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Nodal solutions of nonlocal integral boundary value problems

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ABSTRACT

NODAL SOLUTIONS OF NONLOCAL INTEGRAL BOUNDARY VALUE PROBLEMS

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In this dissertation, we study the nonlinear boundary value problem consisting of the second-order equation $-(p(t)y')' + q(t)y = w(t)f(y)$ on $[a, b]$ and one of two boundary conditions involving a Riemann-Stieltjes integral. Specifically, we consider the boundary value problems consisting of the boundary conditions:

$$\cos\alpha\ y(a) - \sin\alpha\ (py')(a) = 0, \quad \alpha \in [0, \pi),$$
$$y(b) - \int_a^b y(s)d\xi(s) = 0;$$

$$\cos\alpha\ y(a) - \sin\alpha\ (py')(a) = 0, \quad \alpha \in [0, \pi),$$
$$(py')(b) - \int_a^b (py')(s)d\xi(s) = 0.$$

By relating the problems to the the eigenvalues of the corresponding linear Sturm-Liouville problem with a two-point separated boundary condition, we obtain results on the existence and nonexistence of nodal solutions to these problems. The shooting method and a generalized energy function are used to prove the main results. We also discuss the changes in the existence of different types of nodal solutions as the problem

changes. Finally, we examine a more general differential equation with multiple terms on the right-hand side of the form:

$$-(p(t)y')' + q(t)y = \sum_{i=1}^{m} w_i(t)f_i(y), \ \ t \in (a, b),$$

together with one of the aforementioned boundary conditions.

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TABLE OF CONTENTS

CHAPTER 1

INTRODUCTION

1.1 Early History

In 1836-1837, Charles Francois Sturm and Joseph Liouville collaborated and published significant work on the eigenvalue problem of second-order differential equations. The problem studied jointly by Sturm and Liouville in [51] is given by

$$-\frac{d}{dx}\left(k\frac{dV}{dx}\right)+lV=rgV \quad \text{on} \quad \text{the} \quad \text{interval} \quad [\mathbf{x},\mathbf{X}] \tag{1.1.1}$$

with the following separated boundary conditions

$$\begin{aligned} \frac{dV}{dx}-hV&=0 \quad \text{for} \quad x=\mathbf{x}, \\ \frac{dV}{dx}+HV&=0 \quad \text{for} \quad x=\mathbf{X}. \end{aligned} \tag{1.1.2}$$

In Sturm and Liouville's notation k, l, and g are positive coefficient functions on the interval $[\mathbf{x},\mathbf{X}]$, h and H are positive real numbers, and r is a real-valued parameter. Although not mentioned, we assume that Sturm and Liouville had in mind continuous coefficient functions k, l, and g. This began what would become an important and far-reaching area of mathematics known as Sturm-Liouville theory. In physics, many boundary value problems (BVPs) can be separated into ordinary differential equations (ODEs) of second-order. Examples include the heat equation, Laplace's equation, and the wave equation. For example, we might want to solve the wave equation BVP

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2},$$

subject to the boundary condition (BC)

$$y(0,t) = y(l,t) = 0, \quad t > 0.$$

By separation of variables, we assume $y(x,t) = v(x)w(t)$, leading to two ODEs:

$$v''(x) + \lambda v(x) = 0, \quad 0 < x < l,$$

and

$$w''(t) + \mu w(t) = 0, \quad \mu = c^2\lambda, \quad \text{and} \quad t > 0.$$

Both of these yield periodic solutions on the their respective domains. Furthermore, for $n \in \mathbb{N}_0$, we have eigenvalues $\lambda_n = (n+1)^2\pi^2/l^2$ and $\mu_n = c^2\lambda_n$.

Before the joint collaboration of Sturm and Liouville, the primary investigations of (1.1.1), (1.1.2) were dedicated to finding explicit analytic solutions. Furthermore, problems found their basis in physical systems where the BVPs were often derived empirically from experiment and observation. For example, eigenvalue problems were studied by Brook Taylor (1713) and Johann Bernoulli (1728) in papers about the vibrating string and the hanging chain, respectively [6, 53]. Daniel Bernoulli (1733) furthered his father's work and derived the differential equation governing the shape of a vibrating, hanging, homogeneous chain. The shape of the chain, $y(x)$, obeyed the equation

$$\alpha \frac{d}{dx}\left(x\frac{dy}{dx}\right) + y = 0,$$

whose solution is an infinite series of the form

$$y = AJ_0(2\sqrt{x/\alpha}),$$

where J_0 is the the zeroth-order Bessel function [5].

While Taylor and the Bernoullis derived equations of type (1.1.1),(1.1.2) from physical principles, D'Alembert, Fourier, and Poisson derived the the eigenvalue equations from the partial differential equations by separation of variables. In studying the vibrations of a nonhomogeneous string, D'Alembert (1747) showed the existence of at least one eigenvalue of the BVP

$$\frac{\partial^2 y}{\partial x^2} = \mathrm{X}\frac{\partial^2 y}{\partial t^2},$$

with Dirichlet BC

$$y(0,t) = y(a,t),$$

where $\mathrm{X}(x)$ is the density and $y(x,t)$ is the amplitude, [11].

Fourier studied heat conduction in homogeneous materials. Working in cylindrical coordinates and via separation of variables, Fourier (1822) studied the equation

$$\frac{d^2u}{dx^2} + \frac{1}{x}\frac{du}{dx} + \frac{m}{k}u = 0,$$

satisfying the BC

$$hu + \frac{du}{dx} = 0 \quad \text{for} \quad x = \beta,$$

where h, k are constants ($k > 0$), x is the distance from the axis, β the radius of the cylinder, and $e^{-mt}u(x)$ is the temperature of the rod. Fourier correctly (although lacking adequate proof) obtained infinitely many eigenvalues and corresponding eigenfunctions [16].

Finally, Poisson (1826) was able to show the existence of eigenvalues and eigenfunctions for the heat problem in double-layered spheres. Poisson's methods showed the existence of real eigenvalues as the zeros of a transcendental equation and he also demonstrated the orthogonality properties of the corresponding eigenfunctions [45, 46]. He studied the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \mathrm{X}(x)u,$$

with the BC

$$\frac{du}{dx} - hu = 0 \quad \text{for} \quad x = \alpha,$$

$$\frac{du}{dx} + Hu = 0 \quad \text{for} \quad x = \beta.$$

The aforementioned authors set the stage for the monumental joint work of Charles Francois Sturm and Joseph Liouville. What significantly differentiates the work of Sturm and Liouville from others is that they did not necessarily seek explicit analytic solutions to (1.1.1), (1.1.2). Furthermore, their work, while important in application to physical processes, was not motivated by any physical process or experiment. Instead, Sturm and Liouville relied on the properties of the coefficient functions of (1.1.1), (1.1.2) to ascertain qualitative information on the existence and behavior of solutions without regard to the exact form of a solution. This way of thinking was a new concept not before explored until the collaboration of Sturm and Liouville.

Sturm and Liouville's contributions in what is now known as Sturm-Liouville theory can be placed into three areas:

1. properties of eigenvalues,

2. behavior of corresponding eigenfunctions, and

3. expansion of arbitrary functions in an infinite series of eigenfunctions.

The joint work of Sturm and Liouville began the study of the regular separated Sturm-Liouville problem (SLP). Since then, great advances have been made. Herman Weyl, in his 1910 paper [61], began the investigation of the singular SLP. John von Neumann (1929) and Marshall Harvey Stone (1932) independently investigated the spectral properties of the SLP by general theory of unbounded linear operators in Hilbert spaces (see [43, 48]). Titchmarsh, in a series of three papers in 1941 [54–56], applied the theory of functions of a complex variable to both regular and singular SLPs. A great review of post-1836-1837 SLP work can be found in Werner O. Amrein, et. al [3].

1.2 Examples of BVPs

BVPs are pervasive in physics as well as other diverse fields. Areas where BVPs are studied include:

- Fluid dynamics
- Epidemiology
- Accelerator physics

- Population dynamics

- Material science

We list a few examples below. For more examples, we refer the reader to [12, 23, 62] and references therein.

Example 1. Particle Accelerator Physics

In subatomic particle accelerators (like those at Fermi National Laboratory in Batavia, Illinois or at the European Council for Nuclear Research in Geneva, Switzerland), particle trajectories in circular accelerators can be modeled by a BVP:

$$x''(s) + \left(\frac{1}{R^2(s)} - k(s)\right)x(s) - \frac{1}{R(s)}\frac{\Delta p}{p} = 0,$$

$$y''(s) + k(s)y(s) = 0,$$

with the periodic boundary condition

$$x(s) = x(s + L), y(s) = y(s + L).$$

These equations describe the transverse motion of particles in circular particle accelerator with $R(s)$ and $k(s)$ related to the strength of the magnetic dipole and quadrupole fields, respectively. $\Delta p/p$ is the ratio of momentum spread to momentum of a particle circulating in the accelerator.

Example 2. The Spread of Genes in a Population

The propagation of an advantageous gene in a population can be modeled by the following BVP system:

$$-v\frac{dP}{dz} = D\frac{d^2P}{dz^2} + \alpha P(1 - P),$$

$$P(-\infty) = 1, P(\infty) = 0.$$

Here, P is the frequency of the advantageous gene, v is the speed which the gene propagates through the population, D is the dispersion rate, and α is the reproductive rate of the population.

Example 3. Flow of Water in Unsaturated Soil

Horizontal flow of water in unsaturated soil can be modeled by:

$$\frac{B}{2}\frac{d\theta}{dB} = \frac{d}{dB}\left[D(\theta)\frac{d\theta}{dB}\right],$$

$$\theta(0) = \theta_0, \theta(\infty) = 0.$$

Here, B is equal to $x/\sqrt{t}$, θ is the soil wetness, and $D(\theta)$ is the hydraulic diffusivity of the soil.

1.3 Results in the Literature

1.3.1 Linear SLP

The linear SLP has been well studied. For a thorough exposition, see Zettl's monograph on SL theory, [65]. It is well known that the classical SLP consisting of the equation

$$-(p(t)y')' + q(t)y = \lambda w(t)y, \quad t \in (a,b), \tag{1.3.1}$$

with separated boundary conditions (BC)

$$\begin{aligned} &\cos\alpha\ y(a) - \sin\alpha\ (py')(a) = 0, \quad \alpha \in [0,\pi), \\ &\cos\beta\ y(b) - \sin\beta\ (py')(b) = 0, \quad \beta \in (0,\pi], \end{aligned} \tag{1.3.2}$$

has a countable number of eigenvalues $\lambda_i, i = 0, 1, 2, \ldots,$ which are bounded below and unbounded above, and can be arranged such that

$-\infty < \lambda_0 < \lambda_1 < \lambda_2 < \lambda_3 < .... < \lambda_n < \lambda_{n+1} < ...$ and $\lambda_n \to \infty$ as $n \to \infty$.

Furthermore, for each λ_n, the corresponding eigenfunction y_n has exactly n zeros in (a, b). Recently, Kong and Zettl [31] showed that the eigenvalues of regular SLPs are differentiable functions of all the data: coefficient functions $1/p$, q, weight function, w and the BC, except at the Dirichlet BC. Moreover, Kong, Wu and Zettl [30] showed that all the eigenvalues, except possibly the first, approach $+\infty$ as the interval (a, b) shrinks to an endpoint.

1.3.2 Nonlinear Two-Point BC

Naturally, work on the existence of solutions of the associated nonlinear problem:

$$-(p(t)y')' + q(t)y = w(t)f(y), \quad t \in (a,b) \tag{1.3.3}$$

with separated BCs

$$\begin{aligned} &\cos\alpha\ y(a) - \sin\alpha\ (py')(a) = 0, \quad \alpha \in [0,\pi), \\ &\cos\beta\ y(b) - \sin\beta\ (py')(b) = 0, \quad \beta \in (0,\pi], \end{aligned} \tag{1.3.4}$$

has also been of great interest. Many special cases of (1.3.3), (1.3.4) have been studied with a variety of techniques and methods; see, for example [14,15,17,61,66]. In most of these papers, results are about the existence of positive solutions and for special BCs such as Dirichlet and Neumann BCs.

Erbe [13] used the idea of relating (1.3.3), (1.3.4) to its corresponding linear SLP. In his paper, fixed point theory on cones was used to show the existence of positive solutions by comparing the values of $f(y)/y$, $y \in (0, \infty)$, with the smallest eigenvalue, λ_1, of the corresponding linear SLP. However, no results for solutions to (1.3.3), (1.3.4) were found with zeros in (a, b).

More recently, the existence of nodal solutions of two-point separated BCs have been investigated, i.e., solutions with a specific zero-counting property in (a, b), see [26,28,36,39–42]. Naito and Tanaka [41] studied a special case of (1.3.3), (1.3.4) with $p \equiv 1$, $q \equiv 0$ and the Dirichlet BC

$$-y'' = w(t)f(y), \quad y(0) = y(1) = 0.$$

They established, under suitable conditions, there was a solution with exactly n zeros in (a, b) if λ_n is the interior of the range $f(y)/y, y \in (0, \infty)$. Here, λ_n is an eigenvalue of the associated linear SLP. Critical ideas in the proof are:

1. $f(y)/y$ is compared to the eigenvalues of the associated linear SLP.

2. The Prüfer angle and celebrated Sturm-Picone comparison theorem for the number of zeros of a solution on $(0, 1)$ are used.

3. The shooting method and an energy function are used to match BC.

In [28], Kong extended Naito and Tanaka's work in [41] to include $q \in C^1$ and a general separated BC, i.e., (1.3.3), (1.3.4) with $p \equiv 1$.

1.3.3 Nonlinear Nonlocal BC

Problems involving two-point separated BCs found in quantum mechanics, the heat equation, and other applications are known about and well studied. In this vein, work on nonlocal (multipoint/integral) BVPs is of particular interest whenever data is given to a physical problem where two-point boundary conditions fail to adequately describe the boundary data. While it is often impractical or impossible to find an explicit form to solutions of nonlocal BVPs, qualitative information about a solution is helpful in understanding physical phenomena. For these reasons, the study of nonlocal BVPs is of particular interest to mathematicians.

The existence of positive solutions concerning nonlocal BCs including three-point, multipoint and integral BCs has been studied extensively; see, for example, [1, 10, 18–21, 32, 33, 47, 57–60, 64] and the references therein. Riemann-Stieltjes integral formulations of the multipoint BCs have also been considered by many authors, notably by Webb and Infante [57, 59]. In these papers, the existence of positive solutions are studied by using fixed point index theory.

