_1)\\
&= -[\frac{\alpha_1}{1+e^{(\alpha_2-\alpha_1 s)\delta T}-e^{-\alpha_1\delta sT}}+(1-\alpha_1)]<0\\
\Longrightarrow \lim_{T\to\infty} g'(s;T) &= -[\alpha_1+1-\alpha_1]=-1<0,\ if\ \alpha_2<\alpha_1,\ or\\
&= -(1-\alpha_1)<0,\ if\ \alpha_2>\alpha_1 \qquad (2.58)
\end{aligned}$$

From (2.58) we are sure that we need to use Watson's lemma. As $g(s;T)$ is a real valued function on the semi-infinite interval $[0,\infty)$ and in $(0,0+\epsilon]$ with $\epsilon > 0$ this function is continuously differentiable and

$$\sup_{0+\epsilon\leq s\leq\infty} g(s,T) \leq g(0,T) - \psi, \tag{2.59}$$

with $\psi > 0$. Now for $g'(s,T)$ we have $g'(s,T) < 0$ as $\frac{1}{\beta} < 1$ and $s^+ \to 0$. We can also write,

$$g'(s,T) = -as^{r-1} + o(s^{r-1})\ \forall r > 0$$

Now, if we assume $r = 1$ then $g'(s,T) = -a$. From our previous results we know that, $\lim_{s^+\to 0,\ T\to\infty} g'(s,T)$ is either -1 or $-(1-\alpha_1)$, based on the conditions described in (2.58). As we are concentrating on asymmetric copulas we better choose $-(1-\alpha_1)$ in order to maintain some non-exchangeability. Thus, $-a = -(1-\alpha_1)$ or, $a = 1-\alpha_1 > 0$.

Let us assume there is another continuous real valued function $h(s,T) \in [0,\infty)$ such that,

$$h(s) = bs^{m-1} + o(s^{m-1})$$

with $m > 0$. From (2.54) we know, $h(s) = s^{\frac{1}{\gamma}-1}$ in our case. Thus,

$$bs^{m-1} + o(s^{m-1}) = s^{\frac{1}{\gamma}-1} \implies b = 1,\ and\ m = \frac{1}{\gamma} \tag{2.60}$$

Finally, a we are assuming $\int_0^\infty |h(s)| e^{g(s,T)} ds < \infty$. Let us find out the exact condition under which the whole integration becomes finite. After putting the value of $g(s,T)$ in the integrand we have,

$$
\begin{aligned}
&\int_0^\infty |h(s)| e^{g(s,T)} \; ds \\
&= \int_0^\infty |s^{\frac{1}{\gamma}-1}| \; e^{1-\frac{1}{T}[\frac{1}{\delta}\log(e^{\alpha_1 \delta s T}+e^{\alpha_2 \delta T}-1)+(1-\alpha_1)sT+(1-\alpha_2)T]} \; ds \\
&= e^{\alpha_2} \int_0^\infty s^{\frac{1}{\gamma}-1} \; e^{-\frac{1}{\delta T}\log(e^{\alpha_1 \delta s T}+e^{\alpha_2 \delta T}-1)-(1-\alpha_1)s} \; ds \\
&\leq e^{\alpha_2} \left[\int_0^{\frac{\alpha_2}{\alpha_1}} s^{\frac{1}{\gamma}-1} \; e^{-\alpha_2-(1-\alpha_1)s} \; ds \; + \int_{\frac{\alpha_2}{\alpha_1}}^\infty s^{\frac{1}{\gamma}-1} \; e^{-(\alpha_1+1-\alpha_1)s} \; ds \right], \; at \; T \to \infty \\
&= e^{\alpha_2} \Bigg[e^{-\alpha_2} \; e^{-(1-\alpha_1)s} \sum_{i=0}^{\frac{1}{\gamma}-1} (-1)^{\frac{1}{\gamma}-i-1} \frac{(\frac{1}{\gamma}-1)!}{i!(\alpha_1-1)^{\frac{1}{\gamma}-i}} \; s^i \Bigg|_0^{\frac{\alpha_2}{\alpha_1}} \\
&\qquad\qquad + e^{-s} \sum_{i=0}^{\frac{1}{\gamma}-1} (-1)^{\frac{1}{\gamma}-i-1} \frac{(\frac{1}{\gamma}-1)!}{i!} \; s^i \Bigg|_{\frac{\alpha_2}{\alpha_1}}^\infty \Bigg]
\end{aligned}
\tag{2.61}
$$

If we carefully look at (2.61) both the terms on the right hand side is always finite. The main reasons are we have e^{-s} and $\gamma > 0$; which leads us three possibilities, $\gamma \in (0,1)$, $\gamma = 1$ and $\gamma > 1$. Let us discuss each of the cases separately. As we have e^{-s} as the first term, it is always finite as $s \to \infty$. Now, only thing matters is the value of γ. When $\gamma \in (0,1)$, $\gamma^{-1} - 1$ takes the highest value when $\gamma \to 0$. By assumption $\gamma > 0$. So $\gamma^{-1} - 1 < \infty$. Under this case we still possibility to have $s^i \to \infty$ as $s \to \infty$. Therefore, we need more restriction on γ. In this bound of $(0,1)$ s^i is not finite. Furthermore, when $\gamma = 1$, $s^i \to \infty$ as $s \to \infty$. Hence, we need $\gamma > 1$ to make $s^i < \infty$ for any large s.

By Watson's lemma we can write the approximated value of the integral $I(T) = \int_0^\infty e^{Tg(s,T)} ds$ with $T \geq 1$ are all finite and the asymptotic approximation is;

$$
I(T) \sim \Gamma\left(\frac{1}{\gamma}\right) \frac{1}{1-\alpha_1} \left(T^{-\frac{1}{\gamma}}\right) \; e^{Tg(0,T)} \; as, \; T \to \infty
$$

$$I(T) \sim \Gamma\left(\frac{1}{\gamma}\right)\frac{1}{1-\alpha_1}\left(T^{-\frac{1}{\gamma}}\right) \ as, \ T \to \infty \ and \ e^0 = 1 \tag{2.62}$$

From equation (2.54) we know,

$$\mathrm{E}[X_1|X_2 > t] \sim \gamma^{-1}\Gamma\left(\frac{1}{\gamma}\right)\frac{1}{1-\alpha_1} \ as \ t \to \infty \ and \ T = t^{\gamma}, \tag{2.63}$$

where $\alpha_1 \in [0,1]$, $\gamma > 0$ and $t \to \infty$. This completes the proof. □

Claim 14. *In the similar fashion if X_1 and X_2 are two dependent random variables then,* $\mathrm{E}[X_2|X_1 > t] \sim \gamma^{-1}\Gamma\left(\frac{1}{\gamma}\right)\frac{1}{1-\alpha_2}$*; as $t \to \infty$ and $T = t^{\gamma}$, $\forall\ \gamma > 0$.*

Thus, in our case the measure of tail *non-exchangeability* is going to be;

$$\eta_1(t) \sim \frac{1-\alpha_2}{1-\alpha_1}, \tag{2.64}$$

as $(\alpha_1, \alpha_2) \in [0,1]^2$, $\gamma > 0$ and $t \to \infty$.

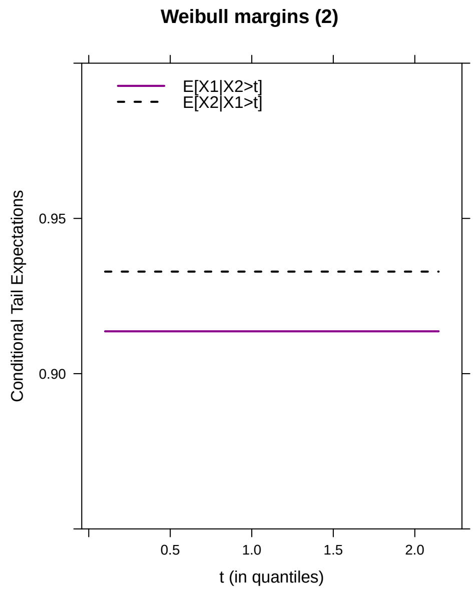
Remark 9. If we do not consider *non-exchangeability* $\lim_{s^{+}\to\ 0,\mathrm{T}\to\infty} g'(s,T) = -1$, which is also a constant . Thus, $-a = -1$ or, $a = 1 > 0$. If we do Watson's lemma then, the expressions of two conditional expectations should be, $\mathrm{E}[X_1|X_2 > t] \sim \gamma^{-1}\Gamma\left(\frac{1}{\gamma}\right)$ and $\mathrm{E}[X_2|X_1 > t] \sim \gamma^{-1}\Gamma\left(\frac{1}{\gamma}\right)$ respectively as $t \to \infty$.

In Figures 2.4a, 2.4b and 2.4c we are comparing simulations obtained from Watson's lemma with the actual conditional expectations. In Figure 2.4a we are using the simulation results from (2.63) and the previous claim (where $\alpha_2 > \alpha_1 s$). In Figure 2.4b we are using the results $\mathrm{E}[X_1|X_2 > t] \sim \gamma^{-1}\Gamma\left(\frac{1}{\gamma}\right)$ and $\mathrm{E}[X_2|X_1 > t] \sim \gamma^{-1}\Gamma\left(\frac{1}{\gamma}\right)$. And finally, in Figure 2.4c we are plotting the two conditional expectations with Weibull margins. In Figure 2.4c we do not find any significant difference between two conditional expectations even at 99th quantile. In this case if we use $\alpha_2 > \alpha_1$, we will come up with Figure 2.4a. Here we see the gap between the two conditional expectations are too high at 99th percentile. Furthermore, both the values are greater than 0.90 which is not true for the actual case. Now, if we compare Figures 2.4b and 2.4c, we can see the behavior of the expectations are very similar. Hence, between Figures 2.4a and 2.4b we would choose Figure 2.4b as better simulation of conditional tail expectation with Weibull margins.

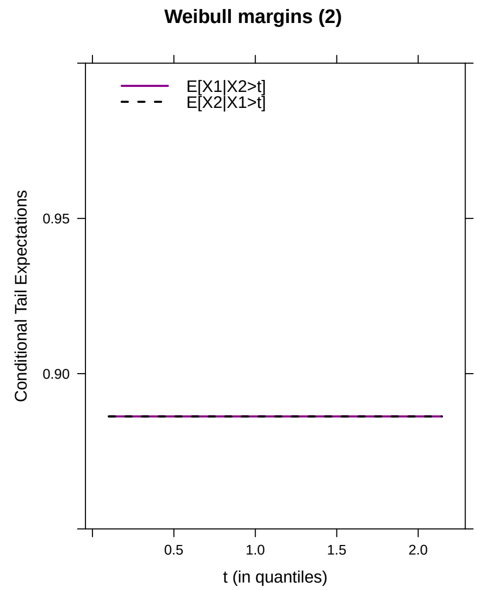
2.1.2.3 Exponential Margins

Finally, we are using Khoudraji (1996) non-exchangeable transformation of Clayton Copula with exponential margins. In the literature this distribution has very important properties like it is an exponential family, it does not have memory and most importantly it is a Poisson process. As it is a Poisson process, at extreme values the value of the distribution behaves as a constant. Our primary objective in this paper is to discuss about the extreme events and find out the probability of occurrence of extreme-events in nature, actuary, econometrics and finance. Furthermore, we are trying to find out a slow variation function which can explain the conditional tail expectations. Here the determinant of *non-exchangeability* is nothing but the ratio of two tail order conditional tail-order expectations [i.e. $\mathrm{E}[X_1|X_2 > t]/\mathrm{E}[X_2|X_1 > t]$]. At the end of this section we do some simulation results under this environment.

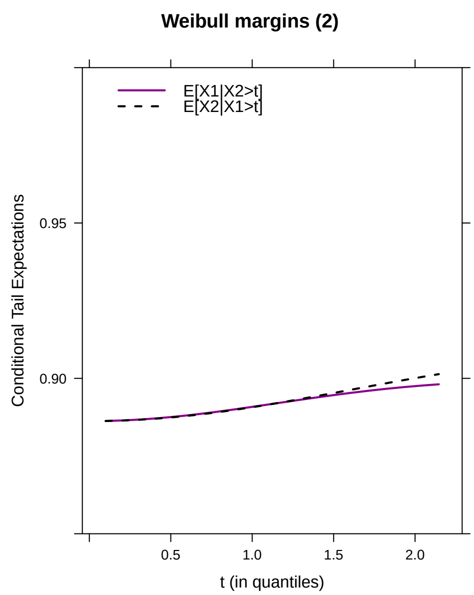
In order to derive tail conditional expectations we are following Hua and Joe (2014) and we are able to show that the integrand function can be decomposed into $g(.)$ and $h(.) > 0$ functions which are *multiplicatively separable*. Here, $g(.)$ is a monotonically decreasing function which satisfies, $g(0, -\log(\overline{F}(t))) = 0$ and $g(\infty, -\log(\overline{F}(t))) = -\infty$ as $t \to \infty$. As this integration does not have any closed form solution, we try to use different simulation



(a) $\alpha_1 = 0.03$, $\alpha_2 = 0.05$, $\gamma = 2$ and $\alpha_2 > \alpha_1$



(b) $\alpha_1 = 0.05$, $\alpha_2 = 0.03$, $\gamma = 2$ and $\alpha_2 < \alpha_1$



(c) $\alpha_1 = 0.03$, $\alpha_2 = 0.05$, $\gamma = 2$

Figure 2.4: Comparison between Simulation of Watson's Lemma and Actual Expectations

methods. We finally end up with Watson's lemma. In this case Laplace approximation does not work as $g'(0, \log(\overline{F}(t))) \not> 0$ but, $g'(s, -\log(\overline{F}(t))) < 0$ as $t \to \infty$; where t is the proxy of X_2 and $\overline{F}(.)$ is the survival function and always in $[0, 1]$. After using Watson's lemma we get our tail *non-exchageability* as;

$$\frac{\mathrm{E}(X_1|X_2 > t)}{\mathrm{E}(X_2|X_1 > t)} \sim \begin{cases} 1 & \text{if } \alpha_2 > \alpha_1; \\ \frac{1-\alpha_2}{1-\alpha_1} & \text{if } \alpha_2 < \alpha_1; \\ \frac{1-\frac{\alpha_2}{2}}{1-\frac{\alpha_1}{2}} & \text{if } \alpha_2 = \alpha_1; \end{cases}$$

where α_1 and α_2 are *non-exchangeable* components under Khoudraji (1996) *non-exchangeable* transformation of Clayton Copula. In the above expression we can easily see that, at extreme values our measure of *non-exchangeability* goes to a constant irrespective of the relation between the non-exchangeable components. At the end of this section as we do the simulation we can see that, our simulation-results are consistent with the results corresponding to numerical integrations with certain levels of error.

We know the cumulative distribution function of exponential type is $F(x) = 1 - e^{-\lambda x}$, $\forall x \in [0, \infty)$. Thus, the survival function becomes, $\overline{F}(x) = 1 - F(x) = e^{-\lambda x}$, $\lambda > 0$, and $\forall\ x \in [0, \infty)$. In the following propositions we are trying to derive a slow variation function of conditional tail expectations.

Proposition 15. *Integrand of the conditional expectations of Khoudraji transformed Clayton Copula with exponential margin is multiplicatively separative of two monotonic functions one of which is strictly positive and other is decreasing.*

Proof. Following Hua and Joe (2014) we transform the survival function $\overline{F}(t) \to e^{-T}$, *or*, $T = -\log \overline{F}(t) = -\log e^{-\lambda t} = \lambda t$. Now, $y = \log \overline{F}(x) = -\log e^{-\lambda x} = \lambda x$. As, $y = \lambda x$ then after totally differentiate this equation n both sides we get $dy = \lambda dx \implies dx = \lambda^{-1} dy$.

Again by the algorithm given by Hua and Joe (2014) we are putting $y = sT$. Thus, $dy = sdT + Tds = Tds$, as we are assuming T is constant.

Now,

$$\begin{aligned}
\mathrm{E}[X_1|X_2 > t] &= \int_0^\infty e^T \widehat{C}(e^{-y}, e^{-T}) \lambda^{-1} dy, \ \forall \lambda > 0 \\
&= \lambda^{-1} T \int_0^\infty e^T \widehat{C}(e^{-sT}, e^{-T}) ds, \ \forall s \in [0, \infty) \\
&= \lambda^{-1} T \int_0^\infty e^{T(1+\frac{1}{T}\log \widehat{C}(e^{-sT}, e^{-T}))} ds, \ \forall s \in [0, \infty) \qquad (2.65)
\end{aligned}$$

where, $g(s;T) = 1 + T^{-1} \log \widehat{C}(e^{-sT}, e^{-T})$ and , $h(s) = 1 > 0$. Clearly, from (2.65) we can say that, the behavior of the conditional expectation depends on $g(s;T)$ function. □

In this case $g(0,T) = 0$ and $g(\infty, T) = -\infty$ as $T \to \infty$. Now we can do either Laplace Approximation or Watson's lemma. If we want to do the first, we need to further check if $g'(0,T) > 0$ as $T \to \infty$. We found that this condition does not hold [see appendix]. So we have to use Watson's lemma. Before applying Watson's lemma we need to check if $g(s,T)$ is decreasing in s.

Lemma 16. *Under Khoudraji transformed Clayton Copula g function is decreasing in s and value depends on non-exchangeable coefficients of the Copula.*

Proof. We know from above;

$$\begin{aligned} g'(s;T) &= -\frac{1}{T}\left[\frac{\alpha_1\delta T e^{\alpha_1\delta sT}}{\delta(e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1)}+(1-\alpha_1)T\right] \\ &= -\left[\frac{\alpha_1}{(e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1)e^{-\alpha_1\delta sT}}+(1-\alpha_1)\right] \\ &= -\left[\frac{\alpha_1}{1+e^{(\alpha_2-\alpha_1 s)\delta T}-e^{-\alpha_1\delta sT}}+(1-\alpha_1)\right]<0 \\ \implies \lim_{T\to\infty} g'(s;T) &= -[\alpha_1+1-\alpha_1]=-1<0,\ if\ \alpha_2<\alpha_1 \\ or\ ,&= -(1-\alpha_1)<0,\ if\ \alpha_2>\alpha_1 \\ or\ ,&= -\left(1-\frac{\alpha_1}{2}\right)<0,\ if\ \alpha_2=\alpha_1 \end{aligned} \tag{2.66}$$

From (2.66) we are sure that we can use Watson's lemma. Thus, g is decreasing in s and the value depends on the coefficients of *non-exchangeability* [i.e. α_1, α_2]. □

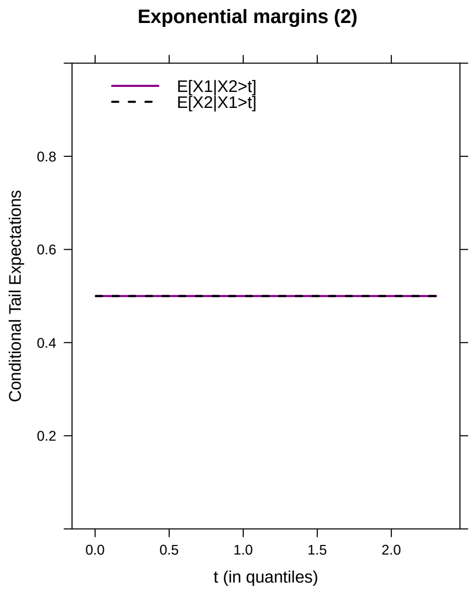
Proposition 17. *Suppose, X_1 and X_2 are two dependent random variables. Conditional tail expectation of KB4 Copula with Exponential margins goes to some constant as $t \to \infty$ and the measure of tail* non-exchangeability *can be written as;*

$$\frac{\mathrm{E}[X_1|X_2>t]}{\mathrm{E}[X_2|X_1>t]} \sim \begin{cases} 1 & \text{if } \alpha_2>\alpha_1; \\ \frac{1-\alpha_2}{1-\alpha_1} & \text{if } \alpha_2<\alpha_1; \\ \frac{1-\frac{\alpha_2}{2}}{1-\frac{\alpha_1}{2}} & \text{if } \alpha_2=\alpha_1. \end{cases}$$

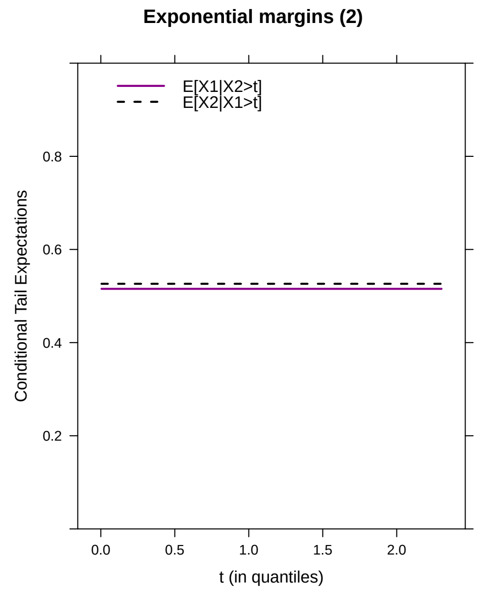
where α_1 and α_2 are non-exchangeable components.

Proof. Please see in the appendix. □

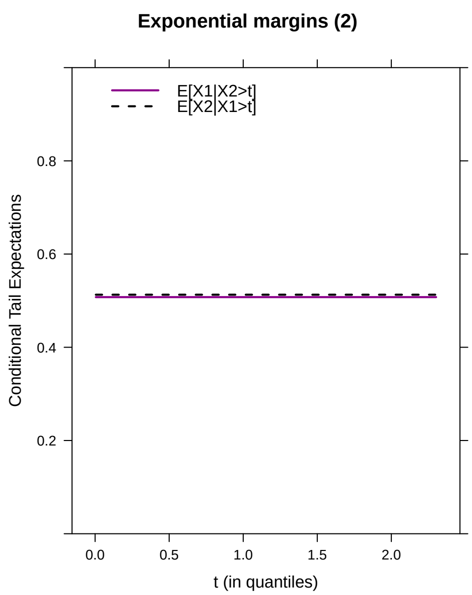
In Figures 2.5a, 2.5b and 2.5c we do the simulation using Watson's lemma for $\alpha_2 > \alpha_1$, $\alpha_2 < \alpha_1$ and $\alpha_2 = \alpha_1$ respectively. In Figure 2.5d we plot the actual simulation using



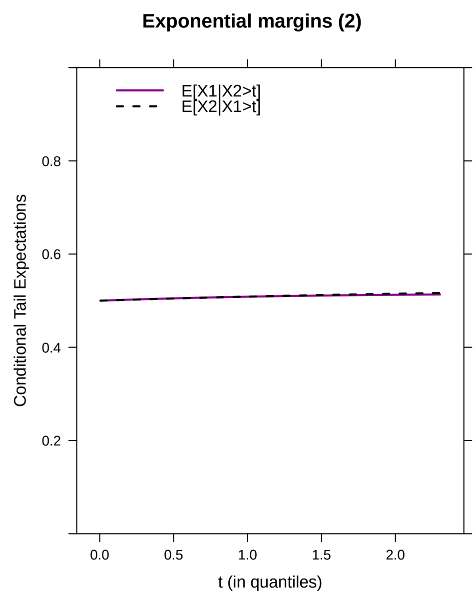
(a) $\alpha_1 = 0.03,\ \alpha_2 = 0.05,\ \lambda = 2$ and $\alpha_2 > \alpha_1$



(b) $\alpha_1 = 0.05,\ \alpha_2 = 0.03,\ \lambda = 2$ and $\alpha_2 < \alpha_1$



(c) $\lambda = 2$ and $\alpha_2 = \alpha_1$



(d) $\alpha_1 = 0.03,\ \alpha_2 = 0.05,\ \lambda = 2$

Figure 2.5: Comparison between Simulation of Watson's Lemma and Actual Expectations

exponential distribution. Through our plotting we assume $\alpha_1 = 0.03$, $\alpha_2 = 0.05$, and $\lambda = 2$. As in Figure 2.5d we do not find any significant tail order *non-exchangeability*, under exponential margins we can assume any of $\alpha_2 > \alpha_1$, $\alpha_2 < \alpha_1$ and $\alpha_2 = \alpha_1$. Here any of Figures 2.5a, 2.5b and 2.5c is going to be a good approximation of conditional expectations. From above discussion we see $\mathrm{E}[X_1|X_2 > t]/E[X_2|X_1 > t] \to 1$. Thus, the system is exchangeable for exponential margins. Here although we use non-exchangeable copula, we finally end-up with some kind of *exchangeability*.

2.2 Case II: $\mathrm{E}[X_1|X_2 = t]$

Another way to determine the tail order non-exchangeability is through $\mathrm{E}[X_1|X_2 = t]$ where t is given. In this section, we are going to discuss this kind of conditional expectations. The main difference between this type of conditional tail expectation with the one described in the previous section is that, in the earlier case we use $\mathrm{E}[X_1|X_2 > t]$ as conditional expectations, but now , we are using $\mathrm{E}[X_1|X_2 = t]$ as conditional tail expectations. In both the cases t is very high. In that way we are able to capture the tail order conditional expectations. Like before, in our case X_1 and X_2 are two random variables. After assuming t is very large we are trying to find out a slow variation function. In this environment one system is non-exchangeable if for any two random variables X_1 and X_2 we have $\mathrm{E}[X_1|X_2 = t] \neq \mathrm{E}[X_2|X_1 = t]$. Furthermore, we define a tail-order non-exchangeable if, $\mathrm{E}[X_1|X_2 = t] \neq \mathrm{E}[X_2|X_1 = t]$; when $t \to \infty$. Throughout this section I define the measure of *non-exchangeaility* as $\eta_2(t) = \mathrm{E}[X_1|X_2 = t]/\mathrm{E}[X_2|X_1 = t]$. If one system is exchangeable we can say, $\eta_2(t) = 1$. Here we are trying to find when *non-exchangeability* occurs.

Definition 2. *A system is tail non-exchangeable if for any two random variables X_1 and X_2 we have, $\eta_2(t) = \mathrm{E}[X_1|X_2 = t]/\mathrm{E}[X_2|X_1 = t] \neq 1$; when $t \to \infty$.*

In order to find out this kind of tail-order conditional expectations, firstly we use the definition of $\mathrm{E}[X_1|X_2 = t]$ provided by Hua and Joe (2014). They represent the definition in terms of factor Copulas. The main thing in their paper is they do analysis of Gumbel Copula with Pareto and Weibull margins respectively. In their analysis, they do Laplace Approximation or Watson's Lemma in order to find a slow variation function. In this paper we are taking any exchangeable Copula and use Khoudraji (1996) non-exchangeable transformation in order to get a non-exchangeable structure. We do not say anything about non-exchangeable Copula because, after making Khoudraji-transformation we might get exchangeable Copula which is not our interest in this paper.

Definition 3. *If* $\mathrm{E}[X_1|X_2 = t]$ *is a conditional expectation for any given* t*, then it can be expressed in terms of non-exchangeable Copula as;*

$$\mathrm{E}[X_1|X_2 = t] = \int_0^\infty \widehat{C}_{1|2}(\overline{F}(x)|\overline{F}(t))\ dx,\ \forall\ t \tag{2.67}$$

In the above case $\widehat{C}_{1|2}(\overline{F}(x)|\overline{F}(t))$ *is copula which has the form ,*

$$\widehat{C}_{1|2}(\overline{F}(x)|\overline{F}(t)) = \frac{\partial}{\partial \overline{F}(t)}[\overline{F}(x)^{1-\alpha_1}\overline{F}(t)^{1-\alpha_2}\widehat{C}^*(\overline{F}(x)^{\alpha_1}, \overline{F}(t)^{\alpha_2})],$$

$\overline{F}(x)$ *and* $\overline{F}(t)$ *are survival functions and* $(\alpha_1, \alpha_2)\ \in [0, 1]^2$.

Clearly, the above integration does not have any closed form solution. We have to use either Laplace approximation or Watson's lemma. Before going to the mathematical results let us understand this scenario intuitively. By the definition of factor Copula we know that, this is nothing but the first order derivative of the non-exchangeable Copula itself. We know, by differentiation we try to linearize a function and try to calculate the slope of that line. Thus, the degree of tail *non-exchangeability* in this case is relatively less than the previous section because, by linearizing a non-linear function we already lose some *exchangeability*. Even when we do the simulation of Khoudraji (1996) non-exchangeable transformation of factor-Clayton Copula with Pareto margin, we see less *non-exchangeability* in this case. In the following sections we first derive tail-order *non-exchageability* by general KT-transformed survival factor Copulas by using either Laplace Approximation or Watson's lemma. Then we use Khoudraji (1996) non-exchangeable transformed factor Clayton Copula with Pareto, Weibull and Exponential margins in order to show that, theoretical results are consistent with the examples. Finally, we do some simulation studies in order to see the tail *exchangeability* of these examples.

2.2.1 Theoretical results

Tail non-exchangeability is one of the important facts in extreme value theory now-a-days. Like in the previous section, we also measure tail-order non-exchangeability by $\mathrm{E}[X_1|X_2 = t]/\mathrm{E}[X_2|X_1 = t]$. Here t is given and has a very large value. Hence, in this case by $\mathrm{E}[X_1|X_2 = t]$ we really find out a slow variation function at the tail. By Hua and Joe (2014) we know the definition of the conditional expectation in terms of survival copulas. Here the integrand is factor survival Copula itself. At first we take general survival Copula. Then we do Khoudraji (1996) non-exchangeable transformation. We further differentiate the *non-exchangeable* KB4 Copula with respect to the second argument. From the theory of differentiation we know that, differentiation is nothing but the linearization of any function. In our case, we have the integrand in the form of $\widehat{C}_{1|2}(\overline{F}(x), \overline{F}(t))$; where this Copula is itself a survival Copula as, $\overline{F}(x)$ and $\overline{F}(t)$ are survival functions. The important fact in this case is the integrand is itself a first order derivative of the KT transformed. We also know that, by differentiating we are actually doing the linearization of the function. Suppose, a function is non-linear at first with certain level of *non-exchangeability*. After doing the first order differentiation we certainly lose some *non-exchangeability*. Thus, if we try to calculate some slow variation function then under this case, we can get less flexibility at the tail compared with $\mathrm{E}[X_1|X_2 > t]$ in the previous section.

The integration of this expectation does not have any closed form solution. So we have to rely on numerical approximations instead of just integrating the whole function. Hua and Joe (2014) suggest that, under this scenario we can do either Laplace Approximation or Watson's lemma. They do it for extreme value Copulas with Pareto and Weibull margins. In our paper we try to use these two approximations under this non-exchangeable structure and find out the conditions under which we can actually do it. Before proceeding to the approximation we convert one exponential and positive functions which are multiplicatively separable to each other. We define the exponential function as $e^{g(s,T)}$ where g is a monotonic decreasing function with $g(0, T) = 0$ and $g(\infty, T) = -\infty$ as $t \to \infty$ where $T = -\log(\overline{F}(t))$. Another function we also defined as $h > 0$ which is also a positive function with $e^{g(s,T)}$ and $h > 0$ are multiplicatively separable. If we do the integration through the approximation process we can find such two exponential functions, one multiplied by α_1 and other by α_2. These two terms are the non-exchangeable parameters under Khoudraji (1996) non-exchangeable transformation. Hence, we can say that as we do the transformation with factor Copulas, we can have a linear combination of two terms. We get the final result depending on which of these two dominating the tail. It is important to note that, in non-exchangeable Gumbel Copula with our three different margins we get two separate terms each of which dominates the tail, which was not observed in the case of *non-exchangeable Clayton Copula*.

In the example section we try to figure out how this generalized slow variation functions look like at the tail. Here as an example we use Khoudraji (1996) transformed Clayton copula with *Pareto, Weibull* and *Exponential* margins respectively. The main objective is to use this kind of Copula function as a representative of *Archimedian* family. The reason behind using *Pareto, Weibull* and *Exponential* margins are; Pareto-margin represents power distribution under which we can take less number of data points and make prediction of the whole system. Weibull distribution represents sub-exponential families. Under this margin we can use less number of data points and make the prediction. Finally, exponential distribution represents *Exponential* families and as all the distribution functions asymptotically converges to the exponential series; this makes this distribution a strong candidate of our discussion. In the Clayton case we find that out of two slow variation functions one dominates the other. As a result, we do have just one slow variation function in order to explain $\mathrm{E}[X_1|X_2 = t]$.

At the end of each example we do simulation studies corresponding to each margins. The results are consistent with the previous section. We can do both the Laplace Approximation and Watson's Lemma in the case of non-exchangeable Clayton Copula with Pareto margins but, *non-exchangebility* is relatively less compared with $\mathrm{E}[X_1|X_2 > t]$. This result is consistent what we explained in the previous paragraph. The main reason behind this is by differentiating a function we are actually linearize it and as the integrand in $\mathrm{E}[X_1|X_2 = t]$ is itself a factor Copula [i.e. first order derivative of non-exchangeable survival Copula], we get less *non-exchangeability* than in the previous section. Finally, from our pictures we see *non-exchageability* only in the case of Pareto margins. In the cases of Weibull and Exponential margins we do not find *non-exchangeability* which is consistent with the result from the previous section [i.e. $\mathrm{E}[X_1|X_2 > t]$]. From the picture we can conclude that $\mathrm{E}[X_1|X_2 > t]$ is better measure for tail-order *non-exchangeability* of Khoudraji (1996) type. In the following section we are discussing these one by one.

2.2.1.1 General cases

In this part we are deriving the tail *non-exchangeability* under $\mathrm{E}[X_1|X_2 = t]$ as $t \to \infty$. In order to derive this we first assume a symmetric Copula. Then we do the Khoudraji (1996) non-exchangeable transformation of if as described in (2.68). In this case as we say earlier that α_1, α_2 are non-exchangeable parameters and $\alpha_1, \alpha_2 \ \in [0,1]^2$. If $\alpha_1 = \alpha_2 = 1$ we get exactly the same result found in Hua and Joe (2014). On the other hand, if $\alpha_1 = \alpha_2 = \alpha$ occurs, we say that the system is exchangeable. If a system is exchangeable at the tail, then we conclude that the environment is not risky at all. Under this case our measure of tail *non-exchangeability* $\eta_2(t) = \mathrm{E}[X_1|X_2 = t]/\mathrm{E}[X_2|X_1 = t] \to 1$ as $t \to \infty$. In other words, $\mathrm{E}[X_1|X_2 = t] \sim \mathrm{E}[X_2|X_1 = t]$ as $t \to \infty$.

Like our whole paper, the *exchangeable* survival Copula $\widehat{C}^*$ be any Copula defined as, $\widehat{C}^*(\overline{F}(x), \overline{F}(t))$. Here we are assuming $\overline{F}(x), \overline{F}(t)$ are survival functions of the distribution functions $F(x)$ and $F(t)$ respectively. As we know any distribution function is monotonically increasing, thus it *asymptotically* follows some exponential functions. As the survival function is defined by $[1 - F(x)]$ and $[1 - F(t)]$ respectively, asymptotically it is a monotonically decreasing function. Following Khoudraji (1996) device Genest et al. (1998), Genest et al. (2011) and finally, Genest and Nešlehová (2013) we define a non-exchangeable copula function;

$$\widehat{C}(\overline{F}(x), \overline{F}(t)) = \overline{F}(x)^{1-\alpha_1}\overline{F}(t)^{1-\alpha_2}\widehat{C}^*(\overline{F}(x)^{\alpha_1}, \overline{F}(t)^{\alpha_2}),\ (\alpha_1, \alpha_2) \ \in [0,1]^2, \tag{2.68}$$

where $\widehat{C}$ is any copula which is *non-exchangeable* in nature.

Now, by using the method provided by Hua and Joe (2014) of copula we are trying to see the tail order Khoudraji (1996) type non-exchangeability of survival Copula defined in

(2.68). Throughout this section we assume X_1 and X_2 are random variables with support in $\mathbb{R}$ and they have same marginal distribution functions where *cumulative distribution function (CDF)* F is properly defined and uniformly continuous on $[0,1]$ and they are independent and identically distributed. Furthermore, we assume that, F, $\widehat{C}$, *and* $\overline{F}$ are Lebesgue measurable $\in [0,1]^3$ with the density function $f(.,.) = F^{'}(.,.) = \partial F(.,.)/\partial(.,.)$. Finally, we assume X_1 has finite mean (i.e $\int_0^\infty \overline{F}(x)dx < \infty$).

In order to find out the slow variation function we follow Hua and Joe (2014) and get the form $\mathrm{E}[X_1|X_2 = t] = \int_0^\infty \widehat{C}_{1|2}(\overline{F}(x)|\overline{F}(t))\ dx,\ \forall\ t$. In this case $\widehat{C}_{1|2}(\overline{F}(x)|\overline{F}(t))$ is the survival Copula. As this integration does not any closed form solution we need to find a *slow variation* function at $t \to \infty$. In order to do so we firstly derive this Copula into the linear combination of $T(1-\alpha_2)e^{-w}\int_0^\infty e^{Tg_1(s,T)}h(s)\ ds$ and $T\alpha_2 e^{-w}\int_0^\infty e^{Tg_2(s,T)}h(s)\ ds$. In this case α_1 and α_2 are the Khoudraji (1996) *non-exchangeable* parameters with $g_i(s,T;\alpha_1,\alpha_2)$; where $i = 1,2$ and $h(s)$ functions. In this case $g_i()$ is a *monotonic decreasing* function with support on the real line and $h()$ is a positive function. Secondly, we try to find out the slow variation function of these two integrations. We do either Laplace Approximation or Watson's lemma to find out the slow variation functions at $t \to \infty$. After doing this we are able to find out the slow variation functions as $T(1-\alpha_2)e^{Tg_1(\gamma,T)}\sqrt{\frac{2\pi}{-T(1-\alpha_2)g_1^{''}(\gamma,T)}}$ and $T\alpha_2 e^{Tg_2(\gamma,T)}\sqrt{\frac{2\pi}{-T\alpha_2 g_2^{''}(\gamma,T)}}$; where $\gamma = \lim_{T\to\infty}\ s_0(T)$ as $t \to \infty$.

Lemma 18. *The conditional expectation in the form of* $\mathrm{E}[X_1|X_2 = t]$ *is a linear combination of two integrations when the integrand itself defined as a non-exchangeable survival Copula.*

Proof. Please see the appendix. □

From above lemma we can clearly see that, the integrands can be written as, $e^{Tg(s,T)}h(s)$ where $h(s) = 1 > 0$. It is important to say that, $Tg(s,T)$ is the dominant term of the integration. Here, as we are using Khoudraji (1996) device, we are assuming $(\alpha_1, \alpha_2) \in [0, 1]$. Now, if we carefully see (A.17), we will be able to see two parts of the expression of $\mathrm{E}[X_1|X_2 = t]$. The main reason behind getting two parts is that, by using Khoudraji device of *non-exchangeable transformation* to copulas we have two multiplicative separable terms with the copula, and when we differentiate with respect to the second argument, we get two terms which are added to each other. Apart from that the functions $g,\ g_1,\ and\ g_2$ depend on $\log \widehat{C}^*$ or $\widehat{C}^*_{1|2}$ or both (i.e in (A.17)). As in (A.17) we have two components, we need to discuss this condition in much more details. In order to do numerical integration in the form of either Laplace approximation or Watson's lemma we need the form like $T\Gamma_1 h(s)$. In (A.17) we have two of these forms $T(1-\alpha_2)e^{-w}\Gamma_1 h(s)\ ds$ and $T\alpha_2 e^{-w} h(s)\ \Gamma_2$ respectively which behave differently from each other. In later sections we will discuss about this. For definition of Γ_1 and Γ_2 please see the appendix of the above lemma.

As these integrations do not have any closed form solutions we have to do numerical integration. Hua and Joe (2014) suggest that under these circumstances we can do either Laplace approximation of Watson's lemma. Before going to those numerical approximations directly we have to check certain conditions.

Lemma 19. *Tail dominant functions g_1 and g_2 in Γ_1 and Γ_2 are monotonically decreasing and satisfy $g_i(0, T) = 0$ and $g_i(\infty, T) = -\infty$; where $i = 1, 2$, $\Gamma_1 = \int_0^\infty e^{Tg_1(s,T)} h(s)\ ds$ and $\Gamma_2 = \int_0^\infty e^{Tg_2(s,T)} h(s)\ ds$.*

Proof. Let us now discuss about the case of $\mathrm{E}[X_1|X_2 = t]$. In (A.17) we had two terms namely $g_1(s, T)$ and $g_2(s, T)$. If we separately show $g_1(0, T) = 0$ and $g_2(0, T) = 0$ and

$g_1(\infty,T)=-\infty$ *and* $g_2(\infty,T)=-\infty$ we can use either Laplace approximation or Watson's lemma. We know,

$$
\begin{aligned}
g_1(s,T) &= -(\alpha_2+s\alpha_1)+\frac{1}{T}[\log\frac{\widehat{C}^*(e^{-\alpha_1 sT},e^{-\alpha_2 T})}{f[F^{-1}(1-e^{-sT})]}+w] \\
\implies g_1(0,T) &= -\alpha_2+\frac{1}{T}[\log\frac{\widehat{C}^*(1,e^{-\alpha_2 T})}{f[F^{-1}(0)]}+w] \qquad (2.69)
\end{aligned}
$$

In order to satisfy condition (2.69) we need $\log\frac{\widehat{C}^*(1,e^{-\alpha_2 T})}{f[F^{-1}(0)]}+w=-\alpha_2 T$. Again,

$$
\begin{aligned}
g_2(s,T;\alpha_1,\alpha_2) &= -s(2-\alpha_1)+\frac{1}{T}[\log\frac{\widehat{C}^*_{1|2}(e^{-\alpha_1 sT}|e^{-\alpha_2 T})}{f[F^{-1}(1-e^{-sT})]}+w] \\
\implies g_2(0,T;\alpha_1,\alpha_2) &= \frac{1}{T}[\log\widehat{C}^*_{1|2}(1|e^{-\alpha_2 T})f[F^{-1}(0)]+w] \qquad (2.70)
\end{aligned}
$$

The above condition holds if $\frac{1}{T}[\log\frac{\widehat{C}^*_{1|2}(1|e^{-\alpha_2 T})}{f[F^{-1}(0)]}+w]=0$, or, $\log\frac{\widehat{C}^*_{1|2}(1|e^{-\alpha_2 T})}{f[F^{-1}(0)]}+w=0$, or, $\log\widehat{C}^*_{1|2}(1|e^{-\alpha_2 T})-\log(f[F^{-1}(0)])+w=0$. We know w is the adjustment constant which is $\log(f[F^{-1}(0)])$. Thus, in the above expression we have only $\log\widehat{C}^*_{1|2}(1|e^{-\alpha_2 T})$. We know, $\widehat{C}^*_{1|2}(1|e^{-\alpha_2 T})=1$. This implies $\log(1)=0$. This condition holds with our survival Clayton Copula with Pareto, Weibull and exponential margins. Again by using the above condition $g_1(\infty,T)=-\alpha_2-\infty+\frac{1}{T}[\log\frac{\widehat{C}^*(0,e^{-\alpha_2 T})}{f[F^{-1}(1)]}+w]=-\alpha_2-\infty+\frac{1}{T}[\log\frac{e^{-\alpha_2 T}}{f[F^{-1}(1)]}+w]=-\infty$. On the other hand, $g_2(\infty,T)=-\infty+\frac{1}{T}[\log\frac{\widehat{C}^*_{1|2}(0|e^{-\alpha_2 T})}{f[F^{-1}(1)]}+w]=-\infty$. Again like before w is the real constant depending on the marginal distribution F, so that one can have $g_1(\infty,T)=-\infty$ and $g_2(\infty,T)=-\infty$ respectively. Hence, first two conditions hold for *Laplace approximation or Watson's lemma.* □

Proposition 20. *First order condition of g function is always positive at the tail.*

Proof. Let us check if $\lim_{T\to\infty} g_1'(0,T;\alpha_1,\alpha_2) > 0$. From (2.69) and (2.70) we know that, $g_1(s,T) = -(\alpha_2 + s\alpha_1) + \frac{1}{T}[\log \frac{\widehat{C}^*(e^{-\alpha_1 sT}, e^{-\alpha_2 T})}{f[F^{-1}(1-e^{-sT})]} + w]$; where all the symbols have their usual meanings throughout this paper.

Hence;

$$\begin{aligned} g_1(s,T) &= -(\alpha_2 + s\alpha_1) + \frac{1}{T}\left[\log \frac{\widehat{C}^*(e^{-\alpha_1 sT}, e^{-\alpha_2 T})}{f[F^{-1}(1-e^{-sT})]} + w\right] \\ &= -(\alpha_2 + s\alpha_1) + \frac{1}{T}\Bigg[\log\{\widehat{C}^*(e^{-\alpha_1 sT}, e^{-\alpha_2 T})\} \\ &\qquad - \log\{f[F^{-1}(1-e^{-sT})]\} + w\Bigg] \\ g_1'(s,T) &= -\alpha_1 + \frac{1}{T}\frac{(-\alpha_1 T)e^{-\alpha_1 sT}\widehat{C}^*_{2|1}(e^{-\alpha_2 T}|e^{-\alpha_1 sT})}{\widehat{C}^*(e^{-\alpha_1 sT}, e^{-\alpha_2 T})} \\ &\qquad - \frac{1}{T}\frac{Te^{-sT}f'[F^{-1}(1-e^{-sT})]}{f^2[F^{-1}(1-e^{-sT})]} \\ &= -\alpha_1 - \frac{\alpha_1 e^{-\alpha_1 sT}\widehat{C}^*_{2|1}(e^{-\alpha_2 T}|e^{-\alpha_1 sT})}{\widehat{C}^*(e^{-\alpha_1 sT}, e^{-\alpha_2 T})} \\ &\qquad - \frac{e^{-sT}f'[F^{-1}(1-e^{-sT})]}{f^2[F^{-1}(1-e^{-sT})]} \\ g_1'(0,T) &= -\alpha_1 - \frac{\alpha_1 \widehat{C}^*_{2|1}(e^{-\alpha_2 T}|1)}{\widehat{C}^*(1, e^{-\alpha_2 T})} - \frac{f'[F^{-1}(0)]}{f^2[F^{-1}(0)]} \\ \lim_{T\to\infty} g_1'(0,T) &= -\alpha_1 - \lim_{T\to\infty}\frac{\alpha_1 \widehat{C}^*_{2|1}(e^{-\alpha_2 T}|1)}{\widehat{C}^*(1, e^{-\alpha_2 T})} - \lim_{T\to\infty}\frac{f'[F^{-1}(0)]}{f^2[F^{-1}(0)]} > 0 \end{aligned} \tag{2.71}$$

This completes the proof. □

Example 2. Let us see how does this work in Khoudraji (1996) *non-exchangeable* (KB4) with Pareto margins. In our case, the marginal cumulative distribution function (CDF) of Pareto is $F(x) = 1 - (1+x)^{-\beta}$. Thus the density function is $f(x) = F'(x) = \beta(1+x)^{-(1+\beta)}$ and $f'(x) = -\beta(1+\beta)(1+x)^{-(2+\beta)}$, $\forall\ \beta > 1$. Thus, the extreme right hand side of (2.71) becomes, $-\lim_{T\to\infty}\frac{f'(F^{-1}(0))}{f^2(F^{-1}(0))} = -\lim_{T\to\infty}\frac{-\beta(1+\beta)}{\beta^2}$. Thus, $-\lim_{T\to\infty}\frac{f'(F^{-1}(0))}{f^2(F^{-1}(0))} = 1 + \frac{1}{\beta} > 1$.