In recent years, the existence of nodal solutions, i.e., solutions with a specific zero-counting property in (a, b), has also attracted much attention in the research of BVPs. Great progress has been made in the study of such solutions for BVPs consisting of Eq. (1.3.3) with certain types of nonlocal BCs. In fact, Ma and O'Regan [38] and Rynne [47] studied the special BVP consisting of the equation

$$y'' + f(y) = 0,\ t \in (0, 1), \tag{1.3.5}$$

and the multipoint BC

$$y(0) = 0, \quad y(1) - \sum_{j=1}^{d} k_j y(\eta_j) = 0. \tag{1.3.6}$$

Here, they used a standard global bifurcation method to establish the existence of nodal solutions of BVP (1.3.5), (1.3.6) by relating it to the eigenvalues of the corresponding linear Sturm-Liouville problem (SLP) with the multipoint BC (1.3.6). However, the establishment of these results relies heavily on the direct computation of the eigenvalues and eigenfunctions of the SLP associated with BVP (1.3.5), (1.3.6), and hence cannot be extended to a general BVP with variable coefficient functions. Motivated by the above work, L. Kong and Q. Kong [25] established the existence of nodal solutions of the BVP consisting of the equation

$$y'' + w(t)f(y) = 0, \ t \in (a, b), \tag{1.3.7}$$

and the multipoint BC

$$\begin{aligned} &\cos\alpha \ y(a) - \sin\alpha \ y'(a) = 0, \quad \alpha \in [0, \pi), \\ &y'(b) - \textstyle\sum_{j=1}^{d} k_j y'(\eta_j) = 0, \end{aligned} \tag{1.3.8}$$

by relating it to the eigenvalues of the corresponding linear SLP with a two-point separated BC:

$$\begin{aligned} &\cos\alpha \ y(a) - \sin\alpha \ y'(a) = 0, \quad \alpha \in [0, \pi), \\ &y(b) = 0. \end{aligned} \tag{1.3.9}$$

Q. Kong et al. [27] continued further by establishing the existence of nodal solutions to Eq. (1.3.7) with the multipoint BC

$$\begin{aligned} &\cos\alpha\ y(a) - \sin\alpha\ y'(a) = 0, \quad \alpha \in [0, \pi), \\ &y(b) - \textstyle\sum_{j=1}^{d} k_j y(\eta_j) = 0. \end{aligned} \tag{1.3.10}$$

This provides a new direction for the research of nonlocal BVPs and the results are significant since eigenvalues are easy to calculate for two-point linear self-adjoint SLPs using standard software packages such as those in [4].

1.4 Outline of Research

In this dissertation, we are concerned with BVPs with nonlocal BCs and we wish to study the existence, nonexistence, and dependence of nodal solutions on the problem. Motivated by the great work of Kong et al. above, we wish to consider a more general form of Eq. (1.3.7) together with different types of nonlocal BCs. Specifically, we want to consider BVPs involving integral BCs. In this vein, we are concerned with the BVP consisting of the equation

$$-(p(t)y')' + q(t)y = w(t)f(y), \quad t \in (a, b), \tag{1.4.1}$$

and the BCs:

$$\begin{aligned} &\cos\alpha\ y(a) - \sin\alpha\ (py')(a) = 0, \quad \alpha \in [0, \pi), \\ &y(b) - \textstyle\int_a^b y(s)d\xi(s) = 0, \end{aligned} \tag{1.4.2}$$

$$\begin{aligned} &\cos\alpha\ y(a) - \sin\alpha\ (py')(a) = 0, \quad \alpha \in [0, \pi), \\ &(py')(b) - \textstyle\int_a^b (py')(s)d\xi(s) = 0. \end{aligned} \tag{1.4.3}$$

Our BVPs depend on f and the coefficient functions p, w, and q together with data on the boundary.

In Chapter 2, since each of the two BCs together with Eq. (1.4.1) represents a different problem, we investigate BVPs (1.4.1), (1.4.2) and (1.4.1), (1.4.3). We study the criteria for the existence, nonexistence and dependence of nodal solutions on the problem for both of these BVPs. Our results extend and improve the results in the literature, specifically in [25, 27, 38, 47]. Prior results utilize the shooting method and an energy function to establish the existence of nodal solutions and match boundary conditions. In particular, our results consider the addition of q in (1.4.1), which may change sign. The shooting method and an energy function are used in our proofs, but certain conditions are imposed on the coefficient functions to establish the existence, nonexistence, and dependance of nodal solutions on the problem.

In Chapter 3, we generalize the above BVPs by considering multiple terms on the right side of Eq. (1.4.1). Specifically, we investigate the BVPs given by

$$-(p(t)y')' + q(t)y = \sum_{i=1}^{m} w_i(t)f_i(y), \; t \in (a, b),$$

and the BCs (1.4.2), (1.4.3). Again, this extends and improves current results in the literature for nonlinear, nonlocal BVPs.

In Chapter 4, we present further problems for study.

CHAPTER 2

THEOREMS AND PROOFS

We assume throughout, and without further mention, that the following conditions hold:

(H1) $p, q, w \in C^1[a,b]$ such that $p(t) > 0$, $w(t) > 0$, and $q'(t) + q^* \leq l(t)(q^* - q(t))$ on $[a,b]$ with

$$q^* := \max_{t\in[a,b]}\{q(t), 0\} \quad \text{and} \quad l(t) := \max\left\{\left(\frac{p'(t)+q^*}{p(t)}\right)_+, \frac{w'_-(t)}{w(t)}\right\},$$

where $h'_-(t) := \max\{0, -h'(t)\}$ and $h_+(t) := \max\{0, h(t)\}$, $k_0 = \displaystyle\int_a^b l(t)\ dt$;

(H2) $f \in C(\mathbb{R})$ such that $yf(y) > 0$ for $y \neq 0$, and f is locally Lipschitz on $(-\infty, 0) \cup (0, \infty)$;

(H3) there exist extended real numbers $f_0, f_\infty \in [0, \infty]$ such that

$$f_0 = \lim_{y\to 0} f(y)/y \qquad \text{and} \qquad f_\infty = \lim_{|y|\to\infty} f(y)/y;$$

(H4) $w(t)\dfrac{f(y)}{y} - q(t) > 0$ for all $t \in [a,b]$, and $f(-y) = -f(y)$ for all $y \neq 0$.

Remark 2.0.1. For $p, w \in C^1[a,b]$, the following are examples of the function classes for q satisfying (H1):

(i) $q \in C^1[a,b]$ such that $q'(t) \leq -q^*$ on $[a,b]$. It is easy to see that any nonpositive, nonincreasing function q belongs to this class. In particular, any nonpositive constants belong to this class.

(ii) $q \in C^1[a,b]$ such that $q'(t) \leq -l(t)q(t)$ on $[a,b]$ with $l(t) \geq 1$. For $c \geq 0$, it is easy to see that

$$q_1(t) = ce^{-kt} \text{ for } t \in [0,1] \text{ with } k \geq l(t) \geq 1 \quad \text{and}$$
$$q_2(t) = -ce^{-kt} \text{ for } t \in [0,1] \text{ with } 0 \leq k \leq l(t) \text{ and } l(t) \geq 1$$

belong to this class.

In addition to (H1)-(H3), (H4) is needed for BVP (1.4.1), (1.4.2). The oddness assumption in (H4) is only for convenience. Our results can be extended to the case where $w(t)\dfrac{f(y)}{y} - q(t) > 0$ for $y \neq 0$, without the oddness assumption. The integrals in BCs (1.4.2), (1.4.3) are Riemann-Stieltjes integrals with respect to $\xi(s)$ with $\xi(s)$ of bounded variation. Note that the function $\xi(s)$ given in the above BCs are of bounded variation on $[a,b]$. Thus, there are two nondecreasing functions $\xi_1(s)$ and $\xi_2(s)$ such that

$$\xi(s) = \xi_1(s) - \xi_2(s), \quad s \in [a,b]. \tag{2.0.1}$$

Let us consider the BVP (1.4.1), (1.4.2):

$$-(p(t)y')' + q(t)y = w(t)f(y), \quad t \in (a,b),$$

and the BC

$$\cos\alpha\ y(a) - \sin\alpha\ (py')(a) = 0, \quad \alpha \in [0,\pi),$$
$$y(b) - \textstyle\int_a^b y(s)d\xi(s) = 0,$$

$a, b \in \mathbb{R}$ with $a < b$. In the case where $\xi(s) = s$, the Riemann-Stieltjes integral in the second line of the BC reduces to the Riemann integral. In the case that $\xi(s) = \sum_{j=1}^{d} k_j \chi(s - \eta_j)$, where $d \geq 1$, $k_j \in \mathbb{R}$, $j = 1, \ldots, d$, $\{\eta_j\}_{j=1}^{d}$ is a strictly increasing sequence of distinct points in (a, b), and $\chi(s)$ is the characteristic function, i.e.,

$$\chi(s) = \begin{cases} 1, & s \geq 0, \\ 0, & s < 0, \end{cases}$$

the second equation reducing to the multipoint BC of the second part of (1.3.10),

$$y(b) - \sum_{j=1}^{d} k_j y(\eta_j) = 0.$$

Here, it is evident that BC (1.4.2) is a generalization of BC (1.3.10) and, similarly, BC (1.4.3) is a generalization of BC (1.3.8).

2.1 Solution Classes $\mathcal{S}_n^\gamma$

We study the nodal solutions of BVP (1.4.1), (1.4.3) in the following classes.

Definition 2.1.1. A solution y of BVP (1.4.1), (1.4.3) is said to belong to class $\mathcal{S}_n^\gamma$ for $n \in \mathbb{N}_0 := \{0, 1, 2, \ldots\}$ and $\gamma \in \{+, -\}$ if

(i) y has exactly n zeros in (a, b),

(ii) $\gamma y(t) > 0$ in a right-neighborhood of a.

Our results on the existence and nonexistence of nodal solutions of BVP (1.4.1),(1.4.3) are established utilizing the eigenvalues of the linear SLP consisting of the equation

$$-(p(t)y')' + q(t)y = \lambda w(t)y, \quad t \in (a, b), \tag{2.1.1}$$

and the two-point BC

$$\begin{aligned} &\cos\alpha \ y(a) - \sin\alpha \ (py')(a) = 0, \quad \alpha \in [0, \pi), \\ &y(b) = 0. \end{aligned} \tag{2.1.2}$$

It is well known that SLP (2.1.1), (2.1.2) has an infinite number of eigenvalues $\{\lambda_n\}_{n=0}^{\infty}$ satisfying

$$-\infty < \lambda_0 < \lambda_1 < \cdots < \lambda_n < \cdots, \quad \text{as} \quad \lambda_n \to \infty \text{ and } n \to \infty,$$

and any eigenfunction associated with λ_n has n simple zeros in (a, b); see [65, Theorem 4.3.2].

Note that the function $\xi(s)$ given in BC (1.4.3) is of bounded variation on $[a, b]$. Thus, there are two nondecreasing functions $\xi_1(s)$ and $\xi_2(s)$ such that

$$\xi(s) = \xi_1(s) - \xi_2(s), \quad s \in [a, b]. \tag{2.1.3}$$

In the following we assume (2.1.3) holds. We now present some results concerning problem (1.4.1), (1.4.3) with the proofs given later after several technical lemmas are derived. The first theorem is about the existence of certain types of nodal solutions.

2.1.1 Main Theorems

Theorem 2.1.1. Assume either (i) $f_0 < \lambda_n$ and $\lambda_{n+1} < f_\infty$, or (ii) $f_\infty < \lambda_n$ and $\lambda_{n+1} < f_0$, for some $n \in \mathbb{N}_0$. Suppose

$$1 - \int_a^b \sqrt{\frac{p(s)}{p(b)}}\; e^{k_0/2}\; d\big(\xi_1(s) + \xi_2(s)\big) > 0. \tag{2.1.4}$$

Then BVP (1.4.1), (1.4.3) has two solutions $y_{n,\gamma} \in \mathcal{S}_{n+1}^{\gamma}$ for $\gamma \in \{+, -\}$.

Remark 2.1.1. (a) Note that for the multipoint case, i.e., BVP (1.4.1), (1.4.3) with the second condition in (1.4.3) replaced by (1.3.8), we have that $\xi(s) = \sum_{i=1}^{m} k_i \chi(s - x_i)$. Thus $\xi(s) = \xi_1(s) - \xi_2(s)$ with

$$\xi_1(s) = \sum_{i=1}^{m} (k_i)_+ \chi(s - x_i) \quad \text{and} \quad \xi_2(s) = \sum_{i=1}^{m} (k_i)_- \chi(s - x_i),$$

where $(k_i)_\pm = \max\{\pm k_i, 0\}$. Hence $\xi_1(s) + \xi_2(s) = \sum_{i=1}^{m} |k_i| \chi(s - x_i)$. It is easy to see that condition (2.1.4) then becomes

$$1 - \sum_{i=1}^{m} |k_i| \sqrt{\frac{p(x_i)}{p(b)}}\; e^{k_0/2} > 0.$$

(b) When $\xi_i \in C^1[a, b]$ for $i = 1, 2$, condition (2.1.4) becomes

$$1 - \int_a^b \sqrt{\frac{p(s)}{p(b)}}\; e^{k_0/2}\; \big(\xi_1'(s) + \xi_2'(s)\big) ds > 0.$$

In particular, if $p(t) \equiv 1$, $q(t) \equiv 0$, $w(t) > 0$ is increasing, and $\xi(t) = t$, then it is reduced to $b - a < 1$.

As a consequence of Theorem 2.1.1, we have the following corollary on the existence of an infinite number of different types of nodal solutions for a special case of BVP (1.4.1), (1.4.3).

Corollary 2.1.1. Consider the special case that $p(t) \equiv 1$ and $q(t) \equiv 0$ on $[a, b]$. Assume (2.1.4) holds and either $f_0 = 0$ and $f_\infty = \infty$, or $f_\infty = 0$ and $f_0 = \infty$. Then there exists $\alpha^* \in (\pi/2, \pi)$ such that

(i) if $\alpha \in [0, \alpha^*)$, then BVP (1.4.1), (1.4.3) has two solutions $y_n^\gamma \in \mathcal{S}_{n+1}^\gamma$ for each $n \geq 0$ and $\gamma \in \{+, -\}$;

(ii) if $\alpha \in [\alpha^*, \pi)$, then BVP (1.4.1), (1.4.3) has two solutions $y_n^\gamma \in \mathcal{S}_{n+1}^\gamma$ for each $n \geq 1$ and $\gamma \in \{+, -\}$.

The next theorem is about the nonexistence of certain types of nodal solutions.

Theorem 2.1.2. (i) Assume $f(y)/y \leq \lambda_n$ for some $n \in \mathbb{N}_0$ and all $y \neq 0$. Then BVP (1.4.1), (1.4.3) has no solution in $\mathcal{S}_i^\gamma$ for all $i \geq n + 1$ and $\gamma \in \{+, -\}$.

(ii) Assume $f(y)/y \geq \lambda_n$ for some $n \in \mathbb{N}_0$ and all $y \neq 0$. Then BVP (1.4.1), (1.4.3) has no solution in $\mathcal{S}_i^\gamma$ for all $i \leq n$ and $\gamma \in \{+, -\}$.

The following Theorems (2.1.3-2.1.7) show that we can "create" or "eliminate" certain types of nodal solutions by changing the interval $[a, b]$; the coefficient functions q, p, w; and the boundary condition angle α. Since the eigenvalues of SLP (2.1.1), (2.1.2) can be easily computed using computer software such as that in [4], we are able to construct specific BVPs (1.4.1), (1.4.3) that have or do not have nodal solutions in $\mathcal{S}_n^\gamma$ for a prescribed $n \in \mathbb{N}_0$.

The first result is about the changes as the interval $[a, b]$ shrinks, more precisely, as $b \to a^+$. We discuss both the cases when one of f_0 and f_∞ is infinite and when both of them are finite.

Theorem 2.1.3. Let Eq. (1.4.1) and BC (1.4.3) be fixed and let (2.1.4) hold.

(i) Assume either $f_0 < \infty$ and $f_\infty = \infty$ or $f_\infty < \infty$ and $f_0 = \infty$. Then for any $n \in \mathbb{N}_0$, there exists $b_n > a$ such that for any $b \in (a, b_n)$ and for any $i \geq n$, BVP (1.4.1), (1.4.3) has two solutions $y_i^\gamma \in \mathcal{S}_{i+1}^\gamma$ for $\gamma \in \{+, -\}$.

(ii) Assume $f_0 < \infty$ and $f_\infty < \infty$. Then for any $n \in \mathbb{N}_0$, there exists $b_n > a$ such that for any $b \in (a, b_n)$ and for any $i \geq n+1$, BVP (1.4.1), (1.4.3) has no solutions in $\mathcal{S}_i^\gamma$ for $\gamma \in \{+, -\}$.

We next present a result on the nonexistence of certain types of nodal solutions of BVP (1.4.1), (1.4.3) as the function w increases in a given direction. More precisely, let $s \geq 0$ and $h \in C^1[a, b]$ such that $h(t) > 0$ on $[a, b]$, and consider the equation

$$-(p(t)y')' + q(t)y = [w(t) + sh(t)]f(y). \tag{2.1.5}$$

Theorem 2.1.4. Let the interval $[a, b]$ and BC (1.4.3) be fixed and let (2.1.4) hold. Assume $f(y)/y \geq f_* > 0$ for all $y \neq 0$. Then for any $n \in \mathbb{N}_0$, there exists $s_n \geq 0$ such that for any $s > s_n$ and for any $i \leq n$, BVP (2.1.5), (1.4.3) has no solution in $\mathcal{S}_i^\gamma$ for $\gamma \in \{+, -\}$.

The next result is on the nonexistence and existence of certain types of nodal solutions of BVP (1.4.1), (1.4.3) as the function q changes in a given direction. More precisely, let $s \in \mathbb{R}$ and $h \in C^1[a, b]$ such that $h(t) > 0$ on $[a, b]$, and consider the equation

$$-(p(t)y')' + \big[q(t) + sh(t)\big]y = w(t)f(y). \tag{2.1.6}$$

Theorem 2.1.5. Let the interval $[a, b]$ and BC (1.4.3) be fixed and let (2.1.4) hold.

(i) For any $n \in \mathbb{N}_0$, there exists $s_n \le 0$ such that for any $s < s_n$ and for any $i \le n$, BVP (2.1.6), (1.4.3) has no solutions in $\mathcal{S}_i^\gamma$ for $\gamma \in \{+,-\}$.

(ii) Assume either $f_0 < \infty$ and $f_\infty = \infty$ or $f_\infty < \infty$ and $f_0 = \infty$. Then for any $n \in \mathbb{N}_0$, there exists $s_n \ge 0$ such that for any $s > s_n$ and for any $i \ge n$, BVP (2.1.6), (1.4.3) has two solutions $y_{i,\gamma} \in \mathcal{S}_{i+1}^\gamma$ for $\gamma \in \{+,-\}$.

(iii) Assume $f_0 < \infty$ and $f_\infty < \infty$. Then for any $n \in \mathbb{N}_0$, there exists $s_* \ge 0$ such that for any $s > s_*$, BVP (2.1.6), (1.4.3) has no solution in $\mathcal{S}_i^\gamma$ for all $i \ge n+1$ and $\gamma \in \{+,-\}$.

Similar to Theorem 2.1.4, we show a result on the nonexistence of certain types of nodal solutions of BVP (1.4.1), (1.4.3) as the function $1/p(t)$ increases in a certain direction. More precisely, let $s \ge 0$ and $h \in C[a,b]$ such that $h(t) > 0$ on $[a,b]$, and consider the equation

$$-\left(\frac{1}{1/p(t)+sh(t)}y'\right)' + q(t)y = w(t)f(y). \tag{2.1.7}$$

Theorem 2.1.6. Let the interval $[a,b]$ and BC (1.4.3) be fixed and let (2.1.4) hold. Define $\hat{q} := \max\{q(t)/w(t) : t \in [a,b]\}$ and assume $f(y)/y \ge f_* > \hat{q}$ for all $y \ne 0$. Then for any $n \in \mathbb{N}_0$, there exists $s_n \ge 0$ such that for any $s > s_n$, BVP (2.1.7), (1.4.3) has no solution in $\mathcal{S}_i^\gamma$ for all $i \le n$ and $\gamma \in \{+,-\}$.

Finally, our last result is on the existence of certain types of nodal solutions of BVP (1.4.1), (1.4.3) as the boundary condition angle α changes.

Theorem 2.1.7. Let Eq. (1.4.1) and the interval $[a,b]$ be fixed and let (2.1.4) hold. Assume either $f_0 = 0$ and $f_\infty = \infty$ or $f_\infty = 0$ and $f_0 = \infty$. For $n \in \mathbb{N}_0$, denote $\lambda_n(\alpha)$ the nth eigenvalue of the SLP (2.1.1), (2.1.2). Suppose k is the first nonnegative integer such that $\lambda_k(\alpha^*) > 0$ for some $\alpha^* \in (0,\pi)$. Then

(i) for $\alpha \in [0, \alpha^*)$, BVP (1.4.1), (1.4.3) has two solutions $y_n^\gamma \in \mathcal{S}_{n+1}^\gamma$ for all $n \geq k$ and $\gamma \in \{+, -\}$;

(ii) for $\alpha \in [\alpha^*, \pi)$, BVP (1.4.1), (1.4.3) has two solutions $y_n^\gamma \in \mathcal{S}_{n+1}^\gamma$ for all $n \geq k+1$ and $\gamma \in \{+, -\}$.

2.1.2 Proofs

To prove Theorem 2.1.1, we need some preliminaries. The lemmas below are on the initial value problems (IVPs) associated with Eq. (1.4.1) and are simple generalizations of [28, Corollary 3.1, Lemmas 4.1, 4.2, 4.4, and 4.5] originally for the case where $p(t) \equiv 1$ with essentially the same proofs. The first one is on the global existence of solutions of IVPs associated with Eq. (1.4.1).

Lemma 2.1.1. Any initial value problem associated with Eq. (1.4.1) has a unique solution which exists on the whole interval $[a, b]$. Consequently, the solution depends continuously on the initial condition.

For $\gamma \in \{+, -\}$, let $y(t, \rho)$ be the solution of Eq. (1.4.1) satisfying

$$y(a) = \gamma\rho \sin\alpha \quad \text{and} \quad (py')(a) = \gamma\rho\cos\alpha, \tag{2.1.8}$$

where $\rho > 0$ is a parameter. Let $\theta(t, \rho)$ be the Prüfer angle of $y(t, \rho)$; i.e., $\theta(\cdot, \rho)$ is a continuous function on $[a, b]$ such that

$$\tan\theta(t, \rho) = y(t, \rho)/(py')(t, \rho) \quad \text{and} \quad \theta(a, \rho) = \alpha.$$

By Lemma 2.1.1, $\theta(t, \rho)$ is continuous in ρ on $(0, \infty)$ for any $t \in [a, b]$.

The next two lemmas provide some estimates for the Prüfer angle.

Lemma 2.1.2. (i) Assume $f_0 < \lambda_n$ for some $n \in \mathbb{N}_0$. Then there exists $\rho_* > 0$ such that $\theta(b, \rho) < (n+1)\pi$ for all $\rho \in (0, \rho_*)$.

(ii) Assume $\lambda_n < f_\infty$ for some $n \in \mathbb{N}_0$. Then there exists $\rho^* > 0$ such that $\theta(b, \rho) > (n+1)\pi$ for all $\rho \in (\rho^*, \infty)$.

Lemma 2.1.3. (i) Assume $f_\infty < \lambda_n$ for some $n \in \mathbb{N}_0$. Then there exists $\rho^* > 0$ such that $\theta(b, \rho) < (n+1)\pi$ for all $\rho \in (\rho^*, \infty)$.

(ii) Assume $\lambda_n < f_0$ for some $n \in \mathbb{N}_0$. Then there exists $\rho_* > 0$ such that $\theta(b, \rho) > (n+1)\pi$ for all $\rho \in (0, \rho_*)$.

Proof of Theorem 2.1.1. We first prove it for the case where $f_0 < \lambda_n$ and $\lambda_{n+1} < f_\infty$. Without loss of generality we assume $\gamma = +$. The case with $\gamma = -$ can be proved in the same way. Let $y(t, \rho)$ be the solution of Eq. (1.4.1) satisfying (2.1.8) with $\gamma = +$ and $\theta(t, \rho)$ its Prüfer angle. By Lemma 2.1.2, there exist $0 < \rho_* < \rho^* < \infty$ such that

$$\theta(b, \rho) < (n+1)\pi \quad \text{for all } \rho \in (0, \rho_*)$$

and

$$\theta(b, \rho) > (n+2)\pi \quad \text{for all } \rho \in (\rho^*, \infty).$$

By the continuity of $\theta(t, \rho)$ in ρ, there exist $\rho_* \le \rho_{n+1} < \rho_{n+2} \le \rho^*$ such that

$$\theta(b, \rho_{n+1}) = (n+1)\pi \quad \text{and} \quad \theta(b, \rho_{n+2}) = (n+2)\pi,$$
$$(n+1)\pi < \theta(b, \rho) < (n+2)\pi \quad \text{for} \quad \rho_{n+1} < \rho < \rho_{n+2}. \tag{2.1.9}$$

Then, for all $t \in [a,b]$ and all $\rho > 0$, we define an energy function $E(t,\rho)$ for $y(t,\rho)$ by

$$E(t,\rho) = \frac{1}{2p(t)}\big(p(t)y'(t,\rho)\big)^2 + \frac{1}{2}\big(q^* - q(t)\big)y^2(t,\rho) + w(t)F\big(y(t,\rho)\big), \qquad (2.1.10)$$

where $F(y) = \int_0^y f(s)ds$. By (H1) and (H2), $F(y) \geq 0$ on $\mathbb{R}$ yielding $E(t,\rho) \geq 0$ on $[a,b]$. For ease of notation, in the following, we use $p = p(t)$, $q = q(t)$, $w = w(t)$, $l = l(t)$, $y = y(t,\rho)$, $E = E(t,\rho)$. Then, by (1.4.1) and (H1), we find that

$$\begin{aligned}
E' &= -\frac{p'}{2p^2}(py')^2 - \frac{1}{2}q'y^2 + q^*yy' + w'F(y) \\
&\geq -\frac{p'}{2p^2}(py')^2 - \frac{1}{2}q'y^2 - \frac{q^*}{2}\big(y^2 + y'^2\big) + w'F(y) \\
&= -\frac{(p' + q^*)}{2p^2}(py')^2 - \frac{1}{2}(q' + q^*)y^2 + w'F(y) \\
&\geq -\Big(\frac{p' + q^*}{p}\Big)_+\Big(\frac{1}{2p}(py')^2\Big) - l\Big(\frac{1}{2}(q^* - q)y^2\Big) - \frac{w'_-}{w}wF(y) \\
&\geq -l\frac{(py')^2}{2p} - l\Big(\frac{1}{2}(q^* - q)y^2\Big) - lwF(y) \\
&= -lE(t,\rho).
\end{aligned}$$

Thus, $E'(t,\rho) + lE(t,\rho) \geq 0$ for all $t \in [a,b]$ and $\rho > 0$. By solving this inequality, we obtain that

$$E(s,\rho) \leq E(b,\rho)e^{k_0}, \quad s \in [a,b]. \qquad (2.1.11)$$

We observe that for $\rho = \rho_{n+1}$ and $\rho = \rho_{n+2}$

$$E(s,\rho) \geq \frac{1}{2p(s)}\left[p(s)y'(s,\rho)\right]^2, \quad s \in [a,b],$$

and

$$E(b,\rho) = \frac{1}{2p(b)}\left[p(b)y'(b,\rho)\right]^2.$$

Thus, for $\rho = \rho_{n+1}$, $\rho = \rho_{n+2}$, and $s \in [a, b]$,

$$|(py')(s,\rho)| \le \sqrt{2p(s)E(s,\rho)} \quad \text{and} \quad |(py')(b,\rho)| = \sqrt{2p(s)E(b,\rho)}. \tag{2.1.12}$$

Define

$$\Gamma(\rho) = (py')(b,\rho) - \int_a^b (py')(s,\rho)d\xi(s). \tag{2.1.13}$$

Assume $n = 2k-1$ with $k \in \mathbb{N}_0$. Since $(py')(b,\rho_{2k}) > 0$ and $(py')(b,\rho_{2k+1}) < 0$, by (2.1.3), (2.1.11), (2.1.12), and (2.1.4) we have

$$\begin{aligned}
&\Gamma(\rho_{2k}) = (py')(b,\rho_{2k}) - \int_a^b (py')(s,\rho_{2k})d\xi(s) \\
\ge\ & \Big|(py')(b,\rho)\Big| - \int_a^b |(py')(s,\rho_{2k})| d\big(\xi_1(s)+\xi_2(s)\big) \\
\ge\ & \sqrt{2p(b)E(b,\rho_{2k})} - \int_a^b \sqrt{2p(s)E(s,\rho_{2k})}\, d\big(\xi_1(s)+\xi_2(s)\big) \\
\ge\ & \sqrt{2p(b)E(b,\rho_{2k})} - \int_a^b \sqrt{2p(s)E(b,\rho_{2k})e^{k_0}}\, d\big(\xi_1(s)+\xi_2(s)\big) \\
=\ & \sqrt{2p(b)E(b,\rho_{2k})}\left(1 - \int_a^b \sqrt{p(s)/p(b)}\, e^{k_0/2} d\big(\xi_1(s)+\xi_2(s)\big)\right) > 0
\end{aligned}$$

and

$$\begin{aligned}
&\Gamma(\rho_{2k+1}) = (py')(b,\rho_{2k+1}) - \int_a^b (py')(s,\rho_{2k+1})d\xi(s) \\
\le\ & -\Big|(py')(b,\rho_{2k+1})\Big| + \int_a^b (|(py')(s,\rho_{2k+1})| d\big(\xi_1(s)+\xi_2(s)\big) \\
\le\ & -\sqrt{2p(b)E(b,\rho_{2k+1})} + \int_a^b \sqrt{2p(s)E(s,\rho_{2k+1})}\ d\big(\xi_1(s)+\xi_2(s)\big) \\
\le\ & -\sqrt{2p(b)E(b,\rho_{2k+1})} + \int_a^b \sqrt{2p(s)E(b,\rho_{2k+1})e^{k_0}}\ d\big(\xi_1(s)+\xi_2(s)\big) \\
=\ & -\sqrt{2p(b)E(b,\rho_{2k+1})}\left(1 - \int_a^b \sqrt{p(s)/p(b)}\ e^{k_0/2}\ d\big(\xi_1(s)+\xi_2(s)\big)\right) < 0.
\end{aligned}$$

By the continuity of $\Gamma(\rho)$, there exists $\bar{\rho} \in (\rho_{2k}, \rho_{2k+1})$ such that $\Gamma(\bar{\rho}) = 0$. Similarly, for $n = 2k$ with $k \in \mathbb{N}_0$, there exists $\bar{\rho} \in (\rho_{2k+1}, \rho_{2k+2})$ such that $\Gamma(\bar{\rho}) = 0$. In both cases, it follows from (2.1.9) that

$$(n+1)\pi < \theta(b, \bar{\rho}) < (n+2)\pi.$$

Since

$$\theta'(t, \rho) = \frac{1}{p(t)} \cos^2 \theta(t, \rho) + w(t) \frac{f(y(t, \rho))y(t, \rho)}{r^2(t, \rho)} - q(t) \sin^2 \theta(t, \rho) \tag{2.1.14}$$

for $t \in [a, b]$, where $r = (y^2 + py')^{1/2}$, we have that $\theta(\cdot, \rho)$ is strictly increasing at the points t where $\theta(t, \rho) = 0 \pmod{\pi}$. We note that $y(t) = 0$ if and only if $\theta(t, \rho) = 0 \pmod{\pi}$. Thus, y has exactly $n + 1$ zeros in (a, b). Initial condition (2.1.8) implies that $y(t, \bar{\rho}) > 0$ in a right-neighborhood of a. Therefore, $y(t, \bar{\rho}) \in \mathcal{S}^+_{n+1}$.