Now, let us solve the second term of the right hand side of equation (2.71). At first let us write the original survival Clayton copula before Khoudraji (1996) transformation. The numerator part is just the first order derivative of Clayton Copula with respect to its first argument and the denominator term is just the copula itself. We know, Clayton Copula is $\widehat{C}^*(u^{\alpha_1}, v^{\alpha_2}) = (u^{-\alpha_1\delta} + v^{-\alpha_2\delta} - 1)^{-1/\delta}\ \forall\ (\alpha_1, \alpha_2)\ \in [0,1]^2$. Hence,

$$\widehat{C}^*_{2|1}(v|u) = \frac{\alpha_1\delta}{\delta}(u^{-\alpha_1\delta} + v^{-\alpha_2\delta} - 1)^{-(1+1/\delta)}u^{-(1+\alpha_1\delta)} \tag{2.72}$$

From (2.72) we know that the middle term is $-\lim_{T\to\infty}\frac{\alpha_1\widehat{C}^*_{2|1}(e^{-\alpha_2 T|1})}{\widehat{C}(1,e^{-\alpha_2 T})}$. Before taking limit and using (2.72) we get;

$$\frac{\alpha_1\widehat{C}^*_{2|1}(v|u)}{\widehat{C}^*(u^{\alpha_1}, v^{\alpha_2})} = \frac{\alpha_1(u^{-\alpha_1\delta} + v^{-\alpha_2\delta} - 1)^{-(1+1/\delta)}u^{-(1+\alpha_1\delta)}}{(u^{-\alpha_1\delta} + v^{-\alpha_2\delta} - 1)^{-1/\delta}} = \frac{\alpha_1 e^{sT(1+\alpha_1\delta)}}{u^{-\alpha_1\delta} + v^{-\alpha_2\delta} - 1}.$$

After putting $u = e^{-sT}$ and $v = e^{-T}$ we get,

$$\begin{aligned} -\lim_{T\to\infty}\frac{\alpha_1\widehat{C}^*_{2|1}(1|e^{-\alpha_2 T})}{\widehat{C}^*(1, e^{-\alpha_2 T})} &= -\lim_{T\to\infty}\frac{\alpha_1 e^{sT(1+\alpha_1\delta)}}{u^{-\alpha_1\delta} + v^{-\alpha_2\delta} - 1} \\ &= -\lim_{T\to\infty}\alpha_1(1 + e^{\alpha_2\delta T} - 1)^{-1} = 0,\ \textit{where } s = 0 \end{aligned} \tag{2.73}$$

Thus, in order to satisfy $g_1'(0,T) > 0$ we need $\alpha_1 + \beta^{-1} > 1$.

Remark 10. In the similar way we can show that, $\lim_{T\to\infty} g_2(s,T) > 0$. With this g_2 function it is true that under *Khoudraji non-exchageable* transformed Clayton Copula with Pareto margin $\alpha_1 + \beta^{-1} > 1$ is true. This result is very consistent with our results in the example section. Again the surprisingly this condition is exactly same with $\mathrm{E}[X_1|X_2 > t]$.

Proposition 21. *Suppose X_1 and X_2 are two dependent random variables. Conditional expectation at the tail is a linear combination two slow variation functions and can be explained by;*

$$\begin{aligned}\mathrm{E}[X_1|X_2=t] \sim & -\log\overline{F}(t)(1-\alpha_2)e^{-\log\overline{F}(t)g_1(\gamma,-\log\overline{F}(t))}\sqrt{\frac{2\pi}{\log\overline{F}(t)(1-\alpha_2)g_1''(\gamma,-\log\overline{F}(t))}} \\ & -\log\overline{F}(t)\alpha_2 e^{-\log\overline{F}(t)g_2(\gamma,-\log\overline{F}(t))}\sqrt{\frac{2\pi}{\log\overline{F}(t)\alpha_2 g_2''(\gamma,T)}},\end{aligned} \tag{2.74}$$

as $t\to\infty$, $(\alpha_1,\alpha_2)\ \in[0,1]^2$, where $\gamma=\lim_{t\to\infty}\ s_0(-\log\overline{F}(t))$.

Proof. Please see the Appendix. □

Proposition 22. *If* Laplace Approximation *is not applicable,* Watson's lemma *is better fit to calculate a slow variation function which might be a good representer of* $\mathrm{E}[X_1|X_2=t]$ *at the tail [i.e. $t\to\infty$].*

Proof. Now suppose, we consider in certain margins with certain conditions Laplace Approximation is not applicable. Under this case we have to use Watson's Lemma. Here we are not using the Watson's lemma directly. We are using a version of Watson's lemma. In order to do the approximation of this integration we are using Theorem 36 of Breitung (1994) [p. 48]. As $g_1(s,T)$ and $g_2(s,T)$ are real functions on the semi-infinite interval $[0,\infty)^2$ and in an interval $(0,0+\epsilon_i]^2$,where $i=1,2$, with $\epsilon_1,\epsilon_2>0$, these functions are continuously differentiable and

$$\begin{aligned}&\sup_{0+\epsilon_1\le s\le\infty} g_1(s,T)\le g_1(0,T)-\psi_1,\ \textit{and}\\ &\sup_{0+\epsilon_2\le s\le\infty} g_2(s,T)\le g_2(0,T)-\psi_2\end{aligned} \tag{2.75}$$

with $\psi_1, \psi_2 > 0$.

Now for $g_1'(s,T)$ and $g_2'(s,T)$ we have $g_1'(s,T) < 0$ and $g_2'(s,T) < 0$ for all $s \in (0, 0 + \max\{\epsilon_1, \epsilon_2\}]$. We can also write $g_1'(s,T) = -as^{r_1-1} + o(s^{r-1})\ \forall r_1 > 0$ and $g_2'(s,T) = -as^{r_2-1} + o(s^{r-1})\ \forall r_2 > 0$. Now if we assume $r = 1$ then $r_1 = r_2 = 1$ and, $g_1'(s,T) = -a_1$ and $g_2'(s,T) = -a_2$. From our previous results we know that,

$$g_1'(s,T) = -\alpha_1 - \frac{\alpha_1 e^{-\alpha_1 sT}\widehat{C}^*_{2|1}(e^{-\alpha_2 T}|e^{-\alpha_1 sT})}{\widehat{C}^*(e^{-\alpha_1 sT}, e^{-\alpha_2 T})} - \frac{e^{-sT} f'[F^{-1}(1-e^{-sT})]}{f^2[F^{-1}(1-e^{-sT})]} = -\Upsilon_1$$

and

$$g_2'(s,T) = -(2-\alpha_1) - \frac{\alpha_1 e^{-\alpha_1 sT}\widehat{C}^*_{2|1,1}(e^{-\alpha_2 T}|e^{-\alpha_1 sT})}{\widehat{C}^*_{2|1}(e^{-\alpha_2 T}|e^{-\alpha_1 sT})} - \frac{e^{-sT} f'[F^{-1}(1-e^{-sT})]}{f^2[F^{-1}(1-e^{-sT})]} = -\Upsilon_2$$

which are constants at $s^+ \to 0$ and $t \to \infty$. Thus, $-a_1 = -\Upsilon_1$ or, $a_1 = \Upsilon_1 > 0$. Similarly, we can say that, $a_2 = \Upsilon_2$.

Let us assume there is another real and continuous function $h(s,T) \in [0,\infty)$ such that, $h_i(s,T) = bs^{m_i-1} + o(s^{m_i-1})$ with $m_i > 0$ and $i = 1,\ 2$. More specifically we assume $h_i(s,T) = 1\ \forall\ i = 1,2$ in our case. Thus,

$$b_i s^{m_i-1} + o(s^{m_i-1}) = 1 \implies b_i = 1 \tag{2.76}$$

where $m_i = 1$ for all $i = 1,2$.

Now, after using this theorem we get;

$$\begin{aligned} E[X_1|X_2=t] &\sim \frac{T(1-\alpha_2)}{\Upsilon_1 T} + \frac{T\alpha_2}{\Upsilon_2 T} \\ &\sim \frac{1-\alpha_2}{\Upsilon_1} + \frac{\alpha_2}{\Upsilon_2},\ as\ t \to \infty, \end{aligned} \tag{2.77}$$

where $T = -\log \overline{F}(t)$, $(\alpha_1, \alpha_2) \in [0,1]^2$, $\Upsilon_1 = \alpha_1 + \frac{\alpha_1 e^{-\alpha_1 sT} \widehat{C}^*_{2|1}(e^{-\alpha_2 T}|e^{-\alpha_1 sT})}{\widehat{C}^*(e^{-\alpha_1 sT}, e^{-\alpha_2 T})} + \frac{e^{-sT} f'[F^{-1}(1-e^{-sT})]}{f^2[F^{-1}(1-e^{-sT})]}$ and $\Upsilon_2 = (2-\alpha_1) + \frac{\alpha_1 e^{-\alpha_1 sT} \widehat{C}^*_{2|1,1}(e^{-\alpha_2 T}|e^{-\alpha_1 sT})}{\widehat{C}^*_{2|1}(e^{-\alpha_2 T}|e^{-\alpha_1 sT})} + \frac{e^{-sT} f'[F^{-1}(1-e^{-sT})]}{f^2[F^{-1}(1-e^{-sT})]}$ for all $\Upsilon_1,\ \Upsilon_2 \in \mathbb{R} \setminus \{0\}$. □

Corollary 23. *Thus, if* $\alpha_1 + \alpha_2 = 1$ *then* $E[X_1|X_2 = t] \sim \frac{\alpha_1}{\Upsilon_1} + \frac{\alpha_2}{\Upsilon_2}$ *as* $s^+ \to 0$ *and* $t \to \infty$ *for all* $\Upsilon_1,\ \Upsilon_2 \in \mathbb{R} \setminus \{0\}$.

Corollary 24. *Under tail non-exchangeability of Khoudraji type* $\mathrm{E}[X_2|X_1 = t] \sim \frac{1-\alpha_1}{\Upsilon'_1} + \frac{\alpha_1}{\Upsilon'_2}$; *where* $(\alpha_1, \alpha_2) \in [0,1]^2$, $\Upsilon'_1 = \alpha_2 + \frac{\alpha_2 e^{-\alpha_2 sT} \widehat{C}^*_{2|1}(e^{-\alpha_2 sT}|e^{-\alpha_1 T})}{\widehat{C}^*(e^{-\alpha_2 sT}, e^{-\alpha_1 T})} + \frac{e^{-sT} f'[F^{-1}(1-e^{-sT})]}{f^2[F^{-1}(1-e^{-sT})]}$ *and* $\Upsilon'_2 = (2-\alpha_2) + \frac{\alpha_2 e^{-\alpha_2 sT} \widehat{C}^*_{2|1,1}(e^{-\alpha_2 sT}|e^{-\alpha_1 T})}{\widehat{C}^*_{2|1}(e^{-\alpha_2 sT}|e^{-\alpha_1 T})} + \frac{e^{-sT} f'[F^{-1}(1-e^{-sT})]}{f^2[F^{-1}(1-e^{-sT})]}$ *as* $s^+ \to 0$ *and* $t \to \infty$.

Corollary 25. *Again, if* $\alpha_1 + \alpha_2 = 1$ *then* $E[X_1|X_2 = t] \sim \frac{\alpha_2}{\Upsilon'_1} + \frac{\alpha_1}{\Upsilon'_2}$ *as* $t \to \infty$ *for all* $\Upsilon'_1,\ \Upsilon'_2 \in \mathbb{R} \setminus \{0\}$.

Remark 11. Important part of the above proposition is that, we take h function as 1. This is not true all the time. This function depends how do we formulate the integrands. In the cases of Clayton Copula with Pareto and Weibull margins we define $h(.) = 1 > 0$ but, for exponential margin we actually formulate an $h(s) > 0$ function.

2.2.2 Examples with Clayton Copula

Like before, we are using a particular form of *Archimedian* copula named Clayton Copula. The probability density function of this survival Copula is $\widehat{C}^*(u,v) = (u^{-\delta} + v^{-\delta} - 1)^{-1/\delta}$, where $(u,v) \in [0,1]^2$ and $\delta \geq 0$. Then we use Khoudraji (1996) non-exchangeable device to transform it into *Khoudraji* transformed Clayton Copula (or KB4). Furthermore, when we discuss the *non-exchangeability* at the tail, we take three distributions as margins. They are *Pareto, Weibull* and *Exponentia*l margins respectively. The main reason behind to choose three marginals is they represent *power, sub-exponential* and *exponential distribution* families respectively which are very important in the theories of statistics. Our objective in this paper is to predict the probability of extreme events. There are two ways, if we look after more datasets, we cannot probably get the exact probability of extremely rare events. On the other hand, if we do calculate the probability of extreme events, we might not get enough dataset at the tail.

A bivariate Khoudraji-transformed Clayton Copula has been studied in Hofert et al. (2012), which takes care about the non-exchangeable structure. In our study we name this as KB4 copula. The Probability density function (PDF) of bivariate KB4 copula can be written as;

$$\widehat{C}(u,v) = (u^{-\alpha_1\delta} + v^{-\alpha_2\delta} - 1)^{-1/\delta} u^{1-\alpha_1} v^{1-\alpha_2}, \quad \delta \geq 0;\ (\alpha_1, \alpha_2) \in [0,1]^2.$$

Now in order to get our conditional expectations we need Copula structure. As by the definition we know that, Copula is the joint density function whose arguments are lying between zero and one. Thus, in our case of $\mathrm{E}[X_1|X_2 = t]$ Copula is the first order derivative with one of its arguments. As it is differentiation, it should be less exchangeable than the previous case. Then we do calculate tail order conditional expectations by using either Laplace Approximation or Watson's Lemma. Again, in this case we try to be consistent with our measure of tail *non-exchangeability*. It can be seen that, in the case of Clayton Copula with Pareto margin we can use both the Laplace Approximation and Watson's Lemma. Furthermore, when we actually do the simulation, we find out that, simulations done by Laplace Approximation gives better result than Watson's Lemma. Even when we do Watson's Lemma we do not find any *non-exchageability* of Khoudraji (1996) type at the tail.

Only Laplace Approximation gives some *non-exchangeability.* On the other hand, in the cases of Weibull and Exponential margins Watson's Lemma gives better approximation at the tail.

We define the conditional Copula function as,

$$\begin{aligned}
\widehat{C}_{1|2}(u|v) &= \frac{\partial \widehat{C}(u,v)}{\partial v} \\
&= (1-\alpha_2)(u^{-\alpha_1\delta} + v^{-\alpha_2\delta} - 1)^{-1/\delta} u^{1-\alpha_1} v^{-\alpha_2} \\
&\qquad + \alpha_2 u^{1-\alpha_1} v^{-\alpha_2(1+\delta)} (u^{-\alpha_1\delta} + v^{-\alpha_2\delta} - 1)^{-(1+\frac{1}{\delta})} \\
&= (u^{-\alpha_1\delta} + v^{-\alpha_2\delta} - 1)^{-1/\delta} u^{1-\alpha_1} v^{-\alpha_2} [(1-\alpha_2) \\
&\qquad + \alpha_2 v^{-\alpha_2\delta} (u^{-\alpha_1\delta} + v^{-\alpha_2\delta} - 1)^{-1}] \\
&= (u^{-\alpha_1\delta} + v^{-\alpha_2\delta} - 1)^{-1/\delta} u^{1-\alpha_1} v^{-\alpha_2} \left[(1-\alpha_2) + \frac{\alpha_2 v^{-\alpha_2\delta}}{u^{-\alpha_1\delta} + v^{-\alpha_2\delta} - 1} \right], \qquad (2.78)
\end{aligned}$$

where $(u, v) \in [0, 1]^2$.

Now, we are interested to check the behavior of $\eta_2(t) = \mathrm{E}[X_1|X_2 = t]/\mathrm{E}[X_2|X_1 = t]$, at $t \to \infty$. In this case $\eta_2(t)$ is the measure of non-exchangeability at the tail. If the value of $\eta_2(t) = 1$ at $t \to \infty$, we say that the system is exchangeable otherwise, not. According to Hua and Joe (2014) the conditional expectation can be written as $E[X_1|X_2 = t] = \int_0^\infty \widehat{C}_{1|2}(\overline{F}(x)|\overline{F}(t))dx, \ \forall\ t$. Throughout this paper we are deriving this above conditional expectations at $t \to \infty$. Here the marginal distributions are the arguments of this kind of factor Copula functions. Furthermore, we do study three cases when margins are Pareto, Weibull and Exponential respectively.

2.2.2.1 Pareto margin

Like before we are using Pareto margin first with Khoudraji (1996) *non-exchangeable* transformed Clayton Copula. As discussed before, Pareto distribution is based upon power law and it is right skewed. The main advantage to use this distribution is that, we do not need a lot of data points at the tail in order to explain tail probability of occurrence of extreme events. The availability of data is rare in these occasions. There is a payoff about this kind of literature. When we are dealing with less data points, we surely lose many information about the character of the dataset. The estimation obtained might be asymptotically biased. Under this situation we can fix some t and make a conditional expectation based on that t.

Like before, let us consider a simple case where X_1, X_2 are random variables follow Pareto distribution with identical cdf $F(x) = 1-(1+x)^{-\beta}, \beta > 1$. Let $\widehat{C}$ be the survival copula of (X_1, X_2). Like before; $F(x) = 1-(1+x)^{-\beta}$, $\beta > 1$. Thus, $\overline{F}(x) = 1 - F(x) = 1-1+(1+x)^{-\beta} = (1+x)^{-\beta}$. Following Hua and Joe (2014) we will transform the survival function $\overline{F}(t) \to e^{-T}$

$or, T = -\log \overline{F}(t) = \beta \log(1+t)$ we have, letting $T = \beta\log(1+t)$

$$\begin{aligned}\mathrm{E}[X_1|X_2=t] &= \frac{T}{\beta}\int_0^\infty e^{T[\frac{s}{\beta}+\frac{1}{T}\log \widehat{C}_{1|2}(e^{-sT}|e^{-T})]}ds \\ &= \frac{T}{\beta}\int_0^\infty e^{Tg(s;T)}h(s)ds\end{aligned} \tag{2.79}$$

where $g(s;T) = \beta^{-1} + T^{-1}\log \widehat{C}_{1|2}(e^{-sT}|e^{-T})$, $h(s) = 1 > 0$.

Again, by using the result obtained from conditional survival copula distribution we have,

$$\widehat{C}_{1|2}(e^{-sT}|e^{-T}) = e^{\alpha_2 T-(1-\alpha_1)sT}(e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1)^{-\frac{1}{\delta}}\left[(1-\alpha_2)+\frac{\alpha_2 e^{\alpha_2\delta t}}{e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1}\right].$$

After putting value of $\widehat{C}_{1|2}(e^{-sT}|e^{-T})$ we get;

$$\begin{aligned} g(s;T) =& \frac{s}{\beta} + \frac{1}{T}\bigg[\alpha_2 T - (1-\alpha_1)sT - \frac{1}{\delta}\log(e^{\alpha_1\delta sT} + e^{\alpha_2\delta T} - 1) \\ &+ \log\bigg[(1-\alpha_2) + \frac{\alpha_2 e^{\alpha_2\delta T}}{e^{\alpha_1\delta sT} + e^{\alpha_2\delta T} - 1}\bigg]\bigg]. \end{aligned}$$

By doing easy calculations we can show that, $g(0,T) = 0$ and $g(\infty, T) = -\infty$. Hence, we can do either Laplace Approximation or Watson's Lemma.

Proposition 26. *If $\beta^{-1} + \alpha_1 > 1$, then g function is continuously increasing around the neighborhood of zero; where $\beta > 1$ and $\alpha_1 \in [0,1]$.*

Proof. We know,

$$\begin{aligned} g(s;T) &= \frac{s}{\beta} + \frac{1}{T}\bigg[\alpha_2 T - (1-\alpha_1)sT - \frac{1}{\delta}\log(e^{\alpha_1\delta sT} + e^{\alpha_2\delta T} - 1) \\ &\quad + \log\bigg[(1-\alpha_2) + \frac{\alpha_2 e^{\alpha_2\delta T}}{e^{\alpha_1\delta sT} + e^{\alpha_2\delta T} - 1}\bigg]\bigg] \\ g'(s;T) &= \frac{1}{\beta} + \frac{1}{T}\bigg[-T(1-\alpha_1) - \frac{\alpha_1\delta T e^{\alpha_1\delta sT}}{\delta(e^{\alpha_1\delta sT} + e^{\alpha_2\delta T} - 1)} \\ &\quad - \frac{\alpha_2 e^{\alpha_2\delta T}(e^{\alpha_1\delta sT} + e^{\alpha_2\delta T} - 1)^{-2}\alpha_1\delta T e^{\alpha_1\delta sT}}{(1-\alpha_2) + \frac{\alpha_2 e^{\alpha_2\delta T}}{e^{\alpha_1\delta sT} + e^{\alpha_2\delta T} - 1}}\bigg] \\ &= \frac{1}{\beta} - (1-\alpha_1) - \frac{\alpha_1 e^{\alpha_1\delta sT}}{e^{\alpha_1\delta sT} + e^{\alpha_2\delta T} - 1} \\ &\quad - \frac{\alpha_1\alpha_2\delta e^{\alpha_1\delta sT + \alpha_2\delta T}}{(e^{\alpha_1\delta sT} + e^{\alpha_2\delta T} - 1)^2\left[1 - \alpha_2 + \frac{\alpha_2 e^{\alpha_2\delta T}}{e^{\alpha_1\delta sT} + e^{\alpha_2\delta T} - 1}\right]} \\ g'(0;T) &= \frac{1}{\beta} - \frac{\alpha_1}{e^{\alpha_2\delta T}} - (1-\alpha_1) - \frac{\alpha_1\alpha_2\delta e^{\alpha_2\delta T}}{e^{2\alpha_2\delta T}\left[1 - \alpha_2 + \frac{\alpha_2 e^{\alpha_2\delta T}}{e^{\alpha_2\delta T}}\right]} \\ &= \frac{1}{\beta} + \alpha_1 - 1 - \frac{\alpha_1}{e^{\alpha_2\delta T}} - \frac{\alpha_1\alpha_2\delta}{e^{\alpha_2\delta T}} \\ \lim_{T\to\infty} g'(0;T) &= \frac{1}{\beta} + \alpha_1 - 1 > 0 \ if\ \frac{1}{\beta} + \alpha_1 > 1 \end{aligned} \tag{2.80}$$

From the calculation in (2.80) we conclude that if condition $\beta^{-1} + \alpha_1 > 1$ holds, g is convex around the neighborhood of 0; where $0 < \alpha_1 < 1$ and $\beta^{-1} \in [0, 1]$ [as by the copula we assume $\beta > 1$]. Furthermore, as $g'(0, T) > 0$ or $\beta^{-1} + \alpha_1 > 1$ we can use Laplace Approximation. □

Remark 12. Now we can use Laplace approximation. In this case asymptotic rate might be $T^{1/2}e^{T\gamma}h(s_0(T))[-g''(s_0(T); T)]^{-1/2}$ where $h(s_0(T)) = 1$ and $\gamma = \lim_{T\to\infty} max_s g(s; T)$, $s_0(T) = argmax_s g(s, T)$.If we carefully look at the result, condition (2.80) is exactly same as the unconditional KB4 case for Pareto margins.

Lemma 27. *In the case of survival Khoudraji (1996) non-exchangeable Clayton factor Copula at very large t, s converges to the ratio of two parameters of non-exchangeability or, $\gamma = \alpha_2\ (\alpha_1)^{-1}$ whenever $\alpha_2 > \alpha_1$ or $\alpha_2 < \alpha_1$ at $t \to \infty$.*

Proof. Please see the Appendix. □

From the above lemma we see that, $s(T)$ slowly converges to the ratio of the Khoudraji non-exchageable parameters [i.e. α_1 and α_2]. The interesting fact about this ratio is this is the same with the $\mathrm{E}[X_1|X_2 > t]$ case. This means that whether the Copula is general or factor non-exchangeable of Khoudraji (1996) type, the ratio does not change in both of these cases as $t \to \infty$.That makes our result more interesting. We assume in this model that $s \in [0, \infty)$. From the above lemma we know that, $\gamma = \lim_{T\to\infty} s(T) = \alpha_2\ (\alpha_1)^{-1}$. This implies that, if we want to have very large s, we need to have very large α_2 [probably closer to 1] and very small α_1 [very close to 0]. On the other hand, if we want to have very small s [close to zero], we need to have very small α_2 [probably closer to 0] and very large α_1 [very close to 1]. Clearly, $|\alpha_2 - \alpha_1|$ is going to be very high in both of the above cases. That means, the system is very *non-exchangeable* of Khoudraji type.

Proposition 28. *In Khoudraji (1996) non-exchangeable transformed survival factor Copula if* $\alpha_2 > \alpha_1$ *then we have* $-g''(s,T) > 0$ *as* $T \to \infty$. *In other words* g *becomes concave at very large* T *for all* $(\alpha_1, \alpha_2) \in (0,1]^2$ *and* $s \in [0,\infty)$. *Furthermore,* $-g''(s,T) \to \frac{\alpha_2(\alpha_1\delta)^2}{2} > 0$ *as* $T \to \infty$*; where* $\delta \geq 0$ *and* $\alpha_1, \alpha_2 > 0$.

Proof. Please see the Appendix. □

Remark 13. The import thing about the above proposition is g function is concave only if $\alpha_2 > \alpha_1$. Otherwise, this function does not have concavity property anymore. If we assume $\alpha_2 < \alpha_1$, all terms of the second order become zero. Thus we can not go any further. If one chooses very large s then, α_1 must be significantly small in order to hold $\alpha_2 > \alpha_1$. That implies there is significant *Khoudraji non-exchangeability.*

Proposition 29. *Conditional Expectation at the tail for Khoudraji (1996) non-exchangeable survival Clayton Copula go to a slow variation function if* $\alpha_2 > \alpha_1$ *and we have the form;*

$$\mathrm{E}[X_1|X_2 = t] \sim \frac{1}{\beta}(1+t)^{\frac{\alpha_2}{\alpha_1} - (1-\alpha_1)\frac{\alpha_2\beta}{\alpha_1}} \sqrt{\frac{4\pi\beta\log(1+t)}{\alpha_2(\alpha_1\delta)^2}}, \ \textit{as } t \to \infty,$$

where X_1 *and* X_2 *are two dependent random variables and furthermore we have,* $\beta > 1$, $\delta \geq 0$ *and* $(\alpha_1, \alpha_2) \in (0,1]^2$.

Proof. Now, finally we have to calculate $g(s;T)$ at $T \to \infty$. We know,

$$\begin{aligned}
g(s;T) &= \frac{s}{\beta} + \alpha_2 - (1-\alpha_1)s - \frac{\log(e^{\alpha_1\delta sT} + e^{\alpha_2\delta T} - 1)}{\delta T} \\
&\quad + \log\left[(1-\alpha_2) + \frac{\alpha_2 e^{\alpha_2\delta T}}{e^{\alpha_1\delta sT} + e^{\alpha_2\delta T} - 1}\right] \\
\lim_{T\to\infty} g(s;T) &= \frac{s}{\beta} + \alpha_2 - (1-\alpha_1)s + \log\{1-\alpha_2+\alpha_2\} \\
&\quad - \lim_{T\to\infty} \frac{\log(e^{\alpha_1\delta sT} + e^{\alpha_2\delta T} - 1)}{\delta T},\ as\ \alpha_2 > \alpha_1 \\
&= \frac{s}{\beta} + \alpha_2 - (1-\alpha_1)s \\
&\quad - \lim_{T\to\infty} \frac{\alpha_2 s e^{\alpha_1\delta sT} + \alpha_2 e^{\alpha_2\delta T}}{e^{\alpha_1\delta sT} + e^{\alpha_2\delta T} - 1},\ by\ L'Hospital\ Rule \\
&= \frac{s}{\beta} + \alpha_2 - (1-\alpha_1)s \\
&\quad - \lim_{T\to\infty} \frac{\alpha_1 s e^{(\alpha_1 s-\alpha_2)\delta T} + \alpha_2}{e^{(\alpha_1 s-\alpha_2)\delta T} + 1 - e^{-\alpha_2\delta T}} \\
&= \frac{s}{\beta} + \alpha_2 - (1-\alpha_1)s - \alpha_2 \\
&\quad - \lim_{T\to\infty} \alpha_1 s e^{(\alpha_1 s-\alpha_2)\delta T},\ as\ \alpha_2 < \alpha_1 \\
&= \frac{s}{\beta} - (1-\alpha_1)s \\
&= \frac{\alpha_2}{\alpha_1\beta} - (1-\alpha_1)\frac{\alpha_2}{\alpha_1},\ as\ \lim_{T\to\infty} s_0(T) = \frac{\alpha_2}{\alpha_1},\ by\ lemma. \qquad (2.81)
\end{aligned}$$

Combining the above conditions of function g we get, $g(0;T) = 0$ and $g(\infty,T) = -\infty$. Thus, $g(s;T)$ is strictly increasing for $s \in (0, s_0]$ and is strictly decreasing for $s \in [s_0;\infty)$.

Now as $T \to \infty$ we can write,

$$\mathrm{E}[X_1|X_2 = t] \sim \frac{1}{\beta} e^{Tg(s_0,\alpha_1,\alpha_2,\delta;T)} \sqrt{\frac{2\pi T}{-g''(s_0,\alpha_1,\alpha_2,\delta;T)}} \qquad (2.82)$$

After putting $T = \beta\log(1+t)$ in equation (2.82) we get,

$$\mathrm{E}[X_1|X_2 = t] \sim \frac{1}{\beta} e^{\beta\log(1+t) g(s_0,\alpha_1,\alpha_2,\delta;\beta\log(1+t))} \sqrt{\frac{2\pi\beta\log(1+t)}{-g''(s_0,\alpha_1,\alpha_2,\delta;T)}}$$

$$\begin{aligned}
\mathrm{E}[X_1|X_2 = t] &\sim \frac{1}{\beta}(1+t)^{\beta g(s_0,\alpha_1'\alpha_2,\delta;\beta\log(1+t))} \sqrt{\frac{2\pi\beta\log(1+t)}{-g''(s_0,\alpha_1,\alpha_2,\delta;T)}} \\
&\sim \frac{1}{\beta}(1+t)^{\beta[\frac{\alpha_2}{\alpha_1\beta} - (1-\alpha_1)\frac{\alpha_2}{\alpha_1}]} \sqrt{\frac{2\pi\beta\log(1+t)}{\frac{\alpha_2(\alpha_1\delta)^2}{2}}}, \ \textit{from } (2.81),\ (A.31) \textit{ and as } \alpha_2 > \alpha_1 \\
&\sim \frac{1}{\beta}(1+t)^{\frac{\alpha_2}{\alpha_1} - (1-\alpha_1)\frac{\alpha_2\beta}{\alpha_1}} \sqrt{\frac{4\pi\beta\log(1+t)}{\alpha_2(\alpha_1\delta)^2}}, \ \textit{as } t\to\infty, \qquad (2.83)
\end{aligned}$$

where all the symbols have their usual meaning. Thus completes the proof. □

Corollary 30. *In the similar way we can get;*

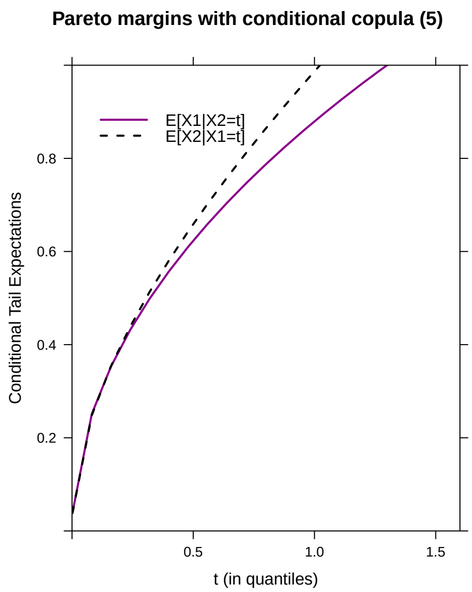
$$\mathrm{E}[X_2|X_1 = t] \sim \frac{1}{\beta}(1+t)^{\frac{\alpha_1}{\alpha_2} - (1-\alpha_2)\frac{\alpha_1\beta}{\alpha_2}} \sqrt{\frac{4\pi\beta\log(1+t)}{\alpha_1(\alpha_2\delta)^2}}, \ \textit{as } t\to\infty, \qquad (2.84)$$

where X_1 and X_2 are two dependent random variables and furthermore we have, $\beta > 1$, $\delta \geq 0$ and $(\alpha_1,\alpha_2) \in (0,1]^2$.

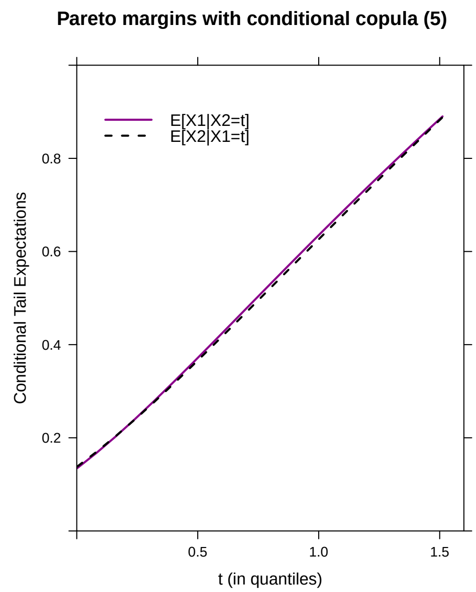
After using (2.83), (2.84) our measure of tail *non-exchangeability* becomes;

$$\begin{aligned}
\eta_2(t) &= \frac{\mathrm{E}[X_1|X_2 = t]}{\mathrm{E}[X_2|X_1 = t]} \\
&\sim \frac{(1+t)^{\frac{\alpha_2}{\alpha_1} - (1-\alpha_1)\frac{\alpha_2\beta}{\alpha_1}}}{(1+t)^{\frac{\alpha_1}{\alpha_2} - (1-\alpha_2)\frac{\alpha_1\beta}{\alpha_2}}}, \ \textit{as } t\to\infty, \qquad (2.85)
\end{aligned}$$

where X_1 and X_2 are two random variables and furthermore we have, $\beta > 1$, $\delta \geq 0$ and $(\alpha_1,\alpha_2) \in (0,1]^2$.



(a) $\alpha_1 = 0.85$, $\alpha_2 = 0.90$, $\beta = 5$ and $\delta = 1$



(b) $\alpha_1 = 0.85$, $\alpha_2 = 0.90$, $\beta = 5$ and $\delta = 1$

Figure 2.6: Comparison between Laplace Approximation and the Actual Conditional Tail Expectations when α_1 and α_2 are different

In Figure 2.6a and 2.6b we try to compare the simulation using Laplace approximation with the actual conditional tail expectations. In Figure 2.6a we are using the simulation results obtained in (2.83) and (2.84). Throughout our simulations we assume $\alpha_1 = 0.85$, $\alpha_2 = 0.90$, $\beta = 5$ and $\delta = 1$. We take $\delta = 1$ because for the higher and lower values we can see uneven fluctuations. We are not able to find any pattern. After fixing the values of the parameters we see in Figure 2.6a there is no significant *non-exchangeability* at around 0, but this *non-exchangeability* increases as we come closer to 90 th percentile. On the other hand, in Figure 2.6b we do not find that much *non-exchangeability* throughout the plot. We can say Laplace approximation might overestimate the small changes between two tail order conditional expectations at higher quantiles. Again, if we carefully look at α_1 and α_2, we find they are not significant different from each other. Even these two parameters are very close to each other we can find higher tail *non-exchangeability*.

Proposition 31. *If* $\alpha_1 + \beta^{-1} \not> 1$ *and* $\alpha_2 > \alpha_1$ *then, conditional tail order expectation* $\mathrm{E}[X_1|X_2 = t]$ *goes to some constant;*

$$\mathrm{E}[X_1|X_2 = t] \sim \frac{1}{\beta[1 - \frac{1}{\beta} - \alpha_1(1 - \frac{\alpha_2\delta}{2})]} \ as, \ \alpha_2 > \alpha_1, \ as \ t \to \infty,$$

where X_1 *and* X_2 *are two dependent random variables and furthermore we have,* $\beta > 1$, $\delta \geq 0$ *and* $(\alpha_1, \alpha_2) \in [0, 1]^2$.

Proof. Let us consider $\alpha_2 > \alpha_1$. From our previous discussions we know that, $g'(s;T) = \frac{1}{\beta} - (1 - \alpha_1) - \frac{\alpha_1 e^{\alpha_1 \delta s T}}{e^{\alpha_1 \delta s T} + e^{\alpha_2 \delta T} - 1} - \frac{\alpha_1 \alpha_2 \delta e^{\alpha_1 \delta s T + \alpha_2 \delta T}}{(e^{\alpha_1 \delta s T} + e^{\alpha_2 \delta T} - 1)^2 \left[1 - \alpha_2 + \frac{\alpha_2 e^{\alpha_2 \delta T}}{e^{\alpha_1 \delta s T} + e^{\alpha_2 \delta T} - 1}\right]}$.

If we use first case then we get;

$$\begin{aligned}
g'(s;T) &= \frac{1}{\beta} - (1 - \alpha_1) - \frac{\alpha_1 e^{\alpha_1 \delta s T}}{e^{\alpha_1 \delta s T} + e^{\alpha_2 \delta T} - 1} \\
&\quad - \frac{\alpha_1 \alpha_2 \delta e^{(\alpha_1 s + \alpha_2)\delta T}}{(e^{\alpha_1 \delta s T} + e^{\alpha_2 \delta T} - 1)^2 \left[1 - \alpha_2 + \frac{\alpha_2 e^{\alpha_2 \delta T}}{e^{\alpha_1 \delta s T} + e^{\alpha_2 \delta T} - 1}\right]} \\
&= \frac{1}{\beta} - (1 - \alpha_1) - \frac{\alpha_1 e^{(\alpha_1 s - \alpha_2)\delta T}}{e^{(\alpha_1 s - \alpha_2)\delta T} + 1 - e^{-\alpha_2 \delta T}} \\
&\quad - \frac{\alpha_1 \alpha_2 \delta e^{(\alpha_1 s + \alpha_2)\delta T}}{(e^{\alpha_1 \delta s T} + e^{\alpha_2 \delta T} - 1)^2 \left[1 - \alpha_2 + \frac{\alpha_2}{e^{(\alpha_1 s - \alpha_2)\delta T} + 1 - e^{-\alpha_2 \delta T}}\right]} \\
\lim_{T \to \infty} g'(s;T) &= \frac{1}{\beta} - (1 - \alpha_1) - \lim_{T \to \infty} \frac{\alpha_1 \alpha_2 \delta e^{(\alpha_1 s + \alpha_2)\delta T}}{(e^{\alpha_1 \delta s T} + e^{\alpha_2 \delta T} - 1)^2}, \ as \ \alpha_2 > \alpha_1, \\
&= \frac{1}{\beta} - (1 - \alpha_1) - \lim_{T \to \infty} \frac{\alpha_1 \alpha_2 \delta e^{\alpha_2 \delta T}}{2(e^{\alpha_1 \delta s T} + e^{\alpha_2 \delta T} - 1)}, \ by \ L'Hospital \ Rule, \\
&= \frac{1}{\beta} - (1 - \alpha_1) - \lim_{T \to \infty} \frac{\alpha_1 \alpha_2 \delta}{2(e^{(\alpha_1 s - \alpha_2)\delta T} + 1 - e^{-\alpha_2 \delta T})} \\
&= \frac{1}{\beta} - (1 - \alpha_1) - \frac{\alpha_1 \alpha_2 \delta}{2}, \ as \ T \to \infty, \\
&= \frac{1}{\beta} - 1 + \alpha_1 (1 - \frac{\alpha_2 \delta}{2}) < 0, \ for \ Watson's \ Lemma
\end{aligned} \tag{2.86}$$

From (2.86) we know that, $(\beta^{-1} - 1) < 0$ all the time because $\beta > 1$. Thus, in order to make $g'(s;T) < 0$ we need $(1 - \frac{\alpha_2\delta}{2}) < 0$. In order to satisfy this condition we need $\frac{\alpha_2\delta}{2} > 1$ or, $\alpha_2\delta > 2$. Thus, we need a very large δ as $\alpha_2 \in [0, 1]$. As $g(s, T)$ is real function on the semi-infinite interval $[0, \infty)$ and in an interval $(0, 0 + \epsilon]$ with $\epsilon > 0$ this function is continuously differentiable and $\sup_{0+\epsilon \leq s \leq \infty} g(s, T) \leq g(0, T) - \psi$ with $\psi > 0$, we can use a version of Watson's Lemma defined in Theorem 36 [p. 48] in Breitung (1994).

Now, for $g'(s, T)$ we have $g'(s, T) < 0$ as $\frac{1}{\beta} < 1$ and $T \to \infty$. We can also write $g'(s, T) = -as^{r-1} + o(s^{r-1})\ \forall\ r > 0$. Now if we assume $r = 1$ then $g'(s, T) = -a$. From our previous results we know that, $\lim_{s^+ \to\ 0,\ T \to \infty} g'(s, T) = \frac{1}{\beta} - 1 + \alpha_1(1 - \frac{\alpha_2\delta}{2})$, which is a constant. Thus, $-a = -\left(1 - \frac{1}{\beta} - \alpha_1\left(1 - \frac{\alpha_2\delta}{2}\right)\right)$ or, $a = \left(1 - \frac{1}{\beta} - \alpha_1\left(1 - \frac{\alpha_2\delta}{2}\right)\right) > 0$. Let us assume there is another real and continuous function $h(s, T) \in [0, \infty)$ such that, $h(s) = bs^{m-1} + o(s^{m-1})$ with $m > 0$. More specifically we assume $h(s) = 1$ in our case. Thus, $bs^{m-1} + o(s^{m-1}) = 1$
$\implies b = 1$; when $m = 1$.

Finally, as we are assuming $\int_0^\infty e^{g(s,T)} ds < \infty$ then by Watson's lemma we can write the approximated value of the integral $I(T) = \int_0^\infty e^{Tg(s,T)} ds$ with $T \geq 1$ are all finite and the asymptotic approximation is;

$$I(T) \sim \frac{T^{-1}}{1 - \frac{1}{\beta} - \alpha_1(1 - \frac{\alpha_2\delta}{2})}\ as,\ T \to \infty \tag{2.87}$$

Using (2.87) we have the conditional tail expectation by using Watson's lemma as;

$$\mathrm{E}[X_1|X_2 = t] \sim \frac{1}{\beta\left[1 - \frac{1}{\beta} - \alpha_1\left(1 - \frac{\alpha_2\delta}{2}\right)\right]}\ as,\ \alpha_2 > \alpha_1,\ as\ t \to \infty \tag{2.88}$$

This completes the proof. □

Corollary 32. *In the similar way we get,*

$$\mathrm{E}[X_2|X_1 = t] \sim \frac{1}{\beta\left[1 - \frac{1}{\beta} - \alpha_2\left(1 - \frac{\alpha_1\delta}{2}\right)\right]} \ as, \ \alpha_2 > \alpha_1, \ and \ t \to \infty, \tag{2.89}$$

where X_1 *and* X_2 *are two dependent random variables and furthermore we have,* $\beta > 1$, $\delta \geq 0$ *and* $(\alpha_1, \alpha_2) \in [0,1]^2$.

Remark 14. Thus, our measure of tail *non-exchangeability* of Khoudraji (1996) type when $\alpha_1 + \frac{1}{\beta} \not> 1$ becomes;

$$\eta_2(t) \sim \frac{1 - \frac{1}{\beta} - \alpha_2(1 - \frac{\alpha_1\delta}{2})}{1 - \frac{1}{\beta} - \alpha_1(1 - \frac{\alpha_2\delta}{2})},$$

as $\alpha_2 > \alpha_1$, and $t \to \infty$.

Corollary 33. *Suppose* X_1 *and* X_2 *are two dependent random variables. If* $\alpha_1 + \beta^{-1} \not> 1$ *and* $\alpha_2 < \alpha_1$ *then, conditional tail expectation* $\mathrm{E}[X_1|X_2 = t]$ *goes to some constant;*

$$\mathrm{E}[X_1|X_2 = t] \sim \frac{1}{\beta[1 - \frac{1}{\beta}]} \ as, \ \alpha_2 < \alpha_1, \ as \ t \to \infty,$$

where $\beta > 1$, $\delta \geq 0$ *and* $(\alpha_1, \alpha_2) \in [0,1]^2$. *Furthermore, our measure of tail* non-exchangeability *by using Watson's Lemma is;* $\eta_2(t) = \mathrm{E}[X_1|X_2 = t]/\mathrm{E}[X_2|X_1 = t] \to 1$ *as* $t \to \infty$.

Corollary 34. *If* $\alpha_1 + \beta^{-1} \not> 1$ *and* $\alpha_2 = \alpha_1$ *then, conditional tail expectation* $\mathrm{E}[X_1|X_2 = t]$ *goes to some constant;*

$$\mathrm{E}[X_1|X_2 = t] \sim \frac{1}{\beta\left[1 - \frac{1}{\beta} - \frac{\alpha_1}{2}\left[1 - \frac{\alpha_1\delta}{2(1 - \frac{\alpha_2}{2})}\right]\right]} \ as, \ \alpha_2 = \alpha_1, \ and \ t \to \infty,$$

where X_1 and X_2 are two dependent random variables and furthermore we have, $\beta > 1$, $\delta \geq 0$ and $(\alpha_1, \alpha_2) \in [0,1]^2$. Furthermore, our measure of tail non-exchangeability by using Watson's Lemma is;

$$\eta_2(t) \sim \frac{1 - \frac{1}{\beta} - \frac{\alpha_2}{2}\left[1 - \frac{\alpha_2\delta}{2(1-\frac{\alpha_1}{2})}\right]}{1 - \frac{1}{\beta} - \frac{\alpha_1}{2}\left[1 - \frac{\alpha_1\delta}{2(1-\frac{\alpha_2}{2})}\right]}$$

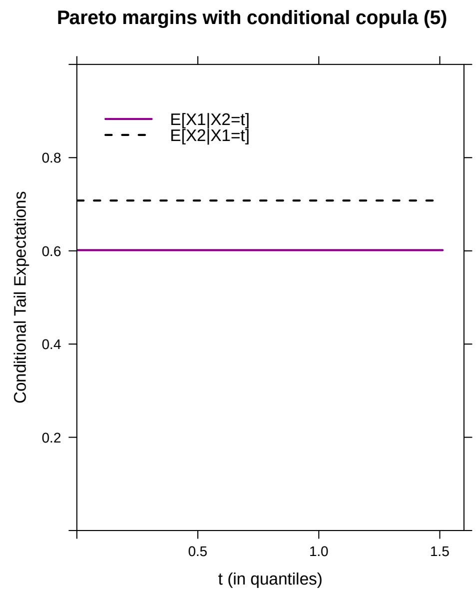
as $t \to \infty$.

Remark 15. *Hence, from the above discussion we know that, measure of conditional tail non-exchangeability of Khoudraji (1996) type under Watson's lemma can be written as;*

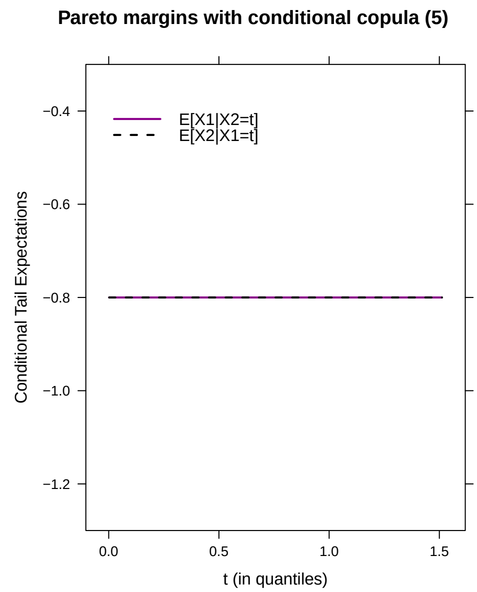
$$\eta_2(t) \sim \begin{cases} \frac{1-\frac{1}{\beta}-\alpha_2(1-\frac{\alpha_1\delta}{2})}{1-\frac{1}{\beta}-\alpha_1(1-\frac{\alpha_2\delta}{2})} & \text{if } \alpha_2 > \alpha_1; \\ 1 & \text{if } \alpha_2 < \alpha_1; \\ \frac{1-\frac{1}{\beta}-\frac{\alpha_2}{2}\left[1-\frac{\alpha_2\delta}{2(1-\frac{\alpha_1}{2})}\right]}{1-\frac{1}{\beta}-\frac{\alpha_1}{2}\left[1-\frac{\alpha_1\delta}{2(1-\frac{\alpha_2}{2})}\right]} & \text{if } \alpha_2 = \alpha_1; \end{cases} \tag{2.90}$$

where X_1 and X_2 are two dependent random variables and furthermore we have, $\beta > 1$, $\delta \geq 0$ and $(\alpha_1, \alpha_2) \in [0,1]^2$. The system is tail non-exchangeable when $\alpha_2 > \alpha_1$ and $\alpha_2 = \alpha_1$.

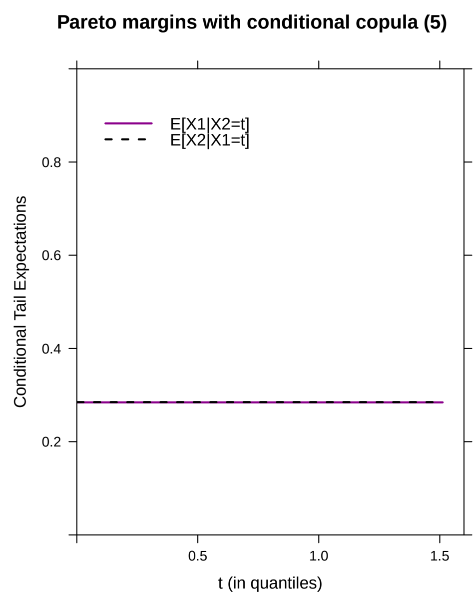
In Figures 2.7a, 2.7b and 2.7c we are plotting the simulation results obtained above. These three plots represent the cases where $\alpha_2 > \alpha_1$, $\alpha_2 < \alpha_1$ and $\alpha_2 = \alpha_1$ respectively. In order to make these results comparable with Laplace approximation we keep all the parameters fixed like in Figures 2.6a and 2.6b. In Figure 2.7d we actually plot conditional tail expectations using our Khoudraji-transfored Clayton copula with Pareto margins. Again in 2.7d we see both the conditional expectations are positively sloped but they are exchangeable. In panels 2.7b and 2.7c we find *exchangeability* between two conditional expectations but here the simulations are not positively sloped like the original case. As we are looking at $\eta_2(t)$ at $t \to \infty$, the slopes and the values of these conditional expectations do not matter. Only



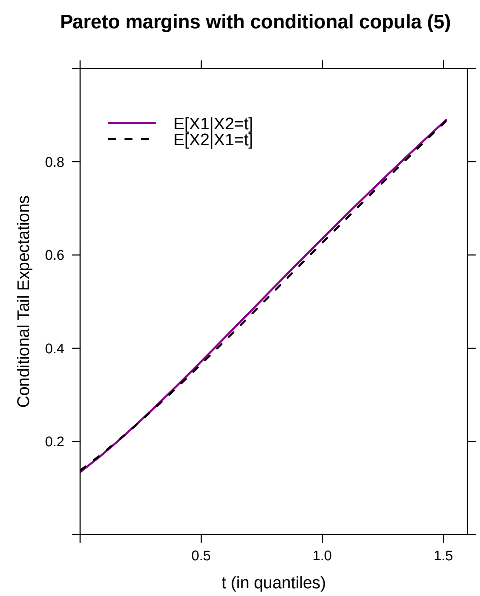
(a) $\alpha_1 = 0.85,\ \alpha_2 = 0.90,\ \beta = 5,\ \delta = 1$ and $\alpha_2 > \alpha_1$



(b) $\alpha_1 = 0.85,\ \alpha_2 = 0.90,\ \beta = 5,\ \delta = 1$ and $\alpha_2 < \alpha_1$



(c) $\alpha_1 = 0.85,\ \alpha_2 = 0.90,\ \beta = 5,\ \delta = 1$ and $\alpha_2 = \alpha_1$



(d) $\alpha_1 = 0.85,\ \alpha_2 = 0.90,\ \beta = 5$ and $\delta = 1$

Figure 2.7: Comparison between Watson's Lemma and the Actual Conditional Tail Expectations when α_1 and α_2 are different

thing matters is their ratio. As $t \to \infty$, $\mathrm{E}[X_1|X_2 = t]/\mathrm{E}[X_2|X_1 = t] \to 1$ in Figures 2.7b and 2.7c, so as in Figure 2.7d. On the other hand the plot in Figure 2.7a shows more *non-exchangeability* than desirable limit. Thus, we choose Watson's lemma when $\alpha_2 < \alpha_1$ and $\alpha_2 = \alpha_1$. Furthermore, if we compare these results with 2.6a, we see Laplace approximation gives over non-exchangeability than Watson's lemma in Figure 2.7a which is good. Hence, in this case, Laplace Approximation is better approximation method than Watson's Lemma in KB4 with Pareto margins.

2.2.2.2 Weibull margin

Now, in this section we are considering tail order *non-exchangeability* of Claytom Copula with Weibull margins. As we told in the previous section, the reason behind to choose this distribution is because of this is in sub-exponential family. The advantage of using this is we can use it distribution with a very small size of sample. In our paper we are trying to develop a model on the extreme value theory. Here we are interested at the extreme, on the other hand, if we have a density function, we must consider the tails of that distribution. In extreme value theory sometimes getting more data at the tail is very expensive. Suppose, let us consider the case of the incidence of losing one's whole money in the stock market. Before building up this environment let us see the common cases. When one person put some money in the stock market, there is some probability to win or lose it or get certain portion of it. In this case, the extreme scenario is either somebody earns double or triple or more of his investment or he loses some or all of it. Second case is more severe than the first because in the later, the person rethinks about his decision to put money in the stock market in future. First scenario does not make severe impact of the person as he calculates some probability to get a larger amount of money from an investment. In the other case, it makes. That is why we determine the second case is more severe than the first one.

In this section our primary objective is to find out the tail order *non-exchangeability of Khoudraji (1996) type*. Firstly, we try to find out the tail order conditional expectations and try t find out what would be the best representative of this integral. Following Hua and Joe (2014) we know that, we can use either Laplace Approximation or Watson's Lemma based on the conditions of the integrands we get. Secondly, we convert the integrand into two multiplicative separable functions one of which is positive and another is monotonically decreasing. After doing some initial tests of that function we find out that,the monotonic function g is not positive around the neighborhood of zero. Thus, we can not use Laplace Approximation anymore. We have to use Watson's lemma.The approximation we get by Watson's lemma is very consistent with our previous results in the case of $\mathrm{E}[X_1|X_2 > t]$. Let us think this case in an intuitive way. In $\mathrm{E}[X_1|X_2 > t]$ we know that only Watson's Lemma is going to work in order to get the tail order conditional expectations. Furthermore, here in order to calculate $\mathrm{E}[X_1|X_2 = t]$ at $t \to \infty$ we are basically integrating the survival factor Copula. As the survival functions are between zero and one, factor Copula is nothing but the first order differentiation of Khoudraji-transformed survival Clayton Copula with respect to the first argument. As differentiation is nothing but the linearizing a function at a point, result getting in $\mathrm{E}[X_1|X_2 = t]$ case is less exchangeable than $\mathrm{E}[X_1|X_2 > t]$ case. Finally, when we the simulation, we find out less non-exchangeability at the tail. We find out that as $t \to \infty$ the slow variation goes to some constant.

We consider X_1 and X_2 are random variables which follow Weibull distribution with identical cumulative distribution functions $F(x) = 1 - e^{-x^{\gamma}}, \ \forall x, \gamma > 0$. Now the survival function should be $\overline{F}(x) = 1 - F(x) = e^{-x^{\gamma}} \ \forall x, \gamma > 0$. Following Hua and Joe (2014) we transform the survival function $\overline{F}(t) \to \ e^{-T} \ or, T = -\log \overline{F}(t) = t^{\gamma}, \ y = -log\overline{F}(x) =$

$x^\gamma \implies x = y^{\frac{1}{\gamma}}$. Differentiating totally both sides of the previous equation we get $dx = \frac{1}{\gamma} y^{\frac{1}{\gamma}-1} dy$.

Again, by using the method provided by Hua and Joe (2014) we get;

$$\mathrm{E}[X_1|X_2 = t] = \gamma^{-1} T^{\frac{1}{\gamma}} \int_0^\infty e^{T \frac{1}{T} \widehat{C}_{1|2}(e^{-sT}|e^{-T})} s^{\frac{1}{\gamma}-1} ds = \gamma^{-1} T^{\frac{1}{\gamma}} \int_0^\infty e^{T g(s;T)} h(s) ds;\ \forall s \in [0, \infty).$$

In our case $g(s;T) = \frac{1}{T} \widehat{C}_{1|2}(e^{-sT}|e^{-T})$ and $h(s) = s^{\frac{1}{\gamma}-1}$. g is a monotonic functions and h is a positive function. If we carefully look at the integrand, we see it can be written in the multiplicative form of two functions g and h respectively. Again, by using the result obtained from survival copula we have,

$$\widehat{C}_{1|2}(e^{-sT}, e^{-T}) = e^{\alpha_2 T - (1-\alpha_1) sT} (e^{\alpha_1 \delta sT} + e^{\alpha_2 \delta T} - 1)^{-\frac{1}{\delta}} \left[(1-\alpha_2) + \frac{\alpha_2 e^{\alpha_2 \delta t}}{e^{\alpha_1 \delta sT} + e^{\alpha_2 \delta T} - 1} \right].$$

Thus, in our case,

$$\begin{aligned} g(s;T) = \frac{1}{T} \Big[& \alpha_2 T - (1-\alpha_1) sT - \frac{1}{\delta} \log(e^{\alpha_1 \delta sT} + e^{\alpha_2 \delta T} - 1) \\ & + \log \left[(1-\alpha_2) + \frac{\alpha_2 e^{\alpha_2 \delta T}}{e^{\alpha_1 \delta sT} + e^{\alpha_2 \delta T} - 1} \right] \Big]. \end{aligned}$$

where $(\alpha_1, \alpha_2) \in [0,1]^2$, $\delta \geq 0$ and $s \in [0, \infty)$.

After doing easy calculations we see that $g(0,T) = 0$ and $g(\infty, T) = -\infty$. Thus, we can use either Laplace Approximation or Watson's Lemma to get the slow variation function at $t \to \infty$. Apart from that we also know that, $h(s) > 0$ in this case.

Claim 35. *In Khoudraji-transformed survival Clayton Copula with Weibull margins Laplace Approximation is not applicable.*

Proof. Please see the Appendix. □

Proposition 36. *If $\alpha_2 > \alpha_1$, then for Khoudraji (1996) transformed survival Clayton Copula with Weibull margin with any $\gamma > 0$ and $\delta \geq 0$ we have;*

$$\mathrm{E}[X_1|X_2 = t] \sim \frac{1}{\gamma}\Gamma\left(\frac{1}{\gamma}\right)\frac{1}{1-\alpha_2\left(1-\frac{\alpha_1\delta}{2}\right)}$$

as $t \to \infty$.

Proof. From our previous discussion we know,

$$g'(s;T) = -\left[(1-\alpha_1) + \frac{\alpha_1 e^{\alpha_1\delta sT}}{e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1} + \frac{\alpha_1\alpha_2\delta e^{(\alpha_1 s+\alpha_2)\delta T}}{(e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1)^2\left[1-\alpha_2+\frac{\alpha_2 e^{\alpha_2\delta T}}{e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1}\right]}\right]$$

Let us assume that $\alpha_2 > \alpha_1$. Now the above first order derivative becomes;

$$\begin{aligned}
g'(s;T) &= -(1-\alpha_1) - \frac{\alpha_1 e^{(\alpha_1 s-\alpha_2)\delta T}}{e^{(\alpha_1 s-\alpha_2)\delta T}+1-e^{-\alpha_2\delta T}} \\
&\quad - \frac{\alpha_1\alpha_2\delta e^{(\alpha_1 s+\alpha_2)\delta T}}{(e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1)^2\left[1-\alpha_2+\frac{\alpha_2}{e^{(\alpha_1 s-\alpha_2)\delta T}+1-e^{-\alpha_2\delta T}}\right]} \\
\lim_{T\to\infty} g'(s;T) &= -(1-\alpha_1) - \lim_{T\to\infty}\frac{\alpha_1\alpha_2\delta e^{(\alpha_1 s+\alpha_2)\delta T}}{(e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1)^2},\ \textit{as}\ \alpha_2 > \alpha_1, \\
&= -(1-\alpha_1) - \lim_{T\to\infty}\frac{\alpha_1\alpha_2\delta e^{\alpha_2\delta T}}{2(e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1)},\ \textit{by L'Hospital Rule}, \\
&= -(1-\alpha_1) - \frac{\alpha_1\alpha_2\delta}{2},\ \textit{as}\ T\to\infty, \\
&= -1+\alpha_1\left(1-\frac{\alpha_2\delta}{2}\right) < 0. \qquad (2.91)
\end{aligned}$$

In (2.91) $-1 < 0$ all the time . Thus, in order to make $g'(s;T) < 0$ we need $(1-\frac{\alpha_2\delta}{2}) < 0$. In order to satisfy this condition we need $\frac{\alpha_2\delta}{2} > 1$ or, $\alpha_2\delta > 2$. Thus, we need a very large δ as $\alpha_2 \in [0,1]$. As $g(s,T)$ is real valued function on the semi-infinite interval $[0,\infty)$ and in $(0, 0+\epsilon]$ with $\epsilon > 0$ this function is continuously differentiable and $\sup_{0+\epsilon\leq s\leq\infty} g(s,T) \leq$

$g(0,T) - \psi$; with $\psi > 0$, we can use a version of Watson's Lemma defined in Theorem 36 [p. 48] in Breitung (1994). Now, for $g'(s,T)$ we have $g'(s,T) < 0$ as $T \to \infty$. We can also write $g'(s,T) = -as^{r-1} + o(s^{r-1})\ \forall\ r > 0$. Now if we assume $r = 1$ then $g'(s,T) = -a$. From our previous results we know that, $\lim_{s^+ \to\ 0,\ T\to\infty} g'(s,T) = -1 + \alpha_1(1 - \frac{\alpha_2\delta}{2})$, which is a constant. Thus, $-a = -(1-\alpha_1(1-\frac{\alpha_2\delta}{2}))$ or, $a = (1-\alpha_1(1-\frac{\alpha_2\delta}{2})) > 0$. Let us assume there is another real and continuous function $h(s) \in [0,\infty)$ such that, $h(s) = bs^{m-1} + o(s^{m-1})$ with $m > 0$. More specifically we assume $h(s) = s^{\frac{1}{\gamma}-1}$ in our case. Thus, $bs^{m-1} + o(s^{m-1}) = s^{\frac{1}{\gamma}-1}$ $\implies b = 1$; where $m = \frac{1}{\gamma}$.

Finally, as we are assuming $\int_0^\infty e^{g(s,T)}ds < \infty$ then by Watson's lemma we can write the approximated value of the integral $I(T) = \int_0^\infty e^{Tg(s,T)}ds$ with $T \geq 1$ are all finite and the asymptotic approximation is;

$$I(T) \sim \Gamma(\frac{1}{\gamma})\frac{1}{1-\alpha_1(1-\frac{\alpha_2\delta}{2})}T^{-\frac{1}{\gamma}},\ as\ g(0;T) = 0\ as\ T \to \infty. \tag{2.92}$$

Thus using (2.91) in the original integration we get,

$$\mathrm{E}[X_1|X_2 = t] \sim \frac{1}{\gamma}\Gamma\left(\frac{1}{\gamma}\right)\frac{1}{1-\alpha_1(1-\frac{\alpha_2\delta}{2})},\ as\ t \to \infty. \tag{2.93}$$

where $(\alpha_1, \alpha_2)\ \in [0,1]^2$, $\gamma > 0$ and $\delta \geq 0$. This completes the proof. □

Corollary 37. *If $\alpha_2 > \alpha_1$, then for Khoudraji (1996) transformed survival Clayton Copula with Weibull margin with any $\gamma > 0$ and $\delta \geq 0$ we have;*

$$\mathrm{E}[X_2|X_1 = t] \sim \frac{1}{\gamma}\Gamma\left(\frac{1}{\gamma}\right)\frac{1}{1-\alpha_1\left(1-\frac{\alpha_2\delta}{2}\right)}$$

as $t \to \infty$.

Remark 16. Using the results from the above proposition and corollary we get our measure of tail non-exchangeability as ,

$$\eta_2(t) \sim \frac{1-\alpha_2(1-\frac{\alpha_1\delta}{2})}{1-\alpha_1(1-\frac{\alpha_2\delta}{2})}, \ where\ \alpha_2 > \alpha_1 \tag{2.94}$$

Corollary 38. *Suppose X_1 and X_2 are two dependent random variables. If $\alpha_2 < \alpha_1$, then for Khoudraji (1996) transformed survival Clayton Copula with Weibull margin with any $\gamma > 0$ and $\delta \geq 0$ we have;*

$$\mathrm{E}[X_1|X_2 = t] \sim \frac{1}{\gamma}\Gamma(\frac{1}{\gamma})$$

as $s^+ \to 0$ and $t \to \infty$.

Proof. Please see Appendix. □

Remark 17. In this case the system is exchangeable. In this situation Khoudraji (1996) non-exchangeable transformation of Clayton Copula gives $\mathrm{E}[X_1|X_2 = t] = \mathrm{E}[X_2|X_1 = t]$.

Corollary 39. *Suppose X_1 and X_2 are two dependent random variables. If $\alpha_2 = \alpha_1$, then for Khoudraji (1996) transformed survival Clayton Copula with Weibull margin with any $\gamma > 0$ and $\delta \geq 0$ we have;*

$$\mathrm{E}[X_1|X_2 = t] \sim \frac{1}{\gamma}\Gamma\left(\frac{1}{\gamma}\right)\frac{1}{1-\frac{\alpha_1}{2}\left[1-\frac{\alpha_1\delta}{2(1-\frac{\alpha_2}{2})}\right]}$$

as $s^+ \to 0$ and $t \to \infty$.

Corollary 40. *Suppose X_1 and X_2 are two dependent random variables. If $\alpha_2 = \alpha_1$, then for Khoudraji (1996) transformed survival Clayton Copula with Weibull margin with any $\gamma > 0$ and $\delta \geq 0$ we have;*

$$\mathrm{E}[X_2|X_1 = t] \sim \frac{1}{\gamma}\Gamma\left(\frac{1}{\gamma}\right)\frac{1}{1-\frac{\alpha_2}{2}\left[1-\frac{\alpha_2\delta}{2(1-\frac{\alpha_1}{2})}\right]}$$

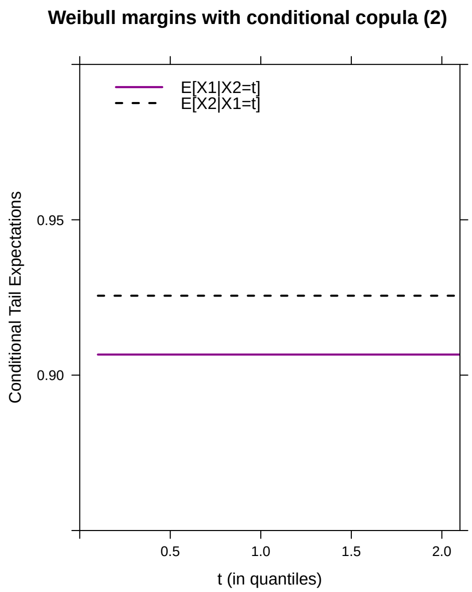
as $s^+ \to 0$ and $t \to \infty$.

Remark 18. Hence, from the above discussion we know that, measure of conditional *tail non-exchangeability* under *Watson's lemma* can be written as;

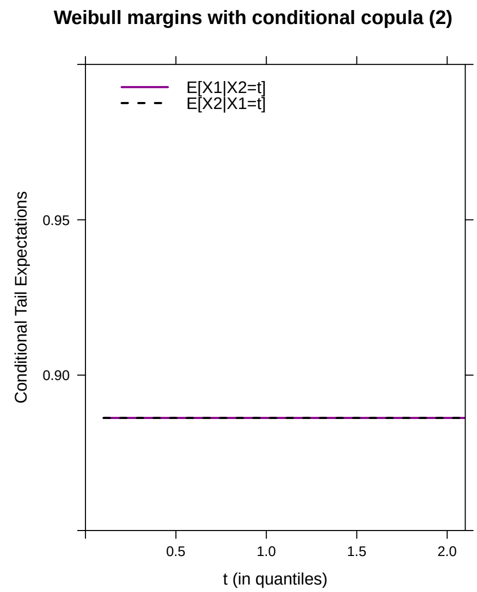
$$\eta_2(t) \sim \begin{cases} \frac{1-\alpha_2(1-\frac{\alpha_1\delta}{2})}{1-\alpha_1(1-\frac{\alpha_2\delta}{2})} & \text{if } \alpha_2 > \alpha_1; \\ 1 & \text{if } \alpha_2 = \alpha_1; \\ \frac{1-\frac{\alpha_2}{2}[1-\frac{\alpha_2\delta}{2(1-\frac{\alpha_1}{2})}]}{1-\frac{\alpha_1}{2}[1-\frac{\alpha_1\delta}{2(1-\frac{\alpha_2}{2})}]} & \text{if } \alpha_2 < \alpha_1; \end{cases} \tag{2.95}$$

The system is non-exchangeable when $\alpha_2 > \alpha_1$ and $\alpha_2 < \alpha_1$.

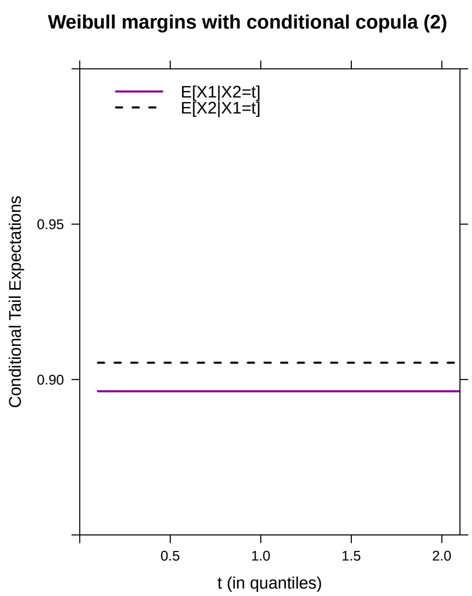
In Figures 2.8a, 2.8b and 2.8c we are plotting the simulation results obtained in (2.95). In Figure 2.8d we plot two conditional expectations using numerical integration method. Like in the case of $\mathrm{E}[X_1|X_2 > t]$ we fix the vertical axis in $[0.85, 1.00]$ to make the pictures comparable. Like in $\mathrm{E}[X_1|X_2 > t]$ case we fix the parameters $\alpha_1 = 0.03$, $\alpha_2 = 0.05$, $\gamma = 2$ and $\delta = 10$. If we carefully see 2.8d, we cannot find much *non-exchangeability* at 99th quantile. In 2.8a we see a lot of *non-exchangeability* throughout the simulation. As we are interested at $t \to \infty$, at 99th quantile we find a lot of *non-exchangeability*. In Figure 2.8b we do not find non-exchageability at all. If we compare Figure 2.8b with Figure 2.8d we can see at 0th quantile both the plots start almost at the same point. As t increases, both the



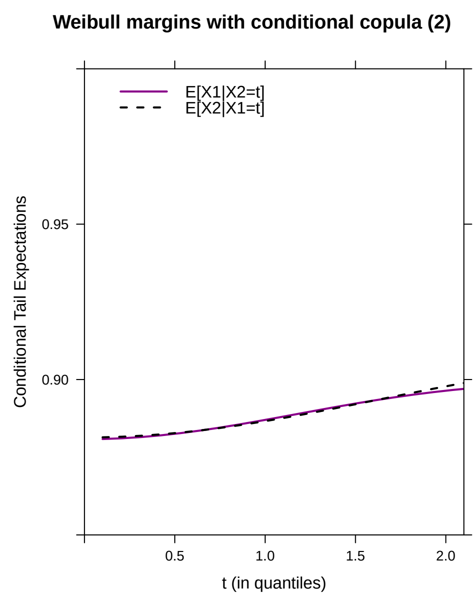
(a) $\alpha_1 = 0.03$, $\alpha_2 = 0.05$, $\gamma = 2$, $\delta = 10$ and $\alpha_2 > \alpha_1$



(b) $\alpha_1 = 0.03$, $\alpha_2 = 0.05$, $\gamma = 2$, $\delta = 10$ and $\alpha_2 < \alpha_1$



(c) $\alpha_1 = 0.03$, $\alpha_2 = 0.05$, $\gamma = 2$, $\delta = 10$ and $\alpha_2 = \alpha_1$



(d) $\alpha_1 = 0.03$, $\alpha_2 = 0.05$, $\gamma = 2$ and $\delta = 10$

Figure 2.8: Comparison between Watson's Lemma and the Actual Conditional Tail Expectations with Weibull Margins when α_1 and α_2 are different

expectations increase slowly in Figure 2.8d unlike to Figure 2.8b but, our measure of tail *non-exchageability* $\eta_2(t)$ is going to be the same in both the cases. Finally in Figure 2.8c we consider the case where $\alpha_2 = \alpha_1$. Although this case is too specific, we can get some *non-exchangeability* here which is smaller than in Figure 2.8b. If we compare Figure 2.8c with Figure 2.8d we see at 99th quantile both the conditional expectations take the value around 0.90. Hence, we can say that, in this case we have better tail *non-exchangeability*.

2.2.2.3 Exponential margin

In this section , we are discussing about Khoudraji (1996) *non-exchangeable* transformation of Clayton survival Copula with exponential margins. This one of the important distribution in statistics because itself is exponential family. In our case of exponential margins we do not find much tail *non-exchangeability* of Khoudraji type. In fact in this case we cannot use Laplace approximation as the regularity condition does not hold. On the other hand regularity condition for Watson's lemma does hold. Again this result is consistent with our study. If we carefully see the definition of $\mathrm{E}[X_1|X_2 = t]$, clearly it is based on Copula and apart from we also know that as survival Copula is nothing but the first order derivative of non-exchangeable survival Clayton Copula. Hence, by the definition of differentiation we know that, in this case we have less tail *non-exchangeability* than the case with $\mathrm{E}[X_1|X_2 > t]$ with exponential margins. In the simulations studies in the previous section we do not find enough *non-exchangeability* at the tail. Before doing the study we expected that in this section, we are going to have less *non-exchangeability*, which actually turns out to be true after the simulation study in this section. Finally we conclude that, only in Pareto margin we can use both the Laplace Approximation and Watson's Lemma as the regularity conditions are satisfied in these cases but, when we simulate our results, we only find out tail *non-exchangeability* in the case of Clayton Copula with exponential margin.

Let us consider the exponential margins with KB4 copula. Cumulative distribution function of exponential distribution is $F(x) = 1 - e^{-\lambda x}$, $\forall x \in [0, \infty)$. Thus, the survival function becomes, $\overline{F}(x) = 1 - F(x) = e^{-\lambda x},\ \lambda > 0$ and $\forall x \in [0, \infty)$. Following Hua and Joe (2014) we transform the survival function $\overline{F}(t) \to e^{-T},\ or,\ T = -\log \overline{F}(t) = -\log e^{-\lambda t} = \lambda t$. Now, $y = \log \overline{F}(x) = -\log e^{-\lambda x} = \lambda x$. As, $y = \lambda x$ then after totally differentiate this equation in both sides we get $dy = \lambda dx \implies dx = \lambda^{-1} dy$. Again, by the method suggested by Hua and Joe (2014) we are putting $y = sT$. Thus, $dy = sdT + Tds = Tds$, as we are assuming T is constant.

Now, $\mathrm{E}[X_1|X_2 = t] = \lambda^{-1} T \int_0^\infty e^{T \frac{1}{T} \log \widehat{C}_{1|2}(e^{-y}|e^{-T})} ds$, $\forall s \in [0, \infty)$ and $\lambda > 0$. Thus $\mathrm{E}[X_1|X_2 = t] = \lambda^{-1} T \int_0^\infty e^{Tg(s;T)} h(s) ds$. In this case, $g(s;T) = \frac{1}{T} \log \widehat{C}_{1|2}(e^{-y}|e^{-T})$ and $h(s) = 1$. Here also we stick to $0 < \alpha_1, \alpha_2 < 1$ and $\lambda > 0$. By using the result obtained from conditional survival copula distribution we have, $g(s;T) = \frac{1}{T}[\alpha_2 T - (1-\alpha_1) sT - \frac{1}{\delta} \log(e^{\alpha_1 \delta s T} + e^{\alpha_2 \delta T} - 1) + \log[(1-\alpha_2) + \frac{\alpha_2 e^{\alpha_2 \delta T}}{e^{\alpha_1 \delta s T} + e^{\alpha_2 \delta T} - 1}]]$, where $(\alpha_1, \alpha_2) \in [0,1]^2$, $\delta \geq 0$ and $s \in [0, \infty)$. From our theoretical results we know that the integrand can be multiplicatively separable with g and h functions, where first one is a monotonic function and the later is a positive function. After doing some easy calculation we are able to show that $g(0, T) = 0$ and $g(\infty, T) = -\infty$. These two conditions lead us to use either Laplace approximation or Watson's Lemma. Now, we are discussing about further regularity conditions to check which approximation is better fit in this case.

Proposition 41. *In Khoudraji (1996) non-exchangeable transformed Clayton Copula with Exponential margins Laplace Approximation is not applicable.*

Proof. As $g(0,T) = 0$ and $g(\infty,T) = -\infty$, we can use either Laplace approximation or Watson's lemma. We know that, before applying Laplace approximation we have to verify if $g'(0;T) > 0$ as $T \to \infty$.

We know,

$$\begin{aligned} g(s;T) &= \frac{1}{T}\Bigg[\alpha_2 T - (1-\alpha_1)sT - \frac{1}{\delta}\log(e^{\alpha_1\delta sT} + e^{\alpha_2\delta T} - 1) \\ &\qquad + \log\left[(1-\alpha_2) + \frac{\alpha_2 e^{\alpha_2\delta T}}{e^{\alpha_1\delta sT} + e^{\alpha_2\delta T} - 1}\right]\Bigg] \\ g'(s;T) &= -(1-\alpha_1) - \frac{\alpha_1 e^{\alpha_1\delta sT}}{e^{\alpha_1\delta sT} + e^{\alpha_2\delta T} - 1} \\ &\qquad - \frac{\alpha_1\alpha_2\delta e^{(\alpha_1 s+\alpha_2)\delta T}}{(e^{\alpha_1\delta sT} + e^{\alpha_2\delta T} - 1)^2\left[1 - \alpha_2 + \frac{\alpha_2 e^{\alpha_2\delta T}}{e^{\alpha_1\delta sT} + e^{\alpha_2\delta T} - 1}\right]} \\ g'(0;T) &= -\frac{\alpha_1}{e^{\alpha_2\delta T}} - (1-\alpha_1) - \frac{\alpha_1\alpha_2\delta e^{\alpha_2\delta T}}{e^{2\alpha_2\delta T}\left[1 - \alpha_2 + \frac{\alpha_2 e^{\alpha_2\delta T}}{e^{\alpha_2\delta T}}\right]} \\ &= \alpha_1 - 1 - \frac{\alpha_1}{e^{\alpha_2\delta T}} - \frac{\alpha_1\alpha_2\delta}{e^{\alpha_2\delta T}} \end{aligned}$$

$$\implies \lim_{T\to\infty} g'(0;T) = \alpha_1 - 1 < 0 \textit{ as } \alpha_1 \in [0,1], \textit{ by assumption.} \tag{2.96}$$

As $g'(0;T) < 0$, we cannot use Laplace approximation. This completes the proof. □

Proposition 42. *Suppose X_1 and X_2 are two dependent random variables. If $\alpha_2 > \alpha_1$, then for Khoudraji (1996) transformed survival Clayton Copula with Weibull margin with any $\lambda > 0$ and $\delta \geq 0$ we have;*

$$\mathrm{E}[X_1|X_2 = t] \sim \frac{1}{\lambda}\left[\frac{1}{1 - \alpha_1(1 - \frac{\alpha_2\delta}{2})}\right],$$

as $s^+ \to 0$, $t \to \infty$ and $(\alpha_1, \alpha_2) \in [0,1]^2$.

Proof. From our previous discussions we know that, $g'(s;T) = -(1-\alpha_1) - \frac{\alpha_1 e^{\alpha_1 \delta s T}}{e^{\alpha_1 \delta s T} + e^{\alpha_2 \delta T} - 1} - \frac{\alpha_1 \alpha_2 \delta e^{(\alpha_1 s + \alpha_2)\delta T}}{(e^{\alpha_1 \delta s T} + e^{\alpha_2 \delta T} - 1)^2 \left[1 - \alpha_2 + \frac{\alpha_2 e^{\alpha_2 \delta T}}{e^{\alpha_1 \delta s T} + e^{\alpha_2 \delta T} - 1}\right]}$.

In this case we have;

$$\begin{aligned}
g'(s;T) &= -(1-\alpha_1) - \frac{\alpha_1 e^{(\alpha_1 s - \alpha_2)\delta T}}{e^{(\alpha_1 s - \alpha_2)\delta T} + 1 - e^{-\alpha_2 \delta T}} \\
&\quad - \frac{\alpha_1 \alpha_2 \delta e^{(\alpha_1 s + \alpha_2)\delta T}}{(e^{\alpha_1 \delta s T} + e^{\alpha_2 \delta T} - 1)^2 \left[1 - \alpha_2 + \frac{\alpha_2}{e^{(\alpha_1 s - \alpha_2)\delta T} + 1 - e^{-\alpha_2 \delta T}}\right]} \\
\lim_{T\to\infty} g'(s;T) &= -(1-\alpha_1) - \lim_{T\to\infty} \frac{\alpha_1 \alpha_2 \delta e^{(\alpha_1 s + \alpha_2)\delta T}}{(e^{\alpha_1 \delta s T} + e^{\alpha_2 \delta T} - 1)^2},\ as\ \alpha_2 > \alpha_1, \\
&= -(1-\alpha_1) - \lim_{T\to\infty} \frac{\alpha_1 \alpha_2 \delta e^{\alpha_2 \delta T}}{2(e^{\alpha_1 \delta s T} + e^{\alpha_2 \delta T} - 1)},\ by\ L'Hospital\ Rule, \\
&= -(1-\alpha_1) - \lim_{T\to\infty} \frac{\alpha_1 \alpha_2 \delta}{2(e^{(\alpha_1 s - \alpha_2)\delta T} + 1 - e^{-\alpha_2 \delta T})} \\
&= -(1-\alpha_1) - \frac{\alpha_1 \alpha_2 \delta}{2},\ as\ T \to \infty, \\
&= -1 + \alpha_1 \left(1 - \frac{\alpha_2 \delta}{2}\right) < 0,\ for\ Watson's\ Lemma \qquad (2.97)
\end{aligned}$$

In (2.97) $-1 < 0$ all the time . Thus, in order to make $g'(s;T) < 0$ we need $(1 - \frac{\alpha_2 \delta}{2}) < 0$. In order to satisfy this condition we need $\frac{\alpha_2 \delta}{2} > 1$ or, $\alpha_2 \delta > 2$. Thus, we need a very large δ as $\alpha_2 \in [0,1]$. As $g(s,T)$ is real function on the semi-infinite interval $[0,\infty)$ and in $(0, 0+\epsilon]$ with $\epsilon > 0$ this function is continuously differentiable and $\sup_{0+\epsilon \le s \le \infty} g(s,T) \le g(0,T) - \psi$, with $\psi > 0$, we can use a version of Watson's Lemma defined in Theorem 36 [p. 48] in Breitung (1994). Now, for $g'(s,T)$ we have $g'(s,T) < 0$ as $T \to \infty$. We can also write $g'(s,T) = -as^{r-1} + o(s^{r-1})\ \forall\ r > 0$. Now if we assume $r = 1$ then $g'(s,T) = -a$. From our previous results we know that, $\lim_{s^+ \to\ 0,\ T\to\infty} g'(s,T) = -1 + \alpha_1(1 - \frac{\alpha_2 \delta}{2})$, which is a constant. Thus, $-a = -(1 - \alpha_1(1 - \frac{\alpha_2 \delta}{2}))$ or, $a = (1 - \alpha_1(1 - \frac{\alpha_2 \delta}{2})) > 0$.

Let us assume there is another real and continuous function $h(s) \in [0,\infty)$ such that, $h(s) = bs^{m-1} + o(s^{m-1})$ with $m > 0$. More specifically we assume $h(s) = 1$ in our case. Thus,

$bs^{m-1} + o(s^{m-1}) = 1 \implies b = 1$ when $m = 1$. Finally, as we are assuming $\int_0^\infty e^{g(s,T)} ds < \infty$, then by Watson's lemma we can write the approximated value of the integral $I(T) = \int_0^\infty e^{Tg(s,T)} ds$ with $T \geq 1$ are all finite and the asymptotic approximation is;

$$I(T) \sim \frac{1}{1-\alpha_1(1-\frac{\alpha_2\delta}{2})} T^{-1}, \ \textit{as } g(0;T) = 0, \ \textit{and as } T \to \infty. \tag{2.98}$$

Thus using (2.97) in the original integration we get,

$$\mathrm{E}[X_1|X_2 = t] \sim \frac{1}{\lambda}\left[\frac{1}{1-\alpha_1(1-\frac{\alpha_2\delta}{2})}\right], \ \textit{as } s^+ \to \ 0 \textit{ and } t \to \infty. \tag{2.99}$$

This completes the proof. □

Corollary 43. *Suppose X_1 and X_2 are two dependent random variables. If $\alpha_2 > \alpha_1$, then for Khoudraji (1996) transformed survival Clayton Copula with Weibull margin with any $\lambda > 0$ and $\delta \geq 0$ we have;*

$$\mathrm{E}[X_2|X_1 = t] \sim \frac{1}{\lambda}\left[\frac{1}{1-\alpha_2(1-\frac{\alpha_1\delta}{2})}\right],$$

as $s^+ \to \ 0$, $t \to \infty$ and $(\alpha_1, \alpha_2) \ \in [0,1]^2$.

Corollary 44. *Suppose X_1 and X_2 are two dependent random variables.If $\alpha_2 < \alpha_1$, then for Khoudraji (1996) transformed survival Clayton Copula with Weibull margin with any $\lambda > 0$ and $\delta \geq 0$ we have;*

$$\mathrm{E}[X_1|X_2 = t] \sim \frac{1}{\lambda},$$

as $s^+ \to \ 0$, $t \to \infty$ and $(\alpha_1, \alpha_2) \ \in [0,1]^2$.

Proof. Please see the Appendix. □

Corollary 45. *Suppose X_1 and X_2 are two dependent random variables. If $\alpha_2 = \alpha_1$, then for Khoudraji (1996) transformed survival Clayton Copula with Weibull margin with any $\lambda > 0$ and $\delta \geq 0$ we have;*

$$\mathrm{E}[X_1|X_2 = t] \sim \frac{1}{\lambda}\left[\frac{1}{1 - \frac{\alpha_1}{2}\left[1 - \frac{\alpha_1\delta}{2(1-\frac{\alpha_2}{2})}\right]}\right],$$

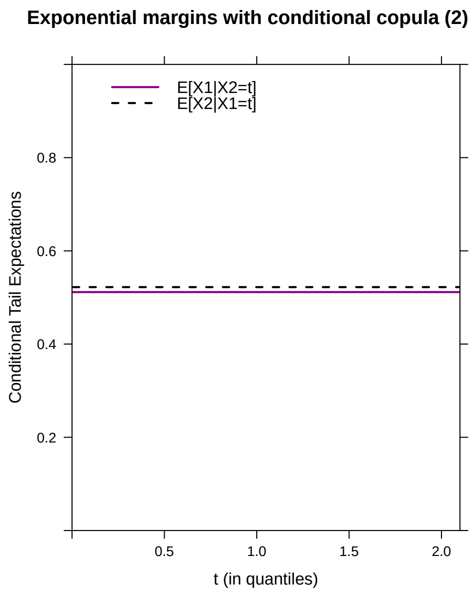
as $s^+ \to 0$, $t \to \infty$ and $(\alpha_1, \alpha_2) \in [0,1]^2$.

Remark 19. From above discussion we get our measure of tail *non-exchangeability* under Watson's lemma ;

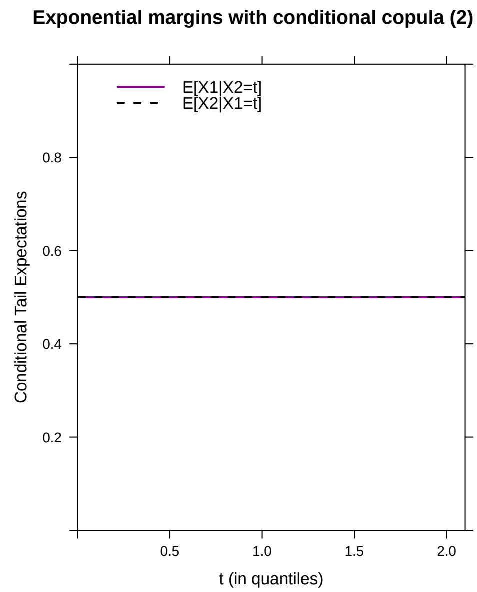
$$\eta_2(t) \sim \begin{cases} \frac{1-\alpha_2(1-\frac{\alpha_1\delta}{2})}{1-\alpha_1(1-\frac{\alpha_2\delta}{2})} & \text{if } \alpha_2 > \alpha_1; \\ 1 & \text{if } \alpha_2 = \alpha_1; \\ \frac{1-\frac{\alpha_2}{2}[1-\frac{\alpha_2\delta}{2(1-\frac{\alpha_1}{2})}]}{1-\frac{\alpha_1}{2}[1-\frac{\alpha_1\delta}{2(1-\frac{\alpha_2}{2})}]} & \text{if } \alpha_2 < \alpha_1; \end{cases} \tag{2.100}$$

Thus, again the system is non-exchangeable when $\alpha_2 > \alpha_1$ and $\alpha_2 = \alpha_1$.

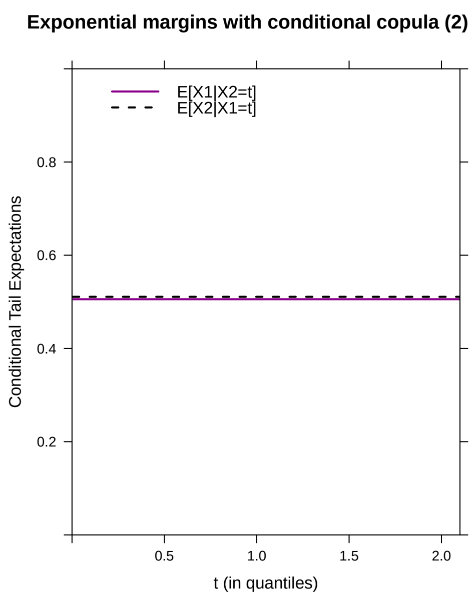
In Figures 2.9a, 2.9b and 2.9c we are plotting the simulation results obtained in (2.95). In Figure 2.9d we plot two conditional expectations using numerical integration method. Like in the case of $\mathrm{E}[X_1|X_2 > t]$ we fix the vertical axis in $[0,1]$ to make the pictures comparable. Like in $\mathrm{E}[X_1|X_2 > t]$ case, we fix the parameters $\alpha_1 = 0.03$, $\alpha_2 = 0.05$, $\lambda = 2$ and $\delta = 10$. In Figure 2.9a we have the highest *non-exchageability.* If we compare this with Figure 2.9d we can have some good approximations. In fact any of the Figures 2.9a, 2.9b and 2.9c should be very good approximation of Figure 2.9d. The main reason behind this is two plots of $\mathrm{E}[X_1|X_2 = t]$ and $\mathrm{E}[X_2|X_1 = t]$ in Figure 2.9d are approximately the same.



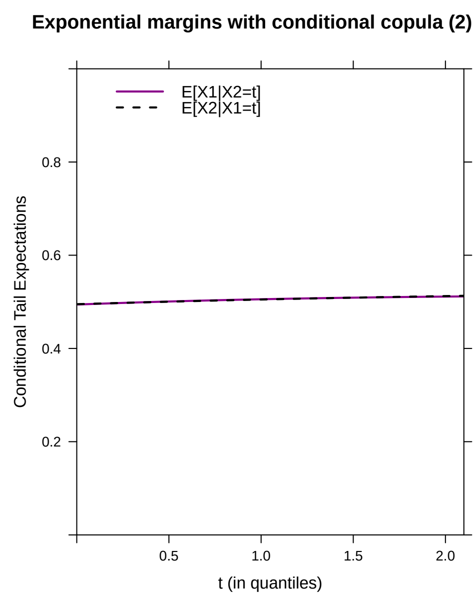
(a) $\alpha_1 = 0.03$, $\alpha_2 = 0.05$, $\lambda = 2$, $\delta = 10$ and $\alpha_2 > \alpha_1$



(b) $\alpha_1 = 0.03$, $\alpha_2 = 0.05$, $\lambda = 2$, $\delta = 10$ and $\alpha_2 < \alpha_1$



(c) $\alpha_1 = 0.03$, $\alpha_2 = 0.05$, $\lambda = 2$, $\delta = 10$ and $\alpha_2 = \alpha_1$



(d) $\alpha_1 = 0.03$, $\alpha_2 = 0.05$, $\lambda = 2$ and $\delta = 10$

Figure 2.9: Comparison between Watson's Lemma and the Actual Conditional Tail Expectations with Exponential Margins when α_1 and α_2 are different

CHAPTER 3

EXTREME VALUE COPULA

Extreme-value Copula is one of the most popular in current days. According to Gudendorf and Segers (2010) in the field of insurance, finance or environmental science extreme events need to be taken care off. Sometimes losing money in the stock market is more harsh effect than earning money from the system. Now let us go further extreme case. Suppose one person completely lose the entire money he invested in the stock market. This is an extreme event. Another example might be the probability of occurrence of EF-5 level of tornado in certain southern parts of the United States. This is an example of occurrence of an extreme natural event. Extreme-value theory deals with these kind of extreme events. Extreme value copula takes care about the *probabilistic* view of the occurrence of these types of extreme events.

Definition 4. *A copula* $\widehat{C}^*$ *is an extreme-value Copula if and only if the Copula satisfies the following.*

$$\widehat{C}^*([u_1]^{\frac{1}{k}}, [u_2]^{\frac{1}{k}}, ...[u_d]^{\frac{1}{k}})^k \to \widehat{C}^*(u_1, u_2, ..., u_d),$$

for all integer $k \geq 1$ *and* $(u_1, u_2, ..., u_d) \in [0, 1]^d$.

Following Genest and Nešlehová (2013) we can say, a survival copula $\widehat{C}^*$ is said to be an extreme value copula iff there exists a function $A : [0, 1] \to [\frac{1}{2}, 1]$ such that for all $[\overline{F}(x_1), \overline{F}(x_2)] \in [0, 1]^2$, we have,

$$\widehat{C}^*(\overline{F}(x_1), \overline{F}(x_2)) = exp\left[\log(\overline{F}(x_1)\overline{F}(x_2))A^*\left\{\frac{\log(\overline{F}(x_2))}{\log(\overline{F}(x_1)\overline{F}(x_2))}\right\}\right]$$

Here the function A is called Pickands (1981) dependence function which is convex and for all $\nu \in [0,1]$, we have $\nu \vee (1-\nu) \leq A^*(\nu) \leq 1$. The bounds $A^*(\nu) \equiv 1$ and $A^*(\nu) = \nu \vee (1-\nu)$ are independence survival copula and Fréchet-Hoeffding upper bound respectively. Durante and Mesiar (2010) suggested that, under certain cases $\widehat{C}^*$ is non-exchangeable. They calculated at the extreme case the measure of *non-exchangeability* might be $\mu_\infty(\widehat{C}^*) \leq 3\frac{4^4}{5^5}$ where $\mu_\infty(\widehat{C}^*)$ is defined as $\mu_\infty(\widehat{C}^*) = 3\left(\max_{(\overline{F}(x_1),\overline{F}(x_2))\in[0,1]^2}\left|\widehat{C}^*(\overline{F}(x_1),\overline{F}(x_2)) - \widehat{C}^*(\overline{F}(x_2),\overline{F}(x_1))\right|\right)$ [see Durante and Mesiar (2010)]. So we can use this survival copula structure as *non-exchangeable* structure first and try to calculate $\mathrm{E}[X_1|X_2 > t]$ first for this case, and secondly, we assume a general case of extreme value copula and use Khoudraji device to make it non-exchangeable and calculate the conditional expectations of this case.