The proof for the case where $f_\infty < \lambda_n$ and $\lambda_{n+1} < f_0$ is essentially the same as above except that the discussion is based on Lemma 2.1.3 instead of Lemma 2.1.2. □

Proof of Corollary 2.1.1. Consider the SLP consisting of Eq. (2.1.1) with $p(t) \equiv 1$, $q(t) \equiv 0$, and the BC

$$\cos\alpha\; y(a) - \sin\alpha\; y'(a) = 0, \qquad \alpha \in [0, \pi),$$
$$\cos\beta\; y(b) - \sin\beta\; y'(b) = 0, \qquad \beta \in (0, \pi].$$

Denote by $\lambda_n(\alpha, \beta)$ the nth eigenvalue of this problem for $n \in \mathbb{N}_0$. It is easy to see that $\lambda_0(\pi/2, \pi/2) = 0$. In fact, $y_0(t) \equiv 1$ is an associated eigenfunction. From [31, Theorem 4.2] and [29, Lemma 3.32], we see that $\lambda_0(\alpha, \beta)$ is a continuous function of (α, β) on $[0, \pi) \times (0, \pi]$ and is strictly decreasing in α and strictly increasing in β. Furthermore, for any $\beta \in (0, \pi]$,

$$\lim_{\alpha \to \pi-} \lambda_0(\alpha, \beta) = -\infty \quad \text{and} \quad \lim_{\alpha \to \pi-} \lambda_{n+1}(\alpha, \beta) = \lambda_n(0, \beta) \text{ for } n \in \mathbb{N}_0.$$

This shows that $\lambda_0(\pi/2, \pi) > 0$, and hence there exists $\alpha^* \in (\pi/2, \pi)$ such that $\lambda_0(\alpha, \pi) > 0$ for $\alpha \in [0, \alpha^*)$, and $\lambda_0(\alpha, \pi) \leq 0$ and $\lambda_1(\alpha, \pi) > 0$ for $\alpha \in [\alpha^*, \pi)$. Note that $\beta = \pi$ if and only if $y(b) = 0$. Then the conclusion follows from Theorem 2.1.1.

□

Proof of Theorem 2.1.2. (i) Assume to the contrary that BVP (1.4.1), (1.4.3) has a solution $y \in \mathcal{S}_i^\gamma$ for some $i \geq n+1$ and $\gamma \in \{+, -\}$. Let $\tilde{w}(t) = w(t)f(y(t))/y(t)$. Then $\tilde{w}(t)$ is continuous on $[a, b]$ by the continuous extension since $f_0 < \infty$. Let $\theta(t)$ be the Prüfer angle of $y(t)$ with $\theta(a) = \alpha$. Then $\theta(b) > i\pi$ and, since $\theta(t)$ satisfies (2.1.14), it is strictly increasing on $[a, b]$. Note, from the assumption that $\tilde{w}(t) \leq \lambda_n w(t) \leq \lambda_{i-1} w(t)$ on $[a, b]$, we have that for $t \in [a, b]$,

$$\begin{aligned} \theta'(t) &= \frac{1}{p(t)} \cos^2 \theta(t) + [\tilde{w}(t) - q(t)] \sin^2 \theta(t, \rho) \\ &\leq \frac{1}{p(t)} \cos^2 \theta(t, \rho) + [\lambda_{i-1} w(t) - q(t)] \sin^2 \theta(t, \rho). \end{aligned}$$

Let $u(t)$ be an eigenfunction of SLP (2.1.1), (2.1.2) associated with the eigenvalue λ_{i-1} and $\phi(t)$ its Prüfer angle with $\phi(a) = \alpha$. Then

$$\phi'(t) = \frac{1}{p(t)} \cos^2 \phi(t) + [\lambda_{i-1} w(t) - q(t)] \sin^2 \phi(t)$$

and $\phi(b) = i\pi$. By the theory of differential inequalities, we find that $\theta(b) \leq \phi(b) = i\pi$. We have reached a contradiction.

(ii) It is similar to (i) and hence omitted. □

The proofs of the subsequent theorems are based on the following lemma for the dependence of the nth eigenvalue of SLP (2.1.1), (2.1.2) on the problem which can be excerpted from [30, Theorems 2.2 and 2.3], [31, Theorem 4.2], and [29, Lemma 3.32].

Lemma 2.1.4. For any $n \in \mathbb{N}_0$, we have the following conclusions:

(a) Consider the nth eigenvalue of SLP (2.1.1), (2.1.2) as a function of b for $b \in (a, \infty)$, denoted by $\lambda_n(b)$. Then $\lambda_n(b) \to \infty$ as $b \to a^+$.

(b) Consider the nth eigenvalue of SLP (2.1.1), (2.1.2) as a function of w for $w \in C^1[a,b]$, denoted by $\lambda_n(w)$. Then $\lambda_n(w)$ is decreasing as long as it is positive; i.e., for $w_1, w_2 \in C^1[a,b]$ such that $w_1(t) \le w_2(t)$ for $t \in [a,b]$, we have $\lambda_n(w_1) \ge \lambda_n(w_2)$ as long as $\min\{\lambda_n(w_1), \lambda_n(w_2)\} \ge 0$.

(c) Consider the nth eigenvalue of SLP (2.1.1), (2.1.2) as a function of q for $q \in C^1[a,b]$, denoted by $\lambda_n(q)$. Then $\lambda_n(q)$ is increasing; i.e., for $q_1, q_2 \in C^1[a,b]$ such that $q_1(t) \le q_2(t)$ for $t \in [a,b]$, we have $\lambda_n(q_1) \le \lambda_n(q_2)$.

(d) Consider the nth eigenvalue of SLP (2.1.1), (2.1.2) as a function of $1/p$ for $1/p \in C^1[a,b]$, denoted by $\lambda_n(1/p)$. Then $\lambda_n(1/p)$ is decreasing; i.e., for $1/p_1, 1/p_2 \in C^1[a,b]$ such that $1/p_1(t) \le 1/p_2(t)$ for $t \in [a,b]$, we have $\lambda_n(1/p_1) \ge \lambda_n(1/p_2)$.

(e) Consider the nth eigenvalue of SLP (2.1.1), (2.1.2) as a function of the boundary condition angle α, denoted by $\lambda_n(\alpha)$. Then $\lambda_n(\alpha)$ is a continuous and decreasing function on $[0, \pi)$. Furthermore,

$$\lim_{\alpha \to \pi^-} \lambda_0(\alpha) = -\infty \quad \text{and} \quad \lim_{\alpha \to \pi^-} \lambda_{n+1}(\alpha) = \lambda_n(0) \ \text{ for } \ n \ge 1.$$

Proof of Theorem 2.1.3. (i) Without loss of generality, assume $f_0 < \infty$ and $f_\infty = \infty$. Let $\lambda_n(b)$ be defined as in Lemma 2.1.4 (a). By Lemma 2.1.4 (a), for any $n \in \mathbb{N}_0$, there exists $b_n > a$ such that for any $b \in (a, b_n)$ we have $f_0 < \lambda_n(b) < f_\infty$ and hence $f_0 < \lambda_i(b) < f_\infty$ for all $i \geq n$. Then the conclusion follows from Theorem 2.1.1.

(ii) By Lemma 2.1.4 (a), for any $n \in \mathbb{N}$, there exists $b_n > a$ such that for any $b \in (a, b_n)$ we have that $\lambda_n(b) > f^* := \sup\{f(y)/y : y \in (0, \infty)\}$. Then the conclusion follows from Theorem 2.1.2 (i). □

Proof of Theorem 2.1.4. For $s \geq 0$ and $i \in \mathbb{N}_0$, we denote by $\lambda_i(s)$ the ith eigenvalue of the SLP consisting of the equation

$$-(p(t)y')' + q(t)y = \lambda\big[w(t) + sh(t)\big]y$$

and BC (2.1.2). Let $h_* = \min\{h(t)/w(t) : t \in [a, b]\}$, and denote by $\mu_i(s)$ the ith eigenvalue of the SLP consisting of the equation

$$-(p(t)y')' + q(t)y = \mu(1 + sh_*)w(t)y$$

and BC (2.1.2). Since

$$w(t) + sh(t) \geq (1 + sh_*)w(t) \quad \text{for } s \geq 0,$$

by Lemma 2.1.4 (b),

$$\lambda_i(s) \leq \mu_i(s) \text{ for all } s \geq 0 \text{ and } i \geq 0, \text{ whenever } \lambda_i(s) \geq 0. \tag{2.1.15}$$

Note that for $i \geq 0$, $\mu_i(s)(1 + sh_*) = \mu_i(0)$, we have

$$\mu_i(s) = \frac{\mu_i(0)}{1 + sh_*} \to 0 \quad \text{as } s \to \infty.$$

This together with (2.1.15) implies that $\lambda_i(s) < f_*$ as $s \to \infty$. Then, for any $n \in \mathbb{N}_0$, there exists $s_n \geq 0$ such that $\lambda_n(s) < f_*$ for $s > s_n$. Therefore, the conclusion follows from Theorem 2.1.2 (ii). □

Proof of Theorem 2.1.5. For $s \in \mathbb{R}$ and $i \in \mathbb{N}_0$, we denote by $\lambda_i(s)$ the ith eigenvalue of the SLP consisting of the equation

$$-(p(t)y')' + \big[q(t) + sh(t)\big]y = \lambda w(t)y$$

and BC (2.1.2). Let $h_* = \min\{\frac{h(t)}{w(t)} : t \in [a, b]\}$, and denote by $\mu_i(s)$ the ith eigenvalue of the SLP consisting of the equation

$$-(p(t)y')' + [q(t) + sh_*(t)w(t)]y = \mu w(t)y \tag{2.1.16}$$

and BC (2.1.2).

(i) Since for $s \leq 0$,

$$q(t) + sh(t) \leq q(t) + sh_*w(t),$$

by Lemma 2.1.4 (c), $\lambda_i(s) \leq \mu_i(s)$ for all $s \leq 0$ and $i \geq 0$. Note that Eq. (2.1.16) yields

$$-(p(t)y')' + q(t)y = (\mu - sh_*)w(t)y.$$

Thus, for $s \leq 0$ and $i \geq 0$, $\mu_i(0) = \mu_i(s) - sh_*$, which implies that

$$\mu_i(s) = \mu_i(0) + sh_* \to -\infty \quad \text{as } s \to -\infty,$$

and hence $\lambda_i(s) \to -\infty$ as $s \to -\infty$ for all $i \geq 0$. Then, for any $n \in \mathbb{N}_0$ there exists $s_n \leq 0$ such that $\lambda_n < 0$ for all $s < s_n$. Therefore, the conclusion follows from Theorem 2.1.2 (ii).

(ii) Without loss of generality, assume $f_0 < \infty$ and $f_\infty = \infty$. Similar to the argument in (i), we have $\lambda_i(s) \to \infty$ as $s \to \infty$ for all $i \geq 0$. Then for any $n \in \mathbb{N}_0$ there exists $s_n \geq 0$ such that for any $s > s_n$ we have $f_0 < \lambda_n(s) < f_\infty$ and hence $f_0 < \lambda_i(s) < f_\infty$ for $i \geq n$. Therefore, the conclusion follows from Theorem 2.1.1.

(iii) As we can see from Part (ii), for any $n \in \mathbb{N}_0$, there exists $s_* \geq 0$ such that for all $s > s_*$ we have $\lambda_n(s) > f^* := \sup\{f(y)/y : y \in (0, \infty)\}$. Thus, the conclusion follows from Theorem 2.1.2 (i). □

Proof of Theorem 2.1.6. For $s \geq 0$ and $i \in \mathbb{N}_0$, we denote by $\lambda_i(s)$ the ith eigenvalue of the SLP consisting of the equation

$$-\left(\frac{1}{1/p(t) + sh(t)} y'\right)' + q(t)y = \lambda w(t) y$$

and BC (2.1.2) with an eigenfunction $u_i(t, s)$. Let $\theta_i(t, s)$ be the Prüfer angle of $u_i(t, s)$ satisfying $\theta_i(a, s) = \alpha$. Then

$$\theta_i'(t, s) = \left[\frac{1}{p(t)} + sh(t)\right] \cos^2 \theta_i(t, s) + [\lambda_i w(t) - q(t)] \sin^2 \theta_i(t, s). \tag{2.1.17}$$

By Lemma 2.1.4 (d), $\lambda_i(s)$ is decreasing and hence

$$\lim_{s \to \infty} \lambda_i(s) = \lambda_i^* \in [-\infty, \infty).$$

We show that $\lambda_i^* < f_*$ and then the conclusion follows from Theorem 2.1.2 (ii). Assume the contrary, i.e., $\lambda_i^* \geq f_*$. Let $w_* = \min\{w(t) : t \in [a, b]\}$. By (2.1.17),

$$\begin{aligned} \theta_i'(t,s) &\geq \left[\frac{1}{p(t)} + sh(t)\right]\cos^2\theta_i(t,s) + [\lambda_i^* w(t) - q(t)]\sin^2\theta_i(t,s) \\ &= \left[\frac{1}{p(t)} + sh(t)\right]\cos^2\theta_i(t,s) + [\lambda_i^* - q(t)/w(t)]w(t)\sin^2\theta_i(t,s) \\ &\geq \left[\frac{1}{p(t)} + sh(t)\right]\cos^2\theta_i(t,s) + [f_* - \hat{q}]w_*\sin^2\theta_i(t,s). \end{aligned}$$

Let $\phi(t, s)$ be the solution of the equation

$$\phi'(t,s) = \left[\frac{1}{p(t)} + sh(t)\right]\cos^2\phi(t,s) + [f_* - \hat{q}]w_*\sin^2\phi(t,s), \tag{2.1.18}$$

satisfying $\phi(a, s) = \alpha$. By the theory of differential inequalities, we have $\phi(t, s) \leq \theta_i(t, s)$. In particular,

$$\phi(b,s) \leq \theta_i(b,s) = (i+1)\pi. \tag{2.1.19}$$

We observe from (2.1.18) that $\phi(t, s)$ is strictly increasing in t and s, and $0 < \phi(t, s) \leq (i+1)\pi$ for $t \in [a, b]$ and $s \geq 0$. Let $\phi^*(t) = \lim_{s\to\infty} \phi(t, s)$. Then $0 < \phi^*(t) \leq (i+1)\pi$ for $t \in [a, b]$. We claim that

$$\phi^*(t) \not\equiv k\pi + \frac{\pi}{2} \quad \text{on} \quad (a, b] \tag{2.1.20}$$

for any $0 \leq k \leq i$. If not, for any $a_1 \in (a, b]$ and $\epsilon > 0$, there exists $s^* > 0$ such that for $s \geq s^*$,

$$\phi(a_1, s) \in (k\pi + \pi/2 - \epsilon, k\pi + \pi/2),$$

which yields that

$$\phi(t,s) \in (k\pi + \pi/2 - \epsilon, k\pi + \pi/2) \quad \text{for} \quad t \in [a_1, b].$$

This implies that

$$0 < \phi(b,s) - \phi(a_1,s) < \epsilon. \tag{2.1.21}$$

However, from (2.1.18), we see that for s sufficiently large,

$$\phi'(t,s) \geq \frac{1}{2}(f_* - \hat{q})w_* \quad \text{for} \quad t \in [a_1, b].$$

This contradicts (2.1.21) and hence verifies (2.1.20).

It is easy to see that $\phi(t,s) \to \phi^*(t)$ uniformly on $[a_1,b]$ as $s \to \infty$. Thus, $\phi^*(t)$ is continuous on $[a_1,b]$. From (2.1.20), we can find a nontrivial closed interval $[c,d] \subset [a,b]$ such that $\cos^2 \phi^*(t) \geq \nu > 0$ for $t \in [c,d]$. Then from (2.1.18),

$$\phi'(t,s) \geq \left[\frac{1}{p(t)} + sh(t)\right]\nu \to \infty \quad \text{uniformly for} \quad t \in [c,d] \quad \text{as} \quad s \to \infty.$$

Therefore,

$$\begin{aligned} \phi(b,s) &\geq \phi(d,s) \geq \phi(c,s) + \int_c^d \left[\frac{1}{p(t)} + sh(t)\right]\nu ds \\ &\geq \int_c^d \left[\frac{1}{p(t)} + sh(t)\right]\nu ds \to \infty \quad \text{as} \quad s \to \infty. \end{aligned}$$

This contradicts (2.1.19) and hence completes the proof. □

Proof of 2.1.7. By assumption, $\lambda_k(\alpha^*) > 0$. Then Lemma 2.1.4 (e) shows that $\lambda_k(\alpha) > 0$ for $\alpha \in [0, \alpha^*]$, and for $\alpha \in (\alpha^*, \pi)$, $\lambda_k(\alpha) < 0$ and $\lambda_{k+1}(\alpha) > 0$. Therefore, the conclusion follows from Theorem 2.1.1. □

2.2 Solution Classes $\mathcal{T}_n^\gamma$

We study the following BVP (1.4.1), (1.4.2) in the following classes.

Definition 2.2.1. A solution y of BVP (1.4.1), (1.4.2) is said to belong to class $\mathcal{T}_k^\gamma$ for $k \in \mathbb{N}_0 := \{0, 1, 2, \dots\}$ and $\gamma \in \{+, -\}$ if

(i) py' has exactly $k+1$ zeros in (a, b) if $\alpha \in [0, \pi/2)$ and has exactly k zeros in (a, b) if $\alpha \in [\pi/2, \pi)$,

(ii) there is exactly one zero of y strictly between any two consecutive zeros of py',

(iii) $\gamma y(t) > 0$ in a right-neighborhood of a.

Our results on the existence of nodal solutions of BVP (1.4.1), (1.4.2) are established using the eigenvalues, $\{\lambda_n\}_{n=0}^\infty$, of the linear SLP consisting of the equation

$$-(p(t)y')' + q(t)y = \lambda w(t)y, \ t \in (a, b), \tag{2.2.1}$$

and the two-point BC

$$\begin{aligned} &\cos\alpha \ y(a) - \sin\alpha \ (py')(a) = 0, \quad \alpha \in [0, \pi), \\ &(py')(b) = 0. \end{aligned} \tag{2.2.2}$$

It is well known that any eigenfunction associated with λ_n has n simple zeros in (a, b); see [65, Theorem 4.3.2]. In the following, let $F(y) = \int_0^y f(\xi)\, d\xi$ for $y \in \mathbb{R}$ and denote

$$H(t, y) := \frac{1}{2}(q^* - q(t))y^2 + w(t)F(y) \quad \text{and} \quad k_0 = \int_a^b l(t)dt.$$

By (H2), F is strictly increasing on $[0, \infty)$. Thus, for any fixed $t \in [a, b]$, $H(t, y)$ is strictly increasing in y on $[0, \infty)$ and, hence, is invertible in y on $[0, \infty)$. We denote by $H_+^{-1}(t, y)$ its inverse. Similarly, $H(t, y)$ has an inverse $H_-^{-1}(t, y)$ in y on $(-\infty, 0]$.

Note that f is odd implies that F is even. Therefore, for each $y > 0$,

$$H_-^{-1}(t, y) = -H_+^{-1}(t, y). \tag{2.2.3}$$

We also note the special case when $q \equiv 0$; i.e., $H(t, y) = w(t)F(y)$,

$$H_\pm^{-1}(t, y) = F_\pm^{-1}\Big(\frac{y}{w(t)}\Big), \tag{2.2.4}$$

in their respective domains.

2.2.1 Main Theorems

We now present our main results on the existence and nonexistence of nodal solutions of BVP (1.4.1), (1.4.2). The first theorem concerns the existence of certain types of nodal solutions.

Theorem 2.2.1. Let λ_n and λ_{n+1} be eigenvalues of BVP (2.2.1), (2.2.2) for some $n \in \mathbb{N}_0$ and assume for all $t \in [a, b]$ either (i) $f_0 < \lambda_n$ and $\lambda_{n+1} < f_\infty$ or (ii) $f_\infty < \lambda_n$ and $\lambda_{n+1} < f_0$. Suppose that for any $c > 0$,

$$\int_a^b H_+^{-1}\left(s, ce^{k_0}\right) d\left(\xi_1(s) + \xi_2(s)\right) < H_+^{-1}(b, c), \tag{2.2.5}$$

where $\xi_1(s)$ and $\xi_2(s)$ are given by (2.1.3). Then BVP (1.4.1), (1.4.2) has two solutions $y_n^\gamma \in \mathcal{T}_n^\gamma$ for $\gamma \in \{+, -\}$.

As consequences of Theorem 2.2.1 we have the following corollaries.

Corollary 2.2.1. Consider the special case of Eq. (1.4.1) with $q \equiv 0$; i.e.,

$$-(p(t)y')' = w(t)f(y). \tag{2.2.6}$$

Assume either $f_0 < \lambda_n$ and $\lambda_{n+1} < f_\infty$ or $f_\infty < \lambda_n$ and $\lambda_{n+1} < f_0$ for some $n \in \mathbb{N}_0$.

(i) If, for any $c > 0$,

$$\int_a^b F_+^{-1}\Big(\frac{ce^{k_0}}{w(s)}\Big) d(\xi_1(s) + \xi_2(s)) < F_+^{-1}\Big(\frac{c}{w(b)}\Big), \tag{2.2.7}$$

then BVP (2.2.6), (1.4.2) has two solutions $y_n^\gamma \in \mathcal{T}_n^\gamma$ for $\gamma \in \{+, -\}$.

(ii) In particular, when $w \equiv w^* > 0$ and $p \equiv 1$ on $[a, b]$, if

$$\int_a^b d(\xi_1(s) + \xi_2(s)) < 1,$$

then BVP (2.2.6), (1.4.2) has two solutions $y_n^\gamma \in \mathcal{T}_n^\gamma$ for $\gamma \in \{+, -\}$.

Remark 2.2.1. Let the second condition of BC (1.4.2) be replaced by the multipoint BC (1.3.10); then we may choose $\xi(s) = \xi_1(s) - \xi_2(s)$ with

$$\xi_1(s) = \sum_{j=1}^{d} (k_j)_+ \chi(s - \eta_j) \quad \text{and} \quad \xi_2(s) = \sum_{j=1}^{d} (k_j)_- \chi(s - \eta_j),$$

where $(k_j)_\pm = \max\{\pm k_j, 0\}$. Hence, $\xi_1(s) + \xi_2(s) = \sum_{j=1}^d |k_j| \chi(s - \eta_j)$. Then, it is easy to see that (2.2.7) reduces to

$$\sum_{j=1}^{d} |k_j| F_+^{-1}\Big(\frac{ce^{k_0}}{w(\eta_j)}\Big) < F_+^{-1}\Big(\frac{c}{w(b)}\Big). \tag{2.2.8}$$

In particular, when $f(y) = |y|^{r-1}y$ for $r > 0$, then (2.2.8) reduces to

$$\sum_{j=1}^{d} |k_j| \Big(\frac{w(b)e^{k_0}}{w(\eta_j)}\Big)^{1/(r+1)} < 1,$$

and when $w(t) \equiv w_0 > 0$, then (2.2.8) reduces to

$$\sum_{j=1}^{d} |k_j| < 1.$$

Therefore, it is easy to see that Theorem 2.2.1 covers the main results in [27] for BVPs with multipoint BCs.

Corollary 2.2.2. Assume (2.2.5) holds and either $f_0 < \lambda_n \quad \text{and} \quad \lambda_{n+1} < f_\infty$ or $f_\infty < \lambda_n \quad \text{and} \quad \lambda_{n+1} < f_0$ with $n = 0$. Then

(i) BVP (1.1.1), (1.1.2) has positive and negative solutions in $\mathcal{T}_0^\gamma$ for $\gamma \in \{+, -\}$ if $\xi(s)$ is increasing on $[a, b]$;

(ii) BVP (1.4.1), (1.4.2) has two solutions in $\mathcal{T}_0^\gamma$ for $\gamma \in \{+, -\}$ with exactly one zero in (a, b) if $\xi(s)$ is decreasing on $[a, b]$ and such that $\int_a^{b-\epsilon} d\xi(s) < 0$ for some small $\epsilon > 0$.

Corollary 2.2.3. Let $\alpha \in [0, \pi/2)$. Assume the special condition that $q(t) \equiv 0$, (2.2.5) holds, and either $f_0 = 0$ and $f_\infty = \infty$ or $f_\infty = 0$ and $f_0 = \infty$. Then for all $n \geq 0$ and $\gamma \in \{+, -\}$, BVP (1.4.1), (1.4.2) has two solutions $y_n^\gamma \in \mathcal{T}_n^\gamma$.

The next theorem is about the nonexistence of certain types of nodal solutions.

Theorem 2.2.2. (i) Assume for some $n \in \mathbb{N}_0$, $f(y)/y < \lambda_n$ for all $t \in [a, b]$ and $y \neq 0$. Then BVP (1.4.1), (1.4.2) has no solution in $\mathcal{T}_j^\gamma$ for all $j \geq n$ and $\gamma \in \{+, -\}$. (ii) Assume for some $n \in \mathbb{N}_0$, $f(y)/y > \lambda_{n+1}$ for all $t \in [a, b]$ and $y \neq 0$. Then BVP (1.4.1), (1.4.2) has no solution in $\mathcal{T}_j^\gamma$ for all $j \leq n$ and $\gamma \in \{+, -\}$.

The following Theorems (2.2.3-2.2.7) show that we can "create" or "eliminate" certain types of nodal solutions by changing the interval $[a, b]$, the coefficient functions q, p, w, and the boundary condition angle α. Since the eigenvalues of SLP (2.2.1), (2.2.2) can be easily computed using computer software such as that in [4], we are able to construct specific BVPs (1.4.1), (1.4.2) which have or do not have nodal solutions in $\mathcal{T}_n^\gamma$ for a prescribed $n \in \mathbb{N}_0$.

The first result is about the changes as the interval $[a, b]$ shrinks, more precisely, as $b \to a^+$. We discuss both the cases when one of f_0 and f_∞ is infinite and when both of them are finite.

Theorem 2.2.3. Let Eq. (1.4.1) and BC (1.4.2) be fixed and let (2.2.5) hold.

(i) Assume either $f_0 < \infty$ and $f_\infty = \infty$ or $f_\infty < \infty$ and $f_0 = \infty$. Then for any $n \in \mathbb{N}_0$, there exists $b_n > a$ such that for any $b \in (a, b_n)$ and for any $i \geq n$, BVP (1.4.1), (1.4.2) has two solutions $y_i^\gamma \in \mathcal{T}_i^\gamma$ for $\gamma \in \{+, -\}$.

(ii) Assume $f_0 < \infty$ and $f_\infty < \infty$. Then for any $n \in \mathbb{N}_0$, there exists $b_n > a$ such that for any $b \in (a, b_n)$ and for any $i \geq n$, BVP (1.4.1), (1.4.2) has no solutions in $\mathcal{T}_i^\gamma$ for $\gamma \in \{+, -\}$.

We next present a result on the nonexistence of certain types of nodal solutions of BVP (1.4.1), (1.4.2) as the function w increases in a given direction. More precisely, let $s \geq 0$ and $h \in C^1[a, b]$ such that $h(t) > 0$ on $[a, b]$, and consider the equation

$$-(p(t)y')' + q(t)y = [w(t) + sh(t)]f(y). \tag{2.2.9}$$

Theorem 2.2.4. Let the interval $[a, b]$ and BC (1.4.2) be fixed and let (2.2.5) hold. Assume $f(y)/y \geq f_* > 0$ for all $y \neq 0$. Then for any $n \in \mathbb{N}_0$, there exists $s_n \geq 0$

such that for any $s > s_n$ and for any $i \le n$, BVP (2.2.9), (1.4.2) has no solution in $\mathcal{T}_i^\gamma$ for $\gamma \in \{+, -\}$.

The next result is on the nonexistence and existence of certain types of nodal solutions of BVP (1.4.1), (1.4.2) as the function q changes in a given direction. More precisely, let $s \in \mathbb{R}$ and $h \in C^1[a, b]$ such that $h(t) > 0$ on $[a, b]$, and consider the equation

$$-(p(t)y')' + \big[q(t) + sh(t)\big]y = w(t)f(y). \tag{2.2.10}$$

Theorem 2.2.5. Let the interval $[a, b]$ and BC (1.4.2) be fixed and let (2.2.5) hold.

(i) For any $n \in \mathbb{N}_0$, there exists $s_n \le 0$ such that for any $s < s_n$ and for any $i \le n$, BVP (2.2.10), (1.4.2) has no solutions in $\mathcal{T}_i^\gamma$ for $\gamma \in \{+, -\}$.

(ii) Assume either $f_0 < \infty$ and $f_\infty = \infty$ or $f_\infty < \infty$ and $f_0 = \infty$. Then for any $n \in \mathbb{N}_0$, there exists $s_n \ge 0$ such that for any $s > s_n$ and for any $i \ge n$, BVP (2.2.10), (1.4.2) has two solutions $y_{i,\gamma} \in \mathcal{T}_{i+1}^\gamma$ for $\gamma \in \{+, -\}$.