3.0.3 Construction of non-exchangeable Copula

In this section we are concentrating on the construction of non-exchangeable extreme value survival Copulas. From above we know that, Pickands (1981) dependence function $A^*(\nu)$ is convex and for all $\nu \in [0,1]$, we have $\nu \vee (1-\nu) \leq A^*(\nu) \leq 1$. We know extreme value Copula can be written as,

$$\widehat{C}^*(\overline{F}(x_1),\overline{F}(x_2)) = exp\left[\log(\overline{F}(x_1)\overline{F}(x_2))A^*\left\{\frac{\log(\overline{F}(x_2))}{\log(\overline{F}(x_1)\overline{F}(x_2))}\right\}\right]$$

or

$$\widehat{C}^*(\overline{F}(x_1),\overline{F}(x_2)) = \left(\overline{F}(x_1)\overline{F}(x_2)\right)^{A^*\left\{\frac{\log(\overline{F}(x_2))}{\log(\overline{F}(x_1)\ \overline{F}(x_2))}\right\}},$$

where $\left(\overline{F}(x_1),\overline{F}(x_2)\right) \in (0,1]^2 \setminus \{(1,1)\}$

Following Durante and Mesiar (2010) we know one extreme value survival Copula in non-exchangeable if $\left|\widehat{C}^*\left(\overline{F}(x_1),\overline{F}(x_2)\right) - \widehat{C}^*\left(\overline{F}(x_2),\overline{F}(x_1)\right)\right| \neq 0$. They define maximum of this absolute distance as and has the form

$$\begin{aligned}\delta_c = & \max_{(\overline{F}(x_1),\overline{F}(x_2))\in[0,1]^2}\left|\widehat{C}^*(\overline{F}(x_1),\overline{F}(x_2)) - \widehat{C}^*(\overline{F}(x_2),\overline{F}(x_1))\right| \\ = & \max_{(\overline{F}(x_1),\overline{F}(x_2))\in(0,1)^2}\left|\left(\overline{F}(x_1)\overline{F}(x_2)\right)^{A^*\left\{\frac{\log(\overline{F}(x_2))}{\log(\overline{F}(x_1)\ \overline{F}(x_2))}\right\}} - \left(\overline{F}(x_1)\overline{F}(x_2)\right)^{A^*\left\{1-\frac{\log(\overline{F}(x_2))}{\log(\overline{F}(x_1)\ \overline{F}(x_2))}\right\}}\right|\end{aligned} \tag{3.1}$$

They also say that, if δ_c is reached at a point $(\overline{F}(x_1),\overline{F}(x_2)\ \in(0,1)^2$ and if we set

$$\nu = \frac{\log\overline{F}(x_2)}{\log\left(\overline{F}(x_1)\overline{F}(x_2)\right)}$$

then with out loss of generality they suggest that, with $0<\overline{F}(x_1)<\overline{F}(x_2)<1$ they get $\nu\in(0,\frac{1}{2})$ and $A^*\{\nu\}\leq A^*\{1-\nu\}$. After using the properties of Pickands (1981) dependence function Durante and Mesiar (2010) come up with the following non-exchangeable Copula form

$$\widehat{C}(\overline{F}(x_1),\ \overline{F}(x_2)) = \begin{cases}\overline{F}(x_2), & \overline{F}(x_1)^2\geq\overline{F}(x_2), \\ \overline{F}(x_1)\ \sqrt{\overline{F}(x_2)}, & \text{otherwise},\end{cases} \tag{3.2}$$

They also find that in this case, δ_c is $4^4/5^5$.

Even it not very hard to create a *non-exchangeable* extreme value survival Copula function. We all know that Pickands (1981) dependence function takes an important role in this Copula family because almost all information of this is inside *Pickands* dependence function. We also know that *extreme value Copula* can be written as, $\widehat{C}^*(\overline{F}(x_1), \overline{F}(x_2)) = \left(\overline{F}(x_1)\overline{F}(x_2)\right)^{A^*\left\{\frac{\log(\overline{F}(x_2))}{\log(\overline{F}(x_1)\ \overline{F}(x_2))}\right\}}$, where $(\overline{F}(x_1), \overline{F}(x_2)) \in (0,1]^2 \setminus \{(1,1)\}$. In this case if

$$A^*\left\{\frac{\log(\overline{F}(x_2))}{\log(\overline{F}(x_1)\ \overline{F}(x_2))}\right\} \neq A^*\left\{1 - \frac{\log(\overline{F}(x_2))}{\log(\overline{F}(x_1)\ \overline{F}(x_2))}\right\},$$

then we can say the given extreme value survival Copula is *non-exchangeable.*

Let us construct a Pickands (1981) dependence function

$$A^*\{\nu\} = \frac{1 + (1-\nu)^2}{2},$$

where

$$\nu = \frac{\log \overline{F}(x_2)}{\log\left(\overline{F}(x_1)\overline{F}(x_2)\right)},$$

where $\nu \in [0,1] \setminus \{\frac{1}{2}\}$ and $(\overline{F}(x_1),\ \overline{F}(x_2)) \in (0,1]^2 \setminus \{(1,1)\}$. It is not very hard to find out that this function is a convex to the origin function. In this case, $A^*\{1-\nu\} = \{1 + (1 - [1-\nu])^2\}/2$. It is important to note that, when we define the domain of ν we exclude $1/2$ from $[0,1]$ because, at $\nu = 1/2$ the extreme value survival Copula is exchangeable [i.e. $A^*\{\nu\} = A^*\{1-\nu\}$].

Proposition 46. *Survival extreme value Copula with Pickands dependence function* $A^*\{\nu\} = \frac{1+(1-\nu)^2}{2}$ *has the non-exchangeability at the highest level when*

$$\overline{F}(x_1)\overline{F}(x_2) = \left[\frac{1 + (1 - [1-\nu])^2}{1 + (1-\nu)^2}\right]^{\frac{2}{1-2\nu}},$$

where

$$\nu = \frac{\log \overline{F}(x_2)}{\log\left(\overline{F}(x_1)\overline{F}(x_2)\right)},$$

where $\nu \in [0,1] \setminus \{\frac{1}{2}\}$ *and* $(\overline{F}(x_1),\ \overline{F}(x_2)) \in (0,1]^2 \setminus \{(1,1)\}$.

Proof. Let us define $\overline{F}(x_1)\overline{F}(x_2) = \tau$. Following Durante and Mesiar (2010) we define the indicator of non-exchangeability as,

$$\begin{aligned}
\delta_c &= \max_{(\overline{F}(x_1),\overline{F}(x_2))\in[0,1]^2} \left|\widehat{C}^*(\overline{F}(x_1),\overline{F}(x_2)) - \widehat{C}^*(\overline{F}(x_2),\overline{F}(x_1))\right| \\
&= \max_{(\overline{F}(x_1),\overline{F}(x_2))\in(0,1)^2} \left|\tau^{A^*\{\nu\}} - \tau^{A^*\{1-\nu\}}\right| \qquad (3.3)
\end{aligned}$$

Now, if we take the maximum distance between $\tau^{A^*\{\nu\}}$ and $\tau^{A^*\{1-\nu\}}$ then we can get the survival Copula with maximum non-exchangeability. Therefore, the maximum distance between $\tau^{A^*\{\nu\}}$ and $\tau^{A^*\{1-\nu\}}$ is nothing but the first order condition of $|\tau^{A^*\{\nu\}} - \tau^{A^*\{1-\nu\}}|$ with respect to ν.

Hence,

$$\begin{aligned}
&\left[1+(1-\nu)^2\right]\tau^{\frac{1+(1-\nu)^2}{2}-1} = \left[1+(1-[1-\nu])^2\right]\tau^{\frac{1+(1-[1-\nu])^2}{2}-1} \\
&\tau^{\frac{1+(1-\nu)^2}{2}-\frac{1+(1-[1-\nu])^2}{2}} = \frac{1+(1-[1-\nu])^2}{1+(1-\nu)^2} \\
&\tau^{\frac{1}{2}[(1-\nu)^2-[1-(1-\nu)^2]]=\frac{1+(1-[1-\nu])^2}{1+(1-\nu)^2}} \\
&\tau^{\frac{1}{2}[2(1-\nu)-1]} = \frac{1+(1-[1-\nu])^2}{1+(1-\nu)^2} \\
&\tau^{\frac{1}{2}[1-2\nu]} = \frac{1+(1-[1-\nu])^2}{1+(1-\nu)^2} \\
&\tau = \left[\frac{1+(1-[1-\nu])^2}{1+(1-\nu)^2}\right]^{\frac{2}{1-2\nu}}, \qquad (3.4)
\end{aligned}$$

where

$$\nu = \frac{\log \overline{F}(x_2)}{\log\left(\overline{F}(x_1)\overline{F}(x_2)\right)},$$

where $\nu \in [0,1]\setminus\{\frac{1}{2}\}$ and $(\overline{F}(x_1),\overline{F}(x_2)) \in (0,1]^2 \setminus \{(1,1)\}$. This completes the proof. □

After doing some easy calculation we get that,

$$\delta_c = \max_{(\overline{F}(x_1),\overline{F}(x_2))\in[0,1]^2\setminus(1,1)} \left| \widehat{C}^*(\overline{F}(x_1), \overline{F}(x_2)) - \widehat{C}^*(\overline{F}(x_2), \overline{F}(x_1)) \right| = \frac{1}{4} > 0.$$

Thus, the Copula we define is non-exchangeable in $\nu \in [0,1] \setminus \{\frac{1}{2}\}$ and $(\overline{F}(x_1), \overline{F}(x_2)) \in (0,1]^2 \setminus \{(1,1)\}$. In the following section we are going to use these kind of *non-exchangeable extreme value survival Copulas* where $\overline{F}(x_1) = \overline{F}(x)$ and $\overline{F}(x_2) = \overline{F}(t)$. Furthermore, If we define $A^*_\alpha\{\nu\} = A^*\{1-\nu\}$ in order to explain tail *non-exchangeability* of conditional expectations. As we have the exact form of the *Pickands* dependence function, we can get the exact value of *non-exchangeability* using these expressions.

3.1 Conditional Expectations when Extreme value Copula is itself non-exchangeable

By using the method of Hua and Joe (2014) of copula we are trying to see the tail *non-exchangeability* of $\widehat{C}^*$ for extreme value Copula case. Firstly, like before, we assume X_1 and X_2 have same marginal distribution function where *cumulative distribution function* (cdf) F is properly defined and *uniformly* continuous on $[0,1]$. Secondly, we also assume that, F is *absolutely* continuous with the density function $f(.,.) = F^{'}(.,.) = \frac{\partial F(.,.)}{\partial(.,.)}$. Thirdly, X_1 has finite mean (i.e $\int_0^\infty \overline{F}(x)dx < \infty$). Let us transform $\overline{F}(t) \to e^{-T}$ (or, $T = -\log \overline{F}(t)$) and $\overline{F}(x) \to e^{-y}$ (or, $y = -\log \overline{F}(x)$).

Proposition 47. *In the case of extreme value Copula with non-exchangeable Pickands dependence function (like in Durante and Mesiar (2010)) the integrand of conditional tail expectation is multiplicatively separable of two non-negative continuous integrable functions.*

Proof. Now, from the definition of conditional expectations we know;

$$\mathrm{E}[X_1|X_2 > t] = \int_0^\infty \frac{\widehat{C}^*(\overline{F}(x), \overline{F}(t))}{\overline{F}(t)}\, dx,\ \forall\ t \tag{3.5}$$

As $y = -\log \overline{F}(x) \implies dy = -\frac{\frac{\partial \overline{F}(x)}{\partial x}}{\overline{F}(x)} dx \implies \overline{F}(x)dy = -\frac{\partial \overline{F}(x)}{\partial x} dx \implies \overline{F}(x)dy = f(F^{-1}(1 - \overline{F}(x)))dx$, after changing of variables we get, $e^{-y}dy = f(F^{-1}(1 - e^{-y}))dx \implies e^{-y}[f(F^{-1}(1 - e^{-y}))]^{-1}dy = dx$. After putting this condition in (3.5) we get,

$$\mathrm{E}[X_1|X_2 > t] = \int_0^\infty e^T \widehat{C}^*(e^{-\alpha_1 y}, e^{-\alpha_2 T}) e^{-y}[f(F^{-1}(1 - e^{-y}))]^{-1}\, dy \tag{3.6}$$

$y = sT$. If we differentiate this equation with respect to s we get $\frac{dy}{ds} = T \implies dy = Tds$. After putting this condition in (3.6) we get;

$$\mathrm{E}[X_1|X_2 > t] = T \int_0^\infty e^{T\left[\frac{1}{T}\left\{\log \frac{\widehat{C}^*(e^{-\alpha_1 sT}, e^{-\alpha_2 T})}{f(F^{-1}(1-e^{-sT}))}\right\}\right]}\, ds \tag{3.7}$$

$$= T \int_0^\infty e^{Tg(s,T;\alpha_2,\alpha_1)} h(s) ds,\ \textit{here}\ h(s) = 1,\ \forall\ s \in [0, \infty) \tag{3.8}$$

In a similar way suggested by Hua and Joe (2014) we can also get;

$$\mathrm{E}[X_1|X_2 = t] = T \int_0^\infty e^{T[\frac{1}{T}\{\log \frac{\widehat{C}^*_{1|2}(e^{-\alpha_1 sT}, e^{-\alpha_2 T})}{f(F^{-1}(1-e^{-sT}))}\}]}\, ds \tag{3.9}$$

$$= T \int_0^\infty e^{Tg(s,T;\alpha_2,\alpha_1)} h(s) ds,\ \textit{here}\ h(s) = 1,\ \forall\ s \in [0, \infty) \tag{3.10}$$

In the above case $\widehat{C}^*_{1|2}(\overline{F}(x)|\overline{F}(t))$ is factor copula which is defined as $\widehat{C}^*_{1|2}(\overline{F}(x)|\overline{F}(t)) = \frac{\partial \widehat{C}^*(\overline{F}(x),\overline{F}(t))}{\partial \overline{F}(x)}$. From the first section we know *non-exchangeable extreme value* copula has the form;

$$\begin{aligned}
\widehat{C}^*(\overline{F}(x),\overline{F}(t)) &= exp\left[\log(\overline{F}(x)\overline{F}(t))A^*\left\{\frac{\log(\overline{F}(t))}{\log(\overline{F}(x)\overline{F}(t))}\right\}\right] \\
\widehat{C}^*(e^{-sT},e^{-T}) &= exp\left[\log(e^{-sT}e^{-T})A^*\left\{\frac{\log(e^{-T})}{\log(e^{-sT}e^{-T})}\right\}\right] \\
&= exp\left[\log(e^{-T(1+s)})A^*\left\{\frac{\log(e^{-T})}{\log(e^{-T(1+s)})}\right\}\right] \\
&= exp\left[-T(1+s)A^*\left\{\frac{1}{1+s}\right\}\right] \qquad (3.11)
\end{aligned}$$

Clearly, from(3.11) we can say that, in our case $\nu = \frac{1}{1+s} \in [0,1]$ and $A^*\{.\}$ is *Pickands* dependence function. □

3.1.1 Case I : $\mathrm{E}[X_1|X_2 > t]$

Proposition 48. *Suppose X_1 and X_2 are two dependent random variables. Under Pareto margins and $\mathrm{E}[X_1|X_2 > t]$ we can use Laplace Approximation if $\beta^{-1} + A^{*'}\{1\} > 1$, where $\beta > 1$.*

Proof. The g function under Pareto margins can be defined as $g(s,T) = 1+\frac{s}{\beta}+\frac{1}{T}\log\widehat{C}^*(e^{-sT}, e^{-T})$. In this case $T = \beta\log(1+t)$. After using the result from (3.11) we get the g function as;

$$\begin{aligned}
g(s,T) &= 1+\frac{s}{\beta}+\frac{1}{T}\log\widehat{C}^*(e^{-sT}, e^{-T}) \\
&= 1+\frac{s}{\beta}+\frac{1}{T}\log e^{-T(1+s)A^*\left\{\frac{1}{1+s}\right\}} \\
&= 1+\frac{s}{\beta}-(1+s)A^*\left\{\frac{1}{1+s}\right\} \qquad (3.12)
\end{aligned}$$

Following Hua and Joe (2014) we know before applying Laplace approximation we have to check if $g(0,T) = 0$ and $g(\infty,T) = -\infty$. From (3.12) we have, $g(0,T) = 1 - A^*\{1\} = 0$, as $A^*\{\}$ is Pickands dependence function.

Again from (3.12) we know that;

$$\begin{aligned}
g(s,T) &= 1+\frac{s}{\beta}-(1+s)A^*\left\{\frac{1}{1+s}\right\} \\
&= 1+\frac{1}{\beta}-A^*\left\{\frac{1}{1+s}\right\}-sA^*\left\{\frac{1}{1+s}\right\} \\
&= 1-A^*\left\{\frac{1}{1+s}\right\}+s\left[\frac{1}{\beta}-A^*\left\{\frac{1}{1+s}\right\}\right] \\
g(\infty,T) &= \lim_{s\to\infty} s\left[\frac{1}{\beta}-1\right] = -\infty,\ \textit{as}\ \beta > 1\ \textit{and}\ A^*\{0\} = 1 \qquad (3.13)
\end{aligned}$$

In order to use Laplace approximation we need another two conditions $g'(0,T) > 0$ and $-g''(s_0(T),T) > 0$. If we do the first order differentiation of (3.11), we clearly get,

$$\begin{aligned}
g'(s,T) &= \frac{1}{\beta}-\left[-(1+s)A^{*'}\left\{\frac{1}{1+s}\right\}\frac{1}{(1+s)^2}+A^*\left\{\frac{1}{1+s}\right\}\right] \\
&= \frac{1}{\beta}-A^*\left\{\frac{1}{1+s}\right\}+A^{*'}\left\{\frac{1}{1+s}\right\}\frac{1}{1+s} \qquad (3.14)
\end{aligned}$$

Thus;

$$g'(0,T) = \frac{1}{\beta} - 1 + A^{*'}\{1\} > 0$$
$$or, \ \frac{1}{\beta} + A^{*'}\{1\} > 1 \tag{3.15}$$

If we check for the second order condition of (3.11), we can see that,

$$\begin{aligned} g''(s,T) &= A^{*'}\left\{\frac{1}{1+s}\right\}\frac{1}{(1+s)^2} + \left[-A^{*'}\left\{\frac{1}{1+s}\right\}\frac{1}{(1+s)^2} - \frac{1}{(1+s)^3}A^{*''}\left\{\frac{1}{1+s}\right\}\right] \\ &= A^{*'}\left\{\frac{1}{1+s}\right\}\frac{1}{(1+s)^2} - \left[A^{*'}\left\{\frac{1}{1+s}\right\}\frac{1}{(1+s)^2} + \frac{1}{(1+s)^3}A^{*''}\left\{\frac{1}{1+s}\right\}\right] \\ &= -\frac{1}{(1+s)^3}A^{*''}\left\{\frac{1}{1+s}\right\} < 0, \ as \ A^{*''}\{\} > 0. \end{aligned} \tag{3.16}$$

From (3.16) it is clear that $-g''(s,T) > 0$ all the time when $s \in [0,\infty)$. Thus,$-g''(s_0(T),T) > 0$ as $T \to \infty$. So, from the above discussion we see only one condition matters here before using Laplance approximation is condition (3.15) which is $\beta^{-1} + A^{*'}\{1\} > 1$. $\square$

If we know the exact form of $A^*\{\}$ we can actually get $\gamma = s_0(T) = \lim_{T\to\infty}\ s_0(T) = \lim_{T\to\infty}\ argmax_s\ g(s,T)$ [as $s_0(T)$ is independent of T]. After doing Laplace Approximation we get,

$$\begin{aligned}
\mathrm{E}[X_1|X_2>t] &\sim T\int_0^\infty exp[Tg(s,T)]ds,\ as\ h(s)=1,\\
&\sim Te^{Tg(\gamma,T)}\sqrt{\frac{2\pi}{-Tg''(\gamma,T)}}\\
&\sim e^{T\left[1+\frac{\gamma}{\beta}-(1+\gamma)A^*\left\{\frac{1}{1+\gamma}\right\}\right]}\sqrt{\frac{2\pi T}{-g''(\gamma,T)}}\\
&\sim e^{\beta\log(1+t)\left[1+\frac{\gamma}{\beta}-(1+\gamma)A^*\left\{\frac{1}{1+\gamma}\right\}\right]}\sqrt{\frac{2\pi\beta\log(1+t)}{\frac{1}{(1+\gamma)^3}A^{*''}\left\{\frac{1}{1+\gamma}\right\}}}\\
&\sim (1+t)^{\beta\left[1+\frac{\gamma}{\beta}-(1+\gamma)A^*\left\{\frac{1}{1+\gamma}\right\}\right]}\sqrt{\frac{(1+t)^{2\pi\beta(1+\gamma)^3}}{A^{*''}\left\{\frac{1}{1+\gamma}\right\}}},\ as\ t\to\infty \qquad (3.17)
\end{aligned}$$

From (3.17) we conclude that, in $\mathrm{E}[X_1|X_2>t]$ the dominating part at the tail (i.e. $t\to\infty$) is $\beta\left[1+\frac{\gamma}{\beta}-(1+\gamma)A^*\left\{\frac{1}{1+\gamma}\right\}\right]$ which determines tail *non-exchangeability* (assuming the convexity of *Pickands* dependence function). In this case our measure of tail order non-exchangeability would be;

$$\begin{aligned}
\eta_1(t) &= \frac{\mathrm{E}[X_1|X_2>t]}{\mathrm{E}[X_2|X_1>t]}\\
&\sim \frac{\left[(1+t)^{\beta\left[1+\frac{\gamma}{\beta}-(1+\gamma)A^*\left\{\frac{1}{1+\gamma}\right\}\right]}\sqrt{\frac{(1+t)^{2\pi\beta(1+\gamma)^3}}{A^{*''}\left\{\frac{1}{1+\gamma}\right\}}}\right]}{\left[(1+t)^{\beta\left[1+\frac{\gamma}{\beta}-(1+\gamma)A_\alpha^*\left\{\frac{1}{1+\gamma}\right\}\right]}\sqrt{\frac{(1+t)^{2\pi\beta(1+\gamma)^3}}{A_\alpha^{*''}\left\{\frac{1}{1+\gamma}\right\}}}\right]} \qquad (3.18)
\end{aligned}$$

as $t\to\infty$. This measure is not zero because, we assume this type of extreme value copula is non-exchangeable. Thus, $A^*\left\{\frac{1}{1+\gamma}\right\}\neq A_\alpha^*\left\{\frac{1}{1+\gamma}\right\}$ and $A^{*''}\left\{\frac{1}{1+\gamma}\right\}\neq A_\alpha^{*''}\left\{\frac{1}{1+\gamma}\right\}$.

Proposition 49. *Suppose X_1 and X_2 are two dependent random variables and $\int_0^\infty e^{g(s,T)}\, ds < \infty$. If survival non-exchangeable extreme value copula has Pareto margins, then for all $\beta > 1$ and $s \in [0, 0+\epsilon)$, $\frac{1}{\beta} + \frac{1}{1+s} A^{*'}\left\{\frac{1}{1+s}\right\} < A^*\left\{\frac{1}{1+s}\right\}$ we have* $\mathrm{E}[X_1|X_2 > t] \sim \frac{1}{-\frac{1}{\beta} - A^{*'}\{1\} + A^*\{1\}}$, *where* $A^*((1+s)^{-1}) = 2^{-1}\left(1 + (1 + (1+s)^{-1})^2\right)$.

Proof. From the above discussion we know that, $g(0,T) = 0$ and $g(\infty, T) = -\infty$. Apart from that we know the g function is $g(s,T) = 1 + \frac{s}{\beta} - (1+s)A^*\left\{\frac{1}{1+s}\right\}$. Thus, $g'(s,T) = \frac{1}{\beta} + \frac{1}{1+s} A^{*'}\left\{\frac{1}{1+s}\right\} - A^*\left\{\frac{1}{1+s}\right\}$. Now if $\frac{1}{\beta} + \frac{1}{1+s} A^{*'}\left\{\frac{1}{1+s}\right\} < A^*\left\{\frac{1}{1+s}\right\}$, then $g'(s,T) < 0$. In other words, $g(s,T)$ is decreasing in s. Then we can use Watson's lemma.

As $g(s;T)$ is a real valued function on the semi-infinite interval $[0, \infty)$ and in $(0, 0+\epsilon]$ with $\epsilon > 0$ this function is continuously differentiable and

$$\sup_{0+\epsilon \le s \le \infty} g(s,T) \le g(0,T) - \psi, \tag{3.19}$$

with $\psi > 0$. Here we are using Theorem 36 [p.48] of Breiting (1994). This theorem is an extension of Watson's lemma and most importantly, this theorem works for semi-infinite intervals. Now we have $g'(s,T) < 0$ as $\frac{1}{\beta} < 1$ and $s \in [0, 0+\epsilon)$.We can also write

$$g'(s,T) = -as^{r-1} + o(s^{r-1}) \ \forall\ r > 0.$$

Now if we assume $r = 1$ then $g'(s,T) = -a$. From our previous results we know that, $\lim_{s^+ \to\ 0,\ T \to \infty} g'(s,T) = \frac{1}{\beta} + \frac{1}{1+s} A^{*'}\left\{\frac{1}{1+s}\right\} - A^*\left\{\frac{1}{1+s}\right\}$, which is a constant. Thus, $-a = -[-\frac{1}{\beta} - A^{*'}\{1\} + A^*\{1\}]$ or, $a = -\frac{1}{\beta} - A^{*'}\{1\} + A^*\{1\}$. Let us assume there is another real and continuous function $h(s,T) \in [0, \infty)$ such that,

$$h(s,T) = bs^{m-1} + o(s^{m-1})$$

with $m > 0$. More specifically we assume $h(s,T) = 1$ in our case. Thus,

$$bs^{m-1} + o(s^{m-1}) = 1 \implies b = 1,$$

where $m = 1$. Finally, we are assuming $\int_0^\infty e^{g(s,T)}ds < \infty$. Let us see how does this finiteness really work in *non-exchangeable* extreme value Copula. Let us take our *Pickands* dependence function $A^*((1+s)^{-1}) = 2^{-1}\left(1 + (1+(1+s)^{-1})^2\right)$. After doing some easy calculations we have, $A^*((1+s)^{-1}) = 1 + 2(1+s)^{-1} + (1+s)^{-2}$. After putting this value into $g(s,T)$ and integrating $\int_0^\infty e^{g(s,T)}ds$ we get,

$$\begin{aligned}
\int_0^\infty e^{g(s,T)}ds &= \int_0^\infty e^{1+\frac{s}{\beta}-(1+s)A^*\left\{\frac{1}{1+s}\right\}}ds \\
&= \int_0^\infty e^{1+\frac{s}{\beta}-(1+s)\left[1+\frac{2}{1+s}+\frac{1}{(1+s)^2}\right]}\ ds,\ as\ A^*\left(\frac{1}{1+s}\right) = 1 + \frac{2}{1+s} + \frac{1}{(1+s)^2} \\
&= e^{-2}\int_0^\infty e^{\frac{s}{\beta}-s-\frac{1}{1+s}}\ ds \\
&\leq e^{-2}\int_0^\infty e^{\frac{s}{\beta}-s}\ ds,\ as\ e^{\frac{s}{\beta}-s-\frac{1}{1+s}} \leq e^{\frac{s}{\beta}-s},\ \forall s \in (0,\infty) \\
&= e^{-2}\int_0^\infty e^{\left[\frac{1}{\beta}-1\right]s}\ ds = e^{-2}\ \frac{e^{\left[\frac{1}{\beta}-1\right]s}}{\frac{1}{\beta}-1} < \infty,\ \forall\ \beta > 1
\end{aligned} \tag{3.20}$$

In (3.20) as $\beta > 1$ which implies $\beta^{-1} - 1 < 0$ and furthermore $e^{-2}\ \frac{e^{\left[\frac{1}{\beta}-1\right]s}}{\frac{1}{\beta}-1} < \infty$. With this result and by *dominated convergence theorem* we conclude that $\int_0^\infty e^{g(s,T)}ds < \infty$.

Now, by Watson's lemma we can write the approximated value of the integral $I(T) = \int_0^\infty e^{Tg(s,T)}ds$ with $T \geq 1$ are all finite and the asymptotic approximation is;

$$I(T) \sim \left(\frac{1}{-\frac{1}{\beta} - A^{*'}\{1\} + A^*\{1\}}\right)(T^{-1})e^{Tg(0,T)}\ as,\quad T \to \infty$$

Thus,

$$\begin{aligned} \mathrm{E}[X_1|X_2>t] &\sim T\int_0^\infty e^{Tg(s,T)}ds \\ &\sim T\left(\frac{1}{-\frac{1}{\beta}-A^{*'}\{1\}+A^*\{1\}}\right)(T^{-1})e^{Tg(0,T)}\ as, \quad T\to\infty \\ &\sim \frac{1}{-\frac{1}{\beta}-A^{*'}\{1\}+A^*\{1\}},\ as\ t\to\infty. \end{aligned} \tag{3.21}$$

□

In this case our measure of tail *non-exchangeability* would be;

$$\begin{aligned} \eta_1(t) &= \frac{\mathrm{E}[X_1|X_2>t]}{\mathrm{E}[X_2|X_1>t]} \\ &\sim \left(\frac{-\frac{1}{\beta}-A_\alpha^{*'}\{1\}+A_\alpha^*\{1\}}{-\frac{1}{\beta}-A^{*'}\{1\}+A^*\{1\}}\right)\neq 1,\ as\ t\to\infty. \end{aligned} \tag{3.22}$$

From (3.22) we can conclude that, Watson's lemma is able to show some *non-exchangeability* at the tail. This measure is not zero because, we assume this type of extreme value copula is non-exchangeable. Thus, $A^*\{1\}\neq A_\alpha^*\{1\}$ and $A^{*''}\{1\}\neq A_\alpha^{*''}\{1\}$.

Proposition 50. *Suppose X_1 and X_2 are two dependent random variables with $\int_0^\infty e^{g(s,T)}\,ds<\infty$. If survival non-exchangeable extreme value copula has Weibull margins, then we have $\mathrm{E}[X_1|X_2>t]\sim\frac{1}{A^*\{1\}-A^{*'}\{1\}}$.*

Proof. The g function under Weibull margins can be defined as $g(s,T) = 1+\frac{1}{T}\log \widehat{C}^{*}(e^{-sT}, e^{-T})$. In this case $T = t^{\tau}$. After using the result from (3.11) we get the g function as;

$$\begin{aligned} g(s,T) &= 1 + \frac{1}{T}\log \widehat{C}^{*}(e^{-sT}, e^{-T}) \\ &= 1 + \frac{1}{T}\log e^{-T(1+s)A^{*}\left\{\frac{1}{1+s}\right\}} \\ &= 1 - (1+s)A^{*}\left\{\frac{1}{1+s}\right\} \end{aligned} \tag{3.23}$$

Following Hua and Joe (2014) we know before applying Laplace approximation we have to check if $g(0,T) = 0$ and $g(\infty,T) = -\infty$. From (3.23) we have, $g(0,T) = 1 - A^{*}\{1\} = 0$, as $A^{*}\{\}$ is Pickands dependence function. Again from (3.12) we know that;

$$\begin{aligned} g(s,T) &= 1 - (1+s)A^{*}\left\{\frac{1}{1+s}\right\} \\ &= 1 - A^{*}\left\{\frac{1}{1+s}\right\} - sA^{*}\left\{\frac{1}{1+s}\right\} \\ g(\infty,T) &= -\infty,\ as\ \beta > 1\ and\ A^{*}\{0\} = 1 \end{aligned} \tag{3.24}$$

In order to use Laplace approximation we need another two conditions $g'(0,T) > 0$ and $-g''(s_0(T),T) > 0$. If we do the first order differentiation of (3.11), we clearly get,

$$\begin{aligned} g'(s,T) &= -\left[-(1+s)A^{*'}\left\{\frac{1}{1+s}\right\}\frac{1}{(1+s)^2} + A^{*}\left\{\frac{1}{1+s}\right\}\right] \\ &= -A^{*}\left\{\frac{1}{1+s}\right\} + A^{*'}\left\{\frac{1}{1+s}\right\}\frac{1}{1+s} \end{aligned} \tag{3.25}$$

Thus;

$$g'(0,T) = -1 + A^{*'}\{1\} = -1 \ngtr 0 \tag{3.26}$$

From (3.26) we clearly see that, $g'(0,T) = -1 + A^{*'}\{1\} = -1 \not> 0$. Thus, we cannot use Laplace approximation anymore. We have to use Watson's lemma. Before using this lemma we have to check if $g(s,T)$ is decreasing in s. We know the g function is $g(s,T) = 1 - (1+s)A^*\left\{\frac{1}{1+s}\right\}$. Thus, $g'(s,T) = \frac{1}{1+s}A^{*'}\left\{\frac{1}{1+s}\right\} - A^*\left\{\frac{1}{1+s}\right\}$. Now if $\frac{1}{1+s}A^{*'}\left\{\frac{1}{1+s}\right\} < A^*\left\{\frac{1}{1+s}\right\}$, then $g'(s,T) < 0$. In other words, $g(s,T)$ is decreasing in s. Now we can use Watson's lemma.

As $g(s;T)$ is a real valued function on the semi-infinite interval $[0,\infty)$ and in $(0, 0+\epsilon]$ with $\epsilon > 0$ this function is continuously differentiable and

$$\sup_{0+\epsilon \leq s \leq \infty} g(s,T) \leq g(0,T) - \psi, \tag{3.27}$$

with $\psi > 0$. Here we are using Theorem 36 [p.48] of Breiting (1994). This theorem is an extension of Watson's lemma and most importantly, this theorem works for semi-infinite intervals. Now we have $g'(s,T) < 0$ and $s \in [0, 0+\epsilon)$.We can also write

$$g'(s,T) = -as^{r-1} + o(s^{r-1}) \ \forall \ r > 0$$

Now if we assume $r = 1$ then $g'(s,T) = -a$. From our previous results we know that, $\lim_{s^+ \to\ 0,\ T\to\infty} g'(s,T) = A^{*'}\{1\} - A^*\{1\}$, which is a constant. Thus, $-a = -[-A^{*'}\{1\} + A^*\{1\}]$ or, $a = -A^{*'}\{1\} + A^*\{1\}$. Let us assume there is another real and continuous function $h(s,T) \in [0,\infty)$ such that,

$$h(s,T) = bs^{m-1} + o(s^{m-1})$$

with $m > 0$. More specifically we assume $h(s,T) = 1$ in our case. Thus,

$$bs^{m-1} + o(s^{m-1}) = 1 \implies b = 1,$$

where $m = 1$. Finally, as we are assuming $\int_0^\infty e^{g(s,T)}ds < \infty$ then by Watson's lemma we can write the approximated value of the integral $I(T) = \int_0^\infty e^{Tg(s,T)}ds$ with $T \geq 1$ are all finite and the asymptotic approximation is;

$$I(T) \sim \left(\frac{1}{A^*\{1\} - A^{*'}\{1\}}\right)(T^{-1})e^{Tg(0,T)} \; as, \quad T \to \infty$$

Thus,

$$\begin{aligned}
\mathrm{E}[X_1|X_2 > t] &\sim T \int_0^\infty e^{Tg(s,T)}ds \\
&\sim T\left(\frac{1}{A^*\{1\} - A^{*'}\{1\}}\right)(T^{-1})e^{Tg(0,T)} \; as, \quad T \to \infty \\
&\sim \frac{1}{A^*\{1\} - A^{*'}\{1\}}, \; as \; T = t^\tau \; and \; t \to \infty. \qquad (3.28)
\end{aligned}$$

□

In this case our measure of tail *non-exchangeability* would be;

$$\begin{aligned}
\eta_1(t) &= \mathrm{E}[X_1|X_2 > t]/\mathrm{E}[X_2|X_1 > t] \\
&\sim \left(\frac{A^*_\alpha\{1\} - A^{*'}_\alpha\{1\}}{A^*\{1\} - A^{*'}\{1\}}\right) \neq 1, \; as \; t \to \infty. \qquad (3.29)
\end{aligned}$$

From (3.29) we can conclude that, Watson's lemma is able to show some *non-exchangeability* at the tail. This measure is not 1 because, we assume this type of extreme value copula is *non-exchangeable*. Thus, $A^*\{1\} \neq A^*_\alpha\{1\}$ and $A^{*''}\{1\} \neq A^{*''}_\alpha\{1\}$.

If we do the same approximation of survival non-exchangeable extreme value copula with exponential margins, we will get exactly same expression of $\eta_1(t)$. Again, like the case of Weibull margins we cannot use Laplace approximation. Here, $T = \lambda t$.

3.1.2 Case II : $\mathrm{E}[X_1|X_2 = t]$

Here we are using the extreme value survival copula in the form of $\widehat{C}^*(u,v) = exp\left[\log(uv)A^*\left\{\frac{\log(v)}{\log(uv)}\right\}\right]$, $\forall\ (u,v) \in [0,1]^2$, where $u = \overline{F}(x_1)$ and $v = \overline{F}(x_2)$.

Now, let us define,

$$
\begin{aligned}
&\widehat{C}^*_{1|2}(u|v)\\
&= \frac{\partial}{\partial v} exp\left[\log(uv)A^*\left\{\frac{\log(v)}{\log(uv)}\right\}\right]\\
&= exp\left[\log(uv)A^*\left\{\frac{\log(v)}{\log(uv)}\right\}\right]\\
&\qquad \left[\frac{u}{uv}A^*\left\{\frac{\log(v)}{\log(uv)}\right\} + \log(uv)A^{*'}\left\{\frac{\log(v)}{\log(uv)}\right\}\frac{\frac{\log(uv)}{v} - \frac{u\log v}{uv}}{[\ln(uv)]^2}\right]\\
&= exp\left[\log(uv)A^*\left\{\frac{\log(v)}{\log(uv)}\right\}\right]\\
&\qquad \left[\frac{1}{v}A^*\left\{\frac{\log(v)}{\log(uv)}\right\} + \log(uv)A^{*'}\left\{\frac{\log(v)}{\log(uv)}\right\}\frac{\frac{\log(uv)}{v} - \frac{\log v}{v}}{[\ln(uv)]^2}\right]\\
&= exp\left[\log(uv)A^*\left\{\frac{\log(v)}{\log(uv)}\right\}\right]\\
&\qquad \left[\frac{1}{v}A^*\left\{\frac{\log(v)}{\log(uv)}\right\} + \log(uv)A^{*'}\left\{\frac{\log(v)}{\log(uv)}\right\}\frac{\log(uv) - \log v}{v[\ln(uv)]^2}\right]
\end{aligned}
\tag{3.30}
$$

After using the result obtained from (3.11) we get;

$$\begin{aligned}
&\widehat{C}^{*}_{1|2}(e^{-sT}|e^{-T}) \\
&= exp\left[-T(1+s)A^{*}\left\{\frac{1}{1+s}\right\}\right] \\
&\quad \left[e^{T}A^{*}\left\{\frac{1}{1+s}\right\} - T(1+s)A^{*'}\left\{\frac{1}{1+s}\right\}\frac{se^{T}}{T(1+s)^{2}}\right] \\
&= exp\left[T - T(1+s)A^{*}\left\{\frac{1}{1+s}\right\}\right] \\
&\quad \left[A^{*}\left\{\frac{1}{1+s}\right\} - TA^{*'}\left\{\frac{1}{1+s}\right\}\frac{s}{T(1+s)}\right] \\
&= exp\left[T - T(1+s)A^{*}\left\{\frac{1}{1+s}\right\}\right] \\
&\quad \left[A^{*}\left\{\frac{1}{1+s}\right\} - A^{*'}\left\{\frac{1}{1+s}\right\}\frac{s}{(1+s)}\right] \qquad (3.31)
\end{aligned}$$

Now, we are interested to check tail *non-exchangeability*, i.e. $\eta_2(t) = \mathrm{E}[X_1|X_2 = t]/\mathrm{E}[X_2|X_1 = t]$. According to Hua and Joe (2014) the conditional tail expectation can be written as; $E[X_1|X_2 = t] = \int_0^\infty \widehat{C}^{*}_{1|2}(\overline{F}(x)|\overline{F}(t))dx, \ \forall\ t$ Like before let us consider a simple case with X_1, X_2 follow Pareto distribution with identical cdf $F(x) = 1 - (1+x)^{-\beta}, \beta > 1$.

Proposition 51. *Suppose X_1 and X_2 are two dependent random variables. The integrand of conditional expectation of extreme value with non-exchangeable Pickands dependence function and Pareto margin can be decomposed into two multiplicative separable functions.*

Proof. Let $\widehat{C}^{*}$ be the survival copula of the set of random variables (X_1, X_2). Like before;

$$F(x) = 1 - (1+x)^{-\beta},\ \beta > 1$$

$$Thus, \overline{F}(x) = 1 - F(x) = 1 - 1 + (1+x)^{-\beta} \ = (1+x)^{-\beta}$$

Following Hua and Joe (2014) we will transform the survival function $\overline{F}(t) \to \ e^{-T}$ $or, T = -\log \overline{F}(t) = \beta \log(1+t)$ we have, letting $T = \beta \log(1+t)$

$$\begin{aligned}
\mathrm{E}[X_1|X_2 = t] &= \frac{1}{\beta}\int_0^\infty T\widehat{C}^*_{1|2}(e^{-sT}|e^{-T})e^{\frac{sT}{\beta}}ds \\
&= \frac{T}{\beta}\int_0^\infty e^{T[\frac{s}{\beta}+\frac{1}{T}\log\widehat{C}^*_{1|2}(e^{-sT}|e^{-T})]}ds \\
&= \frac{T}{\beta}\int_0^\infty e^{T[\frac{s}{\beta}+\frac{1}{T}\log\left\{exp\left[T-T(1+s)A^*\left\{\frac{1}{1+s}\right\}\right]\left[A^*\left\{\frac{1}{1+s}\right\}-A^{*'}\left\{\frac{1}{1+s}\right\}\frac{s}{(1+s)}\right]\right\}]}ds \\
&= \frac{T}{\beta}\int_0^\infty e^{T\left[\frac{s}{\beta}+1-(1+s)A^*\left\{\frac{1}{1+s}\right\}\right]+\log\left[A^*\left\{\frac{1}{1+s}\right\}-A^{*'}\left\{\frac{1}{1+s}\right\}\frac{s}{(1+s)}\right]}ds \\
&= \frac{T}{\beta}\int_0^\infty e^{T\left[1+\frac{s}{\beta}-(1+s)A^*\left\{\frac{1}{1+s}\right\}\right]}\left[A^*\left\{\frac{1}{1+s}\right\}-A^{*'}\left\{\frac{1}{1+s}\right\}\frac{s}{(1+s)}\right]ds
\end{aligned} \tag{3.32}$$

$$= \frac{T}{\beta}\int_0^\infty e^{Tg(s;T)}h(s)ds,$$

where $g(s;T) = 1+\frac{s}{\beta}-(1+s)A^*\left\{\frac{1}{1+s}\right\}$, $h(s) = \left[A^*\left\{\frac{1}{1+s}\right\}-A^{*'}\left\{\frac{1}{1+s}\right\}\frac{s}{(1+s)}\right]$. □

Proposition 52. *Under Pareto margins and* $\mathrm{E}[X_1|X_2 = t]$ *we can use Laplace Approximation if* $\forall s \in [0,\infty)$*, we have* $\frac{1}{\beta}+A^{*'}\{1\} > 1$ *and* $A^*\left\{\frac{1}{1+s}\right\} > A^{*'}\left\{\frac{1}{1+s}\right\}\frac{s}{(1+s)}$*; where* $\beta > 1$*.*

Proof. The g function under Pareto margins with $\mathrm{E}[X_1|X_2 = t]$ can be defined as $g(s,T) = 1+\frac{s}{\beta}+\frac{1}{T}\log\widehat{C}^*(e^{-sT},e^{-T})$. In this case $T = \beta\log(1+t)$. After using the result from (3.32) we get the g function as;

$$\begin{aligned}
g(s,T) &= 1+\frac{s}{\beta}+\frac{1}{T}\log\widehat{C}^*(e^{-sT},e^{-T}) \\
&= 1+\frac{s}{\beta}+\frac{1}{T}\log e^{-T(1+s)A^*\left\{\frac{1}{1+s}\right\}} \\
&= 1+\frac{s}{\beta}-(1+s)A^*\left\{\frac{1}{1+s}\right\}
\end{aligned} \tag{3.33}$$

Following Hua and Joe (2014) we know before applying Laplace approximation we have to check if $g(0,T) = 0$ and $g(\infty, T) = -\infty$. From (3.33) we have, $g(0,T) = 1 - A^*\{1\} = 0$, as $A^*\{\}$ is Pickands dependence function.

Again from (3.33) we know that;

$$
\begin{aligned}
g(s,T) &= 1 + \frac{s}{\beta} - (1+s)A^*\left\{\frac{1}{1+s}\right\} \\
&= 1 + \frac{1}{\beta} - A^*\left\{\frac{1}{1+s}\right\} - sA^*\left\{\frac{1}{1+s}\right\} \\
&= 1 - A^*\left\{\frac{1}{1+s}\right\} + s\left[\frac{1}{\beta} - A^*\left\{\frac{1}{1+s}\right\}\right] \\
g(\infty,T) &= \lim_{s\to\infty}\ s\left[\frac{1}{\beta} - 1\right] = -\infty,\ as\ \beta > 1\ and\ A^*\{0\} = 1
\end{aligned} \tag{3.34}
$$

In order to use Laplace approximation we need another two conditions $g'(0,T) > 0$ and $-g''(s_0(T),T) > 0$. If we do the first order differentiation of (3.32), we clearly get,

$$
\begin{aligned}
g'(s,T) &= \frac{1}{\beta} - \left[-(1+s)A^{*'}\left\{\frac{1}{1+s}\right\}\frac{1}{(1+s)^2} + A^*\left\{\frac{1}{1+s}\right\}\right] \\
&= \frac{1}{\beta} - A^*\left\{\frac{1}{1+s}\right\} + A^{*'}\left\{\frac{1}{1+s}\right\}\frac{1}{1+s}
\end{aligned} \tag{3.35}
$$

Thus;

$$
\begin{aligned}
g'(0,T) &= \frac{1}{\beta} - 1 + A^{*'}\{1\} > 0 \\
or,\ &\frac{1}{\beta} + A^{*'}\{1\} > 1
\end{aligned} \tag{3.36}
$$

If we check for the second order condition of (3.32), we can see that,

$$\begin{aligned} g''(s,T) &= A^{*'}\left\{\frac{1}{1+s}\right\}\frac{1}{(1+s)^2} \\ &\quad + \left[-A^{*'}\left\{\frac{1}{1+s}\right\}\frac{1}{(1+s)^2} - \frac{1}{(1+s)^3}A^{*''}\left\{\frac{1}{1+s}\right\}\right] \\ &= A^{*'}\left\{\frac{1}{1+s}\right\}\frac{1}{(1+s)^2} \\ &\quad - \left[A^{*'}\left\{\frac{1}{1+s}\right\}\frac{1}{(1+s)^2} + \frac{1}{(1+s)^3}A^{*''}\left\{\frac{1}{1+s}\right\}\right] \\ &= -\frac{1}{(1+s)^3}A^{*''}\left\{\frac{1}{1+s}\right\} < 0,\ as\ A^{*''}\{.\} > 0. \end{aligned} \tag{3.37}$$

From (3.16) it is clear that $-g''(s,T) > 0$ all the time when $s \in [0,\infty)$. Thus, $-g''(s_0(T),T) > 0$ as $T \to \infty$. Finally we need $h(s) > 0$. In this case this is possible only when $A^*\left\{\frac{1}{1+s}\right\} > A^{*'}\left\{\frac{1}{1+s}\right\}\frac{s}{(1+s)}$. So, from the above discussion we see two conditions matter here before using Laplance approximation is condition (3.36) which are $\frac{1}{\beta}+A^{*'}\{1\} > 1$ and $A^*\left\{\frac{1}{1+s}\right\} > A^{*'}\left\{\frac{1}{1+s}\right\}\frac{s}{(1+s)}$ respectively. □

In this case,

$$\begin{aligned} &T\int_0^\infty e^{Tg(s,T)}h(s)ds \\ &\sim T\int_0^\infty h(s)exp\left\{T\left[g(\gamma,T) + \frac{1}{2}(s-\gamma)^2 g''(\gamma,T)\right]\right\}\ ds \\ &\sim Te^{Tg(\gamma,T)}h(\gamma)\sqrt{\frac{2\pi}{-Tg''(\gamma,T)}};\ where\ \gamma = \lim_{T\to\infty}\ argmax_s\ g(s,T) \\ &= Te^{T\left[1+\frac{s}{\beta}-(1+s)A^*\left\{\frac{1}{1+s}\right\}\right]}\left[A^*\left\{\frac{1}{1+s}\right\} - A^{*'}\left\{\frac{1}{1+s}\right\}\frac{s}{(1+s)}\right]\sqrt{\frac{2\pi}{T\frac{1}{(1+s)^3}A^{*''}\left\{\frac{1}{1+s}\right\}}} \\ &= e^{T\left[1+\frac{s}{\beta}-(1+s)A^*\left\{\frac{1}{1+s}\right\}\right]}\left[A^*\left\{\frac{1}{1+s}\right\} - A^{*'}\left\{\frac{1}{1+s}\right\}\frac{s}{(1+s)}\right]\sqrt{\frac{2\pi T}{\frac{1}{(1+s)^3}A^{*''}\left\{\frac{1}{1+s}\right\}}} \end{aligned} \tag{3.38}$$

From (3.32) we know that $\mathrm{E}[X_1|X_2=t] = \frac{T}{\beta}\int_0^\infty e^{Tg(s;T)}h(s)ds$. Using (3.38) we get;

$$\begin{aligned}
&\mathrm{E}[X_1|X_2=t]\\
&\sim \frac{T}{\beta}\int_0^\infty e^{Tg(s;T)}h(s)ds\\
&\sim \frac{1}{\beta}e^{T\left[1+\frac{s}{\beta}-(1+s)A^*\left\{\frac{1}{1+s}\right\}\right]}\\
&\qquad \left[A^*\left\{\frac{1}{1+s}\right\} - A^{*\prime}\left\{\frac{1}{1+s}\right\}\frac{s}{(1+s)}\right]\sqrt{\frac{2\pi T}{\frac{1}{(1+s)^3}A^{*\prime\prime}\left\{\frac{1}{1+s}\right\}}}\\
&\sim \frac{1}{\beta}e^{\beta\log(1+t)\left[1+\frac{\gamma}{\beta}-(1+\gamma)A^*\left\{\frac{1}{1+\gamma}\right\}\right]}\\
&\qquad \left[A^*\left\{\frac{1}{1+\gamma}\right\} - A^{*\prime}\left\{\frac{1}{1+\gamma}\right\}\frac{\gamma}{(1+\gamma)}\right]\sqrt{\frac{2\pi\beta\log(1+t)}{\frac{1}{(1+\gamma)^3}A^{*\prime\prime}\left\{\frac{1}{1+\gamma}\right\}}}\\
&\sim \frac{1}{\beta}(1+t)^{\beta\left[1+\frac{\gamma}{\beta}-(1+\gamma)A^*\left\{\frac{1}{1+\gamma}\right\}\right]}\\
&\qquad \left[A^*\left\{\frac{1}{1+\gamma}\right\} - A^{*\prime}\left\{\frac{1}{1+\gamma}\right\}\frac{\gamma}{(1+\gamma)}\right]\sqrt{\frac{2\pi\beta\log(1+t)}{\frac{1}{(1+\gamma)^3}A^{*\prime\prime}\left\{\frac{1}{1+\gamma}\right\}}} \qquad (3.39)\\
&as,\ \gamma = \lim_{T\to\infty} argmax_s\ g(s,T),\ t\to\infty\ and\ T=\beta\log(1+t)
\end{aligned}$$

In this case our measure of tail *non-exchangeability* would be;

$$\begin{aligned}
\eta_2(t) &= \mathrm{E}[X_1|X_2=t]/\mathrm{E}[X_2|X_1=t]\\
&\sim \frac{\left\{(1+t)^{\beta\left[1+\frac{\gamma}{\beta}-(1+\gamma)A^*\left\{\frac{1}{1+\gamma}\right\}\right]}\left[A^*\left\{\frac{1}{1+\gamma}\right\} - A^{*\prime}\left\{\frac{1}{1+\gamma}\right\}\frac{\gamma}{(1+\gamma)}\right]\sqrt{\frac{2\pi\beta\log(1+t)}{\frac{1}{(1+\gamma)^3}A^{*\prime\prime}\left\{\frac{1}{1+\gamma}\right\}}}\right\}}{\left\{(1+t)^{\beta\left[1+\frac{\gamma}{\beta}-(1+\gamma)A^*_\alpha\left\{\frac{1}{1+\gamma}\right\}\right]}\left[A^*_\alpha\left\{\frac{1}{1+\gamma}\right\} - A^{*\prime}_\alpha\left\{\frac{1}{1+\gamma}\right\}\frac{\gamma}{(1+\gamma)}\right]\sqrt{\frac{2\pi\beta\log(1+t)}{\frac{1}{(1+\gamma)^3}A^{*\prime\prime}_\alpha\left\{\frac{1}{1+\gamma}\right\}}}\right\}} \qquad (3.40)\\
&\neq 1,\ as\ t\to\infty.
\end{aligned}$$

From (3.29) we can conclude that, Watson's lemma is able to show some *non-exchangeability* at the tail. This measure is not 1 because, we assume this type of extreme value copula is *non-exchangeable*. Thus, $A^* \left\{\frac{1}{1+\gamma}\right\} \neq A^*_\alpha \left\{\frac{1}{1+\gamma}\right\}$ and $A^{*''} \left\{\frac{1}{1+\gamma}\right\} \neq A^{*''}_\alpha \left\{\frac{1}{1+\gamma}\right\}$.

3.2 Conditional Expectations of exchangeable Extreme value Survival Copula

In this section we are going to use *exchangeable* survival extreme value Copulas with different margins. By *extreme* value Copula we use the same definition used by Gudendorf and Segers (2010). In their paper Gudendorf and Segers (2010) they use Logistic or Gumbel-Hougaard, Galambos or Negative Logistic, Husler-Reiss and t-Extreme Value Copulas as examples of *parametric exchangeable extreme value Copulas*. Here throughout our paper we are using survival functions as the arguments of the Copula functions. Thus, in our case it is nothing but survival Copula functions. Again, as this extreme value Copula is exchangeable, we are transforming into a non-exchangeable structure by using Khoudraji (1996) device. In chapter 5 of their book Genest and Nešlehová (2013) say that, if the survival Copula is exchangeable, *Pickands* dependence function after using Khoudraji (1996) *non-exchangeable* transformation becomes;

$$A(r) = (1-\alpha_2)r + (1-\alpha_1)(1-r) + \{\alpha_2 r + \alpha_1(1-r)\}A^* \left\{\frac{\alpha_2 r}{\alpha_2 r + \alpha_1(1-r)}\right\},$$

where $r = \log v/\log(uv)$, $r \in [0,1]$, $(\alpha_1, \alpha_2) \in [0,1]^2$, and $u = e^{-sT}$ and $v = e^{-T}$.

Proposition 53. *In Khoudraji (1996) non-exchangeable transformed survival Gumbel Copula Picands dependence function is* $A(r) = [1/(1+s)]M$*, where* $M = [1-\alpha_2 + s(1-\alpha_1) + (\alpha_2^\delta + (\alpha_1 s)^\delta)^{1/\delta}]$*,* $r = (1+s)^{-1}$*, for all* $(\alpha_1, \alpha_2) \in [0,1]^2$*,* $s \in [0,\infty)$ *and* $\delta \geq 0$*.*

Proof. From Gumbel distribution we know that, $A^*(r_1) = [r_1^\theta + (1-r_1)^\theta]^{1/\theta}$, where $r_1 = (\alpha_2 r)/[\alpha_2 r + \alpha_1(1-r)]$. Thus, $A^*(r_1) = \left[\left(\frac{\alpha_2 r}{\alpha_2 r+\alpha_1(1-r)}\right)^\theta + \left\{1-\left(\frac{\alpha_2 r}{\alpha_2 r+\alpha_1(1-r)}\right)\right\}^\theta\right]^{1/\theta}$.

Now from the definition we know that, $r = \log v/\log(uv) = \log(e^{-T})/\log(e^{-T(1+s)}) = 1/(1+s)$, where all the symbols have their usual meanings.

Now,

$$\begin{aligned} A(r) &= \frac{1-\alpha_2}{1+s} + (1-\alpha_1)\left(1-\frac{1}{1+s}\right) \\ &\quad + \left\{\frac{\alpha_2}{1+s} + \alpha_1\left(1-\frac{1}{1+s}\right)\right\} A^*\left\{\frac{\alpha_2/(1+s)}{\frac{\alpha_2}{1+s}+\alpha_1\left(1-\frac{1}{1+s}\right)}\right\} \\ &= \frac{1-\alpha_2}{1+s} + \frac{(1-\alpha_1)s}{1+s} + \frac{1}{1+s}(\alpha_1 s+\alpha_2)\ A^*\left\{\frac{\alpha_2}{\alpha_2+\alpha_1 s}\right\} \\ &= \frac{1}{1+s}\left[1-\alpha_2+s(1-\alpha_1)+(\alpha_2+\alpha_1 s)A^*\left\{\frac{\alpha_2}{\alpha_2+\alpha_1 s}\right\}\right] \qquad (3.41) \end{aligned}$$

After putting the value of $A^*\left\{\frac{\alpha_2}{\alpha_2+\alpha_1 s}\right\}$ in 3.41 we get;

$$\begin{aligned} A(r) &= \frac{1}{1+s}\left[1-\alpha_2+s(1-\alpha_1)+(\alpha_2+\alpha_1 s)\left[\left(\frac{\alpha_2}{\alpha_2+\alpha_1 s}\right)^\delta + \left(\frac{\alpha_1 s}{\alpha_2+\alpha_1 s}\right)^\delta\right]^{1/\delta}\right] \\ &= \frac{1}{1+s}\left[1-\alpha_2+s(1-\alpha_1)+(\alpha_2+\alpha_1 s)\left[\frac{1}{\alpha_2+\alpha_1 s}\left(\alpha_2^\delta+(\alpha_1 s)^\delta\right)^{1/\delta}\right]\right] \\ &= \frac{1}{1+s}\left[1-\alpha_2+s(1-\alpha_1)+\left(\alpha_2^\delta+(\alpha_1 s)^\delta\right)^{1/\delta}\right] = \frac{1}{1+s}\ M, \qquad (3.42) \end{aligned}$$

where $M = [1-\alpha_2+s(1-\alpha_1)+(\alpha_2^\delta+(\alpha_1 s)^\delta)^{1/\delta}]$, $r = (1+s)^{-1}$, for all $(\alpha_1,\alpha_2) \in [0,1]^2$, $s \in [0,\infty)$ and $\delta \geq 0$. This completes the proof. □

In the similar way like above we calculate the *Pickands* dependence functions after doing Khoudraji (1996) non-exchangeable transformation in Table 3.1. The second Column in this table is the *Pickands* dependence functions after doing Khoudraji *non-exchangeable*

Copula	$A(r)$ with $r=(1+s)^{-1}$	$A^*(r_1)$
Gumbel	$A(r)=\frac{1}{1+s}\big[1-\alpha_2+s(1-\alpha_1) + \big(\alpha_2^{\delta}+(\alpha_1 s)^{\delta}\big)^{1/\delta}\big]$	$[r_1^{\theta}+(1-r_1)^{\theta}]^{1/\theta}$
Galambos	$A(r)=\frac{1}{1+s}\big[1+s - \Big(\alpha_2^{-1/\delta}+(\alpha_1 s)^{-1/\delta}\Big)^{-\delta}\big]$	$1-\Big\{r_1^{-1/\delta}+(1-r_1)^{-1/\delta}\Big\}^{-\delta}$
Husler	$A(r)=\frac{1}{1+s}\big[1-\alpha_2+s(1-\alpha_1) +\alpha_1 s\Phi\big(\lambda+\frac{1}{2\lambda}\log\frac{\alpha_1 s}{\alpha_2}\big)+\alpha_2\Phi\big(\lambda+\frac{1}{2\lambda}\log\frac{\alpha_2}{\alpha_1 s}\big)\big]$	$(1-r_1)\Phi\big(\lambda+\frac{1}{2\lambda}\log\frac{1-r_1}{r_1}\big) +r_1\Phi\big(\lambda+\frac{1}{2\lambda}\log\frac{r_1}{1-r_1}\big)$
Tawn	$A(r)=\frac{1}{1+s}\big[1-\alpha_2+s(1-\alpha_1)+\alpha_1 s(1-\Psi_1) +(1-\Psi_2)\alpha_2+\big((\alpha_2\Psi_1)^{1/\delta}+(\alpha_1 s\Psi_2)^{1/\delta}\big)^{\delta}\big]$	$(1-\Psi_1)(1-r_1)+(1-\Psi_2)r_1 +\big[(\Psi_1 r_1)^{1/\theta}+\Psi_2(1-r_1)^{1/\theta}\big]^{\theta}$
t-EV	$A(r)=\frac{1}{1+s}\big[1-\alpha_2+s(1-\alpha_1) +\alpha_2 t_{\nu+1}\Big(Z_{\frac{\alpha_2}{\alpha_2+\alpha_1 s}}\Big)+\alpha_1 s t_{\nu+1}\Big(Z_{\frac{\alpha_1 s}{\alpha_2+\alpha_1 s}}\Big)$	$r_1 t_{\nu+1}\left(Z_{r_1}\right)+(1-r_1)\left(Z_{1-r_1}\right)$

Table 3.1: Pickands Dependence functions after Khoudraji (1996) transformation of different exchangeable extreme value Copulas

transformations. Throughout the table we have $(\alpha_1,\alpha_2,r_1)\in[0,1]^3$, $\theta=\delta\geq 0$, $\lambda\in[0,\infty)$, $\nu>0$ and $(\Psi_1,\Psi_2)\in[0,1]^2$. In the third column we put the general *Pickands* dependence functions of different exchangeable extreme value copulas which we get from the literature. The contribution of this table is the second column. In the third column corresponding to Husler Copula Ψ represents the standard normal distribution. Finally, for t-EV Copula we follow Gudendorf and Segers (2010) in order to define Z_{r_1}. They define $Z_{r_1}=(1+\nu)^{1/2}\left[\{r_1/(1-r_1)\}^{1/\nu}-\rho\right](1-\rho^2)^{-1/2}$, where $\rho\in[-1,1]$ is the correlation parameter, $r_1\in[0,1]$ and ν is the degree of freedom of the t distribution.

Now we are going to discuss about the tail *non-exchangeability* of Khoudraji (1996) type in the case of extreme value Copula. In this section we are discussing only the case of Gumbel survival Copula. Other extreme Copulas might be a good research topic in future for us. From the literature we know that Gumbel distribution function looks like, $\widehat{C}^*(u,v)=exp\left\{-\left[[-\log(u)]^{\delta}+[-\log(v)]^{\delta}\right]^{1/\delta}\right\}$, where $(u,v)\in[0,1]^2$ and the other notations have the same meaning we used before. After putting $u = e^{-sT}$ and $v = e^{-T}$ and doing Khoudraji *non-exchangeable* transformation we get;

$$\widehat{C}(e^{-sT}, e^{-T}) = exp\left[-T\left\{s(1-\alpha_1) + (1-\alpha_2) + [(\alpha_1 s)^{\delta} + \alpha_2^{\delta}]^{1/\delta}\right\}\right]$$

where $\delta \geq 0$. Now based on this Copula structure we are going to discuss about the conditional tail expectations in the form of $E[X_1|X_2 > t]$ and $E[X_1|X_2 = t]$ and finally we are going to use our measure of *non-exchangeability* in the form of $\eta_1(t)$ and $\eta_2(t)$ respectively. Like in the previous chapter here we also see that in the case of Gumbel Copula with Pareto margins where we can use both Laplace approximation and Watson's lemma to get the tail approximation of the integral. Again like before, by using Watson's lemma we are not able to get much *non-exchangeability* at the tail. On the other hand, in non-exchangeable Gumbel Copula with Weibull and Exponential margins we can only use *Watson's lemma*. We are going to discuss the whole issue one by one.

3.2.1 Case I : $\mathrm{E}[X_1|X_2 > t]$

Firstly, we are going to discuss about the case of tail *non-exchangeability* of Khoudraji (1996) type. As we tell before, throughout this section, we are going to discuss about the case of survival Gumbel Copula function. Here like before two arguments of the Copula function are two survival functions, hence, their values stand between zero and one. We are considering three different margins Pareto, Weibull and Exponential respectively. We are considering these three margins because, Pareto is a power distribution, Weibull and Exponential distributions are in sub-exponential and exponential families respectively. In this case, our measure of tail non-exchangebility is $\eta_1(t)$ and we check if at large values of t,

$\eta_1(t)$ converges to the unity. Again, like before as these integrations do not have any closed form solutions, we can consider either Laplace Approximation or Watson's Lemma.

Proposition 54. *Suppose X_1 and X_2 are two dependent random variables. In the case of Khoudraji non-exchangeable Gumbel Copula with Pareto margin, if $\alpha_1 + 1/\beta > 1$, Then*

$$\mathrm{E}[X_1|X_2 > t] \sim (1+t)^{\beta M_1} \sqrt{\frac{2\pi \log(1+t)}{M_2 M_3}},$$

where $M_1 = \alpha_2 + \gamma/\beta - \gamma(1-\alpha_1) - [(\alpha_1\gamma)^\delta + \alpha_2^\delta]^{1/\delta}$, $M_2 = \alpha_1^\delta \delta\{(\alpha_1\gamma)^\delta + \alpha_2^\delta\}^{1/\delta - 1}\gamma^{\delta-1}$, and $M_3 = (\delta-1)\gamma^{-1} + \alpha_1^\delta \gamma^{\delta-1}(1/\delta - 1)\{(\alpha_1\gamma)^\delta + \alpha_2^\delta\}^{-1}$, as $t \to \infty$, $(\alpha_1, \alpha_2) \in [0,1]^2$, $\delta \geq 0$ and $\gamma = \lim_{T\to\infty} argmax_s\ g(s,T)$.

Proof. From our previous calculations we know that,

$$\begin{aligned}
\mathrm{E}[X_1|X_2 > t] &= T\int_0^\infty e^{T(1-s)}\widehat{C}(e^{-sT}, e^{-T})\left[f(F^{-1}(1-e^{-sT}))\right]\ ds \\
&= T\int_0^\infty e^{T(1-s)+\log\widehat{C}(e^{-sT}, e^{-T})+\log\left[f(F^{-1}(1-e^{-sT}))\right]}\ ds,\ \forall\ T, \qquad (3.43)\\
&= T\int_0^\infty e^{Tg(s,T)}h(s)\ ds,\ \forall\ T, \qquad (3.44)
\end{aligned}$$

where $T = -\log \overline{F}(t)$, $h(s) = 1$ and $s \in [0, \infty)$.

After doing some calculation g function becomes;

$$
\begin{aligned}
g(s,T) &= 1 + \frac{s}{\beta} + \frac{1}{T}\log \widehat{C}(e^{-sT}, e^{-T}) \\
&= 1 + \frac{s}{\beta} + \frac{1}{T}\log\left[e^{-T\left\{s(1-\alpha_1)+(1-\alpha_2)+[(\alpha_1 s)^\delta+\alpha_2^\delta]^{1/\delta}\right\}}\right] \\
&= 1 + \frac{s}{\beta} - \left\{s(1-\alpha_1) + (1-\alpha_2) + [(\alpha_1 s)^\delta + \alpha_2^\delta]^{1/\delta}\right\} \\
&= 1 + \frac{s}{\beta} - s(1-\alpha_1) - (1-\alpha_2) - [(\alpha_1 s)^\delta + \alpha_2^\delta]^{1/\delta} \\
&= \alpha_2 + \frac{s}{\beta} - s(1-\alpha_1) - [(\alpha_1 s)^\delta + \alpha_2^\delta]^{1/\delta}, \qquad (3.45) \\
&= \alpha_2 + s\left[\frac{1}{\beta} + \alpha_1 - 1\right] - [(\alpha_1 s)^\delta + \alpha_2^\delta]^{1/\delta},
\end{aligned}
$$

where $(\alpha_1, \alpha_2) \in [0,1]^2$, $\delta \geq 0$, $\beta > 1$ and $s \in [0, \infty)$.

After doing some easy calculation and assuming $[(\alpha_1 s)^\delta + \alpha_2^\delta]^{1/\delta}$ dominates the rest of the terms in g. Again, we find out that $g(0,T) = 0$ and $g(\infty, T) = -\infty$. As these two conditions are satisfied we can do either Laplace Approximation or Watson's lemma. Let us do the first one first. If we do the first order derivative of (3.45) with respect to s we have;

$$
g'(s,T) = \left[\frac{1}{\beta} + \alpha_1 - 1\right] - \delta\alpha_1^\delta s^{\delta-1}[(\alpha_1 s)^\delta + \alpha_2^\delta]^{1/\delta - 1}, \qquad (3.46)
$$

where $(\alpha_1, \alpha_2) \in [0,1]^2$, $\delta \geq 0$, $\beta > 1$ and $s \in [0, \infty)$. Thus, from (3.46) we immediately get $g'(0,T) = 1/\beta + \alpha_1 - 1 > 0$ [by assumption] as $T \to \infty$.

Again if we do the second order derivative of (3.45) we have;

$$\begin{aligned} g''(s,T) &= -\delta\alpha_1^\delta\Big[(\delta-1)s^{\delta-2}[(\alpha_1 s)^\delta + \alpha_2^\delta]^{1/\delta-1} \\ &\quad + s^{\delta-1}\left(\frac{1}{\delta}-1\right)[(\alpha_1 s)^\delta + \alpha_2^\delta]^{1/\delta-2}\delta\alpha_1(\alpha_1 s)^{\delta-1}\Big] < 0 \\ -g''(s_0,T) &= \delta\alpha_1^\delta\Big[(\delta-1)s_0^{\delta-2}[(\alpha_1 s_0)^\delta + \alpha_2^\delta]^{1/\delta-1} \\ &\quad + s_0^{\delta-1}\left(\frac{1}{\delta}-1\right)[(\alpha_1 s_0)^\delta + \alpha_2^\delta]^{1/\delta-2}\delta\alpha_1(\alpha_1 s_0)^{\delta-1}\Big] > 0, \end{aligned} \tag{3.47}$$

where $(\alpha_1, \alpha_2) \in [0,1]^2$, $\delta \geq 0$, $\beta > 1$, $s \in [0,\infty)$ and
$\gamma = \lim_{T\to\infty} s_0(T) = \lim_{T\to\infty}\ argmax_s\ g(s,T)$.

Now let us see (3.46) more carefully. If we do the first order condition of this equation, we have;

$$\begin{aligned} \left[\frac{1}{\beta}+\alpha_1-1\right] &= \delta\alpha_1^\delta s^{\delta-1}[(\alpha_1 s)^\delta + \alpha_2^\delta]^{1/\delta-1} \\ \frac{\frac{1}{\beta}+\alpha_1-1}{\delta\alpha_1^\delta} &= s^{\delta-1}[(\alpha_1 s)^\delta + \alpha_2^\delta]^{1/\delta-1} \\ \zeta s^{1-\delta} &= [(\alpha_1 s)^\delta + \alpha_2^\delta]^{\frac{1-\delta}{\delta}},\ where\ \zeta = \frac{\frac{1}{\beta}+\alpha_1-1}{\delta\alpha_1^\delta}, \\ \zeta^{\frac{\delta}{1-\delta}} s^\delta &= (\alpha_1 s)^\delta + \alpha_2^\delta \\ s^\delta &= \left(\frac{\zeta^{\frac{\delta}{1-\delta}}}{\alpha_1}\right)^\delta - \left(\frac{\alpha_2}{\alpha_1}\right)^\delta \\ s^\delta &= \left(\frac{\alpha_2}{\alpha_1}\right)^\delta\left[\left(\frac{\zeta^{\frac{1}{1-\delta}}}{\alpha_1}\right)^\delta - 1\right]^{-1} \end{aligned} \tag{3.48}$$

After solving (3.48) further and putting value of ζ we have; $s_0(T) = (\alpha_2/\alpha_1)$ $\left[((\frac{1}{\beta}+\alpha_1-1)/\delta\alpha_1^\delta)/\alpha_1 - 1\right]^{-1/\delta}$. As in this expression we do not have any term related to either T or s, we can directly say that, $\gamma = s_0(T)$ in this case.

Following Hua and Joe (2014) if we use the Laplace approximation method the value of conditional tail expectations become;

$$\begin{aligned}
&\mathrm{E}[X_1|X_2 > t] \\
&\sim e^{Tg(\gamma,T)}\sqrt{\frac{2\pi T}{-g''(\gamma,T)}} \\
&\sim e^{\beta\log(1+t)\left[\alpha_2+\frac{\gamma}{\beta}-\gamma(1-\alpha_1)-[(\alpha_1\gamma)^\delta+\alpha_2^\delta]^{1/\delta}\right]} \\
&\quad\sqrt{\frac{2\pi\beta\log(1+t)}{\alpha_1^\delta\delta\{(\alpha_1\gamma)^\delta+\alpha_2^\delta\}^{1/\delta-1}\gamma^{\delta-1}\left[(\delta-1)\gamma^{-1}+\alpha_1^\delta\gamma^{\delta-1}(1/\delta-1)\{(\alpha_1\gamma)^\delta+\alpha_2^\delta\}^{-1}\right]}} \\
&\sim (1+t)^{\beta M_1}\sqrt{\frac{2\pi\log(1+t)}{M_2M_3}}, \qquad (3.49)
\end{aligned}$$

where $M_1 = \alpha_2 + \gamma/\beta - \gamma(1-\alpha_1) - [(\alpha_1\gamma)^\delta + \alpha_2^\delta]^{1/\delta}$, $M_2 = \alpha_1^\delta\delta\{(\alpha_1\gamma)^\delta + \alpha_2^\delta\}^{1/\delta-1}\gamma^{\delta-1}$, and $M_3 = (\delta-1)\gamma^{-1} + \alpha_1^\delta\gamma^{\delta-1}(1/\delta-1)\{(\alpha_1\gamma)^\delta + \alpha_2^\delta\}^{-1}$, as $t \to \infty$, $(\alpha_1, \alpha_2) \in [0,1]^2$, $T = \beta\log(1+t)$, $\delta \geq 0$ and $\gamma = \lim_{T\to\infty} argmax_s\ g(s,T)$. □

Corollary 55. *Suppose X_1 and X_2 are two dependent random variables. In the case of Khoudraji non-exchangeable Gumbel Copula with Pareto margin, if $\alpha_1 + 1/\beta > 1$, Then*

$$\mathrm{E}[X_2|X_1 > t] \sim (1+t)^{\beta M_1'}\sqrt{\frac{2\pi\log(1+t)}{M_2'M_3'}},$$

where $M_1' = \alpha_1 + \gamma/\beta - \gamma(1-\alpha_2) - [(\alpha_2\gamma)^\delta + \alpha_1^\delta]^{1/\delta}$, $M_2' = \alpha_2^\delta\delta\{(\alpha_2\gamma)^\delta + \alpha_1^\delta\}^{1/\delta-1}\gamma^{\delta-1}$, and $M_3' = (\delta-1)\gamma^{-1} + \alpha_2^\delta\gamma^{\delta-1}(1/\delta-1)\{(\alpha_2\gamma)^\delta + \alpha_1^\delta\}^{-1}$, as $t \to \infty$, $(\alpha_1, \alpha_2) \in [0,1]^2$, $T = \beta\log(1+t)$, $\delta \geq 0$ and $\gamma = \lim_{T\to\infty} argmax_s\ g(s,T)$.

Remark 20. In this case as $\mathrm{E}[X_1|X_2 > t] \neq \mathrm{E}[X_2|X_1 > t]$ as $t \to \infty$, we conclude that $\eta_1(t) \neq 1$ as $t \to \infty$. Thus, the system is *non-exchangeable* of Khoudraji (1996) type.

Proposition 56. *Suppose X_1 and X_2 are two dependent random variables. In the case of Khoudraji non-exchangeable Gumbel Copula with Pareto margin, if $\alpha_1 + 1/\beta \ngtr 1$ and $\int_0^\infty e^{g(s,T)}\ |h(s)|\ ds < \infty$, Then*

$$\mathrm{E}[X_1|X_2 > t] \sim \frac{\alpha_1}{\beta \log(1+t)} \left(\frac{1}{1 - \alpha_1 - \frac{1}{\beta}} \right)^2,$$

as $t \to \infty$, $\alpha_1 \in [0, 1]$, $\beta > 1$ and at $\alpha_2 \to 0$.

Proof. In this case let us divide the integrand of the conditional expectation into two multiplicative separable functions. In other words, let us redefine $g(s,T) = \alpha_2 + s\left[\frac{1}{\beta} - 1 + \alpha_1\right]$ and $h(s) = [(\alpha_1 s)^\delta + \alpha_2^\delta]^{1/\delta}$, where $(\alpha_1, \alpha_2) \in [0,1]^2$, $\delta \geq 0$ and $\beta > 1$. If we look at the g function carefully, $g(0,T) \to 0$ as $\alpha_2 \to 0$. That means, in order to g function to be applicable to use Watson's lemma we need $\alpha_2 \in [0, 0+\epsilon)$.

As $g(s;T)$ is a real valued function on the semi-infinite interval $[0,\infty)$ and in $(0, 0+\epsilon]$ with $\epsilon > 0$ this function is continuously differentiable and $\sup_{0+\epsilon \leq s \leq \infty} g(s,T) \leq g(0,T) - \psi$, with $\psi > 0$. Here we are using Theorem 36 [p.48] of Breitung (1994). This theorem is an extension of Watson's lemma and most importantly, this theorem works for semi-infinite intervals. Now we have $g'(s,T) < 0$ and $s \in [0, 0+\epsilon)$. We can also write $g'(s,T) = -as^{r-1} + o(s^{r-1})\ \forall\ r > 0$ Now if we assume $r = 1$ then $g'(s,T) = -a$. From our previous results we know that, $\lim_{s^+ \to\ 0,\ T \to \infty} g'(s,T) = 1/\beta - 1 + \alpha_1$, which is a constant. Thus, $-a = -[1/\beta - 1 + \alpha_1]$ or, $a = 1 - \alpha_1 - 1/\beta$. Let us assume there is another real and continuous function $h(s) \in [0,\infty)$ such that,

$$h(s) = bs^{m-1} + o(s^{m-1})$$

with $m > 0$. More specifically we assume $h(s) = [(\alpha_1 s)^\delta + \alpha_2^\delta]^{1/\delta}$ in our case. Thus,

$$\begin{aligned} [(\alpha_1 s)^\delta + \alpha_2^\delta]^{1/\delta} &= bs^{m-1} + o(s^{m-1}) \\ (\alpha_1 s)^\delta + \alpha_2^\delta &= b^\delta s^{\delta(m-1)} + o(s^{m-1}) \\ \alpha_1^\delta s^\delta &= b^\delta s^{\delta(m-1)}, \; as \; \alpha_2^\delta \sim o(s^{m-1}) \end{aligned} \tag{3.50}$$

Thus, we have $b = \alpha_1$ and $m = 2$ [as $\delta = \delta(m-1)$, the powers of s on the both sides of (3.50) are the same].

Finally, as we are assuming $\int_0^\infty e^{g(s,T)} \; |h(s)| \; ds < \infty$ then by Watson's lemma we can write the approximated value of the integral $I(T) = \int_0^\infty e^{Tg(s,T)} \; h(s) \; ds$ with $T \geq 1$ are all finite and the asymptotic approximation is;

$$\begin{aligned} I(T) &\sim \alpha_1 \Gamma(2) \left(\frac{1}{1 - \alpha_1 - \frac{1}{\beta}} \right)^2 T^{-2} e^{Tg(0,T)} \\ &\sim \alpha_1 \left(\frac{1}{1 - \alpha_1 - \frac{1}{\beta}} \right)^2 T^{-2}, \; as \; g(0,T) = 0, \; \Gamma(2) = 1 \end{aligned} \tag{3.51}$$

Now, using (3.51) we get the conditional expectation as ;

$$\mathrm{E}[X_1 | X_2 > t] \sim \frac{\alpha_1}{\beta \log(1+t)} \left(\frac{1}{1 - \alpha_1 - \frac{1}{\beta}} \right)^2 \tag{3.52}$$

as $t \to \infty$, $\alpha_1 \in [0, 1]$ and $\beta > 1$. This completes the proof. □

Corollary 57. *Suppose X_1 and X_2 are two dependent random variables. In the case of Khoudraji non-exchangeable Gumbel Copula with Pareto margin, if $\alpha_1 + 1/\beta \not> 1$ and $\int_0^\infty e^{g(s,T)}\ |h(s)|\ ds < \infty$, Then*

$$\mathrm{E}[X_2|X_1 > t] \sim \frac{\alpha_2}{\beta \log(1+t)} \left(\frac{1}{1-\alpha_2 - \frac{1}{\beta}} \right)^2,$$

as $t \to \infty$, $\alpha_2 \in [0,1]$, $\beta > 1$ and at $\alpha_2 \to 0$.

Remark 21. In the case of Watson's Lemma our measure of non-exchangeability becomes;

$$\eta_1(t) = \frac{\mathrm{E}[X_1|X_2 > t]}{\mathrm{E}[X_2|X_1 > t]} \sim \frac{\alpha_1}{\alpha_2} \left(\frac{1-\alpha_2-\frac{1}{\beta}}{1-\alpha_1-\frac{1}{\beta}} \right)^2 \neq 1,\ if\ \alpha_1 \neq \alpha_2,\ as\ t \to \infty,$$

where $(\alpha_1, \alpha_2)\ \in [0,1]^2$, $\alpha_2 \neq 0$, $\beta > 1$ and at $\alpha_2 \to 0$. $\eta_1(t) = 1$ iff $\alpha_1 = \alpha_2$, as $t \to \infty$. The system is non-exchangeable of Khoudraji (1996) type as long as $\alpha_1 \neq \alpha_2$ at $t \to \infty$.

Proposition 58. *Suppose X_1 and X_2 are two dependent random variables. In Khoudraji (1996) non-exchangeable transformed Gumbel Survival Copula with Weibull margin, the conditional tail expectation can be written as;*

$$\mathrm{E}[X_1|X_2 > t] \sim \frac{\alpha_2}{\gamma}\ \Gamma(\frac{1}{\gamma}) \left(\frac{1}{1-\alpha_1} \right)^{\frac{1}{\gamma}},$$

as $t \to \infty$; where $\int_0^\infty e^{g(s,T)}\ |h(s)|\ ds < \infty$, $\gamma > 0$ and $(\alpha_1, \alpha_2) \in [0,1]^2$ with $\alpha_2 \to 0$ and $\alpha_1 \neq 1$.

Proof. From previous chapter and following the method provided by Hua and Joe (2014) we know that the conditional expectation of survival Copula with Weibull margin can be written as;

$$
\begin{aligned}
\mathrm{E}[X_1|X_2 > t] &= \gamma^{-1}T^{\frac{1}{\gamma}} \int_0^\infty e^{T[1+\frac{1}{T}\log\widehat{C}(e^{-sT},e^{-T})]} s^{\frac{1}{\gamma}-1}\, ds \\
&= \gamma^{-1}T^{\frac{1}{\gamma}} \int_0^\infty e^{T\left[1+\frac{1}{T}\log\left[e^{-T\left\{s(1-\alpha_1)+(1-\alpha_2)+[(\alpha_1 s)^\delta+\alpha_2^\delta]^{1/\delta}\right\}}\right]\right]} s^{\frac{1}{\gamma}-1}\, ds \\
&= \gamma^{-1}T^{\frac{1}{\gamma}} \int_0^\infty e^{T\left[1+\frac{1}{T}\log\left[e^{-T\{s(1-\alpha_1)+(1-\alpha_2)\}}\right]\right]} [(\alpha_1 s)^\delta+\alpha_2^\delta]^{1/\delta} s^{\frac{1}{\gamma}-1}\, ds \qquad (3.53)\\
&= \gamma^{-1}T^{\frac{1}{\gamma}} \int_0^\infty e^{Tg(s,T)}\, h(s)\, ds,\ \forall\, T = t^\gamma,
\end{aligned}
$$

where $g(s,T) = 1 + \frac{1}{T}\log\left[e^{-T\{s(1-\alpha_1)+(1-\alpha_2)\}}\right] = 1 - s(1-\alpha_1) - (1-\alpha_2)$ and $h(s) = [(\alpha_1 s)^\delta + \alpha_2^\delta]^{1/\delta} s^{\frac{1}{\gamma}-1}$. Here we are assuming $\gamma > 0$, $(\alpha_1, \alpha_2) \in [0,1]^2$, $\delta \geq 0$ and $s \in [0,\infty)$.

If we further solve the g function we get, $g(s,T) = \alpha_2 - s(1-\alpha_1)$. From simple calculations we can easily verify that $g(0,T) \to 0$ at $\alpha_2 \to 0$ and $g(\infty,T) = -\infty$. Thus, we can use either Laplace Approximation or Watson's lemma. But as $g'(s,T) = -(1-\alpha_1) < 0$ irrespective of any s we have $g'(0,T) < 0$, we can never use Laplace Approximation. In this case , we have to use Watson's lemma.

As $g(s;T)$ is a real valued function on the semi-infinite interval $[0,\infty)$ and in $(0, 0+\epsilon]$ with $\epsilon > 0$ this function is continuously differentiable and $\sup_{0+\epsilon \leq s \leq \infty} g(s,T) \leq g(0,T) - \psi$, with $\psi > 0$. Here we are using Theorem 36 [p.48] of Breitung (1994). This theorem is an extension of Watson's lemma and most importantly, this theorem works for semi-infinite intervals. Now we have $g'(s,T) < 0$ and $s \to 0^+$.We can also write $g'(s,T) = -as^{r-1} + o(s^{r-1})\ \forall\ r > 0$ Now if we assume $r = 1$ then $g'(s,T) = -a$. From our previous results we know that, $\lim_{s^+ \to\, 0,\ T\to\infty} g'(s,T) = -(1-\alpha_1)$, which is a constant. Thus, $-a = -[1-\alpha_1]$ or, $a = 1 - \alpha_1$. Let us assume there is another real and continuous function $h(s) \in [0,\infty)$ such that,

$h(s) = bs^{m-1} + o(s^{m-1})$ with $m > 0$. More specifically we assume $h(s) = [(\alpha_1 s)^\delta + \alpha_2^\delta]^{1/\delta} s^{\frac{1}{\gamma}-1}$ in our case.

Thus,

$$\begin{aligned}
[(\alpha_1 s)^\delta + \alpha_2^\delta]^{1/\delta} s^{\frac{1}{\gamma}-1} &= bs^{m-1} + o(s^{m-1}) \\
\left\{ s^{\delta\left[\frac{1}{\gamma}-1\right]} [(\alpha_1 s)^\delta + \alpha_2^\delta] \right\}^{1/\delta} &= bs^{m-1} + o(s^{m-1}) \\
s^{\delta\left[\frac{1}{\gamma}-1\right]} [(\alpha_1 s)^\delta + \alpha_2^\delta] &= b^\delta s^{\delta(m-1)} + o(s^{m-1}) \\
s^{\left[\frac{1}{\gamma}-1\right]} [(\alpha_1 s)^\delta + \alpha_2^\delta] &= b^\delta s^{(m-1)} + o(s^{m-1}) \\
(\alpha_1 s)^\delta s^{\left[\frac{1}{\gamma}-1\right]} + \alpha_2^\delta s^{\left[\frac{1}{\gamma}-1\right]} &= b^\delta s^{(m-1)} + o(s^{m-1}) \qquad (3.54)
\end{aligned}$$

Now, for any $\delta > 1/\gamma$ the most left expression [i.e. $(\alpha_1 s)^\delta s^{\left[\frac{1}{\gamma}-1\right]} \to o(s^{m-1})$ as $s^+ \to 0$ and $T \to \infty$]. Thus, we have $b = \alpha_2$ and $m = 1/\gamma$.

Using the above information and by Watson's lemma we get,

$$I(T) \sim \alpha_2 \Gamma\left(\frac{1}{\gamma}\right) \left(\frac{1}{1-\alpha_1}\right)^{\frac{1}{\gamma}} T^{-\frac{1}{\gamma}}, \ as\ T \to \infty. \qquad (3.55)$$

Combining (3.53) and (3.55) we get the conditional tail expectation as;

$$\mathrm{E}[X_1 | X_2 > t] \sim \frac{\alpha_2}{\gamma}\ \Gamma\left(\frac{1}{\gamma}\right) \left(\frac{1}{1-\alpha_1}\right)^{\frac{1}{\gamma}}, \ as\ t \to \infty, \qquad (3.56)$$

where $\gamma > 0$ and $(\alpha_1, \alpha_2) \in [0,1]^2$ with $\alpha_2 \to 0$. This completes the proof. $\square$

Corollary 59. *Suppose X_1 and X_2 are two dependent random variables with $\int_0^\infty e^{g(s,T)}\ |h(s)|\ ds < \infty$. In Khoudraji (1996) non-exchangeable transformed Gumbel Survival Copula with Weibull margin, the conditional tail expectation can be written as;*

$$\mathrm{E}[X_2|X_1 > t] = \frac{\alpha_1}{\gamma}\ \Gamma\left(\frac{1}{\gamma}\right)\left(\frac{1}{1-\alpha_2}\right)^{\frac{1}{\gamma}},$$

as $t \to \infty$; where $\gamma > 0$ and $(\alpha_1, \alpha_2) \in [0,1]^2$.

Remark 22. In the case of Watson's Lemma our measure of non-exchangeability becomes;

$$\eta_1(t) = \frac{\mathrm{E}[X_1|X_2 > t]}{\mathrm{E}[X_2|X_1 > t]} \sim \frac{\alpha_2}{\alpha_1}\left(\frac{1-\alpha_2}{1-\alpha_1}\right)^{\frac{1}{\gamma}} \neq 1,\ if\ \alpha_1 \neq \alpha_2,\ as\ t \to \infty,$$

where $(\alpha_1, \alpha_2) \in [0,1]^2$, $\alpha_1 \neq 0$ and $\gamma > 0$. $\eta_1(t) = 1$ iff $\alpha_1 = \alpha_2 \neq 1$, as $t \to \infty$. The system is non-exchangeable of Khoudraji (1996) type as long as $\alpha_1 \neq \alpha_2$ at $t \to \infty$.

Corollary 60. *Suppose X_1 and X_2 are two dependent random variables with $\int_0^\infty e^{g(s,T)}\ |h(s)|\ ds < \infty$. In Khoudraji (1996) non-exchangeable transformed Gumbel Survival Copula with Exponential margin, the conditional tail expectation can be written as;*

$$\mathrm{E}[X_1|X_2 > t] \sim \frac{\alpha_1}{\lambda^2 t}\ \left(\frac{1}{1-\alpha_1}\right)^2,$$

as $t \to \infty$; where $T = \lambda t$, $\lambda > 0$ and $\alpha_1 \in [0,1)$.

Proof. Please see in the Appendix. □

Corollary 61. *Suppose X_1 and X_2 are two dependent random variables. In Khoudraji (1996) non-exchangeable transformed Gumbel Survival Copula with Exponential margin, the conditional tail expectation can be written as;*

$$\mathrm{E}[X_2|X_1 > t] \sim \frac{\alpha_2}{\lambda^2 t} \left(\frac{1}{1-\alpha_2}\right)^2,$$

as $t \to \infty$; where $T = \lambda t$, $\int_0^\infty e^{g(s,T)}\ |h(s)|\ ds < \infty$, $\lambda > 0$ and $\alpha_2 \in [0, 1)$.

Remark 23. In the case of Khoudraji-transformed Survival Gumbel Copula with Exponential margin our measure of non-exchangeability becomes;

$$\eta_1(t) = \frac{\mathrm{E}[X_1|X_2 > t]}{\mathrm{E}[X_2|X_1 > t]} \sim \frac{\alpha_1}{\alpha_2} \left(\frac{1-\alpha_2}{1-\alpha_1}\right)^2 \neq 1,\ if\ \alpha_1 \neq \alpha_2,\ as\ t \to \infty,$$

where $(\alpha_1, \alpha_2) \in [0, 1]^2$, $\alpha_1 \neq 0$ and $\lambda > 0$. $\eta_1(t) = 1$ iff $\alpha_1 = \alpha_2 \neq 1$, as $t \to \infty$. The system is non-exchangeable of Khoudraji (1996) type as long as $\alpha_1 \neq \alpha_2$ at $t \to \infty$.

3.2.2 Case II : $\mathrm{E}[X_1|X_2 = t]$

Now we are going to work with another type of conditional expectation $\mathrm{E}[X_1|X_2 = t]$. Firstly, we are trying to calculate this expectation at the tail. Then use our definition to check out the tail order non-exchangeability of Khoudraji type. Like before we are using the Survival Gumbel copula function, but here we are using factor-Copula instead of just using the survival Copula itself. Again, like before we are considering three margins namely Pareto, Weibull and Exponential respectively. As the integrands of the conditional expectations don not have any closed form, we have to use either Laplace

Approximation or Watson's lemma. Then, after getting conditional tail expectations of $\mathrm{E}[X_1|X_2 = t]$ and $\mathrm{E}[X_2|X_1 = t]$ we are checking if $\eta_2(t) \to 1$ as $t \to \infty$. If $\eta_2(t) \to 1$ as $t \to \infty$, we see the system is exchangeable otherwise not. Throughout our paper we are assuming the Khoudraji (1996) non-exchangeable transformed survival Copula is $\widehat{C}(u,v) = u^{1-\alpha_1} v^{1-\alpha_2} \widehat{C}^*(u^{\alpha_1}, v^{\alpha_2})$, where $u = \overline{F}(x)$ and $v = \overline{F}(t)$ are two survival functions [hence, their values are always between 0 and 1].Thus for the Gumbel case it becomes, $\widehat{C}(u,v) = u^{1-\alpha_1} v^{1-\alpha_2}\ exp\left\{-\left[(-\log u^{\alpha_1})^{\delta} + (-\log v^{\alpha_2})^{\delta}\right]^{\frac{1}{\delta}}\right\}$.