(iii) Assume $f_0 < \infty$ and $f_\infty < \infty$. Then for any $n \in \mathbb{N}_0$, there exists $s_* \ge 0$ such that for any $s > s_*$, BVP (2.2.10), (1.4.2) has no solution in $\mathcal{T}_i^\gamma$ for all $i \ge n + 1$ and $\gamma \in \{+, -\}$.

Similar to Theorem 2.2.4, we show a result on the nonexistence of certain types of nodal solutions of BVP (1.4.1), (1.4.2) as the function $1/p(t)$ increases in a certain direction. More precisely, let $s \ge 0$ and $h \in C[a, b]$ such that $h(t) > 0$ on $[a, b]$, and consider the equation

$$-\left(\frac{1}{1/p(t) + sh(t)}y'\right)' + q(t)y = w(t)f(y). \tag{2.2.11}$$

Theorem 2.2.6. Let the interval $[a, b]$ and BC (1.4.2) be fixed and let (2.2.5) hold. Define $\hat{q} := \max\{q(t)/w(t) : t \in [a, b]\}$ and assume $f(y)/y \geq f_* > \hat{q}$ for all $y \neq 0$. Then for any $n \in \mathbb{N}_0$, there exists $s_n \geq 0$ such that for any $s > s_n$, BVP (2.2.11), (1.4.2) has no solution in $\mathcal{T}_i^\gamma$ for all $i \leq n$ and $\gamma \in \{+, -\}$.

Finally, our last result is on the existence of certain types of nodal solutions of BVP (1.4.1), (1.4.2) as the boundary condition angle α changes.

Theorem 2.2.7. Let Eq. (1.4.1) and the interval $[a, b]$ be fixed and let (2.2.5) hold. Assume either $f_0 = 0$ and $f_\infty = \infty$ or $f_\infty = 0$ and $f_0 = \infty$. For $n \in \mathbb{N}_0$ denote $\lambda_n(\alpha)$ the nth eigenvalue of the SLP (2.2.1), (2.2.2). Suppose k is the first nonnegative integer such that $\lambda_k(\alpha^*) > 0$ for some $\alpha^* \in (0, \pi)$. Then

(i) for $\alpha \in [0, \alpha^*)$, BVP (1.4.1), (1.4.2) has two solutions $y_n^\gamma \in \mathcal{T}_n^\gamma$ for all $n \geq k$ and $\gamma \in \{+, -\}$;

(ii) for $\alpha \in [\alpha^*, \pi)$, BVP (1.4.1), (1.4.2) has two solutions $y_n^\gamma \in \mathcal{T}_n^\gamma$ for all $n \geq k+1$ and $\gamma \in \{+, -\}$.

2.2.2 Proofs

To prove Theorem 2.2.1, we need some preliminaries. The first lemma is a combination of Propositions 3.1 and 3.2 and Corollary 3.1 in [28].

Lemma 2.2.1. Any initial value problem associated with Eq. (1.4.1) has a unique solution which exists on the whole interval $[a, b]$. Consequently, the solution depends continuously on the initial condition.

For $\gamma \in \{+,-\}$, let $y(t,\rho)$ be the solution of Eq. (1.4.1) satisfying

$$y(a) = \gamma\rho\sin\alpha \quad \text{and} \quad (py')(a) = \gamma\rho\cos\alpha, \tag{2.2.12}$$

where $\rho > 0$ is a parameter. Let $\theta(t,\rho)$ be the Prüfer angle of $y(t,\rho)$; i.e., $\theta(\cdot,\rho)$ is a continuous function on $[a,b]$ such that

$$\tan\theta(t,\rho) = y(t,\rho)/(py')(t,\rho) \quad \text{and} \quad \theta(a,\rho) = \alpha.$$

By Lemma 2.2.1, $\theta(t,\rho)$ is continuous in ρ on $(0,\infty)$ for any $t \in [a,b]$. The following results are from Lemmas 4.1, 4.2, 4.4, and 4.5 in [28].

Lemma 2.2.2. (i) Assume $f_0 < \lambda_n$ for some $n \in \mathbb{N}_0$. Then there exists $\rho_* > 0$ such that $\theta(b,\rho) < n\pi + \pi/2$ for all $\rho \in (0,\rho_*)$.

(ii) Assume $\lambda_n < f_\infty$ for some $n \in \mathbb{N}_0$. Then there exists $\rho^* > 0$ such that $\theta(b,\rho) > n\pi + \pi/2$ for all $\rho \in (\rho^*,\infty)$.

Lemma 2.2.3. (i) Assume $f_\infty < \lambda_n$ for some $n \in \mathbb{N}_0$. Then there exists $\rho^* > 0$ such that $\theta(b,\rho) < n\pi + \pi/2$ for all $\rho \in (\rho^*,\infty)$.

(ii) Assume $\lambda_n < f_0$ for some $n \in \mathbb{N}_0$. Then there exists $\rho_* > 0$ such that $\theta(b,\rho) > n\pi + \pi/2$ for all $\rho \in (0,\rho_*)$.

Proof of Theorem 2.2.1. We first prove it for the case where $f_0 < \lambda_n$ and $\lambda_{n+1} < f_\infty$. Without loss of generality we assume $\gamma = +$. The case with $\gamma = -$ can be proved in the same way. Let $y(t,\rho)$ be the solution of Eq. (1.4.1) satisfying (2.2.12) with $\gamma = +$ and $\theta(t,\rho)$ its Prüfer angle. By Lemma 2.2.2, there exist $0 < \rho_* < \rho^* < \infty$ such that

$$\theta(b,\rho) < n\pi + \pi/2 \;\; \text{for all } \rho \in (0,\rho_*)$$

and

$$\theta(b,\rho) > (n+1)\pi + \pi/2 \ \text{ for all } \rho \in (\rho^*, \infty).$$

By the continuity of $\theta(t,\rho)$ in ρ, there exist $\rho_* \le \rho_n < \rho_{n+1} \le \rho^*$ such that

$$\theta(b,\rho_n) = n\pi + \pi/2 \quad \text{ and } \quad \theta(b,\rho_{n+1}) = (n+1)\pi + \pi/2, \tag{2.2.13}$$

and

$$n\pi + \pi/2 < \theta(b,\rho) < (n+1)\pi + \pi/2 \ \text{ for } \rho_n < \rho < \rho_{n+1}. \tag{2.2.14}$$

Define an energy function for $y(t,\rho)$ by

$$E(t,\rho) = \frac{1}{2p(t)}\big(p(t)y'(t,\rho)\big)^2 + \frac{1}{2}\big(q^* - q(t)\big)y^2(t,\rho) + w(t)F\big(y(t,\rho)\big). \tag{2.2.15}$$

For ease of notation, in the following we use $p = p(t)$, $q = q(t)$, $w = w(t)$, $l = l(t)$, $y = y(t,\rho)$, $E = E(t,\rho)$. Then, by (1.4.1) and (H1), we find that

$$\begin{aligned}
E' &= -\frac{p'}{2p^2}(py')^2 - \frac{1}{2}q'y^2 + q^*yy' + w'F(y) \\
&\ge -\frac{p'}{2p^2}(py')^2 - \frac{1}{2}q'y^2 - \frac{q^*}{2}\big(y^2 + y'^2\big) + w'F(y) \\
&= -\frac{(p'+q^*)}{2p^2}(py')^2 - \frac{1}{2}(q'+q^*)y^2 + w'F(y) \\
&\ge -\Big(\frac{p'+q^*}{p}\Big)_+\Big(\frac{1}{2p}(py')^2\Big) - l\Big(\frac{1}{2}(q^*-q)y^2\Big) - \frac{w'_-}{w}wF(y) \\
&\ge -l\frac{(py')^2}{2p} - l\Big(\frac{1}{2}(q^*-q)y^2\Big) - lwF(y) \\
&= -lE(t,\rho).
\end{aligned}$$

Thus, $E'(t,\rho) + lE(t,\rho) \geq 0$ for all $t \in [a,b]$ and $\rho > 0$. By solving this inequality, we obtain that

$$E(s,\rho) \leq E(b,\rho)e^{\int_s^b l(\tau)d\tau} \leq E(b,\rho)e^{k_0}, \quad s \in [a,b]. \tag{2.2.16}$$

We observe that for $\rho = \rho_n$ and $\rho = \rho_{n+1}$,

$$E(s,\rho) \geq \frac{1}{2}(q^* - q(s))y(s,\rho)^2 + w(s)F(y(s,\rho)), \quad s \in [a,b], \tag{2.2.17}$$

and

$$E(b,\rho) = \frac{1}{2}(q^* - q(b))y(b,\rho)^2 + w(b)F(y(b,\rho)). \tag{2.2.18}$$

We note that H_+^{-1} is increasing and H_-^{-1} is decreasing. Thus from (2.2.17) we see that for $\rho = \rho_n$ and $\rho = \rho_{n+1}$ and $s \in [a,b]$,

$$y(s,\rho) \leq H_+^{-1}(s, E(s,\rho)) \quad \text{ if } y(s,\rho) \geq 0$$

and

$$-y(s,\rho) \leq -H_-^{-1}(s, E(s,\rho)) \quad \text{ if } y(s,\rho) \leq 0.$$

Therefore, by (2.2.3),

$$|y(s,\rho)| \leq H_+^{-1}(s, E(s,\rho)). \tag{2.2.19}$$

Define

$$\Gamma(\rho) = y(b,\rho) - \int_a^b y(s,\rho)d(\xi(s)). \tag{2.2.20}$$

Let $n = 2k$ with $k \in \mathbb{N}_0$. Since $y(b, \rho_{2k}) > 0$ and $y(b, \rho_{2k+1}) < 0$, by (2.2.16), (2.2.19), and (2.2.5):

$$\begin{aligned} \Gamma(\rho_{2k}) &= y(b, \rho_{2k}) - \int_a^b y(s, \rho_{2k}) d(\xi(s)) \\ &\geq y(b, \rho_{2k}) - \int_a^b |y(s, \rho_{2k})| d(\xi_1(s) + \xi_2(s)) \\ &\geq H_+^{-1}(b, E(b, \rho_{2k})) - \int_a^b H_+^{-1}(s, E(s, \rho_{2k})) d(\xi_1(s) + \xi_2(s)) \\ &\geq H_+^{-1}(b, E(b, \rho_{2k})) - \int_a^b H_+^{-1}(s, E(b, \rho_{2k}) e^{k_0}) d(\xi_1(s) + \xi_2(s)) > 0, \end{aligned} \tag{2.2.21}$$

and

$$\begin{aligned} \Gamma(\rho_{2k+1}) &= y(b, \rho_{2k+1}) - \int_a^b y(s, \rho_{2k+1}) d(\xi(s)) \\ &\leq y(b, \rho_{2k+1}) + \int_a^b |y(s, \rho_{2k+1})| d(\xi_1(s) + \xi_2(s)) \\ &\leq H_-^{-1}(b, E(b, \rho_{2k+1})) + \int_a^b H_+^{-1}(s, E(s, \rho_{2k+1})) d(\xi_1(s) + \xi_2(s)) \\ &\leq -H_+^{-1}(b, E(b, \rho_{2k+1})) + \int_a^b H_+^{-1}(s, E(b, \rho_{2k+1}) e^{k_0}) d(\xi_1(s) + \xi_2(s)) < 0. \end{aligned} \tag{2.2.22}$$

By the continuity of $\Gamma(\rho)$, there exists $\bar{\rho} \in (\rho_{2k}, \rho_{2k+1})$ such that $\Gamma(\bar{\rho}) = 0$. Similarly, for $n = 2k+1$ with $k \in \mathbb{N}_0$, there exists $\bar{\rho} \in (\rho_{2k+1}, \rho_{2k+2})$ such that $\Gamma(\bar{\rho}) = 0$. In both cases, from (2.2.14),

$$n\pi + \pi/2 < \theta(b, \bar{\rho}) < (n+1)\pi + \pi/2.$$

Since

$$\theta'(t,\rho) = \frac{1}{p(t)}\cos^2\theta(t,\rho) + w(t)\frac{f(y(t,\rho))y(t,\rho)}{r^2(t,\rho)} - q(t)\sin^2\theta(t,\rho), \tag{2.2.23}$$

where $r = (y^2+(py')^2)^{1/2}$, by (H4), we have that $\theta(\cdot,\rho)$ is strictly increasing on $[a,b]$. We note that $y(t)=0$ if and only if $\theta(t,\rho)=0 \pmod{\pi}$ and $(py')(t)=0$ if and only if $\theta(t,\rho)=\pi/2 \pmod{\pi}$. Thus, py' has exactly $n+1$ zeros in (a,b) if $\alpha\in[0,\pi/2)$ and n zeros in (a,b) if $\alpha\in[\pi/2,\pi)$, and y has exactly one zero strictly between any two consecutive zeros of py'. Initial condition (2.2.12) implies that $y(t,\bar\rho)>0$ in a right-neighborhood of a. Therefore, $y(t,\bar\rho)\in\mathcal{T}_n^+$.

The proof for the case where $f_\infty<\lambda_n$ and $\lambda_{n+1}<f_0$ is essentially the same as above except that the discussion is based on Lemma 2.2.3 instead of Lemma 2.2.2. □

Proof of Corollary 2.2.1. (i) Since $q\equiv 0$, by (2.2.4), $H_+^{-1}(t,ce^{k_0}) = F_+^{-1}(ce^{k_0}/w(t))$. Thus, (2.2.5) reduces to

$$\int_a^b F_+^{-1}\Big(\frac{ce^{k_0}}{w(s)}\Big)d(\xi_1(s)+\xi_2(s)) < F_+^{-1}\Big(\frac{c}{w(b)}\Big).$$

Then the conclusion follows from Theorem 2.2.1. (ii) In addition, when $w\equiv w^*>0$ and $p\equiv 1$, from the definition of $l(t)$, $l(t)\equiv 0$. Hence, $k_0=\int_a^b l(t)dt = 0$ and, by (2.2.4), we have $H_+^{-1}(t,ce^{k_0}) = F_+^{-1}(c/w_0)$. In this case, (2.2.4) reduces to

$$\int_a^b d(\xi_1(s)+\xi_2(s)) < 1.$$

Then the conclusion follows from Theorem 2.2.1. □

Proof of Corollary 2.2.2. Without loss of generality we let $\gamma = +$. The case for $\gamma = -$ can be proved in the same way. Let $y(t,\rho)$ be the solution of Eq. (1.4.1) satisfying (2.2.12) with $\gamma = +$ and $\theta(t,\rho)$ its Prüfer angle. Let ρ_0 and ρ_1 be given in (2.2.13) with $n = 0$. Then for the function $\Gamma(\rho)$ defined by (2.2.20), from (2.2.21) and (2.2.22) we have $\Gamma(\rho_0) > 0$ and $\Gamma(\rho_1) < 0$.