After doing the first order differentiation of this we have,

$$\begin{aligned}\widehat{C}_{1|2}(u|v) = (1-\alpha_2) u^{1-\alpha_1} v^{-\alpha_2}\ exp\left\{-\left[(-\log u^{\alpha_1})^{\delta} + (-\log v^{\alpha_2})^{\delta}\right]^{\frac{1}{\delta}}\right\} \\ + \alpha_2 u^{1-\alpha_1} v^{-\alpha_2}\ exp\left\{-\left[(-\log u^{\alpha_1})^{\delta} + (-\log v^{\alpha_2})^{\delta}\right]^{\frac{1}{\delta}}\right\} \\ \left[(-\log u^{\alpha_1})^{\delta} + (-\log v^{\alpha_2})^{\delta}\right]^{\frac{1}{\delta}-1} (\log v^{\alpha_2})^{\delta-1},\ \forall\ (u,v) \in [0,1]^2 \end{aligned} \tag{3.57}$$

Now, if we put $u = e^{-sT}$ and $v = e^{-T}$ in (3.57) and solve further we get;

$$\begin{aligned}\widehat{C}_{1|2}(e^{-sT}|e^{-T}) &= (1-\alpha_2)\ e^{-T\{s(1-\alpha_1)-\alpha_2\}}\ exp\left\{-\left[(\alpha_1 sT)^{\delta} + (\alpha_2 T)^{\delta}\right]^{\frac{1}{\delta}}\right\} \\ &\quad + \alpha_2 T(1-\delta)\left[(\alpha_1 sT)^{\delta} + (\alpha_2 T)^{\delta}\right]^{\frac{1}{\delta}-1} \\ &\quad\quad e^{-T\{s(1-\alpha_1)-\alpha_2\}}\ exp\left\{-\left[(\alpha_1 sT)^{\delta} + (\alpha_2 T)^{\delta}\right]^{\frac{1}{\delta}}\right\} \\ &= (1-\alpha_2)\ e^{-T\{s(1-\alpha_1)-\alpha_2\}+\left[(\alpha_1 s)^{\delta}+(\alpha_2)^{\delta}\right]^{\frac{1}{\delta}}} \\ &\quad + \alpha_2 T^{2-\delta}(1-\delta)\left[(\alpha_1 s)^{\delta} + (\alpha_2)^{\delta}\right]^{\frac{1}{\delta}-1} \\ &\quad\quad e^{-T\{s(1-\alpha_1)-\alpha_2\}+\left[(\alpha_1 s)^{\delta}+(\alpha_2)^{\delta}\right]^{\frac{1}{\delta}}}, \end{aligned} \tag{3.58}$$

where $T = -\log \overline{F}(t)$, $(\alpha_1, \alpha_2) \in [0,1]^2$, $\delta \geq 0$ and $s \in [0, \infty)$.

From the survival Copula in (3.58) we can calculate the tail order conditional expectations on *Pareto, Weibull* and *Exponential* margins respectively. Again, like before, we only can use *Laplace* Approximation only in the case of Gumbel Copula with Pareto margin.

Proposition 62. *Suppose X_1 and X_2 are two dependent random variables. In the case of Khoudraji non-exchangeable Gumbel Copula with Pareto margin, if $\alpha_1 + 1/\beta > 1$, Then,*

$$
\begin{aligned}
\mathrm{E}[X_1|X_2=t] \sim & \frac{(1-\alpha_1)}{\beta}(1+t)^{\beta M_1}\sqrt{\frac{2\pi\log(1+t)}{M_2M_3}} \\
& + \frac{\alpha_2(1-\delta)(\beta\log(1+t))^{2-\delta}}{\beta}\left[(\alpha_1\gamma)^\delta+(\alpha_2)^\delta\right]^{\frac{1}{\delta}-1}(1+t)^{\beta M_1}\sqrt{\frac{2\pi\log(1+t)}{M_2M_3}},
\end{aligned}
$$

where $M_1 = \alpha_2 + \gamma/\beta - \gamma(1-\alpha_1) - [(\alpha_1\gamma)^\delta + \alpha_2^\delta]^{1/\delta}$, $M_2 = \alpha_1^\delta\delta\{(\alpha_1\gamma)^\delta + \alpha_2^\delta\}^{1/\delta-1}\gamma^{\delta-1}$, and $M_3 = (\delta-1)\gamma^{-1} + \alpha_1^\delta\gamma^{\delta-1}(1/\delta-1)\{(\alpha_1\gamma)^\delta+\alpha_2^\delta\}^{-1}$, as $t\to\infty$, $(\alpha_1,\alpha_2) \in [0,1]^2$, $\delta \geq 0$ and $\gamma = \lim_{T\to\infty} argmax_s\ g(s,T)$.

Proof. From the discussion from the previous chapter and following method by Hua and Joe (2014) we know that the conditional expectation of survival non-exchangeable factor Gumbel Copula with Pareto margin can be written as;

$$
\begin{aligned}
\mathrm{E}[X_1|X_2=t] &= \frac{T}{\beta}\int_0^\infty \widehat{C}_{1|2}(e^{-sT}|e^{-T})e^{\frac{sT}{\beta}}ds \\
&= \frac{T}{\beta}\int_0^\infty \Big[(1-\alpha_2)\ e^{-T\{s(1-\alpha_1)-\alpha_2+\left[(\alpha_1 s)^\delta+(\alpha_2)^\delta\right]^{\frac{1}{\delta}}\}} \\
&\quad + \alpha_2 T^{2-\delta}(1-\delta)\left[(\alpha_1 s)^\delta+(\alpha_2)^\delta\right]^{\frac{1}{\delta}-1} e^{-T\{s(1-\alpha_1)-\alpha_2+\left[(\alpha_1 s)^\delta+(\alpha_2)^\delta\right]^{\frac{1}{\delta}}\}}\Big]\ e^{\frac{sT}{\beta}}ds \\
&= \frac{(1-\alpha_1)T}{\beta}\int_0^\infty e^{T\{\frac{s}{\beta}-s(1-\alpha_1)+\alpha_2-\left[(\alpha_1 s)^\delta+(\alpha_2)^\delta\right]^{\frac{1}{\delta}}\}}\ ds \\
&\quad + \frac{\alpha_2(1-\delta)T^{3-\delta}}{\beta}\int_0^\infty e^{T\{\frac{s}{\beta}-s(1-\alpha_1)+\alpha_2-\left[(\alpha_1 s)^\delta+(\alpha_2)^\delta\right]^{\frac{1}{\delta}}\}}\ \left[(\alpha_1 s)^\delta+(\alpha_2)^\delta\right]^{\frac{1}{\delta}-1}\ ds
\end{aligned} \tag{3.59}
$$

$$
= \frac{(1-\alpha_1)T}{\beta}\int_0^\infty e^{Tg(s,T)}ds + \frac{\alpha_2(1-\delta)T^{3-\delta}}{\beta}\int_0^\infty e^{Tg(s,T)}\ h(s)\ ds,
$$

From (3.59) we are clear that we have to do two separate Laplace approximations of the two integrals. Here $g(s,T) = \frac{s}{\beta} - s(1-\alpha_1) + \alpha_2 - \left[(\alpha_1 s)^\delta + (\alpha_2)^\delta\right]^{\frac{1}{\delta}}$ and
$h(s) = \left[(\alpha_1 s)^\delta + (\alpha_2)^\delta\right]^{\frac{1}{\delta}-1} > 0$.

We can write the $g(s,T) = \alpha_2 + s[1/\beta + \alpha_1 - 1] - [(\alpha_1 s)^\delta + (\alpha_2)^\delta]^{\frac{1}{\delta}}$, where $(\alpha_1, \alpha_2) \in [0,1]^2$, $\delta \geq 0$, $\beta > 1$ and $s \in [0, \infty)$. After doing some easy calculation and assuming $[(\alpha_1 s)^\delta + \alpha_2^\delta]^{1/\delta}$ dominates the rest of the terms in g we find out that $g(0,T) = 0$ and $g(\infty, T) = -\infty$. As these two conditions are satisfied we can do either Laplace Approximation or Watson's lemma. Let us do the first one first. If we do the first order derivative of g as explained above with respect to s we have;

$$g'(s,T) = \left[\frac{1}{\beta} + \alpha_1 - 1\right] - \delta\alpha_1^\delta s^{\delta-1}[(\alpha_1 s)^\delta + \alpha_2^\delta]^{1/\delta-1}, \tag{3.60}$$

where $(\alpha_1, \alpha_2) \in [0,1]^2$, $\delta \geq 0$, $\beta > 1$ and $s \in [0, \infty)$. Thus, from (3.60) we immediately get $g'(0,T) = 1/\beta + \alpha_1 - 1 > 0$ [by assumption] as $T \to \infty$.

Again if we do the second order derivative of the g function we defined earlier, we get;

$$\begin{aligned} g''(s,T) &= -\delta\alpha_1^\delta \Big[(\delta-1)s^{\delta-2}[(\alpha_1 s)^\delta + \alpha_2^\delta]^{1/\delta-1} \\ &\quad + s^{\delta-1}\left(\frac{1}{\delta}-1\right)[(\alpha_1 s)^\delta + \alpha_2^\delta]^{1/\delta-2}\delta\alpha_1(\alpha_1 s)^{\delta-1}\Big] < 0 \\ -g''(s_0,T) &= \delta\alpha_1^\delta \Big[(\delta-1)s_0^{\delta-2}[(\alpha_1 s_0)^\delta + \alpha_2^\delta]^{1/\delta-1} \\ &\quad + s_0^{\delta-1}\left(\frac{1}{\delta}-1\right)[(\alpha_1 s_0)^\delta + \alpha_2^\delta]^{1/\delta-2}\delta\alpha_1(\alpha_1 s_0)^{\delta-1}\Big] > 0, \end{aligned} \tag{3.61}$$

where $(\alpha_1, \alpha_2) \in [0,1]^2$, $\delta \geq 0$, $\beta > 1$, $s \in [0, \infty)$ and
$\gamma = \lim_{T\to\infty} s_0(T) = \lim_{T\to\infty}\ argmax_s\ g(s,T)$.

Now let us see (3.60) more carefully. If we do the first order condition of this equation, we have;

$$\left[\frac{1}{\beta}+\alpha_1-1\right] = \delta\alpha_1^\delta s^{\delta-1}[(\alpha_1 s)^\delta+\alpha_2^\delta]^{1/\delta-1}$$
$$\frac{\frac{1}{\beta}+\alpha_1-1}{\delta\alpha_1^\delta} = s^{\delta-1}[(\alpha_1 s)^\delta+\alpha_2^\delta]^{1/\delta-1}$$
$$\zeta s^{1-\delta} = [(\alpha_1 s)^\delta+\alpha_2^\delta]^{\frac{1-\delta}{\delta}},\ where\ \zeta = \frac{\frac{1}{\beta}+\alpha_1-1}{\delta\alpha_1^\delta},$$
$$\zeta^{\frac{\delta}{1-\delta}} s^\delta = (\alpha_1 s)^\delta+\alpha_2^\delta$$
$$s^\delta = \left(\frac{\zeta^{\frac{\delta}{1-\delta}}}{\alpha_1}\right)^\delta - \left(\frac{\alpha_2}{\alpha_1}\right)^\delta$$
$$s^\delta = \left(\frac{\alpha_2}{\alpha_1}\right)^\delta\left[\left(\frac{\zeta^{\frac{1}{1-\delta}}}{\alpha_1}\right)^\delta-1\right]^{-1} \tag{3.62}$$

After solving (3.62) further and putting value of ζ we have;
$s_0(T) = (\alpha_2/\alpha_1)\left[((\frac{1}{\beta}+\alpha_1-1)/\delta\alpha_1^\delta)/\alpha_1-1\right]^{-1/\delta}$. As in this expression we do not have any term related to either T or s, we can directly say that, $\gamma = s_0(T)$ in this case.

Following Hua and Joe (2014) if we use the Laplace approximation method the value of conditional tail expectations become;

$$\begin{aligned}
&\mathrm{E}[X_1|X_2=t]\\
&\sim \frac{(1-\alpha_1)}{\beta}e^{Tg(\gamma,T)}\sqrt{\frac{2\pi T}{-g''(\gamma,T)}}+\frac{\alpha_2(1-\delta)T^{2-\delta}}{\beta}\left[(\alpha_1\gamma)^\delta+(\alpha_2)^\delta\right]^{\frac{1}{\delta}-1}e^{Tg(\gamma,T)}\sqrt{\frac{2\pi T}{-g''(\gamma,T)}}\\
&\sim \frac{(1-\alpha_1)}{\beta}e^{\beta\log(1+t)\left[\alpha_2+\frac{\gamma}{\beta}-\gamma(1-\alpha_1)-[(\alpha_1\gamma)^\delta+\alpha_2^\delta]^{1/\delta}\right]}\\
&\quad\sqrt{\frac{2\pi\beta\log(1+t)}{\alpha_1^\delta\delta\{(\alpha_1\gamma)^\delta+\alpha_2^\delta\}^{1/\delta-1}\gamma^{\delta-1}\left[(\delta-1)\gamma^{-1}+\alpha_1^\delta\gamma^{\delta-1}(1/\delta-1)\{(\alpha_1\gamma)^\delta+\alpha_2^\delta\}^{-1}\right]}}\\
&\qquad+\frac{\alpha_2(1-\delta)(\beta\log(1+t))^{2-\delta}}{\beta}\left[(\alpha_1\gamma)^\delta+(\alpha_2)^\delta\right]^{\frac{1}{\delta}-1}\\
&\qquad e^{\beta\log(1+t)\left[\alpha_2+\frac{\gamma}{\beta}-\gamma(1-\alpha_1)-[(\alpha_1\gamma)^\delta+\alpha_2^\delta]^{1/\delta}\right]}\\
&\qquad\quad\sqrt{\frac{2\pi\beta\log(1+t)}{\alpha_1^\delta\delta\{(\alpha_1\gamma)^\delta+\alpha_2^\delta\}^{1/\delta-1}\gamma^{\delta-1}\left[(\delta-1)\gamma^{-1}+\alpha_1^\delta\gamma^{\delta-1}(1/\delta-1)\{(\alpha_1\gamma)^\delta+\alpha_2^\delta\}^{-1}\right]}}
\end{aligned}\tag{3.63}$$

$$\begin{aligned}
&\sim \frac{(1-\alpha_1)}{\beta}(1+t)^{\beta M_1}\sqrt{\frac{2\pi\log(1+t)}{M_2M_3}}\\
&\qquad+\frac{\alpha_2(1-\delta)(\beta\log(1+t))^{2-\delta}}{\beta}\left[(\alpha_1\gamma)^\delta+(\alpha_2)^\delta\right]^{\frac{1}{\delta}-1}(1+t)^{\beta M_1}\sqrt{\frac{2\pi\log(1+t)}{M_2M_3}},
\end{aligned}\tag{3.64}$$

where $M_1=\alpha_2+\gamma/\beta-\gamma(1-\alpha_1)-[(\alpha_1\gamma)^\delta+\alpha_2^\delta]^{1/\delta}$, $M_2=\alpha_1^\delta\delta\{(\alpha_1\gamma)^\delta+\alpha_2^\delta\}^{1/\delta-1}\gamma^{\delta-1}$, and $M_3=(\delta-1)\gamma^{-1}+\alpha_1^\delta\gamma^{\delta-1}(1/\delta-1)\{(\alpha_1\gamma)^\delta+\alpha_2^\delta\}^{-1}$, as $t\to\infty$, $(\alpha_1,\alpha_2)\in[0,1]^2$, $\delta\geq 0$ and $\gamma=\lim_{T\to\infty} argmax_s\ g(s,T)$. This completes the proof. □

Corollary 63. *Suppose X_1 and X_2 are two dependent random variables. In the case of Khoudraji (1996) non-exchangeable Gumbel Copula with Pareto margin, if $\alpha_1 + 1/\beta > 1$, Then,*

$$\mathrm{E}[X_2|X_1 = t] \sim \frac{(1-\alpha_2)}{\beta}(1+t)^{\beta M_1'}\sqrt{\frac{2\pi\log(1+t)}{M_2' M_3'}} + \frac{\alpha_1(1-\delta)(\beta\log(1+t))^{2-\delta}}{\beta}\left[(\alpha_2\gamma)^\delta + (\alpha_1)^\delta\right]^{\frac{1}{\delta}-1}(1+t)^{\beta M_1'}\sqrt{\frac{2\pi\log(1+t)}{M_2' M_3'}},$$

where $M_1' = \alpha_1 + \gamma/\beta - \gamma(1-\alpha_2) - [(\alpha_2\gamma)^\delta + \alpha_1^\delta]^{1/\delta}$, $M_2' = \alpha_2^\delta\delta\{(\alpha_2\gamma)^\delta + \alpha_1^\delta\}^{1/\delta-1}\gamma^{\delta-1}$, and $M_3' = (\delta-1)\gamma^{-1} + \alpha_2^\delta\gamma^{\delta-1}(1/\delta-1)\{(\alpha_2\gamma)^\delta + \alpha_1^\delta\}^{-1}$, as $t \to \infty$, $\alpha_1, \alpha_2 \in [0,1]$, $\delta \geq 0$ and $\gamma = \lim_{T\to\infty} argmax_s\ g(s,T)$.

Remark 24. In the case of Khoudraji-transformed Survival factor Gumbel Copula with Pareto margin our measure of non-exchangeability becomes;

$$\eta_2(t) = \frac{\mathrm{E}[X_1|X_2 = t]}{\mathrm{E}[X_2|X_1 = t]} \neq 1,\ if\ \alpha_1 \neq \alpha_2,\ as\ t \to \infty,$$

where $(\alpha_1, \alpha_2) \in [0,1]^2$, $\alpha_1 \neq 0$ and $\lambda > 0$. $\eta_1(t) = 1$ iff $\alpha_1 = \alpha_2 \neq 1$, as $t \to \infty$. The system is non-exchangeable of Khoudraji (1996) type as long as $\alpha_1 \neq \alpha_2$ at $t \to \infty$. We get the values of two conditional tail expectations from previous two proposition and corollary respectively.

Remark 25. In the same way we can calculate the expression for $\mathrm{E}[X_1|X_2 = t]$ when $\alpha_1 + 1/\beta < 1$. In this case we can not use Laplace approximation as $g'(0,T) \ngtr 0$. We will have to proceed in the same way we explained this method before. Again in this case if $\alpha_1 \neq \alpha_2$ then clearly we are going to get a non-exchangeable structure. Thus, in this case $\eta_2(t) \neq 1$ as $t \to \infty$.

Proposition 64. *Suppose X_1 and X_2 are two dependent random variables and $\int_0^\infty e^{g(s,T)}\ |h_1(s)|\ ds < \infty$ and $\int_0^\infty e^{g(s,T)}\ |h_2(s)|\ ds < \infty$. In the case of Khoudraji (1996) non-exchangeable transformed survival Gumbel Copula with Weibull margin, Then,*

$$\mathrm{E}[X_1|X_2 = t] \sim \frac{N}{\gamma}\left[1 - \alpha_2 + \alpha_2^2(1-\delta)t^{\gamma(2-\delta)}\right],$$

where $N = \Gamma\left(\frac{1}{\gamma}\right)\left(\frac{1}{1-\alpha_1}\right)^{\frac{1}{\gamma}}$, $\gamma > 0$, as $t \to \infty$, $s^+ \to\ 0$, $(\alpha_1, \alpha_2)\ \in [0,1]^2$ and $\delta \geq 0$.

Proof. From the previous chapter we clearly know for the case of survival Copula with Weibull margins, the conditional expectation can be written as;

$$\begin{aligned}
&\mathrm{E}[X_1|X_2 = t] \\
&= \frac{T^{1/\gamma}}{\gamma}\int_0^\infty \widehat{C}_{1|2}(e^{-sT}|e^{-T})s^{1/\gamma-1}\ ds \\
&= \frac{T^{1/\gamma}}{\gamma}\int_0^\infty \Big[(1-\alpha_2)\ e^{-T\{s(1-\alpha_1)-\alpha_2+\left[(\alpha_1 s)^\delta+(\alpha_2)^\delta\right]^{\frac{1}{\delta}}\}} \\
&\qquad + \alpha_2 T^{2-\delta}(1-\delta)\left[(\alpha_1 s)^\delta + (\alpha_2)^\delta\right]^{\frac{1}{\delta}-1} e^{-T\{s(1-\alpha_1)-\alpha_2+\left[(\alpha_1 s)^\delta+(\alpha_2)^\delta\right]^{\frac{1}{\delta}}\}}\Big] s^{1/\gamma-1}\ ds \\
&= \frac{T^{1/\gamma}}{\gamma}(1-\alpha_2)\int_0^\infty s^{1/\gamma-1}e^{-T\{s(1-\alpha_1)-\alpha_2+\left[(\alpha_1 s)^\delta+(\alpha_2)^\delta\right]^{\frac{1}{\delta}}\}}\ ds \\
&\qquad + \alpha_2(1-\delta)\frac{T^{2+1/\gamma-\delta}}{\gamma}\int_0^\infty s^{1/\gamma-1}\left[(\alpha_1 s)^\delta + (\alpha_2)^\delta\right]^{\frac{1}{\delta}-1} e^{-T\{s(1-\alpha_1)-\alpha_2+\left[(\alpha_1 s)^\delta+(\alpha_2)^\delta\right]^{\frac{1}{\delta}}\}}\ ds
\end{aligned} \tag{3.65}$$

$$= \frac{T^{1/\gamma}}{\gamma}(1-\alpha_2)\int_0^\infty h_1(s)e^{Tg(s,T)}ds + \alpha_2(1-\delta)\frac{T^{2+1/\gamma-\delta}}{\gamma}\int_0^\infty h_2(s)e^{Tg(s,T)}ds,$$

where $g(s,T) = s(1-\alpha_1) + \alpha_2 - \left[(\alpha_1 s)^\delta + (\alpha_2)^\delta\right]^{\frac{1}{\delta}}$, $h_1(s) = s^{1/\gamma-1}$ and $h_2(s) = s^{1/\gamma-1}\left[(\alpha_1 s)^\delta + (\alpha_2)^\delta\right]^{\frac{1}{\delta}-1}$. Again here like before we have $(\alpha_1, \alpha_2) \in [0,1]^2$, $\gamma > 0$, $s, T \in [0,\infty)^2$, and $\delta \geq 0$. After doing some easy calculations we get $g(0,T) = 0$ and $g(\infty, T) = -\infty$. Thus, we can use either Laplace approximation or Watson's lemma.

Let us do the first one first. If we do the first order derivative of g as explained above with respect to s we have;

$$g'(s,T) = [\alpha_1 - 1] - \delta\alpha_1^\delta s^{\delta-1}[(\alpha_1 s)^\delta + \alpha_2^\delta]^{1/\delta-1}, \tag{3.66}$$

where $\alpha_1, \alpha_2 \in [0,1]^2$, $\delta \geq 0$ and $s \in [0,\infty)$. Thus, from (3.66) we immediately get $g'(0,T) = \alpha_1 - 1 < 0$ [as by assumption $\alpha_1 \in [0,1]$] as $T \to \infty$. In order to do Laplace approximation we need $g'(0,T) > 0$ which clearly not this case. Thus, we can not use this method. According to Hua and Joe (2014) we have to use Watson's lemma.

Before using Watson's lemma we have to check if $g(s,T)$ is decreasing in s. After doing the first order differentiation of the g function we get [from (3.66)], $g'(s,T) = [\alpha_1 - 1] - \delta\alpha_1^\delta s^{\delta-1}[(\alpha_1 s)^\delta + \alpha_2^\delta]^{1/\delta-1} < 0$. Now, we can use Watson's lemma separately of the two parts of conditional expectation.

As $g(s;T)$ is a real valued function on the semi-infinite interval $[0,\infty)$ and in $(0, 0+\epsilon]$ with $\epsilon > 0$ this function is continuously differentiable and $\sup_{0+\epsilon \leq s \leq \infty} g(s,T) \leq g(0,T) - \psi$, with $\psi > 0$. Here we are using Theorem 36 [p.48] of Breitung (1994). This theorem is an extension of Watson's lemma and most importantly, this theorem works for semi-infinite intervals. Now we have $g'(s,T) < 0$ and $s \to 0^+$.We can also write $g'(s,T) = -as^{r-1} + o(s^{r-1})$ $\forall$ $r > 0$ Now if we assume $r = 1$ then $g'(s,T) = -a$. From our previous results we know that, $\lim_{s^+ \to\ 0,\ T\to\infty} g'(s,T) = -(1-\alpha_1) - \delta\alpha_1^\delta s^{\delta-1}[(\alpha_1 s)^\delta + \alpha_2^\delta]^{1/\delta-1}$, which converges to a constant as $T \to \infty$. Thus, $-a = -[1-\alpha_1]$ or, $a = 1 - \alpha_1$.

Let us assume there is another real and continuous function $h_1(s) \in [0,\infty)$ such that, $h_1(s) = bs^{m-1} + o(s^{m-1})$ with $m > 0$. More specifically we assume $h_1(s) = s^{\frac{1}{\gamma}-1}$ in our case. Thus, $bs^{m-1} + o(s^{m-1}) = s^{1/\gamma-1}$. This implies $b = 1$ and $m = 1/\gamma$.

Using the above information and by Watson's lemma we get,

$$I_1(T) \sim \Gamma\left(\frac{1}{\gamma}\right)\left(\frac{1}{1-\alpha_1}\right)^{\frac{1}{\gamma}} T^{-\frac{1}{\gamma}},\ as\ T \to \infty. \tag{3.67}$$

Combining (3.65) and (3.67) we get the first integration as;

$$\begin{aligned} \mathbb{A} &= \frac{1-\alpha_2}{\gamma}\ \Gamma\left(\frac{1}{\gamma}\right)\left(\frac{1}{1-\alpha_1}\right)^{\frac{1}{\gamma}},\ as\ t \to \infty, \\ &= \frac{1-\alpha_2}{\gamma}\ N, \end{aligned} \tag{3.68}$$

where $N = \Gamma\left(\frac{1}{\gamma}\right)\left(\frac{1}{1-\alpha_1}\right)^{\frac{1}{\gamma}}$, $\gamma > 0$ and $(\alpha_1, \alpha_2)\ \in [0,1]^2$.

Let us assume there is another real and continuous function $h_2(s) \in [0, \infty)$ such that, $h_2(s) = bs^{m-1} + o(s^{m-1})$ with $m > 0$. More specifically we assume $h_2(s) = [(\alpha_1 s)^\delta + \alpha_2^\delta]^{1/\delta} s^{\frac{1}{\gamma}-1}$ in our case.

Thus,

$$\begin{aligned} [(\alpha_1 s)^\delta + \alpha_2^\delta]^{1/\delta} s^{\frac{1}{\gamma}-1} &= bs^{m-1} + o(s^{m-1}) \\ \left\{ s^{\delta\left[\frac{1}{\gamma}-1\right]}[(\alpha_1 s)^\delta + \alpha_2^\delta] \right\}^{1/\delta} &= bs^{m-1} + o(s^{m-1}) \\ s^{\delta\left[\frac{1}{\gamma}-1\right]}[(\alpha_1 s)^\delta + \alpha_2^\delta] &= b^\delta s^{\delta(m-1)} + o(s^{m-1}) \\ s^{\left[\frac{1}{\gamma}-1\right]}[(\alpha_1 s)^\delta + \alpha_2^\delta] &= b^\delta s^{(m-1)} + o(s^{m-1}) \\ (\alpha_1 s)^\delta s^{\left[\frac{1}{\gamma}-1\right]} + \alpha_2^\delta s^{\left[\frac{1}{\gamma}-1\right]} &= b^\delta s^{(m-1)} + o(s^{m-1}) \end{aligned} \tag{3.69}$$

Now, for any $\delta > 1/\gamma$ the most left expression [i.e. $(\alpha_1 s)^\delta s^{\left[\frac{1}{\gamma}-1\right]} \to o(s^{m-1})$ as $s^+ \to\ 0$ and $T \to \infty$]. Thus, we have $b = \alpha_2$ and $m = 1/\gamma$.

Using the above information and by Watson's lemma we get,

$$I(T) \sim \alpha_2 \Gamma\left(\frac{1}{\gamma}\right)\left(\frac{1}{1-\alpha_1}\right)^{\frac{1}{\gamma}} T^{-\frac{1}{\gamma}},\ as\ T \to \infty. \tag{3.70}$$

Combining (3.65) and (3.70) we get the second integration as;

$$\begin{aligned} \mathbb{B} &= \alpha_2^2(1-\delta)\frac{t^{\gamma(2-\delta)}}{\gamma}\ \Gamma\left(\frac{1}{\gamma}\right)\left(\frac{1}{1-\alpha_1}\right)^{\frac{1}{\gamma}},\ as\ t \to \infty\ and\ s^+ \to\ 0, \\ &= \alpha_2^2(1-\delta)\frac{t^{\gamma(2-\delta)}}{\gamma}\ N, \end{aligned} \tag{3.71}$$

where $N = \Gamma\left(\frac{1}{\gamma}\right)\left(\frac{1}{1-\alpha_1}\right)^{\frac{1}{\gamma}}$, $\gamma > 0$ and $(\alpha_1, \alpha_2) \in [0,1]^2$.

Combining the expressions in $\mathbb{A}$ and $\mathbb{B}$ we get the conditional tail expectation as;

$$\mathrm{E}[X_1|X_2 = t] \sim \frac{N}{\gamma}\left[1 - \alpha_2 + \alpha_2^2(1-\delta)t^{\gamma(2-\delta)}\right], \tag{3.72}$$

where $N = \Gamma\left(\frac{1}{\gamma}\right)\left(\frac{1}{1-\alpha_1}\right)^{\frac{1}{\gamma}}$, $\gamma > 0$ and $(\alpha_1, \alpha_2) \in [0,1]^2$, $\delta > 0$ as $t \to \infty$ and $s^+ \to\ 0$. This completes the proof. □

Corollary 65. *Suppose X_1 and X_2 are two dependent random variables with $\int_0^\infty e^{g(s,T)}\ |h_1(s)|\ ds < \infty$ and $\int_0^\infty e^{g(s,T)}\ |h_2(s)|\ ds < \infty$. In the case of Khoudraji (1996) non-exchangeable transformed survival Gumbel Copula with Weibull margin, Then,*

$$\mathrm{E}[X_2|X_1 = t] \sim \frac{N'}{\gamma}\left[1 - \alpha_1 + \alpha_1^2(1-\delta)t^{\gamma(2-\delta)}\right],$$

where $N' = \Gamma\left(\frac{1}{\gamma}\right)\left(\frac{1}{1-\alpha_2}\right)^{\frac{1}{\gamma}}$, $\gamma > 0$, as $s^+ \to\ 0$, $t \to \infty$, $(\alpha_1, \alpha_2) \in [0,1]^2$ and $\delta \geq 0$.

CHAPTER 4

TESTING OF HYPOTHESIS

We have done simulations by using either Laplace approximation or Watson's lemma with different non-exchangeable survival Copulas with Pareto, Weibull or Exponential margins respectively. In this chapter we are going to test *non-exchangeability* empirically. There has been a lot of research in this field and different kinds of estimators have been used in hypothesis testing of this literature. Fermanian et al. (2004) defines that any empirical Copula process converges to a weak Gaussian process. A detailed discussion in weak Gaussian process takes place in Van Der Vaart and Wellner (1996). Furthermore, Fermanian et al. (2004) mention Theorem 3.9.4 of Van Der Vaart and Wellner (1996) in order to prove Theorem 3 in their paper. In this thesis we are trying to show our measure of *non-exchangeability* weakly goes to normal distribution under certain conditions.

From previous sections we know that, if (X_1, X_2) be a bivariate random vector with identically distributed marginals, supported on $[0, \infty)^2$, then the random vector (X_1, X_2) is said to be *tail exchangeable* of Type I if the following condition holds:

$$\text{Condition I:} \quad \lim_{t\to\infty} \eta_1(t) := \lim_{t\to\infty} \frac{\mathrm{E}[X_1|X_2 > t]}{\mathrm{E}[X_2|X_1 > t]} = 1,$$

and is *tail exchangeable* of Type II if the following condition holds:

$$\text{Condition II:} \quad \lim_{t\to\infty} \eta_2(t) := \lim_{t\to\infty} \frac{\mathrm{E}[X_1|X_2 = t]}{\mathrm{E}[X_2|X_1 = t]} = 1.$$

At first we are going to discuss about $\eta_1(t)$ then later of this section we are going to discuss about $\eta_2(t)$. A system is *tail exchangeable* if $\eta_1(t) = 1$ or $\eta_2(t) = 1$, non-exchangeable otherwise. On the other hand, if $E[X_1|X_2 > t] \sim E[X_2|X_1 > t]$ or $E[X_1|X_2 = t] \sim E[X_2|X_1 = t]$ as $t \to \infty$ the system is *exchangeable*, otherwise not. At the end of this section we are trying to simulate the empirical functions and check how these conditional expectations behave. In our *empirical* test we are constructing the test statistic for all t as we can not show limiting property because, when we are taking bigger t then we are going to some extreme data points which leads to lose information of the dataset. In order to define the empirical function we follow the traditional way.

Definition 5. *We define empirical version of the survival copula $\widehat{C}^*(\overline{F}(x), \overline{F}(t))$ as*

$$\widehat{C}^*_n(\overline{F}(x), \overline{F}(t)) = \frac{1}{n}\sum_{j=1}^{n} \mathbb{I}(U_{1j} \le \overline{F}(x), U_{2j} \le \overline{F}(t))$$

where, $\overline{F}(x)$ and $\overline{F}(t)$ are associated any continuous marginal distributions such that $[\overline{F}(x), \overline{F}(t)] \in [0,1]^2$; $\forall\ j = 1, 2, 3, ..., n,$ and at $t \to \infty$.

Following Hua and Joe (2014) we know that if $\{X_1, X_2\}$ is a set of random variables which can be represented by $\overline{F}(x)$ and $\overline{F}(t)$, respectively, $\mathrm{E}[X_1|X_2 > t] = \int_0^\infty \frac{\widehat{C}^*_n(\overline{F}(x), \overline{F}(t))}{\overline{F}(t)} dx$, $\forall\ t$. Thus the *empirical* version of the conditional expectation at the tail is going to be $I_n \triangleq \int_0^\infty n^{-1} \sum_{j=1}^{n} \mathbb{I}[(U_{1j} \le \overline{F}(x), U_{2j} \le \overline{F}(t))/\overline{F}(t)]\ dx$ at $\forall\ t$. Again, here we further assume $\overline{F}(t)$ is given and fixed at sufficiently large value of t.

Proposition 66. *If $\boldsymbol{\Sigma}$ is positive definite, $\boldsymbol{d}(\boldsymbol{\theta}) \neq \mathbf{0}$ and $\boldsymbol{d}(\boldsymbol{\theta})$ is continuous in a neighborhood of $\boldsymbol{\theta}$, then*

$$n^{1/2}\left[h(\bar{\boldsymbol{V}}_n) - h(\boldsymbol{\theta})\right] = n^{1/2}\left(\frac{I_n}{J_n} - \frac{\boldsymbol{\theta}_1}{\boldsymbol{\theta}_2}\right) \xrightarrow{w} \boldsymbol{Z},$$

where $\boldsymbol{Z} \sim N[0, \boldsymbol{d'(\theta)}\ \boldsymbol{\Sigma}\ \boldsymbol{d(\theta)}]$ *with* $\boldsymbol{\theta} = E[\boldsymbol{V}_j]$, $\boldsymbol{\Sigma} = Var[\boldsymbol{V}_j]$, $\boldsymbol{\theta}_1 = \boldsymbol{\theta}_2$, $\boldsymbol{d(\theta)} = \partial h(\boldsymbol{\theta})/\partial(\boldsymbol{\theta})$ *and*

$$\bar{\boldsymbol{V}}_n = n^{-1}\sum_{j=1}^{n} \boldsymbol{V}_j = \begin{bmatrix} I_n \\ J_n \end{bmatrix},$$

where I_n *and* J_n *are empirical versions of* $\mathrm{E}[X_1|X_2 > t]$ *and* $\mathrm{E}[X_2|X_1 > t]$ *respectively.*

Proof. From above we know that, the conditional expectation can be defined as, $\mathrm{E}[X_1|X_2 > t] = \int_0^\infty \frac{\widehat{C}_n^*(\overline{F}(x),\overline{F}(t))}{\overline{F}(t)} dx$. Furthermore as we are assuming $\overline{F}(t)$ is constant the behavior of this conditional expectation depends only on the integration part of the expectation or $\int_0^\infty n^{-1}\sum_{j=1}^{n} \mathbb{I}[U_{1j} \le \overline{F}(x), U_{2j} \le \overline{F}(t)]\ dx$

Let us define,

$$\begin{aligned} I_n &= \int_0^\infty n^{-1}\sum_{j=1}^{n} \mathbb{I}[U_{1j} \le \overline{F}(x), U_{2j} \le \overline{F}(t)]\ dx \\ &= n^{-1}\int_0^\infty \sum_{j=1}^{n} \mathbb{I}[U_{1j} \le \overline{F}(x), U_{2j} \le \overline{F}(t)]\ dx \\ &= n^{-1}\sum_{j=1}^{n}\int_0^\infty \mathbb{I}[U_{1j} \le \overline{F}(x)]\ \mathbb{I}[U_{2j} \le \overline{F}(t)]\ dx \\ &= n^{-1}\sum_{j=1}^{n} \mathbb{I}[U_{2j} \le \overline{F}(t)]\int_0^\infty \mathbb{I}[U_{1j} \le \overline{F}(x)]\ dx,\ \forall\ t, \end{aligned} \tag{4.1}$$

where $[\overline{F}(x), \overline{F}(t)] \in [0,1]^2$.

Furthermore, Serfling (2009) [pg.3] implies that $F(x) \ge y$ if and only if $x \ge F^{-1}(y)$ where

$$F^{-1}(y) = inf_{x\in\mathbb{R}}\ \{x : F(x) \ge y\}.$$

Therefore,

$$\begin{aligned} U_{1j} \leq \overline{F}(x) &\iff U_{1j} \leq \ 1 - F(x) \\ &\iff 1 - U_{1j} \geq F(x) \\ &\iff x \leq F^{-1}(1 - U_{1j}) \end{aligned}$$

and

$$\begin{aligned} I_n &= n^{-1} \sum_{j=1}^{n} \mathbb{I}[U_{2j} \leq \overline{F}(t)] \int_0^{\infty} \mathbb{I}[x \leq F^{-1}(1 - U_{1j})] \ dx \\ &= n^{-1} \sum_{j=1}^{n} \mathbb{I}[U_{2j} \leq \overline{F}(t)] \int_0^{F^{-1}(1-U_{1j})} dx \\ &= n^{-1} \sum_{j=1}^{n} \mathbb{I}[U_{2j} \leq \overline{F}(t)] \ F^{-1}(1 - U_{1j}) \\ &= n^{-1} \sum_{j=1}^{n} B_j Y_j, \end{aligned} \tag{4.2}$$

where $B_j = \mathbb{I}[U_{2j} \leq \overline{F}(t)]$ and $Y_j = F^{-1}(1 - U_{1j})$. In this case, $1 - U_{1j} \sim$ Uniform$(0, 1)$ so that $Y_j = F^{-1}(1 - U_{1j}) \sim \ F$ and that $B_j = \mathbb{I}[U_{2j} \leq \overline{F}(t)] \sim$ Bernoulli(π) where $\pi = P[U_{2j} \leq \overline{F}(t)]$.

In the similar fashion we can define,

$$\begin{aligned} J_n &= \int_0^{\infty} n^{-1} \sum_{j=1}^{n} \mathbb{I}[U_{2j} \leq \overline{F}(x), U_{1j} \leq \overline{F}(t)] \ dx \\ &= n^{-1} \sum_{j=1}^{n} \mathbb{I}[U_{1j} \leq \overline{F}(t)] \ F^{-1}(1 - U_{2j}) \\ &= n^{-1} \sum_{j=1}^{n} C_j Z_j, \end{aligned} \tag{4.3}$$

where $Z_j = F^{-1}(1-U_{2j}) \sim F$ and $C_j = \mathbb{I}[U_{1j} \leq \overline{F}(t)] \sim$ BERNOULLI(ρ) where $\rho = P[U_{1j} \leq \overline{F}(t)]$.

Now, let us define,

$$\boldsymbol{V}_j = \begin{bmatrix} B_jY_j \\ C_jZ_j \end{bmatrix}$$

for $j \in \{1, 2, 3, .., n\}$ and it is important to note that,

$$\bar{\boldsymbol{V}}_n = n^{-1}\sum_{j=1}^{n} \boldsymbol{V}_j = \begin{bmatrix} I_n \\ J_n \end{bmatrix}$$

Now, let us define $\boldsymbol{\theta} = E[\boldsymbol{V}_j]$ and $\boldsymbol{\Sigma} = Var\ [\boldsymbol{V}_j]$. Suppose that $\boldsymbol{\Sigma}$ has all finite elements, then MULTIVARIATE CENTRAL LIMIT THEOREM [i.e Theorem 4.22 of Polansky (2011)] implies that, $n^{1/2}(\bar{\boldsymbol{V}}_n - \boldsymbol{\theta}) \xrightarrow{w} \boldsymbol{Z}$, where $\boldsymbol{Z} \sim \boldsymbol{N}(\boldsymbol{0}, \boldsymbol{\Sigma})$. Now, consider the function $h(\boldsymbol{u}) = h(u_1, u_2) = u_2^{-1}u_1$ and define

$$\boldsymbol{d}(\boldsymbol{u}) = \frac{\partial}{\partial \boldsymbol{u}} h(\boldsymbol{u}) = \begin{bmatrix} u_2^{-1} \\ -u_2^{-1}u_1 \end{bmatrix}$$

Therefore, If $\boldsymbol{\Sigma}$ is positive definite, $\boldsymbol{d(\theta)} \neq \boldsymbol{0}$ and $\boldsymbol{d(\theta)}$ is continuous in a neighborhood of $\boldsymbol{\theta}$, then by Theorem 6.5 of Polansky (2011) implies that,

$$n^{1/2}\left[h(\bar{\boldsymbol{V}}_n) - h(\boldsymbol{\theta})\right] = n^{1/2}\left(\frac{I_n}{J_n} - \frac{\boldsymbol{\theta}_1}{\boldsymbol{\theta}_2}\right) \xrightarrow{w} \boldsymbol{Z},$$

where $\boldsymbol{Z} \sim N[0, \boldsymbol{d'(\theta)}\ \boldsymbol{\Sigma}\ \boldsymbol{d(\theta)}]$. This completes the proof. □

Remark 26. *In order to get the above result we have be more careful as U_{1j} and U_{2j} are not independent. Their dependence is defined through the Copula structure. In this case if* *we do not know the structure of* $\boldsymbol{\Sigma}$, *we can not get above proposition. That is why we need to make a heuristic assumption that we do know the structure of* $\boldsymbol{\Sigma}$ *and we can get this by using Monte Carlo method. In our case* $\boldsymbol{\theta}_1 = \mathrm{E}[X_1|X_2 > t]$ *and* $\boldsymbol{\theta}_2 = \mathrm{E}[X_2|X_1 > t]$. *In the previous chapters we know the exact values of these two parameters for Clayton and Gumbel Copulas. We can use those expressions as the representatives of* $\boldsymbol{\theta}_1$ *and* $\boldsymbol{\theta}_2$.

As from above we see the ratio of two empirical representations of conditional expectations weakly follows normal distribution with mean zero and some variance covariance matrix, we can create a hypothesis testing environment on it. In our case, testing null hypothesis (i.e $\mathscr{H}_0$, say) amounts to check that the variables X_1 and X_2 are dependent and the dependence structure of two conditional expectations $\mathrm{E}[X_1|X_2 > t]$ and $\mathrm{E}[X_2|X_1 > t]$ is *exchangeable.* Furthermore, we are more interested in $\eta_1(t)$.

The main objective of this section is to propose a test of hypothesis for a given t sufficiently large,

$$\mathscr{H}_0: \ \eta_1(t) = \frac{\mathrm{E}[X_1|X_2 > t]}{\mathrm{E}[X_1|X_2 > t]} = 1,$$

against the general alternative

$$\mathscr{H}_1: \ \eta_1(t) = \frac{\mathrm{E}[X_1|X_2 > t]}{\mathrm{E}[X_1|X_2 > t]} \neq 1,$$

From the previous proposition we know that, the proportion of empirical forms of conditional expectations weakly follows normal distribution.

Thus, the test statistic should be,

$$\mathscr{Z}_n = \left(\frac{n^{1/2}\left[\frac{I_n}{J_n} - 1\right]}{(\boldsymbol{d'(\theta)}\ \boldsymbol{\Sigma}\ \boldsymbol{d(\theta)})^{1/2}} \right) \overset{w}{\sim} N[0,1]$$

as $n \to \infty$ where $\boldsymbol{\theta} = E[\boldsymbol{V}_j]$, $\boldsymbol{\Sigma} = Var[\boldsymbol{V}_j]$, $\boldsymbol{d(\theta)} = \partial h(\boldsymbol{\theta})/\partial(\boldsymbol{\theta})$ and

$$\bar{\boldsymbol{V}}_n = n^{-1} \sum_{j=1}^{n} \boldsymbol{V}_j = \begin{bmatrix} I_n \\ J_n \end{bmatrix},$$

with I_n and J_n are empirical versions of $\mathrm{E}[X_1|X_2 > t]$ and $\mathrm{E}[X_2|X_1 > t]$ respectively.

CHAPTER 5

CONCLUSION AND FUTURE RESEARCH

Our primary objective of study throughout this paper is tail *non-exchangeability*. In order to do so firstly, we take an *exchangeable* Copula. Then we do Khoudraji (1996) *non-exchangeable* transformation of our *exchangeable* Copula. Then we construct conditional tail expectation. As this integration does not have any closed form solution, following Hua and Joe (2014) we use numerical approximation either by *Laplace Approximation* or *Watson's Lemma*. First we theoretically develop tail *non-exchangeability* by above two methods and derive the general conditions under which we are able to get some forms of approximation at the tail. Then we take two special Copulas [i.e. Clayton and Gumbel] with *Pareto, Weibull* and *Exponential* margins. In the case of Clayton Copula we use *Laplace Approximation* or *Watson's lemma* based on the conditions satisfied by different margins. *Laplace Approximation* and *Watson's Lemma* work only for Khoudraji (1996) *non-exchangeable* Clayton survival Copula with *Pareto* margin. We show that, when $\alpha_1 + 1/\beta > 1$ we can use *Laplace Approximation* method, otherwise, we are using *Watson's Lemma*. In the case of *non-exchangeable* survival Clayton Copula with *Weibull* and *Exponential* margins we can *only* use *Watson's Lemma*. At the end of this section with *non-exchangeable* Clayton Copula we plot our simulation results and compare with the existing process.

In the chapter with *non-exchangeable* extreme value Copula such as Gumbel Copula, we try to define a separate kind of *non-exchangeability* without using *Khoudraji non-exchangeable device*. We follow the way provided by Durante and Mesiar (2010) and give two examples of these Copulas; one from Durante and Mesiar (2010) directly and other one constructed by ourselves. Then we derive tail *non-exchangeability* by two numerical approximations described above. After that, we use Khoudraji (1996) *non-exchangeable* transformation and do find out tail *non-exchangeability* by using the same numerical approximations.

In both the Clayton and Gumbel survival Copulas we find tail *non-exchangeability*. In the case of survival Clayton Copula with *Pareto* margin we can use *Laplace Approximation* if $\alpha_1 + 1/\beta > 1$ otherwise, we use *Watson's Lemma*. Furthermore, if we carefully look at the conditional expectations, we see the expectations do not change at the tail. In the plots [see figure 2.8a, 2.8b, 2.8c, 2.8d etc.] we see in this case the distance between the conditional expectations are same as $t \to \infty$ but, if we use the software we can find conditional expectations become more *non-exchangeable* as t increases in the case of survival Clayton Copula with *Pareto* margin [like in figure 2.2a and 2.2b.]. On the other hand, if we do *Laplace* Approximation, we get more *non-exchangeability*. Thus, our method is valid only if $\alpha_1 + 1/\beta > 1$.

In the case of survival Clayton Copula with *Weibull* or *Exponential* margins, we find out that $g'(0, T) \not> 0$. Thus, we cannot use *Laplace Approximation* in these cases; we only use *Watson's Lemma*. As discussed before *Watson's Lemma* does not give much *non-exchangeability* at the tail. Then we use the existing software to verify if our approximation is valid. we find out that, by doing numerical integration in R we do not get much *non-exchangeability* at the tail [i.e., $t \to \infty$]. In the figures [see figure 2.8a, 2.8b, 2.8d etc.]we get the same results. In this case we conclude that, we can never use *Laplace* Approximation in survival Clayton Copula under the presence of *Weibull* and *Exponential* distributions; the reason may be that Weibull and Exponential margins are dominating the tail and dependence becomes less visible. On the other hand, we can use *Laplace Approximation* in the case with *Pareto* margin if $\alpha_1 + 1/\beta > 1$ otherwise we cannot use any approximation as *Watson's*

Lemma does not give *non-exchangeability* at the tail. We get the similar results for both the $\mathrm{E}[X_1|X_2 > t]$ and $\mathrm{E}[X_1|X_2 = t]$ cases.