(i) Assume $\xi(s)$ is increasing on $[a,b]$. Then $\xi(s) = \xi_1(s) - \xi_2(s)$ implies that we may take $\xi_2(s) \equiv 0$, and hence $d(\xi(s)) \geq 0$ on $[a,b]$. By the mean value theorem for the Riemann-Stieltjes integral and the continuity of solution y, there exists $t_0 \in [a,b]$ such that $\int_a^b y(s)d(\xi(s)) = y(t_0)\int_a^b d(\xi(s))$. By the continuity of $\theta(t,\rho)$ in ρ, there exists $\tilde{\rho} \in (\rho_0,\rho_1)$ such that $\theta(b,\tilde{\rho}) = \pi$ and $\theta(b,\rho) < \pi$ for $\rho \in (\rho_0,\tilde{\rho})$. Since $\theta(s,\tilde{\rho}) < \pi$ for $s \in [a,b]$, we see that

$$\Gamma(\tilde{\rho}) = -\int_a^b y(s,\tilde{\rho})d(\xi(s)) = -y(t_0,\tilde{\rho})\int_a^b d(\xi(s)) \leq 0.$$

Therefore, there exists $\bar{\rho} \in (\rho_0,\tilde{\rho}]$ such that $\Gamma(\bar{\rho}) = 0$. This means that $y(t,\bar{\rho}) \in \mathcal{T}_0^+$ and is a positive solution.

(ii) Assume $\xi(s)$ is decreasing on $[a,b]$ such that $\int_a^{b-\epsilon} d\xi(s) < 0$ for some $\epsilon > 0$. Since $\xi(s)$ is decreasing, we may take $\xi_1(s) \equiv 0$. As in (i), there exists $\tilde{\rho} \in (\rho_0,\rho_1)$ such that $\theta(b,\tilde{\rho}) = \pi$ and $\theta(b,\rho) > \pi$ for $\rho \in (\tilde{\rho},\rho_1)$. Furthermore, for some $t_0 \in [a,b]$,

$$\Gamma(\tilde{\rho}) = -\int_a^b y(s,\tilde{\rho})d(\xi(s)) = -y(t_0,\tilde{\rho})\int_a^b d(\xi(s)) \geq -y(t_0,\tilde{\rho})\int_a^{b-\epsilon} d(\xi(s)) > 0.$$

Therefore, there exists $\bar{\rho} \in (\tilde{\rho},\rho_1)$ such that $\Gamma(\bar{\rho}) = 0$. This means that $\theta(t,\bar{\rho}) \in (\pi, 3\pi/2)$ and hence $y(t,\bar{\rho}) \in \mathcal{T}_0^+$ and has exactly one zero in (a,b). $\square$

Proof of Corollary 2.2.3. It is easy to see that $\lambda_0 = 0$ is the first eigenvalue of the BVP consisting of Eq. (2.2.1) and the BC

$$(py')(a) = (py')(b) = 0,$$

i.e., the BC (2.2.2) with $\alpha = \pi/2$. In fact, $y_0(t) \equiv 1$ is an associated eigenfunction. From Theorem 4.2 in [31] we see that λ_0 as a function of α is strictly decreasing. This shows that $\lambda_0 > 0$ for $\alpha \in [0, \pi/2)$. By Theorem 2.2.1 we see that BVP (1.4.1), (1.4.2) has a solution $y_n^\gamma \in \mathcal{T}_n^\gamma$ for all $n \geq 0$ and $\gamma \in \{+, -\}$. $\square$

Proof of Theorem 2.2.2. (i) Assume to the contrary that BVP (1.4.1), (1.4.2) has a solution $y \in \mathcal{T}_i^\gamma$ for some $i \geq n$ and $\gamma \in \{+, -\}$. Let $\tilde{w}(t) = w(t)f(y(t))/y(t)$. Then $\tilde{w}(t)$ is continuous on $[a, b]$ by the continuous extension since $f_0 < \infty$. Let $\theta(t)$ be the Prüfer angle of $y(t)$ with $\theta(a) = \alpha$. Then $\theta(t)$ satisfies Eq. (2.2.23) and hence is strictly increasing on $[a, b]$. Note from the assumption that $\tilde{w}(t) < \lambda_i w(t)$ on $[a, b]$, we have

$$\theta'(t) < \frac{1}{p(t)} \cos^2 \theta(t) + \lambda_i w(t) \sin^2 \theta(t) - q(t) \sin^2 \theta(t).$$

Let $u(t)$ be an eigenfunction of SLP (2.2.1), (2.2.2) associated with the eigenvalue λ_i and $\phi(t)$ its Prüfer angle with $\phi(a) = \alpha$. Then

$$\phi'(t) = \frac{1}{p(t)} \cos^2 \phi(t) + \lambda_i w(t) \sin^2 \phi(t) - q(t) \sin^2 \phi(t)$$

and $\phi(b) = i\pi + \pi/2$. By the theory of differential inequalities we find that $\theta(b) < \phi(b)$. This contradicts the assumption that $y \in \mathcal{T}_i^\gamma$ since condition (ii) of the definition of $\mathcal{T}_i^\gamma$ is violated.

(ii) It is similar to (i) and hence is omitted. □

Proofs of Theorems 2.2.3–2.2.7. These proofs, which use Lemma 2.1.4 and Theorems 2.2.1 and 2.2.2, are essentially the same proofs of Theorems 2.1.3–2.1.7 and hence are omitted.

CHAPTER 3

GENERALIZED PROBLEMS

We are concerned with BVP consisting of the equation

$$-(p(t)y')' + q(t)y = \sum_{i=1}^{m} w_i(t) f_i(y), \ t \in (a,b), \tag{3.0.1}$$

and the BCs (1.4.2), (1.4.3). In what follows, conditions (H1)–(H3) are necessary for BVP (3.0.1), (1.4.3), and BVP (3.0.1), (1.4.2) also requires assumption (H4). We assume throughout, and without further mention, that the following hold:

(H1) $p, q, w_i \in C^1[a,b]$ for each $i = 1 : m$ such that $p(t) > 0$, $w_i(t) > 0$, and $q'(t) + q^* \le l(q^* - q(t))$ on $[a,b]$ with

$$q^* := \max_{t\in[a,b]}\{q(t), 0\} \quad \text{and} \quad l(t) := \max_{i=1:m}\left\{\left(\frac{p'(t)+q^*}{p(t)}\right)_+, \frac{(w_i')_-(t)}{w_i(t)}\right\},$$

where $h'_-(t) := \max\{0, -h'(t)\}$ and $h_+(t) := \max\{0, h(t)\}$, $k_0 = \displaystyle\int_a^b l(t)\ dt$;

(H2) $f_i \in C(\mathbb{R})$ such that $yf_i(y) > 0$ for $y \neq 0$, and f_i is locally Lipschitz on $(-\infty, 0) \cup (0, \infty)$ for each $i = 1 : m$;

(H3) for $i = 1 : m$, there exist extended real numbers $(f_i)_0, (f_i)_\infty \in [0, \infty]$ such that

$$(f_i)_0 = \lim_{y\to 0} f_i(y)/y \quad \text{and} \quad (f_i)_\infty = \lim_{|y|\to\infty} f_i(y)/y.$$

(H4) $\displaystyle\sum_{i=1}^{m} w_i(t)\frac{f_i(y)}{y} - q(t) > 0$ for all $t \in [a,b]$, and $y \neq 0$.

3.1 Solution Classes $\mathcal{S}_n^\gamma$

Our results on the existence and nonexistence of nodal solutions of BVP (3.0.1), (1.4.3) are established utilizing the eigenvalues of the linear SLP consisting of the equation

$$-(p(t)y')' + q(t)y = \lambda_n \sum_{i=1}^{m} w_i(t)y, \quad t \in (a,b), \tag{3.1.1}$$

and the two-point BC

$$\begin{aligned} &\cos\alpha\ y(a) - \sin\alpha\ (py')(a) = 0, \quad \alpha \in [0,\pi), \\ &y(b) = 0. \end{aligned} \tag{3.1.2}$$

It is well known that SLP (3.0.1), (3.1.2) has an infinite number of eigenvalues $\{\lambda_n\}_{n=0}^{\infty}$ satisfying

$$-\infty < \lambda_0 < \lambda_1 < \cdots < \lambda_n < \cdots, \quad \text{and} \quad \lambda_n \to \infty \text{ and } n \to \infty,$$

and any eigenfunction associated with λ_n has n simple zeros in (a,b); see [65, Theorem 4.3.2].

3.1.1 Main Theorems

We now present our main results with the proofs given later in this section after several technical lemmas are derived. The first theorem is about the existence of certain types of nodal solutions.

Theorem 3.1.1. Assume either

$$\text{(i)} \quad \sum_{i=1}^{m} w_i(t)(f_i)_0 < \lambda_n \sum_{i=1}^{m} w_i(t) \quad \text{and} \quad \lambda_{n+1} \sum_{i=1}^{m} w_i(t) < \sum_{i=1}^{m} w_i(t)(f_i)_\infty$$

or

$$\text{(ii)} \quad \sum_{i=1}^{m} w_i(t)(f_i)_\infty < \lambda_n \sum_{i=1}^{m} w_i(t) \quad \text{and} \quad \lambda_{n+1} \sum_{i=1}^{m} w_i(t) < \sum_{i=1}^{m} w_i(t)(f_i)_0$$

for some $n \in \mathbb{N}_0$. Suppose

$$1 - \int_a^b \sqrt{\frac{p(s)}{p(b)}}\; e^{k_0/2}\; d\big(\xi_1(s) + \xi_2(s)\big) > 0, \tag{3.1.3}$$

where $\xi_1(s)$ and $\xi_2(s)$ are given by (2.1.3). Then BVP (3.0.1), (1.4.3) has two solutions $y_{n,\gamma} \in \mathcal{S}_{n+1}^{\gamma}$ for $\gamma \in \{+, -\}$.

As a consequence of Theorem 3.1.1, we have the following two corollaries on the existence of an infinite number of different types of nodal solutions in the two Classes $\mathcal{S}_n^\gamma$ and $\mathcal{T}_n^\gamma$. The first corollary is for a special case of BVP (3.0.1), (1.4.3).

Corollary 3.1.1. Consider the special case that $p(t) \equiv 1$ and $q(t) \equiv 0$ on $[a, b]$. Assume (3.1.3) holds and either

$$(f_i)_0 = 0 \text{ for all } i \in \{1, 2, ..., m\} \text{ and } (f_j)_\infty = \infty \text{ for some } j \in \{1, 2, ..., m\},$$

or

$$(f_i)_\infty = 0 \text{ for all } i \in \{1, 2, ..., m\}, \text{ and } (f_j)_0 = \infty \text{ for some } j \in \{1, 2, ..., m\}.$$

Then there exists $\alpha^* \in (\pi/2, \pi)$ such that

(i) if $\alpha \in [0, \alpha^*)$, then BVP (1.1.1), (1.1.2) has a solution $y_n^\gamma \in \mathcal{S}_{n+1}^\gamma$ for each $n \geq 0$ and $\gamma \in \{+, -\}$;

(ii) if $\alpha \in [\alpha^*, \pi)$, then BVP (1.1.1), (1.1.2) has a solution $y_n^\gamma \in \mathcal{S}_{n+1}^\gamma$ for each $n \geq 1$ and $\gamma \in \{+, -\}$.

Corollary 3.1.2. Assume the special condition that

$$\sum_{i=1}^{m} w_i(t) \frac{f_i(y)}{y} - q(t) > 0 \tag{3.1.4}$$

for all $t \in [a, b]$, $y \neq 0$, and (3.1.3) holds. Then BVP (3.0.1), (1.4.3) has two solutions $y_{n,\gamma} \in \mathcal{T}_{n+1}^\gamma$ for $\gamma \in \{+, -\}$.

Remark 3.1.1. The number α^* in Corollary 3.1.1 can be explicitly computed using the fundamental solutions of (3.1.1); see [7, Theorem 2.2] for details.

The next theorem is about the nonexistence of certain types of nodal solutions.

Theorem 3.1.2.

(i) Assume $\sum_{i=1}^{m} \dfrac{f_i(y) w_i(t)}{y} \leq \lambda_n \sum_{i=1}^{m} w_i(t)$ for some $n \in \mathbb{N}_0$ and all $y \neq 0$. Then

BVP (3.0.1), (1.4.3) has no solution in $\mathcal{S}_j^\gamma$ for all $j \geq n+1$ and $\gamma \in \{+,-\}$.

(ii) Assume $\sum_{i=1}^m \dfrac{f_i(y)w_i(t)}{y} \geq \lambda_n \sum_{i=1}^m w_i(t)$ for some $n \in \mathbb{N}_0$ and all $y \neq 0$. Then BVP (3.0.1), (1.4.3) has no solution in $\mathcal{S}_j^\gamma$ for all $j \leq n$ and $\gamma \in \{+,-\}$.

3.1.2 Proofs

To prove Theorem 3.1.1, we need some preliminaries. The lemmas below are on the IVPs associated with Eq. (3.0.1) and are simple generalizations of [28, Corollary 3.1, Lemmas 4.1, 4.2, 4.4, and 4.5] originally for the case where $p(t) \equiv 1$ with essentially the same proofs. The first one is on the global existence of solutions of IVPs associated with Eq. (3.0.1).

Lemma 3.1.1. Any initial value problem associated with Eq. (3.0.1) has a unique solution which exists on the whole interval $[a,b]$. Consequently, the solution depends continuously on the initial condition.

For $\gamma \in \{+,-\}$, let $y(t,\rho)$ be the solution of Eq. (3.0.1) satisfying

$$y(a) = \gamma\rho \sin\alpha \quad \text{and} \quad (py')(a) = \gamma\rho\cos\alpha, \tag{3.1.5}$$

where $\rho > 0$ is a parameter. Let $\theta(t,\rho)$ be the Prüfer angle of $y(t,\rho)$; i.e., $\theta(\cdot,\rho)$ is a continuous function on $[a,b]$ such that

$$\tan\theta(t,\rho) = y(t,\rho)/(py')(t,\rho) \quad \text{and} \quad \theta(a,\rho) = \alpha.$$

By Lemma 3.1.1, $\theta(t,\rho)$ is continuous in ρ on $(0,\infty)$ for any $t \in [a,b]$.

The next two lemmas provide some estimates for the Prüfer angle.

Lemma 3.1.2.

(i) Assume $\sum_{i=1}^{m} w_i(t)(f_i)_0 < \lambda_n \sum_{i=1}^{m} w_i(t)$ for some $n \in \mathbb{N}_0$. Then there exists $\rho_* > 0$ such that $\theta(b, \rho) < (n+1)\pi$ for all $\rho \in (0, \rho_*)$.

(ii) Assume $\lambda_n \sum_{i=1}^{m} w_i(t) < \sum_{i=1}^{m} w_i(t)(f_i)_\infty$ for some $n \in \mathbb{N}_0$. Then there exists $\rho^* > 0$ such that $\theta(b, \rho) > (n+1)\pi$ for all $\rho \in (\rho^*, \infty)$.

Lemma 3.1.3.

(i) Assume $\sum_{i=1}^{m} w_i(t)(f_i)_\infty < \lambda_n \sum_{i=1}^{m} w_i(t)$ for some $n \in \mathbb{N}_0$. Then there exists $\rho^* > 0$ such that $\theta(b, \rho) < (n+1)\pi$ for all $\rho \in (\rho^*, \infty)$.

(ii) Assume $\lambda_n \sum_{i=1}^{m} w_i(t) < \sum_{i=1}^{m} w_i(t)(f_i)_0$ for some $n \in \mathbb{N}_0$. Then there exists $\rho_* > 0$ such that $\theta(b, \rho) > (n+1)\pi$ for all $\rho \in (0, \rho_*)$.