When we use the same methods of numerical approximations for *non-exchangeable* extreme value survival Copulas, we do not get any new surprising result. In the first part of the chapter of *Extreme-Value* Copulas, we provide our own *non-exchangeable* extreme value survival Copula and do use *Laplace Approximation* and *Watson's Lemma* in the presence of *Pareto, Weibull* and *Exponential* margins respectively. All the time when we do *Watson's Lemma* we do get more *exchangeability* at the tail. We find *non-exchangeability* only in the case of *non-exchangeable* extreme value survival Copula with *Pareto* margin. In the later part of that chapter we consider *exchangeable* Gumbel Copula and do the Khoudraji (1996) *non-exchangeable* transformation on it and, finally do our two approximation methods used throughout this paper. Again, like before *Laplace Approximation* works only for *non-exchangeable* survival Gumbel Copula with *Pareto* margin. In the case with *Weibull* and *Exponential* margins we are only able to use *Watson's Lemma*; hence, we are not able to get much tail *non-exchangeability* here.

While deriving *conditional tail expectation* for $\mathrm{E}[X_1|X_2 = t]$ with *non-exchangeable* Gumbel Copula, we do find two separate terms which are equally powerful at the tail. Thus, we have to do *Laplace Approximation* or *Watson's Lemma* two times for each expression. This result is consistent with the theoretical results shown before. In survival Clayton Copula we do not see these kind of two separate terms because, one term is dominated by the other. In this paper we just do the *non-exchangeable* transformation of only survival Clayton and Gumbel Copulas. If we do this numerical approximations with different Copulas, we might get some other results.

Finally, in the chapter of *Testing of Hypothesis* we develop a test statistic based on the empirical survival Copulas. As we know empirically we cannot show the limiting properties because, when we consider extreme values, we concentrate on fewer data points, as a result, we are losing information of the whole data set. Therefore, we only consider whole set of data while testing *non-exchangeability* empirically. We show that, our test statistic weakly follow standard normal distribution under certain conditions. As the data have dependence structure through empirical Copula, our proposition regarding test statistic works if we know the structure of variance-covariance matrix [i.e., $\mathbf{\Sigma}$] in *Proposition 91.*

Firstly, in our future research we can do further study on the structure of variance-covariance matrix. If we assume the structure is known, we can do *Monte Carlo* simulation in order to get this. If we assume this structure is unknown we cannot use our proposition and the proof is not valid at all. In this case we can use *boot-strap* method in order to estimate $\mathbf{\Sigma}$. Again, in the chapter of *Testing of Hypothesis* we are only able to estimate the empirical test for $\mathrm{E}[X_1|X_2 > t]$ not $\mathrm{E}[X_1|X_2 = t]$ because, for later finding out an empirical test statistics is way more harder.

Secondly, throughout this paper we only consider two *non-exchangeable* survival Copulas, Clayton and Gumbel Copulas. We can do study on other *exchangeable Extreme-value* Copulas like *Galambos, t-EV, Husler, Tawn etc.* In table 3.1 we do calculate the Khoudraji (1996) *non-exchangeable* structures of *Pickands* dependence functions of these *Extreme-value* Copulas. Apart from that we can directly take *non-exchangeable* Copula like *Archimax Copula* directly. Furthermore, in this paper we only try to get a mathematical derivation of *non-exchangeability* but we do not check how the degree of *dependence* in the tail affects the measure of *tail non-exchangeability.* In this paper, we only consider *non-exchangeability* in the presence of *positive* dependence, we never consider *non-exchangeability* with *negative* dependence. We do not know how this looks like. Probably, this is going to be a very good future research.

Finally, in our paper we define, the random vector (X_1, X_2) is said to be *tail exchangeable* of Type I if the following condition holds:

$$\text{Condition I:} \quad \lim_{t\to\infty} \eta_1(t) := \lim_{t\to\infty} \frac{\mathrm{E}[X_1|X_2 > t]}{\mathrm{E}[X_2|X_1 > t]} = 1,$$

and is *tail exchangeable* of Type II if the following condition holds:

$$\text{Condition II:} \quad \lim_{t\to\infty} \eta_2(t) := \lim_{t\to\infty} \frac{\mathrm{E}[X_1|X_2 = t]}{\mathrm{E}[X_2|X_1 = t]} = 1.$$

But, we do not know anything regarding the relationship between these two conditions. We cannot say, if one system is *non-exchangeable* under *Condition I* then, it is *non-exchangeable* under *Condition II* or the other way. To verify the relationship between them, some *regularity* conditions such as *stochastically increasing* may be needed.

APPENDIX

Proposition 67. *For all* $(\alpha_1, \alpha_2) \in [0,1]^2$ *and very large value of* T*, the integrand function* g *varies slowly and takes the value;*

$$g_a(s,T) = \begin{cases} 1 + (1-\alpha_1)s & \textit{if } \alpha_2 > \alpha_1; \\ 1 + s - \alpha_2 & \textit{if } \alpha_2 < \alpha_1; \\ (1 - \frac{\alpha_2}{2}) + (1 - \frac{\alpha_1}{2})s & \textit{if } \alpha_2 = \alpha_1. \end{cases}$$

Proof. In our case $g(s;T) = 1 + s\beta^{-1} - T^{-1}[\delta^{-1}\log(e^{\alpha_1\delta sT} + e^{\alpha_2\delta T} - 1) + (1-\alpha_1)sT + (1-\alpha_2)T]$. Let us consider the part of the above expression multiplied by $\frac{1}{T}$ in the right hand side of the g function. We define it as g_a. Then,

$$\begin{aligned} g_a(s;T) &= \frac{1}{T}[\frac{1}{\delta}\log(e^{\alpha_1\delta sT} + e^{\alpha_2\delta T} - 1) + (1-\alpha_1)sT + (1-\alpha_2)T] \\ &= \frac{1}{T}[\frac{1}{\delta}\log(e^{\alpha_1\delta sT} + e^{\alpha_2\delta T} - 1)] + (1-\alpha_1)s + (1-\alpha_2) \end{aligned} \tag{A.1}$$

Taking the limit of T in the both sides we get,

$$\lim_{T\to\infty} g_a(s;T) = \lim_{T\to\infty}\left[\frac{1}{T}\left[\frac{1}{\delta}\log(e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1)\right]\right.$$
$$\left. + (1-\alpha_1)s + (1-\alpha_2)\right]$$
$$= \lim_{T\to\infty}\frac{1}{T}\left[\frac{1}{\delta}\log(e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1)\right]$$
$$+ (1-\alpha_1)s + (1-\alpha_2)$$
$$= \frac{1}{\delta}\left[\lim_{T\to\infty}\frac{e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1}{T}\right]$$
$$+ (1-\alpha_1)s + (1-\alpha_2)$$
$$= \frac{1}{\delta}\left[\lim_{T\to\infty}\frac{\alpha_1\delta s e^{\alpha_1\delta sT}+\alpha_2\delta e^{\alpha_2\delta T}}{e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1}\right]$$
$$+ (1-\alpha_1)s + (1-\alpha_2)\ \forall\ (\alpha_1,\alpha_2)\in[0,1]^2$$
$$= \frac{1}{\delta}\left[\lim_{T\to\infty}\frac{\alpha_1\delta s e^{\alpha_1\delta sT-\alpha_2\delta T}+\alpha_2\delta}{e^{\alpha_1\delta sT-\alpha_2\delta T}+1+\frac{1}{e^{\alpha_2\delta T}}}\right]$$
$$+ (1-\alpha_1)s + (1-\alpha_2),\ by\ dividing\ e^{\alpha_2\delta T}$$
$$= \frac{1}{\delta}\left[\lim_{T\to\infty}\frac{\alpha_1\delta s e^{(\alpha_1\delta s-\alpha_2\delta)T}+\alpha_2\delta}{e^{(\alpha_1\delta s-\alpha_2\delta)T}+1+\frac{1}{e^{\alpha_2\delta T}}}\right]$$
$$+ (1-\alpha_1)s + (1-\alpha_2)$$
$$= \frac{\alpha_2\delta}{\delta} + (1-\alpha_1)s + (1-\alpha_2),\ if\ \alpha_1 < \alpha_2$$
$$= \alpha_2 + (1-\alpha_1)s + (1-\alpha_2) = 1 + (1-\alpha_1)s \tag{A.2}$$

or

$$= \frac{1}{\delta}[\lim_{T\to\infty}\frac{\alpha_1\delta s+\alpha_2\delta e^{(\alpha_2\delta-\alpha_1\delta s)T}}{1+e^{(\alpha_2\delta-\alpha_1\delta s)T}-\frac{1}{e^{\alpha_1\delta sT}}}] + (1-\alpha_1)s + (1-\alpha_2),\ by\ dividing\ e^{\alpha_1\delta sT}$$
$$= \frac{\alpha_1\delta s}{\delta} + (1-\alpha_1)s + (1-\alpha_2)$$
$$= \alpha_1 s + (1-\alpha_1)s + (1-\alpha_2) = 1 + s - \alpha_2\ if\ \alpha_1 > \alpha_2 \tag{A.3}$$

From the second order condition to be satisfied we need $\alpha_1 = \alpha_2$. This is good news for us as, we do not need to consider two cases as described above. We know our g_a function as,

$$
\begin{aligned}
\lim_{T\to\infty} g_a(s;T) &= \lim_{T\to\infty}[\frac{1}{T}[\frac{1}{\delta}\log(e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1)]+(1-\alpha_1)s+(1-\alpha_2)] \\
&= \lim_{T\to\infty}\frac{1}{T}[\frac{1}{\delta}\log(e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1)]+(1-\alpha_1)s+(1-\alpha_2) \\
&= \frac{1}{\delta}[\lim_{T\to\infty}\frac{e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1}{T}]+(1-\alpha_1)s+(1-\alpha_2)\forall\ \alpha_1,\ \alpha_2\in[0,1]\ and\ \delta>0 \\
&= \frac{1}{\delta}[\lim_{T\to\infty}\frac{\alpha_1\delta s e^{\alpha_1\delta sT}+\alpha_2\delta e^{\alpha_2\delta T}}{e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1}]+(1-\alpha_1)s+(1-\alpha_2),\ [\ L'Hospital's\ rule] \\
&= \frac{1}{\delta}[\lim_{T\to\infty}\frac{\alpha_1\delta s e^{\alpha_1\delta sT-\alpha_2\delta T}+\alpha_2\delta}{e^{\alpha_1\delta sT-\alpha_2\delta T}+1+\frac{1}{e^{\alpha_2\delta T}}}]+(1-\alpha_1)s+(1-\alpha_2),\ by\ dividing\ e^{\alpha_2\delta T} \\
&= \frac{1}{\delta}[\lim_{T\to\infty}\frac{\alpha_1\delta s e^{(\alpha_1\delta s-\alpha_2\delta)T}+\alpha_2\delta}{e^{(\alpha_1\delta s-\alpha_2\delta)T}+1+\frac{1}{e^{\alpha_2\delta T}}}]+(1-\alpha_1)s+(1-\alpha_2) \\
&= \frac{1}{\delta}[\frac{\alpha_1\delta s+\alpha_2\delta}{2}]+(1-\alpha_1)s+(1-\alpha_2),\ [\ as\ \alpha_1=\alpha_2] \\
&= \frac{(\alpha_1 s+\alpha_2)\delta}{2\delta}+(1-\alpha_1)s+(1-\alpha_2) \\
&= \frac{\alpha_1 s+\alpha_2}{2}+(1-\alpha_1)s+(1-\alpha_2) \\
&= \frac{\alpha_1 s}{2}+\frac{\alpha_2}{2}+s-\alpha_1 s+1-\alpha_2 \\
&= (1-\frac{\alpha_2}{2})+(1-\frac{\alpha_1}{2})s,\ \forall\ (\alpha_1,\ \alpha_2)\ \in\ [\ 0,\ 1\ ]^2 \qquad \text{(A.4)}
\end{aligned}
$$

From (A.2) and (A.4) we get our desirable results. □

Claim 68. *Khoudraji transformed Clayton Copula does not have any solution of tail order conditional expectations by Laplace Approximation.*

Proof. Now in the case of Clayton Copula with Weibull margin we have,

$$
\begin{aligned}
g(s;T) &= 1 - \frac{1}{T}[\frac{1}{\delta}\log(e^{\alpha_1 \delta s T} + e^{\alpha_2 \delta T} - 1) + (1-\alpha_1)sT + (1-\alpha_2)T] \\
Thus,\ g'(s;T) &= -\frac{1}{T}[\frac{\alpha_1 \delta T e^{\alpha_1 \delta s T}}{\delta(e^{\alpha_1 \delta s T} + e^{\alpha_2 \delta T} - 1)} + (1-\alpha_1)T] \\
\Longrightarrow\ g'(0;T) &= -\frac{1}{T}[\frac{\alpha_1 \delta T}{\delta e^{\alpha_2 \delta T}} + (1-\alpha_1)T] \\
&= -\frac{\alpha_1}{e^{\alpha_2 \delta T}} - (1-\alpha_1) \qquad \text{(A.5)}
\end{aligned}
$$

$$
\Longrightarrow \lim_{T\to\infty} g'(0,T) = -(1-\alpha_1) \not> 0,\ as\ 0 < \alpha_1 < 1 \qquad \text{(A.6)}
$$

So we can not use Laplace approximation. □

Claim 69. *Under Khoudraji (1996) transformed Clayton Copula Laplace Approximation does not work as $T \to \infty$.*

Proof. As,

$$
\begin{aligned}
g(s;T) &= 1 - \frac{1}{T}[\frac{1}{\delta}\log(e^{\alpha_1 \delta s T} + e^{\alpha_2 \delta T} - 1) + (1-\alpha_1)sT + (1-\alpha_2)T] \\
Thus,\ g'(s;T) &= -\frac{1}{T}[\frac{\alpha_1 \delta T e^{\alpha_1 \delta s T}}{\delta(e^{\alpha_1 \delta s T} + e^{\alpha_2 \delta T} - 1)} + (1-\alpha_1)T] \\
\Longrightarrow\ g'(0;T) &= -\frac{1}{T}[\frac{\alpha_1 \delta T}{\delta e^{\alpha_2 \delta T}} + (1-\alpha_1)T] \\
&= -\frac{\alpha_1}{e^{\alpha_2 \delta T}} - (1-\alpha_1) \qquad \text{(A.7)}
\end{aligned}
$$

$$
\Longrightarrow \lim_{T\to\infty} g'(0,T) = -(1-\alpha_1) \not> 0,\ as\ 0 < \alpha_1 < 1, \qquad \text{(A.8)}
$$

we can not use Laplace approximation. We have to check if the condition for Watson's lemma works. □

Proposition 70. *Conditional tail expectation of KB4 Copula with Exponential margins goes to some constant as $t \to \infty$ and the measure of tail order non-exchangeability can be written as;*

$$\mathrm{E}(X_1|X_2 > t)/\mathrm{E}(X_2|X_1 > t) \sim \begin{cases} 1 & \text{if } \alpha_2 > \alpha_1; \\ \frac{1-\alpha_2}{1-\alpha_1} & \text{if } \alpha_2 < \alpha_1; \\ \frac{1-\frac{\alpha_2}{2}}{1-\frac{\alpha_1}{2}} & \text{if } \alpha_2 = \alpha_1. \end{cases}$$

where α_1 and α_2 are non-exchangeable components.

Proof. As $g(s;T)$ is a real function on the semi-infinite interval $[0, \infty)$ and, in an interval $(0, 0+\epsilon]$ with $\epsilon > 0$ this function is continuously differentiable and

$$\sup_{0+\epsilon \le s \le \infty} g(s,T) \le g(0,T) - \psi$$

with $\psi > 0$. Now for $g'(s,T)$ we have $g'(s,T) < 0$ as $T \to \infty$. We can also write

$$g'(s,T) = -as^{r-1} + o(s^{r-1}) \ \forall r > 0 \ .$$

Now, if we assume $r = 1$ then $g'(s,T) = -a$. From (2.66) we know that, $\lim_{s^+ \to \ 0, \ T \to \infty} g'(s,T) = -1, \ or \ -(1-\alpha_1) \ or \ -(1-\frac{\alpha_1}{2})$, based on three conditions explained before. We are describing the approximations one by one of these three cases.

If we consider $\lim_{T\to\infty} g'(s,T) = -1$, then, by following theorem 38 [p.48] by Breitung (1994) we get, $-a = -1$ or, $a = 1 > 0$. Let us assume there is another real and continuous function $h(s) \in [0, \infty)$ such that,

$$h(s) = bs^{m-1} + o(s^{m-1})$$

with $m > 0$. Here, $h(s) = 1$ is a constant function.

$$bs^{m-1} + o(s^{m-1}) = 1 \implies b = 1,\ and\ m = 1 \tag{A.9}$$

Finally, as we are assuming $\int_0^\infty e^{g(s,T)}ds < \infty$ then, by Watson's lemma we can write the approximated value of the integral $I(T) = \int_0^\infty e^{Tg(s,T)}ds$ with $T \geq 1$ are all finite and the asymptotic approximation is;

$$\begin{aligned} I(T) &= \int_0^\infty e^{Tg(s,T)}ds \\ &\sim T^{-1}e^{Tg(0;T)},\ as\ T \to \infty \\ &= T^{-1},\ as\ g(0,T) = 0\ and,\ T \to \infty \end{aligned} \tag{A.10}$$

Hence,

$$\begin{aligned} \mathrm{E}(X_1|X_2 > t) &= \int_0^\infty e^T \widehat{C}(e^{-y}, e^{-T})\lambda^{-1}dy,\ \forall \lambda > 0 \\ &= \lambda^{-1}T \int_0^\infty e^{T(1+\frac{1}{T}\log \widehat{C}(e^{-sT}, e^{-T})}ds,\ \forall s \in [0,\infty) \\ &\sim \lambda^{-1}TT^{-1} = \lambda^{-1},\ \forall \lambda > 0\ and, \forall s \in [0,\infty) \end{aligned} \tag{A.11}$$

Similarly, we can get,

$$\begin{aligned} \mathrm{E}(X_2|X_1 > t) &= \int_0^\infty e^T \widehat{C}(e^{-y}, e^{-T})\lambda^{-1}dy,\ \forall \lambda > 0 \\ &= \lambda^{-1},\ \forall \lambda > 0\ and, \forall s \in [0,\infty) \end{aligned} \tag{A.12}$$

In the similar fashion if we allow some asymmetry, then we can assume either $a = 1 - \alpha_1$ or $a = 1 - \frac{\alpha_1}{2}$. In both the cases $\mathrm{E}(X_1|X_2 > t)$ is either $\lambda^{-1}\frac{1}{1-\alpha_1}$ or $\lambda^{-1}\frac{1}{1-\frac{\alpha_1}{2}}$ respectively $\forall\ (\alpha_1, \alpha_2) \in [0,1]^2\ and\ \lambda > 0$. Similarly, in these two cases, $\mathrm{E}(X_2|X_1 > t)$ is either

$\lambda^{-1}\frac{1}{1-\alpha_2}$ or $\lambda^{-1}\frac{1}{1-\frac{\alpha_2}{2}}$ respectively $\forall\ (\alpha_1, \alpha_2)\ \in [0,1]^2$ *and* $\lambda > 0$. Thus, measure of tail non-exchangeability can be measured as;

$$\mathrm{E}(X_1|X_2 > t)/\mathrm{E}(X_2|X_1 > t) \sim \begin{cases} 1 & \text{if } \alpha_2 > \alpha_1; \\ \frac{1-\alpha_2}{1-\alpha_1} & \text{if } \alpha_2 < \alpha_1; \\ \frac{1-\frac{\alpha_2}{2}}{1-\frac{\alpha_1}{2}} & \text{if } \alpha_2 = \alpha_1; \end{cases} \tag{A.13}$$

□

Lemma 71. *The conditional expectation in the form of* $\mathrm{E}[X_1|X_2 = t]$ *is a linear combination of two integrations when the integrand itself defined as a factor non-exchangeable survival Copula.*

Proof. From given by Hua and Joe (2014) we know that,

$$\begin{aligned} \mathrm{E}[X_1|X_2 = t] &= \int_0^\infty \overline{F}_{1|2}(x|t)\ dx,\ \forall\ t \\ &= \int_0^\infty \widehat{C}_{1|2}(\overline{F}(x)|\overline{F}(t))\ dx,\ \forall\ t \end{aligned} \tag{A.14}$$

In the above case $\widehat{C}_{1|2}(\overline{F}(x)|\overline{F}(t))$ is factor copula which has the form like before is, $\widehat{C}_{1|2}(\overline{F}(x)|\overline{F}(t)) = \frac{\partial}{\partial \overline{F}(t)}[\overline{F}(x)^{1-\alpha_1}\overline{F}(t)^{1-\alpha_2}\widehat{C}^*(\overline{F}(x)^{\alpha_1}, \overline{F}(t)^{\alpha_2})]$.

Now, after using Khoudraji's device to transform into non-exchangeable copula we know,

$$\begin{aligned}
&\widehat{C}_{1|2}(\overline{F}(x)|\overline{F}(t))\\
&= \frac{\partial}{\partial \overline{F}(t)}[\overline{F}(x)^{1-\alpha_1}\overline{F}(t)^{1-\alpha_2}\widehat{C}^*(\overline{F}(x)^{\alpha_1}, \overline{F}(t)^{\alpha_2})]\\
&= (1-\alpha_2)\overline{F}(x)^{1-\alpha_1}\overline{F}(t)^{-\alpha_2}\widehat{C}^*(\overline{F}(x)^{\alpha_1}, \overline{F}(t)^{\alpha_2}) + \alpha_2\overline{F}(x)^{1-\alpha_1}\widehat{C}^*_{1|2}(\overline{F}(x)^{\alpha_1}|\overline{F}(t)^{\alpha_2})\\
&= (1-\alpha_2)e^{-(1-\alpha_1)y}e^{\alpha_2 T}\widehat{C}^*(e^{-\alpha_1 y}, e^{-\alpha_2 T}) + \alpha_2 e^{-(1-\alpha_1)y}\widehat{C}^*_{1|2}(e^{-\alpha_1 y}|e^{-\alpha_2 T}), \qquad \text{(A.15)}\\
&as\ \overline{F}(x) = e^{-y}, \overline{F}(t) = e^{-T}
\end{aligned}$$

Using definition in (A.14) we get,

$$\begin{aligned}
\text{E}[X_1|X_2 = t] &= \int_0^\infty \widehat{C}_{1|2}(\overline{F}(x)|\overline{F}(t))\ dx,\ \forall\ t\\
&= \int_0^\infty [(1-\alpha_2)\overline{F}(x)^{1-\alpha_1}\overline{F}(t)^{-\alpha_2}\widehat{C}^*(\overline{F}(x)^{\alpha_1}, \overline{F}(t)^{\alpha_2})\\
&+ \alpha_2\overline{F}(x)^{1-\alpha_1}\widehat{C}^*_{1|2}(\overline{F}(x)^{\alpha_1}|\overline{F}(t)^{\alpha_2})]\ dx \qquad \text{(A.16)}
\end{aligned}$$

After transforming $\overline{F}(x)$ into e^{-y} we get $y = -\log \overline{F}(x) \implies y = -\log[1 - F(x)] \implies -y = \log[1 - F(x)] \implies e^{-y} = 1 - F(x) \implies 1 - e^{-y} = F(x) \implies x = F^{-1}(1 - e^{-y})$. After differentiating x with respect to y we get, $e^{-y}f[F^{-1}(1 - e^{-y})]\ dy = dx$ Thus, after transformation the integration in (A.16) becomes;

$$
\begin{aligned}
&\mathrm{E}[X_1|X_2 = t] \\
&= \int_0^\infty [(1-\alpha_2) e^{-(1-\alpha_1)y} e^{\alpha_2 T} \widehat{C}^*(e^{-\alpha_1 y}, e^{-\alpha_2 T}) \\
&\qquad + \alpha_2 e^{-(1-\alpha_1)y} \widehat{C}^*_{1|2}(e^{-\alpha_1 y}|e^{-\alpha_2 T})] e^{-y} [f[F^{-1}(1-e^{-y})]]^{-1} \, dy \\
&= T \int_0^\infty [(1-\alpha_2) e^{-(1-\alpha_1)sT} e^{\alpha_2 T} \widehat{C}^*(e^{-\alpha_1 sT}, e^{-\alpha_2 T}) \\
&\qquad + \alpha_2 e^{-(1-\alpha_1)sT} \widehat{C}^*_{1|2}(e^{-\alpha_1 sT}|e^{-\alpha_2 T})] e^{-sT} [f[F^{-1}(1-e^{-sT})]]^{-1} \, ds, \ \textit{after letting } y = sT, \\
&= T(1-\alpha_2) \int_0^\infty e^{-T(s+\alpha_2-s+s\alpha_1)} \widehat{C}^*(e^{-\alpha_1 sT}, e^{-\alpha_2 T}) [f[F^{-1}(1-e^{-sT})]]^{-1} \, ds \\
&\qquad + T\alpha_2 \int_0^\infty e^{-sT(2-\alpha_1)} \widehat{C}^*_{1|2}(e^{-\alpha_1 sT}|e^{-\alpha_2 T}) [f[F^{-1}(1-e^{-sT})]]^{-1} \, ds \\
&= T(1-\alpha_2) \int_0^\infty e^{-T(\alpha_2+s\alpha_1)} \widehat{C}^*(e^{-\alpha_1 sT}, e^{-\alpha_2 T}) [f[F^{-1}(1-e^{-sT})]]^{-1} \, ds \\
&\qquad + T\alpha_2 \int_0^\infty e^{-sT(2-\alpha_1)} \widehat{C}^*_{1|2}(e^{-\alpha_1 sT}|e^{-\alpha_2 T}) [f[F^{-1}(1-e^{-sT})]]^{-1} \, ds \\
&= T(1-\alpha_2) \int_0^\infty e^{T[-(\alpha_2+s\alpha_1)+\frac{1}{T}\log \widehat{C}^*(e^{-\alpha_1 sT}, e^{-\alpha_2 T})[f[F^{-1}(1-e^{-sT})]]^{-1}]} \, ds \\
&\qquad + T\alpha_2 \int_0^\infty e^{T[-s(2-\alpha_1)+\frac{1}{T}\log \widehat{C}^*_{1|2}(e^{-\alpha_1 sT}|e^{-\alpha_2 T})[f[F^{-1}(1-e^{-sT})]]^{-1}]} \, ds \\
&= T(1-\alpha_2)e^{-w} \int_0^\infty e^{T[-(\alpha_2+s\alpha_1)+\frac{1}{T}[\log \frac{\widehat{C}^*(e^{-\alpha_1 sT}, e^{-\alpha_2 T})}{[f[F^{-1}(1-e^{-sT})]]}+w]]} \, ds \\
&\qquad + T\alpha_2 e^{-w} \int_0^\infty e^{T[-s(2-\alpha_1)+\frac{1}{T}[\log \frac{\widehat{C}^*_{1|2}(e^{-\alpha_1 sT}|e^{-\alpha_2 T})}{f[F^{-1}(1-e^{-sT})]}+w]]} \, ds \\
&= T(1-\alpha_2)e^{-w} \int_0^\infty e^{Tg_1(s,T;\alpha_2,\alpha_1)} h(s) \, ds + T\alpha_2 e^{-w} \int_0^\infty e^{Tg_2(s,T;\alpha_2,\alpha_1)} h(s) \, ds \qquad \text{(A.17)} \\
&= Te^{-w} \left[(1-\alpha_2)\Gamma_1 + \alpha_2 \Gamma_2\right], \qquad \text{(A.18)}
\end{aligned}
$$

where $\Gamma_1 = \int_0^\infty e^{Tg_1(s,T;\alpha_2,\alpha_1)} h(s) \, ds$ and $\Gamma_2 = \int_0^\infty e^{Tg_2(s,T;\alpha_2,\alpha_1)} h(s) \, ds$. Again, like in $\mathrm{E}[X_1|X_2 > t]$ case, $w = \log[f[F^{-1}(0)]]$ is a real constant depending on the marginal distribution F. □

Proposition 72. *Suppose X_1 and X_2 are two dependent random variables. Tail conditional expectation is the linear combination two slow variation functions and can be explained by;*

$$\begin{aligned} \mathrm{E}[X_1|X_2=t] \sim & -\log \overline{F}(t)(1-\alpha_2)e^{-\log \overline{F}(t) g_1(\gamma, -\log \overline{F}(t))}\sqrt{\frac{2\pi}{\log \overline{F}(t)(1-\alpha_2)g_1''(\gamma, -\log \overline{F}(t))}} \\ & -\log \overline{F}(t)\alpha_2 e^{-\log \overline{F}(t) g_2(\gamma, -\log \overline{F}(t))}\sqrt{\frac{2\pi}{\log \overline{F}(t)\alpha_2 g_2''(\gamma, T)}} \end{aligned} \tag{A.19}$$

as $t \to \infty$, $(\alpha_1, \alpha_2) \in [0,1]^2$, where $\gamma = s_0(-\log \overline{F}(t))$.

Proof. Let us see the behavior of these two second order conditions one by one.

$$\begin{aligned} g_1'(s,T) &= -[\alpha_1 + \frac{\alpha_1 e^{-\alpha_1 sT}\widehat{C}^*_{2|1}(e^{-\alpha_2 T}|e^{-\alpha_1 sT})}{\widehat{C}^*(e^{-\alpha_1 sT}, e^{-\alpha_2 T})} + \frac{e^{-sT} f'(F^{-1}(1-e^{-sT}))}{f^2(F^{-1}(1-e^{-sT}))}] \\ -g_1'(s,T) &= \alpha_1 + \frac{\alpha_1 e^{-\alpha_1 sT}\widehat{C}^*_{2|1}(e^{-\alpha_2 T}|e^{-\alpha_1 sT})}{\widehat{C}^*(e^{-\alpha_1 sT}, e^{-\alpha_2 T})} + \frac{e^{-sT} f'(F^{-1}(1-e^{-sT}))}{f^2(F^{-1}(1-e^{-sT}))} \end{aligned} \tag{A.20}$$

If we further do the second order, we get;

$$
\begin{aligned}
&- g_1''(s,T) \\
&= \frac{\widehat{C}^*(e^{-\alpha_1 sT}, e^{-\alpha_2 T})[(-\alpha_1 T)e^{-\alpha_1 sT}\widehat{C}^*_{2|1}(e^{-\alpha_2 T}|e^{-\alpha_1 sT}) + (-\alpha_1 T)e^{-2\alpha_1 sT}\widehat{C}^*_{2|1,1}(e^{-\alpha_2 T}|e^{-\alpha_1 sT})]}{\widehat{C}^{*2}(e^{-\alpha_1 sT}, e^{-\alpha_2 T})} \\
&\quad + \frac{(-\alpha_1 T)e^{-\alpha_1 sT}\widehat{C}^{*2}_{2|1}(e^{-\alpha_2 T}|e^{-\alpha_1 sT})}{\widehat{C}^{*2}(e^{-\alpha_1 sT}, e^{-\alpha_2 T})} \\
&\qquad + \frac{f^2(F^{-1}(1-e^{-sT}))[(-T)e^{-sT}f'(F^{-1}(1-e^{-sT}))}{f^4(F^{-1}(1-e^{-sT}))} \\
&\qquad\quad + \frac{Te^{-2sT}f''(F^{-1}(1-e^{-sT}))[f(F^{-1}(1-e^{-sT}))]^{-1}]}{f^4(F^{-1}(1-e^{-sT}))} \\
&\qquad\qquad + \frac{2Te^{-2sT}f'^2(F^{-1}(1-e^{-sT}))f(F^{-1}(1-e^{-sT}))}{f^5(F^{-1}(1-e^{-sT}))} \\
&= \frac{(-\alpha_1 T)e^{-\alpha_1 sT}[\widehat{C}^*_{2|1}(e^{-\alpha_2 T}|e^{-\alpha_1 sT}) + e^{-\alpha_1 sT}\widehat{C}^*_{2|1,1}(e^{-\alpha_2 T}|e^{-\alpha_1 sT})]}{\widehat{C}^*(e^{-\alpha_1 sT}, e^{-\alpha_2 T})} \\
&\quad + \frac{(-\alpha_1 T)e^{-\alpha_1 sT}\widehat{C}^{*2}_{2|1}(e^{-\alpha_2 T}|e^{-\alpha_1 sT})}{\widehat{C}^{*2}(e^{-\alpha_1 sT}, e^{-\alpha_2 T})} \\
&\quad + \frac{Te^{-sT}[e^{-sT}f''(F^{-1}(1-e^{-sT}))[f(F^{-1}(1-e^{-sT}))]^{-1} - f'(F^{-1}(1-e^{-sT}))]}{f^2(F^{-1}(1-e^{-sT}))} \\
&\qquad + \frac{2Te^{-2sT}f'^2(F^{-1}(1-e^{-sT}))}{f^4(F^{-1}(1-e^{-sT}))} \qquad \text{(A.21)}
\end{aligned}
$$

Thus, when $T \to \infty$, (A.21) becomes;

$$\begin{aligned}
&\lim_{T\to\infty} -g_1''(s,T) \\
&= \lim_{T\to\infty}[\frac{(-\alpha_1 T)[\widehat{C}^*_{2|1}(1|1) + \widehat{C}^{*\prime}_{2|1}(1|1)]}{\widehat{C}^*(1,1)} + \frac{(-\alpha_1 T)\widehat{C}^{*2}_{2|1}(1|1)}{\widehat{C}^{*2}(1,1)}] \\
&\qquad + \lim_{T\to\infty} \frac{T[f''(F^{-1}(0))[f(F^{-1}(0))]^{-1} - f'(F^{-1}(0))]}{f^2(F^{-1}(0))} + \lim_{T\to\infty} \frac{2Tf'^2(F^{-1}(0))}{f^4(F^{-1}(0))} \\
&= -\lim_{T\to\infty} T[\alpha_1\{\frac{[\widehat{C}^*_{2|1}(1|1) + \widehat{C}^{*\prime}_{2|1}(1|1)]}{\widehat{C}^*(1,1)} + \frac{\widehat{C}^{*2}_{2|1}(1|1)}{\widehat{C}^{*2}(1,1)}\}] \\
&\qquad + \lim_{T\to\infty} T\{\frac{f''(F^{-1}(0))[f(F^{-1}(0))]^{-1} - f'(F^{-1}(0))}{f^2(F^{-1}(0))} + \frac{2f'^2(F^{-1}(0))}{f^4(F^{-1}(0))}\} \qquad (A.22)
\end{aligned}$$

In a similar way we get,

$$\begin{aligned}
&-g_2''(s,T) \\
&= \frac{(-\alpha_1 T)e^{-\alpha_1 sT}[\widehat{C}^*_{2|1,1}(e^{-\alpha_2 T}|e^{-\alpha_1 sT}) + e^{-\alpha_1 sT}\widehat{C}^*_{2|1,1,1}(e^{-\alpha_2 T}|e^{-\alpha_1 sT})]}{\widehat{C}^*_{2|1}(e^{-\alpha_2 T}|e^{-\alpha_1 sT})} \\
&\qquad + \frac{(-\alpha_1 T)e^{-\alpha_1 sT}\widehat{C}^{*2}_{2|1,1}(e^{-\alpha_2 T}|e^{-\alpha_1 sT})}{\widehat{C}^{*2}_{2|1}(e^{-\alpha_2 T}|e^{-\alpha_1 sT})} \\
&\qquad\quad + \frac{Te^{-sT}[e^{-sT}f''(F^{-1}(1-e^{-sT}))[f(F^{-1}(1-e^{-sT}))]^{-1} - f'(F^{-1}(1-e^{-sT}))]}{f^2(F^{-1}(1-e^{-sT}))} \\
&\qquad\qquad + \frac{2Te^{-2sT}f'^2(F^{-1}(1-e^{-sT}))}{f^4(F^{-1}(1-e^{-sT}))} \qquad (A.23)
\end{aligned}$$

Thus, when $T \to \infty$, (A.23) becomes;

$$\begin{aligned}
&\lim_{T\to\infty} -g_2''(s,T) \\
&= -\lim_{T\to\infty} T[\alpha_1\{\frac{[\widehat{C}^*_{2|1,1}(1|1) + \widehat{C}^*_{2|1,1,1}(1|1)]}{\widehat{C}^*_{2|1}(1,1)} + \frac{\widehat{C}^{*2}_{2|1,1}(1|1)}{\widehat{C}^{*2}_{2|1}(1,1)}\}] \\
&\qquad + \lim_{T\to\infty} T\{\frac{f''(F^{-1}(0))[f(F^{-1}(0))]^{-1} - f'(F^{-1}(0))}{f^2(F^{-1}(0))} + \frac{2f'^2(F^{-1}(0))}{f^4(F^{-1}(0))}\}, \qquad (A.24)
\end{aligned}$$

where $T = -\log \overline{F}(t)$ and all the other symbols have their usual meanings. $\square$

Lemma 73. *In the case of survival Khoudraji (1996) non-exchangeable Clayton factor Copula at very large t, s converges to the ratio of two parameters of non-exchangeability or, $\gamma = \frac{\alpha_2}{\alpha_1}$ whenever $\alpha_2 > \alpha_1$ or $\alpha_2 < \alpha_1$ at $t \to \infty$.*

Proof. We know, $\gamma = \lim_{T\to\infty} max_s g(s;T)$. From 2.80 we know that, $g'(s,T) = \frac{1}{\beta} - (1 - \alpha_1) - \frac{\alpha_1 e^{\alpha_1 \delta s T}}{e^{\alpha_1 \delta s T} + e^{\alpha_2 \delta T} - 1} - \frac{\alpha_1 \alpha_2 \delta e^{\alpha_1 \delta s T + \alpha_2 \delta T}}{(e^{\alpha_1 \delta s T} + e^{\alpha_2 \delta T} - 1)^2 [1 - \alpha_2 + \frac{\alpha_2 e^{\alpha_2 \delta T}}{e^{\alpha_1 \delta s T} + e^{\alpha_2 \delta T} - 1}]}$.

After using the first order condition we have,

$$\begin{aligned}
&g'(s,T) = 0 \\
&\frac{1}{\beta} - (1-\alpha_1) - \frac{\alpha_1 e^{\alpha_1 \delta s T}}{e^{\alpha_1 \delta s T} + e^{\alpha_2 \delta T} - 1} \\
&\qquad - \frac{\alpha_1 \alpha_2 \delta e^{\alpha_1 \delta s T + \alpha_2 \delta T}}{(e^{\alpha_1 \delta s T} + e^{\alpha_2 \delta T} - 1)^2 [1 - \alpha_2 + \frac{\alpha_2 e^{\alpha_2 \delta T}}{e^{\alpha_1 \delta s T} + e^{\alpha_2 \delta T} - 1}]} = 0 \\
&\frac{1}{\beta} - (1-\alpha_1) = \frac{\alpha_1 e^{\alpha_1 \delta s T}}{e^{\alpha_1 \delta s T} + e^{\alpha_2 \delta T} - 1} \\
&\qquad + \frac{\alpha_1 \alpha_2 \delta e^{(\alpha_1 s + \alpha_2) \delta T}}{(e^{\alpha_1 \delta s T} + e^{\alpha_2 \delta T} - 1)^2 [1 - \alpha_2 + \frac{\alpha_2 e^{\alpha_2 \delta T}}{e^{\alpha_1 \delta s T} + e^{\alpha_2 \delta T} - 1}]} \\
&\frac{1 - \beta(1-\alpha_1)}{\alpha_1 \beta} = \frac{e^{\alpha_1 \delta s T}}{e^{\alpha_1 \delta s T} + e^{\alpha_2 \delta T} - 1} + \frac{\alpha_2 \delta e^{(\alpha_1 s + \alpha_2) \delta T}}{(e^{\alpha_1 \delta s T} + e^{\alpha_2 \delta T} - 1)^2 [1 - \alpha_2 + \frac{\alpha_2 e^{\alpha_2 \delta T}}{e^{\alpha_1 \delta s T} + e^{\alpha_2 \delta T} - 1}]} \\
&\frac{1 - \beta(1-\alpha_1)}{\alpha_1 \beta} = \frac{1}{1 + e^{(\alpha_2 - \alpha_1 s)\delta T} - e^{-\alpha_1 \delta s T}} \\
&\qquad + \frac{\alpha_2 \delta e^{(\alpha_1 s + \alpha_2) \delta T}}{(e^{\alpha_1 \delta s T} + e^{\alpha_2 \delta T} - 1)^2 [1 - \alpha_2 + \frac{\alpha_2}{e^{(\alpha_1 s - \alpha_2)\delta T} + 1 - e^{-\alpha_2 \delta T}}]}
\end{aligned} \tag{A.25}$$

Now, let us assume, $\alpha_2 > \alpha_1$. After letting $T \to \infty$ in both sides of (A.25) we get,

$$
\begin{aligned}
\frac{1-\beta(1-\alpha_1)}{\alpha_1\beta} &= \frac{\alpha_2\delta e^{(\alpha_1 s+\alpha_2)\delta T}}{(e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1)^2[1-\alpha_2+\alpha_2]} \\
\frac{1-\beta(1-\alpha_1)}{\alpha_1\beta} &= \frac{\alpha_2\delta e^{(\alpha_1 s+\alpha_2)\delta T}}{(e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1)^2} \\
\frac{1-\beta(1-\alpha_1)}{\alpha_1\beta} &= \frac{\alpha_2\delta e^{\alpha_2\delta T}}{2(e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1)}, \ \textit{using L'Hospital Rule} \\
\frac{2[1-\beta(1-\alpha_1)]}{\alpha_1\beta} &= \frac{\alpha_2\delta}{e^{-\alpha_2\delta T}(e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1)} \\
\frac{1}{1+e^{-\alpha_2\delta T}(e^{\alpha_1\delta sT}-1)} &= \frac{2[1-\beta(1-\alpha_1)]}{\alpha_1\alpha_2\beta\delta} \\
1+e^{-\alpha_2\delta T}(e^{\alpha_1\delta sT}-1) &= \frac{\alpha_1\alpha_2\beta\delta}{2[1-\beta(1-\alpha_1)]} \\
e^{-\alpha_2\delta T}(e^{\alpha_1\delta sT}-1) &= \frac{\alpha_1\alpha_2\beta\delta}{2[1-\beta(1-\alpha_1)]}-1 \\
e^{\alpha_1\delta sT}-1 &= e^{\alpha_2\delta T}[\frac{\alpha_1\alpha_2\beta\delta}{2[1-\beta(1-\alpha_1)]}-1] \\
e^{\alpha_1\delta sT} &= \{e^{\alpha_2\delta T}[\frac{\alpha_1\alpha_2\beta\delta}{2[1-\beta(1-\alpha_1)]}-1]\}+1 \\
e^{\alpha_1\delta sT} &= \{e^{\alpha_2\delta T}[\frac{\alpha_1\alpha_2\beta\delta-2[1-\beta(1-\alpha_1)]}{2[1-\beta(1-\alpha_1)]}]\}+1 \\
e^{\alpha_1\delta sT} &= \frac{e^{\alpha_2\delta T}(\alpha_1\alpha_2\beta\delta-2[1-\beta(1-\alpha_1)])+2[1-\beta(1-\alpha_1)]}{2[1-\beta(1-\alpha_1)]} \\
\alpha_1\delta sT &= \log\frac{e^{\alpha_2\delta T}(\alpha_1\alpha_2\beta\delta-2[1-\beta(1-\alpha_1)])+2[1-\beta(1-\alpha_1)]}{2[1-\beta(1-\alpha_1)]} \qquad \text{(A.26)}
\end{aligned}
$$

Thus, from (A.25) we get;

$$\begin{aligned}
\alpha_1\delta sT &= \log\{e^{\alpha_2\delta T}(\alpha_1\alpha_2\beta\delta - 2[1-\beta(1-\alpha_1)]) \\
&\qquad + 2[1-\beta(1-\alpha_1)]\} - \log\{2[1-\beta(1-\alpha_1)]\} \\
s_0(T) &= \frac{1}{\alpha_1\delta T}[\log\{e^{\alpha_2\delta T}(\alpha_1\alpha_2\beta\delta - 2[1-\beta(1-\alpha_1)]) \\
&\qquad + 2[1-\beta(1-\alpha_1)]\} - \log\{2[1-\beta(1-\alpha_1)]\}] \\
&= \frac{1}{\alpha_1\delta T}[\log\{e^{\alpha_2\delta T}(\alpha_1\alpha_2\beta\delta - 2[1-\beta(1-\alpha_1)]) \\
&\qquad + 2[1-\beta(1-\alpha_1)]\}] - \frac{1}{\alpha_1\delta T}[\log\{2[1-\beta(1-\alpha_1)]\}] \qquad \text{(A.27)}
\end{aligned}$$

Now,

$$\begin{aligned}
\gamma &= \lim_{T\to\infty} s_0(T) \\
&= \lim_{T\to\infty} \frac{1}{\alpha_1\delta T}[\log\{e^{\alpha_2\delta T}(\alpha_1\alpha_2\beta\delta - 2[1-\beta(1-\alpha_1)]) + 2[1-\beta(1-\alpha_1)]\}] \\
&\qquad - \lim_{T\to\infty} \frac{1}{\alpha_1\delta T}[\log\{2[1-\beta(1-\alpha_1)]\}] \\
&= \lim_{T\to\infty} \frac{1}{\alpha_1\delta T}[\log\{e^{\alpha_2\delta T}(\alpha_1\alpha_2\beta\delta - 2[1-\beta(1-\alpha_1)]) + 2[1-\beta(1-\alpha_1)]\}] \\
&= \lim_{T\to\infty} \frac{\alpha_2\delta e^{\alpha_2\delta T}(\alpha_1\alpha_2\beta\delta - 2[1-\beta(1-\alpha_1)])}{\alpha_1\delta[e^{\alpha_2\delta T}(\alpha_1\alpha_2\beta\delta - 2[1-\beta(1-\alpha_1)]) + 2[1-\beta(1-\alpha_1)]]}, \ \textit{using L'Hospital Rule} \\
&= \lim_{T\to\infty} \frac{1}{\alpha_1\delta T}[\log\{e^{\alpha_2\delta T}(\alpha_1\alpha_2\beta\delta - 2[1-\beta(1-\alpha_1)])\}] \\
&= \lim_{T\to\infty} \frac{\alpha_2 e^{\alpha_2\delta T}(\alpha_1\alpha_2\beta\delta - 2[1-\beta(1-\alpha_1)])}{\alpha_1[e^{\alpha_2\delta T}(\alpha_1\alpha_2\beta\delta - 2[1-\beta(1-\alpha_1)]) + 2[1-\beta(1-\alpha_1)]]} \\
&= \lim_{T\to\infty} \frac{\alpha_2(\alpha_1\alpha_2\beta\delta - 2[1-\beta(1-\alpha_1)])}{\alpha_1[(\alpha_1\alpha_2\beta\delta - 2[1-\beta(1-\alpha_1)]) + 2e^{-\alpha_2\delta T}[1-\beta(1-\alpha_1)]]} \\
&= \frac{\alpha_2(\alpha_1\alpha_2\beta\delta - 2[1-\beta(1-\alpha_1)])}{\alpha_1(\alpha_1\alpha_2\beta\delta - 2[1-\beta(1-\alpha_1)])} = \frac{\alpha_2}{\alpha_1} \qquad \text{(A.28)}
\end{aligned}$$

Now let us assume $\alpha_1 > \alpha_2$ and check if the value of γ matches with the case $\alpha_1 s < \alpha_2$.After letting $T \to \infty$ in both sides of (A.25) we get,

$$
\begin{aligned}
\frac{1-\beta(1-\alpha_1)}{\alpha_1\beta} &= 1 + \frac{\alpha_2\delta e^{(\alpha_1 s+\alpha_2)\delta T}}{(e^{\alpha_1\delta sT} + e^{\alpha_2\delta T} - 1)^2[1-\alpha_2]} \\
\frac{1-\beta(1-\alpha_1)}{\alpha_1\beta} &= 1 + \frac{\alpha_2\delta e^{(\alpha_1 s+\alpha_2)\delta T}}{(e^{\alpha_1\delta sT} + e^{\alpha_2\delta T} - 1)^2[1-\alpha_2]} \\
\frac{1-\beta(1-\alpha_1)}{\alpha_1\beta} &= \frac{\alpha_2\delta e^{\alpha_2\delta T}}{2[1-\alpha_2](e^{\alpha_1\delta sT} + e^{\alpha_2\delta T} - 1)}, \; \textit{using L'Hospital Rule} \\
\frac{2(1-\alpha_2)[1-\beta(1-\alpha_1)]}{\alpha_1\beta} &= \frac{\alpha_2\delta}{e^{-\alpha_2\delta T}(e^{\alpha_1\delta sT} + e^{\alpha_2\delta T} - 1)} \\
\frac{1}{1+e^{-\alpha_2\delta T}(e^{\alpha_1\delta sT} - 1)} &= \frac{2(1-\alpha_2)[1-\beta(1-\alpha_1)]}{\alpha_1\alpha_2\beta\delta} \\
1+e^{-\alpha_2\delta T}(e^{\alpha_1\delta sT} - 1) &= \frac{\alpha_1\alpha_2\beta\delta}{2(1-\alpha_2)[1-\beta(1-\alpha_1)]} \\
e^{-\alpha_2\delta T}(e^{\alpha_1\delta sT} - 1) &= \frac{\alpha_1\alpha_2\beta\delta}{2(1-\alpha_2)[1-\beta(1-\alpha_1)]} - 1 \\
e^{\alpha_1\delta sT} - 1 &= e^{\alpha_2\delta T}[\frac{\alpha_1\alpha_2\beta\delta}{2(1-\alpha_2)[1-\beta(1-\alpha_1)]} - 1] \\
e^{\alpha_1\delta sT} &= \{e^{\alpha_2\delta T}[\frac{\alpha_1\alpha_2\beta\delta}{2(1-\alpha_2)[1-\beta(1-\alpha_1)]} - 1]\} + 1 \\
e^{\alpha_1\delta sT} &= \{e^{\alpha_2\delta T}[\frac{\alpha_1\alpha_2\beta\delta - 2(1-\alpha_2)[1-\beta(1-\alpha_1)]}{2(1-\alpha_2)[1-\beta(1-\alpha_1)]}]\} + 1 \\
e^{\alpha_1\delta sT} &= \frac{e^{\alpha_2\delta T}(\alpha_1\alpha_2\beta\delta - 2(1-\alpha_2)[1-\beta(1-\alpha_1)])}{2(1-\alpha_2)[1-\beta(1-\alpha_1)]} \\
&\quad + \frac{2(1-\alpha_2)[1-\beta(1-\alpha_1)]}{2(1-\alpha_2)[1-\beta(1-\alpha_1)]} \\
\alpha_1\delta sT &= \log\Bigg[\frac{e^{\alpha_2\delta T}(\alpha_1\alpha_2\beta\delta - 2(1-\alpha_2)[1-\beta(1-\alpha_1)])}{2(1-\alpha_2)[1-\beta(1-\alpha_1)]} \\
&\quad + \frac{2(1-\alpha_2)[1-\beta(1-\alpha_1)]}{2(1-\alpha_2)[1-\beta(1-\alpha_1)]}\Bigg] \qquad \text{(A.29)}
\end{aligned}
$$

Thus, from (A.29) we get;

$$\alpha_1\delta sT = \log\{e^{\alpha_2\delta T}(\alpha_1\alpha_2\beta\delta - 2(1-\alpha_2)[1-\beta(1-\alpha_1)]) + 2(1-\alpha_2)[1-\beta(1-\alpha_1)]\}$$
$$- \log\{2(1-\alpha_2)[1-\beta(1-\alpha_1)]\}$$
$$s_0(T) = \frac{1}{\alpha_1\delta T}[\log\{e^{\alpha_2\delta T}(\alpha_1\alpha_2\beta\delta - 2(1-\alpha_2)[1-\beta(1-\alpha_1)]) + 2(1-\alpha_2)[1-\beta(1-\alpha_1)]\}$$
$$- \log\{2(1-\alpha_2)[1-\beta(1-\alpha_1)]\}]$$
$$= \frac{1}{\alpha_1\delta T}[\log\{e^{\alpha_2\delta T}(\alpha_1\alpha_2\beta\delta - 2(1-\alpha_2)[1-\beta(1-\alpha_1)]) + 2(1-\alpha_2)[1-\beta(1-\alpha_1)]\}]$$
$$- \frac{1}{\alpha_1\delta T}[\log\{2(1-\alpha_2)[1-\beta(1-\alpha_1)]\}] \tag{A.30}$$

Now,

$$\gamma$$
$$= \lim_{T\to\infty} s_0(T)$$
$$= \lim_{T\to\infty} \frac{1}{\alpha_1\delta T}[\log\{e^{\alpha_2\delta T}(\alpha_1\alpha_2\beta\delta - 2(1-\alpha_2)[1-\beta(1-\alpha_1)]) + 2(1-\alpha_2)[1-\beta(1-\alpha_1)]\}]$$
$$- \lim_{T\to\infty} \frac{1}{\alpha_1\delta T}[\log\{2(1-\alpha_2)[1-\beta(1-\alpha_1)]\}]$$
$$= \lim_{T\to\infty} \frac{1}{\alpha_1\delta T}[\log\{e^{\alpha_2\delta T}(\alpha_1\alpha_2\beta\delta - 2(1-\alpha_2)[1-\beta(1-\alpha_1)]) + 2(1-\alpha_2)[1-\beta(1-\alpha_1)]\}]$$
$$= \lim_{T\to\infty} \frac{\alpha_2\delta e^{\alpha_2\delta T}(\alpha_1\alpha_2\beta\delta - 2(1-\alpha_2)[1-\beta(1-\alpha_1)])}{\alpha_1\delta[e^{\alpha_2\delta T}(\alpha_1\alpha_2\beta\delta - 2(1-\alpha_2)[1-\beta(1-\alpha_1)]) + 2(1-\alpha_2)[1-\beta(1-\alpha_1)]]},\ L'Hospital\ Rule$$
$$= \lim_{T\to\infty} \frac{1}{\alpha_1\delta T}[\log\{e^{\alpha_2\delta T}(\alpha_1\alpha_2\beta\delta - 2(1-\alpha_2)[1-\beta(1-\alpha_1)])\}]$$
$$= \lim_{T\to\infty} \frac{\alpha_2 e^{\alpha_2\delta T}(\alpha_1\alpha_2\beta\delta - 2(1-\alpha_2)[1-\beta(1-\alpha_1)])}{\alpha_1[e^{\alpha_2\delta T}(\alpha_1\alpha_2\beta\delta - 2(1-\alpha_2)[1-\beta(1-\alpha_1)]) + 2(1-\alpha_2)[1-\beta(1-\alpha_1)]]}$$
$$= \lim_{T\to\infty} \frac{\alpha_2(\alpha_1\alpha_2\beta\delta - 2(1-\alpha_2)[1-\beta(1-\alpha_1)])}{\alpha_1[(\alpha_1\alpha_2\beta\delta - 2(1-\alpha_2)[1-\beta(1-\alpha_1)]) + 2e^{-\alpha_2\delta T}[1-\beta(1-\alpha_1)]]}$$
$$= \frac{\alpha_2(\alpha_1\alpha_2\beta\delta - 2(1-\alpha_2)[1-\beta(1-\alpha_1)])}{\alpha_1(\alpha_1\alpha_2\beta\delta - 2(1-\alpha_2)[1-\beta(1-\alpha_1)])} = \frac{\alpha_2}{\alpha_1} \tag{A.31}$$

Hence, from (A.28) and (A.31) we see that $\gamma = \frac{\alpha_2}{\alpha_1}$ whenever $\alpha_2 > \alpha_1$ or $\alpha_2 < \alpha_1$. Finally, before applying Laplace Approximation we have to check if $g''(s_0(T), T) > 0$ as $T \to \infty$. $\square$

Proposition 74. *In Khoudraji (1996) non-exchangeable transformed survival factor Copula if $\alpha_2 > \alpha_1 s$ then we have $-g''(s,T) > 0$ as $T \to \infty$. In other words g becomes concave at very large T for all $\alpha_1, \alpha_2 \in [0,1]^2$ and $s \in [0,\infty)$. Furthermore, $-g''(s,T) \to \frac{\alpha_2(\alpha_1\delta)^2}{2} > 0$ as $T \to \infty$; where $\delta \geq 0$.*

Proof. We know from our earlier calculations;

$$\begin{aligned}
g(s;T) &= \frac{s}{\beta} + \frac{1}{T}[\alpha_2 T - (1-\alpha_1)sT - \frac{1}{\delta}\log(e^{\alpha_1\delta sT} + e^{\alpha_2\delta T} - 1) \\
&\qquad + \log[(1-\alpha_2) + \frac{\alpha_2 e^{\alpha_2\delta T}}{e^{\alpha_1\delta sT} + e^{\alpha_2\delta T} - 1}]] \\
g'(s;T) &= \frac{1}{\beta} - (1-\alpha_1) - \frac{\alpha_1 e^{\alpha_1\delta sT}}{e^{\alpha_1\delta sT} + e^{\alpha_2\delta T} - 1} \\
&\qquad - \frac{\alpha_1\alpha_2\delta e^{\alpha_1\delta sT + \alpha_2\delta T}}{(e^{\alpha_1\delta sT} + e^{\alpha_2\delta T} - 1)^2[1 - \alpha_2 + \frac{\alpha_2 e^{\alpha_2\delta T}}{e^{\alpha_1\delta sT} + e^{\alpha_2\delta T} - 1}]} \\
g''(s;T) &= \frac{\partial}{\partial s}[g'(s;T)] \\
&= \frac{\partial}{\partial s}[\frac{1}{\beta} - (1-\alpha_1) - \frac{\alpha_1 e^{\alpha_1\delta sT}}{e^{\alpha_1\delta sT} + e^{\alpha_2\delta T} - 1} \\
&\qquad - \frac{\alpha_1\alpha_2\delta e^{\alpha_1\delta sT + \alpha_2\delta T}}{(e^{\alpha_1\delta sT} + e^{\alpha_2\delta T} - 1)^2[1 - \alpha_2 + \frac{\alpha_2 e^{\alpha_2\delta T}}{e^{\alpha_1\delta sT} + e^{\alpha_2\delta T} - 1}]}] \\
&= \frac{\partial}{\partial s}[\frac{1}{\beta} - (1-\alpha_1) + A + B] \qquad \text{(A.32)}
\end{aligned}$$

where,

$$A = -\frac{\alpha_1 e^{\alpha_1\delta sT}}{e^{\alpha_1\delta sT} + e^{\alpha_2\delta T} - 1}$$

and,

$$B = -\frac{\alpha_1\alpha_2\delta e^{\alpha_1\delta sT + \alpha_2\delta T}}{(e^{\alpha_1\delta sT} + e^{\alpha_2\delta T} - 1)^2[1 - \alpha_2 + \frac{\alpha_2 e^{\alpha_2\delta T}}{e^{\alpha_1\delta sT} + e^{\alpha_2\delta T} - 1}]}.$$

Then,

$$
\begin{aligned}
\frac{\partial A}{\partial s} &= -\frac{\partial}{\partial s}[\frac{\alpha_1 e^{\alpha_1 \delta s T}}{e^{\alpha_1 \delta s T} + e^{\alpha_2 \delta T} - 1}] \\
&= -[\frac{\alpha_1^2 \delta T e^{\alpha_1 \delta s T}(e^{\alpha_1 \delta s T} + e^{\alpha_2 \delta T} - 1)}{(e^{\alpha_1 \delta s T} + e^{\alpha_2 \delta T} - 1)^2} - \frac{\alpha_1 e^{\delta s T}(\alpha_1 \delta T e^{\alpha_1 \delta s T})}{(e^{\alpha_1 \delta s T} + e^{\alpha_2 \delta T} - 1)^2}] \\
&= -[\frac{\alpha_1^2 \delta T e^{\alpha_1 \delta s T}(e^{\alpha_1 \delta s T} + e^{\alpha_2 \delta T} - 1)}{(e^{\alpha_1 \delta s T} + e^{\alpha_2 \delta T} - 1)^2} - \frac{\alpha_1 e^{\delta s T}(\alpha_1 \delta T e^{\alpha_1 \delta s T})}{(e^{\alpha_1 \delta s T} + e^{\alpha_2 \delta T} - 1)^2}] \\
&= -[\frac{\alpha_1^2 \delta T e^{\alpha_1 \delta s T}}{e^{\alpha_1 \delta s T} + e^{\alpha_2 \delta T} - 1} - \frac{\alpha_1 e^{\delta s T}(\alpha_1 \delta T e^{\alpha_1 \delta s T})}{(e^{\alpha_1 \delta s T} + e^{\alpha_2 \delta T} - 1)^2}] \\
&= -\frac{\alpha_1^2 \delta T e^{\alpha_1 \delta s T}}{e^{\alpha_1 \delta s T} + e^{\alpha_2 \delta T} - 1}(1 - \frac{e^{\alpha_1 \delta s T}}{e^{\alpha_1 \delta s T} + e^{\alpha_2 \delta T} - 1})
\end{aligned}
\tag{A.33}
$$

Taking limit of T in both sides of (A.33) we get,

$$
\begin{aligned}
\lim_{T \to \infty} \frac{\partial A}{\partial s} &= \lim_{T \to \infty} \frac{\alpha_1^2 \delta T e^{\alpha_1 \delta s T}}{e^{\alpha_1 \delta s T} + e^{\alpha_2 \delta T} - 1}(1 - \frac{e^{\alpha_1 \delta s T}}{e^{\alpha_1 \delta s T} + e^{\alpha_2 \delta T} - 1}) \\
&= \lim_{T \to \infty} \frac{\alpha_1^2 \delta}{e^{-\alpha_1 \delta s T}(e^{\alpha_1 \delta s T} + e^{\alpha_2 \delta T} - 1)}(1 - \frac{1}{e^{-\alpha_1 \delta s T}(e^{\alpha_1 \delta s T} + e^{\alpha_2 \delta T} - 1)}) \\
&= \lim_{T \to \infty} \frac{\alpha_1^2 \delta}{1 + e^{\delta T(\alpha_2 - \alpha_1 s)} - e^{-\alpha_1 \delta s T}}(1 - \frac{1}{1 + e^{\delta T(\alpha_2 - \alpha_1 s)} - e^{-\alpha_1 \delta s T}}) \\
&= 0,\ \textit{whether}\ \alpha_2 > \alpha_1\ \textit{or}\ \alpha_2 < \alpha_1
\end{aligned}
\tag{A.34}
$$

Until now, we have every component of $g''(s, T) = 0$. Let us check the B component of this second order derivative where $T \to \infty$. In other words, we need to check;

$$
\frac{\partial B}{\partial s} = -\frac{\partial}{\partial s}\left[\frac{\alpha_1 \alpha_2 \delta e^{(\alpha_1 s + \alpha_2)\delta T}}{(e^{\alpha_1 \delta s T} + e^{\alpha_2 \delta T} - 1)^2[1 - \alpha_2 + \frac{\alpha_2 e^{\alpha_2 \delta T}}{e^{\alpha_1 \delta s T} + e^{\alpha_2 \delta T} - 1}]}\right]
\tag{A.35}
$$

The partial derivative of the denominator of (A.35) becomes;

$$\begin{aligned}
&\frac{\partial}{\partial s}(e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1)^2\left[1-\alpha_2+\frac{\alpha_2 e^{\alpha_2\delta T}}{e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1}\right]\\
&=2\alpha_1\delta T e^{\alpha_1\delta sT}(e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1)^2\left[1-\alpha_2+\frac{\alpha_2 e^{\alpha_2\delta T}}{e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1}\right]\\
&\qquad-(e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1)^2\frac{\alpha_1\alpha_2\delta T e^{(\alpha_1 s+\alpha_2)\delta T}}{(e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1)^2}\\
&=2\alpha_1\delta T e^{\alpha_1\delta sT}(e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1)^2\left[1-\alpha_2+\frac{\alpha_2 e^{\alpha_2\delta T}}{e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1}\right]-\alpha_1\alpha_2\delta T e^{(\alpha_1 s+\alpha_2)\delta T}
\end{aligned} \tag{A.36}$$

Now,

$$\begin{aligned}
\frac{\partial B}{\partial s} &= -\frac{\partial}{\partial s}\Big[\frac{\alpha_1\alpha_2\delta e^{(\alpha_1 s+\alpha_2)\delta T}}{(e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1)^2[1-\alpha_2+\frac{\alpha_2 e^{\alpha_2\delta T}}{e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1}]}\Big] \\
&= -\Big[\frac{\alpha_1^2\alpha_2\delta^2 e^{(\alpha_1 s+\alpha_2)\delta T}(e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1)^2[1-\alpha_2+\frac{\alpha_2 e^{\alpha_2\delta T}}{e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1}]}{(e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1)^4[1-\alpha_2+\frac{\alpha_2 e^{\alpha_2\delta T}}{e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1}]^2} \\
&\quad - \frac{\alpha_1\alpha_2\delta e^{(\alpha_1 s+\alpha_2)\delta T}[2\alpha_1\delta T(e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1)[1-\alpha_2+\frac{\alpha_2 e^{\alpha_2\delta T}}{e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1}]]}{(e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1)^4[1-\alpha_2+\frac{\alpha_2 e^{\alpha_2\delta T}}{e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1}]^2} \\
&\quad - \frac{\frac{(e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1)^2\alpha_1\alpha_2\delta T e^{(\alpha_1 s+\alpha_2)\delta T}}{(e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1)^2}}{(e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1)^4[1-\alpha_2+\frac{\alpha_2 e^{\alpha_2\delta T}}{e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1}]^2}\Big] \\
&= -\Big[\frac{\alpha_1^2\alpha_2\delta^2 e^{(\alpha_1 s+\alpha_2)\delta T}(e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1)^2[1-\alpha_2+\frac{\alpha_2 e^{\alpha_2\delta T}}{e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1}]}{(e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1)^4[1-\alpha_2+\frac{\alpha_2 e^{\alpha_2\delta T}}{e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1}]^2} \\
&\quad - \frac{\alpha_1\alpha_2\delta e^{(\alpha_1 s+\alpha_2)\delta T}[2\alpha_1\delta T(e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1)[1-\alpha_2+\frac{\alpha_2 e^{\alpha_2\delta T}}{e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1}]]}{(e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1)^4[1-\alpha_2+\frac{\alpha_2 e^{\alpha_2\delta T}}{e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1}]^2} \\
&\quad - \frac{\alpha_1\alpha_2\delta T e^{(\alpha_1 s+\alpha_2)\delta T}}{(e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1)^4[1-\alpha_2+\frac{\alpha_2 e^{\alpha_2\delta T}}{e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1}]^2}\Big] \\
&= -\Big[\frac{\alpha_1^2\alpha_2\delta^2 e^{(\alpha_1 s+\alpha_2)\delta T}}{(e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1)^2[1-\alpha_2+\frac{\alpha_2 e^{\alpha_2\delta T}}{e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1}]} \\
&\quad - \frac{\alpha_1\alpha_2\delta T e^{(\alpha_1 s+\alpha_2)\delta T}}{(e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1)^4[1-\alpha_2+\frac{\alpha_2 e^{\alpha_2\delta T}}{e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1}]^2} \\
&\quad - \frac{\alpha_1\alpha_2\delta e^{(\alpha_1 s+\alpha_2)\delta T}[2\alpha_1\delta T(e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1)[1-\alpha_2+\frac{\alpha_2 e^{\alpha_2\delta T}}{e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1}]]}{(e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1)^4[1-\alpha_2+\frac{\alpha_2 e^{\alpha_2\delta T}}{e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1}]^2}\Big]
\end{aligned} \tag{A.37}$$

Now, let us define each of the three terms in (A.37) and check the limit of each of them when $T \to \infty$. Suppose,

$$c_1 = \frac{\alpha_1^2\alpha_2\delta^2 e^{(\alpha_1 s+\alpha_2)\delta T}}{(e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1)^2[1-\alpha_2+\frac{\alpha_2 e^{\alpha_2\delta T}}{e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1}]},$$

$$c_2 = \frac{\alpha_1\alpha_2\delta T e^{(\alpha_1 s+\alpha_2)\delta T}}{(e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1)^4[1-\alpha_2+\frac{\alpha_2 e^{\alpha_2\delta T}}{e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1}]^2}$$

and finally,

$$c_3 = \frac{\alpha_1\alpha_2\delta e^{(\alpha_1 s+\alpha_2)\delta T}[2\alpha_1\delta T(e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1)[1-\alpha_2+\frac{\alpha_2 e^{\alpha_2\delta T}}{e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1}]]}{(e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1)^4[1-\alpha_2+\frac{\alpha_2 e^{\alpha_2\delta T}}{e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1}]^2}.$$

Let us further assume, $\alpha_2 > \alpha_1$ and see what are the behaviors of c_1, c_2 and c_3 as $T \to \infty$. Let us first start with c_1;

$$\begin{aligned}
c_1 &= \frac{\alpha_1^2\alpha_2\delta^2 e^{(\alpha_1 s+\alpha_2)\delta T}}{(e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1)^2[1-\alpha_2+\frac{\alpha_2 e^{\alpha_2\delta T}}{e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1}]} \\
&= \frac{\alpha_1^2\alpha_2\delta^2 e^{(\alpha_1 s+\alpha_2)\delta T}}{(e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1)^2[1-\alpha_2+\frac{\alpha_2}{e^{(\alpha_1 s-\alpha_2)\delta T}+1-e^{-\alpha_2\delta T}}]} \\
\lim_{T\to\infty} c_1 &= \lim_{T\to\infty} \frac{\alpha_1^2\alpha_2\delta^2 e^{(\alpha_1 s+\alpha_2)\delta T}}{(e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1)^2},\ as\ \alpha_2 > \alpha_1 \\
&= \lim_{T\to\infty} \frac{\alpha_1^2\alpha_2\delta^2 e^{\alpha_2\delta T}}{2(e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1)},\ by\ L'Hospital\ Rule \\
&= \lim_{T\to\infty} \frac{\alpha_1^2\alpha_2\delta^2}{e^{(\alpha_1 s-\alpha_2)\delta T}+1-e^{-\alpha_2\delta T}} = \frac{\alpha_2(\alpha_1\delta)^2}{2}
\end{aligned} \tag{A.38}$$

Now,

$$\begin{aligned}
c_2 &= \frac{\alpha_1\alpha_2\delta T e^{(\alpha_1 s+\alpha_2)\delta T}}{(e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1)^4[1-\alpha_2+\frac{\alpha_2 e^{\alpha_2\delta T}}{e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1}]^2} \\
&= \frac{\alpha_1\alpha_2\delta T e^{(\alpha_1 s+\alpha_2)\delta T}}{(e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1)^4[1-\alpha_2+\frac{\alpha_2}{e^{(\alpha_1 s-\alpha_2)\delta T}+1-e^{-\alpha_2\delta T}}]}
\end{aligned}$$

Thus,

$$\begin{aligned}
\lim_{T\to\infty} c_2 &= \lim_{T\to\infty} \frac{\alpha_1\alpha_2\delta T e^{(\alpha_1 s+\alpha_2)\delta T}}{(e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1)^4} \\
&= \lim_{T\to\infty} \frac{\alpha_1\alpha_2\delta T e^{\alpha_2\delta T}}{4(e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1)^3},\ \textit{by } L'Hospital\ Rule \\
&= \lim_{T\to\infty} \frac{\alpha_1\alpha_2\delta[\alpha_2\delta T e^{\alpha_2\delta T}+e^{\alpha_2\delta T}]}{12(e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1)^2[\alpha_1\delta s e^{\alpha_1\delta sT}+\alpha_2\delta e^{\alpha_2\delta T}]},\ \textit{by } L'Hospital\ Rule \\
&= \lim_{T\to\infty} \frac{\alpha_1\alpha_2(1+\alpha_2\delta T)e^{\alpha_2\delta T}}{12(e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1)^2[\alpha_1\delta s e^{\alpha_1\delta sT}+\alpha_2\delta e^{\alpha_2\delta T}]} \\
&= \lim_{T\to\infty} \frac{\alpha_1\alpha_2(1+\alpha_2\delta T)}{12(e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1)^2[\alpha_1 s e^{(\alpha_1 s-\alpha_2)\delta T}+\alpha_2]} \\
&= \lim_{T\to\infty} \frac{\alpha_1\alpha_2(1+\alpha_2\delta T)e^{\alpha_2\delta T}}{12(e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1)^2[\alpha_1\delta s e^{\alpha_1\delta sT}+\alpha_2\delta e^{\alpha_2\delta T}]} \\
&= \lim_{T\to\infty} \frac{\alpha_1\alpha_2(1+\alpha_2\delta T)}{12\alpha_2(e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1)^2} \\
&= \lim_{T\to\infty} \frac{\alpha_1(1+\alpha_2\delta T)}{12(e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1)^2} \\
&= \lim_{T\to\infty} \frac{\alpha_1\alpha_2\delta}{24\delta(e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1)[\alpha_1 s e^{\alpha_1\delta sT}+\alpha_2 e^{\alpha_2\delta T}]} = 0,\ \textit{by } L'Hospital\ Rule.
\end{aligned} \tag{A.39}$$

Again;

$$
\begin{aligned}
c_3 &= \frac{\alpha_1\alpha_2\delta e^{(\alpha_1 s+\alpha_2)\delta T}[2\alpha_1\delta T(e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1)[1-\alpha_2+\frac{\alpha_2 e^{\alpha_2\delta T}}{e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1}]]}{(e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1)^4[1-\alpha_2+\frac{\alpha_2 e^{\alpha_2\delta T}}{e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1}]^2} \\
&= \frac{\alpha_1\alpha_2\delta e^{(\alpha_1 s+\alpha_2)\delta T}2\alpha_1\delta T}{(e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1)^3[1-\alpha_2+\frac{\alpha_2 e^{\alpha_2\delta T}}{e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1}]} \\
&= \frac{\alpha_1\alpha_2\delta e^{(\alpha_1 s+\alpha_2)\delta T}2\alpha_1\delta T}{(e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1)^3[1-\alpha_2+\frac{\alpha_2}{e^{(\alpha_1 s-\alpha_2)\delta T}+1-e^{-\alpha_2\delta T}}]} \\
\lim_{T\to\infty} c_3 &= \lim_{T\to\infty}\frac{\alpha_1\alpha_2\delta e^{(\alpha_1 s+\alpha_2)\delta T}2\alpha_1\delta T}{(e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1)^3},\ \textit{as } \alpha_2>\alpha_1 s \\
&= \lim_{T\to\infty}\frac{2\alpha_1^2\alpha_2\delta T e^{\alpha_2\delta T}}{3(e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1)^2},\ \textit{by L'Hospital Rule} \\
&= \lim_{T\to\infty}\frac{\alpha_1^2\alpha_2\delta(T\alpha_2\delta e^{\alpha_2\delta T}+e^{\alpha_2\delta T})}{3(e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1)\delta(\alpha_1 s e^{\alpha_1\delta sT}+\alpha_2 e^{\alpha_2\delta T})},\ \textit{by L'Hospital Rule} \\
&= \lim_{T\to\infty}\frac{\alpha_1^2\alpha_2 e^{\alpha_2\delta T}(1+\alpha_2\delta T)}{3(e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1)(\alpha_1 s e^{\alpha_1\delta sT}+\alpha_2 e^{\alpha_2\delta T})} \\
&= \lim_{T\to\infty}\frac{\alpha_1^2\alpha_2(1+\alpha_2\delta T)}{3(e^{(\alpha_1 s-\alpha_2)\delta T}+1-e^{-\alpha_2\delta T})(\alpha_1 s e^{\alpha_1\delta sT}+\alpha_2 e^{\alpha_2\delta T})} \\
&= \lim_{T\to\infty}\frac{\alpha_1^2\alpha_2(1+\alpha_2\delta T)}{3(\alpha_1 s e^{\alpha_1\delta sT}+\alpha_2 e^{\alpha_2\delta T})} \\
&= \lim_{T\to\infty}\frac{\alpha_1^2\alpha_2^2\delta}{3((\alpha_1 s)^2\delta e^{\alpha_1\delta sT}+\alpha_2^2\delta e^{\alpha_2\delta T})} = 0,\ \textit{by L'Hospital Rule.}
\end{aligned} \tag{A.40}
$$

Thus, from (A.38), (A.39) and (A.40) we know that, $\lim_{T\to\infty} -g''(s;T) = \frac{\alpha_2(\alpha_1\delta)^2}{2}$ when $\alpha_2 > \alpha_1$. Furthermore, if we try to calculate $\lim_{T\to\infty} g''(s;T)$ in the case where $\alpha_2 < \alpha_1$ we get all the terms equal to zero. That is why we rule out this case. This completes the proof. □

Claim 75. *In Khoudraji-transformed survival Clayton Copula with Weibull margins Laplace Approximation is not applicable.*

Proof. From Hua and Joe (2014) we know that, before applying Laplace approximation we have to verify if $g'(0;T) > 0$ as $T \to \infty$.

We know,

$$
\begin{aligned}
g(s;T) &= \frac{1}{T}[\alpha_2 T - (1-\alpha_1)sT - \frac{1}{\delta}\log(e^{\alpha_1\delta sT} + e^{\alpha_2\delta T} - 1) \\
&\qquad + \log[(1-\alpha_2) + \frac{\alpha_2 e^{\alpha_2\delta T}}{e^{\alpha_1\delta sT} + e^{\alpha_2\delta T} - 1}]] \\
g'(s;T) &= -(1-\alpha_1) - \frac{\alpha_1\delta e^{\alpha_1\delta sT}}{\delta(e^{\alpha_1\delta sT} + e^{\alpha_2\delta T} - 1)} \\
&\qquad - \frac{\alpha_1\alpha_2\delta e^{(\alpha_1 s+\alpha_2)\delta T}}{(e^{\alpha_1\delta sT} + e^{\alpha_2\delta T} - 1)^2[1-\alpha_2 + \frac{\alpha_2 e^{\alpha_2\delta T}}{e^{\alpha_1\delta sT} + e^{\alpha_2\delta T} - 1}]} \\
&= -(1-\alpha_1) - \frac{\alpha_1 e^{\alpha_1\delta sT}}{e^{\alpha_1\delta sT} + e^{\alpha_2\delta T} - 1} \\
&\qquad - \frac{\alpha_1\alpha_2\delta e^{(\alpha_1 s+\alpha_2)\delta T}}{(e^{\alpha_1\delta sT} + e^{\alpha_2\delta T} - 1)^2[1-\alpha_2 + \frac{\alpha_2 e^{\alpha_2\delta T}}{e^{\alpha_1\delta sT} + e^{\alpha_2\delta T} - 1}]} \\
g'(0;T) &= -\frac{\alpha_1}{e^{\alpha_2\delta T}} - (1-\alpha_1) - \frac{\alpha_1\alpha_2\delta e^{\alpha_2\delta T}}{e^{2\alpha_2\delta T}[1-\alpha_2 + \frac{\alpha_2 e^{\alpha_2\delta T}}{e^{\alpha_2\delta T}}]} \\
&= \alpha_1 - 1 - \frac{\alpha_1}{e^{\alpha_2\delta T}} - \frac{\alpha_1\alpha_2\delta}{e^{\alpha_2\delta T}} \\
\lim_{T\to\infty} g'(0;T) &= \alpha_1 - 1 < 0 \textit{ as } \alpha_1 \in [0,1] \textit{ and } \delta \geq 0 \textit{ by assumption.}
\end{aligned} \tag{A.41}
$$

As $g'(0;T) < 0$, we cannot use Laplace approximation. □

Corollary 76. *If $\alpha_2 < \alpha_1$, then for Khoudraji (1996) transformed survival Clayton Copula with Weibull margin with any $\gamma > 0$ and $\delta \geq 0$ we have;*

$$
\mathrm{E}[X_1|X_2 = t] \sim \frac{1}{\gamma}\Gamma(\frac{1}{\gamma})
$$

as $t \to \infty$.

Proof. Let us consider $\alpha_2 < \alpha_1$. From our previous discussions we know that, $g'(s;T) = -(1-\alpha_1) - \frac{\alpha_1 e^{\alpha_1\delta sT}}{e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1} - \frac{\alpha_1\alpha_2\delta e^{\alpha_1\delta sT+\alpha_2\delta T}}{(e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1)^2\left[1-\alpha_2+\frac{\alpha_2 e^{\alpha_2\delta T}}{e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1}\right]}$.

If we use this case, we have;

$$
\begin{aligned}
g'(s;T) &= -(1-\alpha_1) - \frac{\alpha_1}{1+e^{(\alpha_2-\alpha_1 s)\delta T} - e^{-\alpha_1 \delta s T}} \\
&- \frac{\alpha_1\alpha_2\delta e^{(\alpha_1 s+\alpha_2)\delta T}}{(e^{\alpha_1\delta s T}+e^{\alpha_2\delta T}-1)^2[1-\alpha_2+\frac{\alpha_2 e^{(\alpha_2-\alpha_1 s)\delta T}}{1+e^{(\alpha_2-\alpha_1 s)\delta T}-e^{-\alpha_1\delta s T}}]} \\
\lim_{T\to\infty} g'(s;T) &= -(1-\alpha_1) - \alpha_1 - \lim_{T\to\infty}\frac{\alpha_1\alpha_2\delta e^{(\alpha_1 s+\alpha_2)\delta T}}{(1-\alpha_2)(e^{\alpha_1\delta s T}+e^{\alpha_2\delta T}-1)^2},\ as\ \alpha_2 < \alpha_1, \\
&= -1 - \lim_{T\to\infty}\frac{\alpha_1\alpha_2\delta e^{(\alpha_1 s+\alpha_2)\delta T}}{(1-\alpha_2)(e^{\alpha_1\delta s T}+e^{\alpha_2\delta T}-1)^2} \\
&= -1 - \lim_{T\to\infty}\frac{\alpha_1\alpha_2\delta e^{\alpha_2\delta T}}{2(1-\alpha_2)(e^{\alpha_1\delta s T}+e^{\alpha_2\delta T}-1)},\ by\ L'Hospital\ Rule \\
&= -1 - \lim_{T\to\infty}\frac{\alpha_1\alpha_2\delta}{2(1-\alpha_2)(e^{(\alpha_1 s-\alpha_2)\delta T}+1-e^{-\alpha_2\delta T})} \\
&= -1 < 0,\ as\ T\to\infty \qquad \text{(A.42)}
\end{aligned}
$$

In this case $\lim_{T\to\infty} g'(s;T) < 0$ holds all the time. Hence, we can use Watson's lemma. As $g(s,T)$ is real function on the semi-infinite interval $[0,\infty)$ and in an interval $(0, 0+\epsilon]$ with $\epsilon > 0$ this function is continuously differentiable and $\sup_{0+\epsilon\le s\le\infty} g(s,T) \le g(0,T) - \psi$,

with $\psi > 0$, we can use a version of Watson's Lemma defined in theorem 36 [p. 48] in Breitung (1994). Now, for $g'(s,T)$ we have $g'(s,T) < 0$ as $T\to\infty$. We can also write $g'(s,T) = -as^{r-1} + o(s^{r-1})\ \forall\ r > 0$. Now if we assume $r = 1$ then $g'(s,T) = -a$. From our previous results we know that, $\lim_{s^+\to\ 0,\ T\to\infty} g'(s,T) = -1$, which is a constant. Thus, $-a = -1$ or, $a = 1 > 0$. Let us assume there is another real and continuous function $h(s) \in [0,\infty)$ such that, $h(s) = bs^{m-1} + o(s^{m-1})$ with $m > 0$. More specifically we assume $h(s) = s^{\frac{1}{\gamma}-1}$ in our case. Thus; $bs^{m-1} + o(s^{m-1}) = s^{\frac{1}{\gamma}-1}$
$\implies b = 1$; where $m = \frac{1}{\gamma}$.

Finally, as we are assuming $\int_0^\infty e^{g(s,T)}ds < \infty$ then by Watson's lemma we can write the approximated value of the integral $I(T) = \int_0^\infty e^{Tg(s,T)}ds$ with $T \geq 1$ are all finite and the asymptotic approximation is;

$$I(T) \sim \Gamma\left(\frac{1}{\gamma}\right) T^{-\frac{1}{\gamma}},\ as\ g(0;T) = 0\ and\ as\ T \to \infty. \tag{A.43}$$

Thus using (A.42) in the original integration we get,

$$\mathrm{E}[X_1|X_2 = t] \sim \frac{1}{\gamma}\Gamma(\frac{1}{\gamma}),\ as\ t \to \infty. \tag{A.44}$$

□

Corollary 77. *If $\alpha_2 < \alpha_1$, then for Khoudraji (1996) transformed survival Clayton Copula with Weibull margin with any $\lambda > 0$ and $\delta \geq 0$ we have;*

$$\mathrm{E}[X_1|X_2 = t] \sim \frac{1}{\lambda},$$

as $t \to \infty$ and $(\alpha_1, \alpha_2)\ \in [0,1]^2$.

Proof. From our previous discussions we know that, $g'(s;T) = -(1-\alpha_1) - \frac{\alpha_1 e^{\alpha_1 \delta s T}}{e^{\alpha_1 \delta s T} + e^{\alpha_2 \delta T} - 1} - \frac{\alpha_1 \alpha_2 \delta e^{\alpha_1 \delta s T + \alpha_2 \delta T}}{(e^{\alpha_1 \delta s T} + e^{\alpha_2 \delta T} - 1)^2 [1 - \alpha_2 + \frac{\alpha_2 e^{\alpha_2 \delta T}}{e^{\alpha_1 \delta s T} + e^{\alpha_2 \delta T} - 1}]}$.

Here we have;

$$
\begin{aligned}
g'(s;T) &= -(1-\alpha_1) - \frac{\alpha_1 e^{\alpha_1 \delta s T}}{e^{\alpha_1 \delta s T} + e^{\alpha_2 \delta T} - 1} \\
&\quad - \frac{\alpha_1 \alpha_2 \delta e^{(\alpha_1 s + \alpha_2)\delta T}}{(e^{\alpha_1 \delta s T} + e^{\alpha_2 \delta T} - 1)^2 [1 - \alpha_2 + \frac{\alpha_2 e^{\alpha_2 \delta T}}{e^{\alpha_1 \delta s T} + e^{\alpha_2 \delta T} - 1}]} \\
&= -(1-\alpha_1) - \frac{\alpha_1}{1 + e^{(\alpha_2 - \alpha_1 s)\delta T} - e^{-\alpha_1 \delta s T}} \\
&\quad - \frac{\alpha_1 \alpha_2 \delta e^{(\alpha_1 s + \alpha_2)\delta T}}{(e^{\alpha_1 \delta s T} + e^{\alpha_2 \delta T} - 1)^2 [1 - \alpha_2 + \frac{\alpha_2 e^{(\alpha_2 - \alpha_1 s)\delta T}}{1 + e^{(\alpha_2 - \alpha_1 s)\delta T} - e^{-\alpha_1 \delta s T}}]} \\
\lim_{T \to \infty} g'(s;T) &= -(1-\alpha_1) - \alpha_1 - \lim_{T \to \infty} \frac{\alpha_1 \alpha_2 \delta e^{(\alpha_1 s + \alpha_2)\delta T}}{(1-\alpha_2)(e^{\alpha_1 \delta s T} + e^{\alpha_2 \delta T} - 1)^2}, \ as \ \alpha_2 < \alpha_1, \\
&= -1 - \lim_{T \to \infty} \frac{\alpha_1 \alpha_2 \delta e^{(\alpha_1 s + \alpha_2)\delta T}}{(1-\alpha_2)(e^{\alpha_1 \delta s T} + e^{\alpha_2 \delta T} - 1)^2} \\
&= -1 - \lim_{T \to \infty} \frac{\alpha_1 \alpha_2 \delta e^{\alpha_2 \delta T}}{2(1-\alpha_2)(e^{\alpha_1 \delta s T} + e^{\alpha_2 \delta T} - 1)}, \ by \ L'Hospital \ Rule \\
&= -1 - \lim_{T \to \infty} \frac{\alpha_1 \alpha_2 \delta}{2(1-\alpha_2)(e^{(\alpha_1 s - \alpha_2)\delta T} + 1 - e^{-\alpha_2 \delta T})} \\
&= -1 < 0, \ as \ T \to \infty \qquad \text{(A.45)}
\end{aligned}
$$

In this case $\lim_{T \to \infty} g'(s;T) < 0$ holds all the time. Hence, we can use Watson's lemma. As $g(s,T)$ is real function on the semi-infinite interval $[0, \infty)$ and in an interval $(0, 0+\epsilon]$ with $\epsilon > 0$ this function is continuously differentiable and

$$\sup_{0+\epsilon \le s \le \infty} g(s,T) \le g(0,T) - \psi,$$

with $\psi > 0$, we can use a version of Watson's Lemma defined in theorem 36 [p. 48] in Breting (1994). Now, for $g'(s,T)$ we have $g'(s,T) < 0$ as $T \to \infty$. We can also write

$$g'(s,T) = -as^{r-1} + o(s^{r-1}) \ \forall \ r > 0.$$

Now if we assume $r = 1$ then $g'(s,T) = -a$. From our previous results we know that, $\lim_{s^+ \to\ 0,\ T\to\infty} g'(s,T) = -1$, which is a constant. Thus, $-a = -1$ or, $a = 1 > 0$. Let us assume there is another real and continuous function $h(s) \in [0, \infty)$ such that,

$$h(s) = bs^{m-1} + o(s^{m-1})$$

with $m > 0$. More specifically we assume $h(s) = 1$ in our case. Thus,

$$bs^{m-1} + o(s^{m-1}) = 1 \implies b = 1$$

when $m = 1$. Finally, as we are assuming $\int_0^\infty e^{g(s,T)} ds < \infty$ then by Watson's lemma we can write the approximated value of the integral $I(T) = \int_0^\infty e^{Tg(s,T)} ds$ with $T \geq 1$ are all finite and the asymptotic approximation is;

$$I(T) \sim T^{-1},\ as\ g(0;T) = 0 \tag{A.46}$$

Thus using A.45 in the original integration we get,

$$\mathrm{E}[X_1|X_2 = t] \sim \frac{1}{\lambda},\ as\ t \to \infty\ and\ s^+ \to\ 0. \tag{A.47}$$

□

Corollary 78. *In Khoudraji (1996) non-exchangeable transformed Gumbel Survival Copula with Exponential margin, the conditional tail expectation can be written as;*

$$\mathrm{E}[X_1|X_2 > t] \sim \frac{\alpha_1}{\lambda^2 t} \left(\frac{1}{1-\alpha_1}\right)^2,$$

as $t \to \infty$; where $\lambda > 0$ and $(\alpha_1, \alpha_2) \in [0,1]^2$.

Proof. From previous chapter and following the method provided by Hua and Joe (2014) we know that the conditional expectation of survival Copula with Exponential margin can be written as;

$$\begin{aligned} \mathrm{E}[X_1|X_2 > t] &= \lambda^{-1} T \int_0^\infty e^{T[1+\frac{1}{T}\log \widehat{C}(e^{-sT}, e^{-T})]}\ ds \\ &= \lambda^{-1} T \int_0^\infty e^{T\left[1+\frac{1}{T}\log\left[e^{-T\left\{s(1-\alpha_1)+(1-\alpha_2)+[(\alpha_1 s)^\delta+\alpha_2^\delta]^{1/\delta}\right\}}\right]\right]}\ ds \\ &= \lambda^{-1} T \int_0^\infty e^{T\left[1+\frac{1}{T}\log\left[e^{-T\{s(1-\alpha_1)+(1-\alpha_2)\}}\right]\right]}[(\alpha_1 s)^\delta + \alpha_2^\delta]^{1/\delta}\ ds \qquad \text{(A.48)} \\ &= \lambda^{-1} T \int_0^\infty e^{Tg(s,T)}\ h(s)\ ds,\ \forall\ T = \lambda t, \end{aligned}$$

where $g(s,T) = 1 + \frac{1}{T}\log\left[e^{-T\{s(1-\alpha_1)+(1-\alpha_2)\}}\right] = 1 - s(1-\alpha_1) - (1-\alpha_2)$ and $h(s) = [(\alpha_1 s)^\delta + \alpha_2^\delta]^{1/\delta}$. Here we are assuming $\lambda > 0$, $\alpha_1, \alpha_2 \in [0,1]$, $\delta \geq 0$ and $s \in [0, \infty)$.

If we further solve the g function we get, $g(s,T) = \alpha_2 - s(1-\alpha_1)$. From simple calculations we can easily verify that $g(0,T) = 0$ and $g(\infty, T) = -\infty$. Thus, we can use either Laplace Approximation or Watson's lemma. But as $g'(s,T) = -(1-\alpha_1) < 0$ irrespective of any s we have $g'(0,T) < 0$, we can never use Laplace Approximation. In this case , we have to use Watson's lemma.

As $g(s;T)$ is a real valued function on the semi-infinite interval $[0, \infty)$ and in $(0, 0+\epsilon]$ with $\epsilon > 0$ this function is continuously differentiable and $\sup_{0+\epsilon \leq s \leq \infty} g(s,T) \leq g(0,T) - \psi$, with $\psi > 0$. Here we are using theorem 36 [p.48] of Breitung (1994). This theorem is an extension of Watson's lemma and most importantly, this theorem works for semi-infinite intervals. Now we have $g'(s,T) < 0$ and $s \to \infty$.We can also write $g'(s,T) = -as^{r-1} + o(s^{r-1})\ \forall\ r > 0$ Now if we assume $r = 1$ then $g'(s,T) = -a$. From our previous results we know that, $\lim_{s^+ \to\ 0,\ T \to \infty} g'(s,T) = -(1-\alpha_1)$, which is a constant. Thus, $-a = -[1-\alpha_1]$ or, $a = 1 - \alpha_1 > 0$.

Let us assume there is another real and continuous function $h(s) \in [0, \infty)$ such that, $h(s) = bs^{m-1} + o(s^{m-1})$ with $m > 0$. More specifically we assume $h(s) = [(\alpha_1 s)^\delta + \alpha_2^\delta]^{1/\delta}$ in our case. Thus,

$$\begin{aligned} [(\alpha_1 s)^\delta + \alpha_2^\delta]^{1/\delta} &= bs^{m-1} + o(s^{m-1}) \\ (\alpha_1 s)^\delta + \alpha_2^\delta &= b^\delta s^{\delta(m-1)} + o(s^{m-1}) \\ \alpha_1^\delta s^\delta &= b^\delta s^{\delta(m-1)}, \; as \; \alpha_2^\delta \sim o(s^{m-1}) \end{aligned} \tag{A.49}$$

Thus, we have $b = \alpha_1$ and $m = 2$ [as $\delta = \delta(m-1)$, the powers of s on the both sides of (A.49) are the same].

Finally, as we are assuming $\int_0^\infty e^{g(s,T)} \; h(s) \; ds < \infty$ then by Watson's lemma we can write the approximated value of the integral $I(T) = \int_0^\infty e^{Tg(s,T)} \; h(s) \; ds$ with $T \geq 1$ are all finite and the asymptotic approximation is;

$$\begin{aligned} I(T) &\sim \alpha_1 \Gamma(2) \left(\frac{1}{1-\alpha_1} \right)^2 T^{-2} e^{Tg(0,T)} \\ &\sim \alpha_1 \left(\frac{1}{1-\alpha_1} \right)^2 T^{-2}, \; as \; g(0,T) = 0, \; \Gamma(2) = 1 \end{aligned} \tag{A.50}$$

Now, using (A.48) and (A.50) we get the conditional expectation as ;

$$\mathrm{E}[X_1 | X_2 > t] \sim \frac{\alpha_1}{\lambda^2 t} \left(\frac{1}{1-\alpha_1} \right)^2 \tag{A.51}$$

as $t \to \infty$, $T = \lambda t$, $\alpha_1 \in [0, 1]$ and $\lambda > 0$. This completes the proof. □

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