Proof of Theorem 3.1.1. We first prove it for the case where $\sum_{i=1}^{m} w_i(t)(f_i)_0 < \lambda_n \sum_{i=1}^{m} w_i(t)$ and $\lambda_{n+1} \sum_{i=1}^{m} w_i(t) < \sum_{i=1}^{m} w_i(t)(f_i)\infty$. Without loss of generality we assume $\gamma = +$. The case with $\gamma = -$ can be proved in the same way. Let $y(t, \rho)$ be the solution of Eq. (3.0.1) satisfying (3.1.5) with $\gamma = +$ and $\theta(t, \rho)$ its Prüfer angle. By Lemma 3.1.2, there exist $0 < \rho_* < \rho^* < \infty$ such that

$$\theta(b, \rho) < (n+1)\pi \quad \text{for all } \rho \in (0, \rho_*)$$

and

$$\theta(b, \rho) > (n+2)\pi \quad \text{for all } \rho \in (\rho^*, \infty).$$

By the continuity of $\theta(t,\rho)$ in ρ, there exist $\rho_* \le \rho_{n+1} < \rho_{n+2} \le \rho^*$ such that

$$\theta(b,\rho_{n+1}) = (n+1)\pi \quad \text{and} \quad \theta(b,\rho_{n+2}) = (n+2)\pi \tag{3.1.6}$$

and

$$(n+1)\pi < \theta(b,\rho) < (n+2)\pi \;\text{ for }\; \rho_{n+1} < \rho < \rho_{n+2}. \tag{3.1.7}$$

Then, for all $t \in [a,b]$ and all $\rho > 0$, we define an energy function $E(t,\rho)$ for $y(t,\rho)$ by

$$E(t,\rho) = \frac{1}{2p(t)}\big(p(t)y'(t,\rho)\big)^2 + \frac{1}{2}\big(q^* - q(t)\big)y^2(t,\rho) + \sum_{i=1}^{m} w_i(t)F_i\big(y(t,\rho)\big), \tag{3.1.8}$$

where $F_i(y) = \int_0^y f_i(s)ds$ for each $i = 1..m$. By (H1) and (H2), $F_i(y) \ge 0$ on $\mathbb{R}$ for each $i = 1..m$, yielding $E(t,\rho) \ge 0$ on $[a,b]$. For ease of notation, in the following, we use $p = p(t)$, $q = q(t)$, $w_i = w_i(t)$, $y = y(t,\rho)$, $E = E(t,\rho)$. Then, by (3.0.1) and (H1), we find that

$$\begin{aligned}
E' &= -\frac{p'}{2p^2}(py')^2 - \frac{1}{2}q'y^2 + q^*yy' + \sum_{i}^{m} w_i'F_i(y) \\
&\ge -\frac{p'}{2p^2}(py')^2 - \frac{1}{2}q'y^2 - \frac{q^*}{2}\big(y^2 + y'^2\big) + \sum_{i=1}^{m} w_i'F_i(y) \\
&= -\frac{(p'+q^*)}{2p^2}(py')^2 - \frac{1}{2}(q'+q^*)y^2 + \sum_{i=1}^{m} w_i'F_i(y) \\
&\ge -\Big(\frac{p'+q^*}{p}\Big)_+\Big(\frac{1}{2p}(py')^2\Big) - l\Big(\frac{1}{2}(q^*-q)y^2\Big) - \sum_{i=1}^{m}\frac{w'_{i_-}}{w_i}w_iF_i(y) \\
&\ge -l\frac{(py')^2}{2p} - l\Big(\frac{1}{2}(q^*-q)y^2\Big) - l\sum_{i=1}^{m} w_iF_i(y) \\
&= -lE(t,\rho).
\end{aligned}$$

Thus, $E'(t,\rho) + lE(t,\rho) \geq 0$ for all $t \in [a,b]$ and $\rho > 0$. By solving this inequality, we obtain that

$$E(s,\rho) \leq E(b,\rho)e^{k_0}, \quad s \in [a,b]. \tag{3.1.9}$$

We observe that for $\rho = \rho_{n+1}$ and $\rho = \rho_{n+2}$,

$$E(s,\rho) \geq \frac{1}{2p(s)} \left[p(s)y'(s,\rho)\right]^2, \quad s \in [a,b],$$

and

$$E(b,\rho) = \frac{1}{2p(b)} \left[p(b)y'(b,\rho)\right]^2.$$

Thus, for $\rho = \rho_{n+1}$, $\rho = \rho_{n+2}$, and $s \in [a,b]$,

$$|(py')(s,\rho)| \leq \sqrt{2p(s)E(s,\rho)} \;\text{ and }\; |(py')(b,\rho)| = \sqrt{2p(s)E(b,\rho)}. \tag{3.1.10}$$

Define

$$\Gamma(\rho) = (py')(b,\rho) - \int_a^b (py')(s,\rho)d\xi(s). \tag{3.1.11}$$

Assume $n = 2k-1$ with $k \in \mathbb{N}_0$. Since $(py')(b,\rho_{2k}) > 0$ and $(py')(b,\rho_{2k+1}) < 0$, by (2.1.3), (3.1.9), (3.1.10), and (3.1.3) we have

$$\begin{aligned}
&\Gamma(\rho_{2k}) = (py')(b,\rho_{2k}) - \int_a^b (py')(s,\rho_{2k})d\xi(s)\\
\geq\ & \left|(py')(b,\rho)\right| - \int_a^b |(py')(s,\rho_{2k})| d\big(\xi_1(s)+\xi_2(s)\big)\\
\geq\ & \sqrt{2p(b)E(b,\rho_{2k})} - \int_a^b \sqrt{2p(s)E(s,\rho_{2k})}d\big(\xi_1(s)+\xi_2(s)\big)\\
\geq\ & \sqrt{2p(b)E(b,\rho_{2k})} - \int_a^b \sqrt{2p(s)E(b,\rho_{2k})e^{k_0}}d\big(\xi_1(s)+\xi_2(s)\big)\\
=\ & \sqrt{2p(b)E(b,\rho_{2k})}\left(1 - \int_a^b \sqrt{p(s)/p(b)}e^{k_0/2}d\big(\xi_1(s)+\xi_2(s)\big)\right) > 0
\end{aligned}$$

and

$$\Gamma(\rho_{2k+1}) = (py')(b,\rho_{2k+1}) - \int_a^b (py')(s,\rho_{2k+1})d\xi(s)$$

$$\begin{aligned}
&\leq -\Big|(py')(b,\rho_{2k+1})\Big| + \int_a^b (|(py')(s,\rho_{2k+1})|d\big(\xi_1(s)+\xi_2(s)\big)\\
&\leq -\sqrt{2p(b)E(b,\rho_{2k+1})} + \int_a^b \sqrt{2p(s)E(s,\rho_{2k+1})}\ d\big(\xi_1(s)+\xi_2(s)\big)\\
&\leq -\sqrt{2p(b)E(b,\rho_{2k+1})} + \int_a^b \sqrt{2p(s)E(b,\rho_{2k+1})e^{k_0}}\ d\big(\xi_1(s)+\xi_2(s)\big)\\
&= -\sqrt{2p(b)E(b,\rho_{2k+1})}\left(1 - \int_a^b \sqrt{p(s)/p(b)}\ e^{k_0/2}\ d\big(\xi_1(s)+\xi_2(s)\big)\right) < 0.
\end{aligned}$$

By the continuity of $\Gamma(\rho)$, there exists $\bar{\rho} \in (\rho_{2k}, \rho_{2k+1})$ such that $\Gamma(\bar{\rho}) = 0$. Similarly, for $n = 2k$ with $k \in \mathbb{N}_0$, there exists $\bar{\rho} \in (\rho_{2k+1}, \rho_{2k+2})$ such that $\Gamma(\bar{\rho}) = 0$. In both cases, from (1.3.4),

$$(n+1)\pi < \theta(b,\bar{\rho}) < (n+2)\pi.$$

Since

$$\theta'(t,\rho) = \frac{1}{p(t)}\cos^2\theta(t,\rho) + \sum_{i=1}^m w_i(t)\frac{f_i(y(t,\rho))y(t,\rho)}{r^2(t,\rho)} - q(t)\sin^2\theta(t,\rho) \qquad (3.1.12)$$

for $t \in [a,b]$, where $r = (y^2 + (py')^2)^{1/2}$, we have that $\theta(\cdot,\rho)$ is strictly increasing at the points t where $\theta(t,\rho) = 0$ (mod π). We note that $y(t) = 0$ if and only if $\theta(t,\rho) = 0$ (mod π). Thus, y has exactly $n+1$ zeros in (a,b). Initial condition (3.1.2) implies that $y(t,\bar{\rho}) > 0$ in a right-neighborhood of a. Therefore, $y(t,\bar{\rho}) \in \mathcal{S}_{n+1}^+$.

The proof for the case where $\sum_{i=1}^m w_i(t)(f_i)_\infty < \lambda_n \sum_{i=1}^m w_i(t)$ together with $\lambda_{n+1}\sum_{i=1}^m w_i(t) < \sum_{i=1}^m w_i(t)(f_i)_0$ is essentially the same as above except that the discussion is based on Lemma 3.1.3 instead of Lemma 3.1.2. □

Proof of Corollary 3.1.1. Consider the SLP consisting of Eq. (3.1.1) with $p(t) \equiv 1$, $q(t) \equiv 0$, and the BC

$$\cos\alpha\ y(a) - \sin\alpha\ y'(a) = 0, \qquad \alpha \in [0,\pi),$$
$$\cos\beta\ y(b) - \sin\beta\ y'(b) = 0, \qquad \beta \in (0,\pi].$$

Denote by $\lambda_n(\alpha,\beta)$ the nth eigenvalue of this problem for $n \in \mathbb{N}_0$. It is easy to see that $\lambda_0(\pi/2,\pi/2) = 0$. In fact, $y_0(t) \equiv 1$ is an associated eigenfunction. From [31, Theorem 4.2] and [29, Lemma 3.32], we see that $\lambda_0(\alpha,\beta)$ is a continuous function of (α,β) on $[0,\pi)\times(0,\pi]$ and is strictly decreasing in α and strictly increasing in β. Furthermore, for any $\beta \in (0,\pi]$,

$$\lim_{\alpha\to\pi-} \lambda_0(\alpha,\beta) = -\infty \quad \text{and} \quad \lim_{\alpha\to\pi-} \lambda_{n+1}(\alpha,\beta) = \lambda_n(0,\beta) \text{ for } n \in \mathbb{N}_0.$$

This shows that $\lambda_0(\pi/2,\pi) > 0$, and hence there exists $\alpha^* \in (\pi/2,\pi)$ such that $\lambda_0(\alpha,\pi) > 0$ for $\alpha \in [0,\alpha^*)$, and $\lambda_0(\alpha,\pi) \le 0$ and $\lambda_1(\alpha,\pi) > 0$ for $\alpha \in [\alpha^*,\pi)$. Note that $\beta = \pi$ if and only if $y(b) = 0$. Then the conclusion follows from Theorem 3.1.1. □

Proof of Corollary 3.1.2. We seek to satisfy conditions (i)-(iv) of Definition 2.2.1.

(i) Assume Eq. (3.0.1) has a nontrivial solution with a nonsimple zero $t_0 \in [a,b]$; i.e., $y(t_0) = py'(t_0) = 0$. Then from Lemma 3.1.1, $y(t) \equiv 0$ on $[a,b]$, a contradiction. Similarly, assume py' has a nonsimple zero $t_0 \in [a,b]$; i.e., $py'(t_0) = (py')'(t_0) = 0$. From Eq. (3.0.1) and condition (3.1.4), we have that $y(t_0) = 0$. As above, $y(t) \equiv 0$ on $[a,b]$, a contradiction.

(ii)-(iv) By Theorem 6.1, BVP (3.0.1), (1.4.3) has two solutions $y_{n,\gamma} \in \mathcal{S}^{\gamma}_{n+1}$ for $\gamma \in \{+,-\}$. We take $y = y_{n,\gamma}$. Owing to Eq. (3.1.10) and condition (3.1.4), the

Prüfer angle $\theta(t)$ is strictly increasing at the points t where $\theta(t) = 0 \pmod{\pi}$ and where $\theta(t) = \pi/2 \pmod{\pi}$. We note that $y(t) = 0$ if and only if $\theta(t) = 0$ (mod π) and $py'(t) = 0$ if and only if $\theta(t) = \pi/2 \pmod{\pi}$. Thus, py' has exactly $n+1$ zeros in (a, b) if $\alpha \in [0, \pi/2)$ and n zeros in (a, b) if $\alpha \in [\pi/2, \pi)$, and y has exactly one zero strictly between any two consecutive zeros of py'. Finally, $y = y_{n,\gamma} \in \mathcal{S}_{n+1}^{\gamma}$ implies that $y(t) > 0$ in a right-neighborhood of a. □

Proof of Theorem 3.1.2. (i) Assume to the contrary that BVP (3.0.1), (1.4.3) has a solution $y \in \mathcal{S}_j^{\gamma}$ for some $j \geq n+1$ and $\gamma \in \{+, -\}$. Let $\tilde{w}(t) = \sum_{i=1}^{m} w_i(t) f_i(y(t))/y(t)$. Then $\tilde{w}(t)$ is continuous on $[a, b]$ by the continuous extension since $\sum_{i=1}^{m} w_i(t)(f_i)_0 < \infty$. Let $\theta(t)$ be the Prüfer angle of $y(t)$ with $\theta(a) = \alpha$. Then $\theta(t)$ satisfies Eq. (3.1.12) and $\theta(b) > j\pi$. Note, from the assumption that $\tilde{w}(t) \leq \lambda_n \sum_{i=1}^{m} w_i(t) \leq \lambda_{j-1} \sum_{i=1}^{m} w_i(t)$ on $[a, b]$, we have that for $t \in [a, b]$,

$$\begin{aligned} \theta'(t) &= \frac{1}{p(t)} \cos^2 \theta(t) + [\tilde{w}(t) - q(t)] \sin^2 \theta(t, \rho) \\ &\leq \frac{1}{p(t)} \cos^2 \theta(t, \rho) + \left[\lambda_{j-1} \sum_{i=1}^{m} w_i(t) - q(t) \right] \sin^2 \theta(t, \rho). \end{aligned}$$

Let $u(t)$ be an eigenfunction of SLP (3.1.1), (3.1.2) associated with the eigenvalue λ_{j-1} and $\phi(t)$ its Prüfer angle with $\phi(a) = \alpha$. Then

$$\phi'(t) = \frac{1}{p(t)} \cos^2 \phi(t) + \left[\lambda_{j-1} \sum_{i=1}^{m} w_i(t) - q(t) \right] \sin^2 \phi(t)$$

and $\phi(b) = j\pi$. By the theory of differential inequalities, we find that $\theta(b) \leq \phi(b) = j\pi$. We have reached a contradiction.

(ii) It is similar to (i) and hence omitted. □

3.2 Solution Classes $\mathcal{T}_n^\gamma$

In this section we study the BVP (3.0.1), (1.4.2). Our results on the existence of nodal solutions of BVP (3.0.1), (1.4.2) are established using the eigenvalues, $\{\lambda_n\}_{n=0}^{\infty}$, of the linear SLP consisting of the equation

$$-(p(t)y')' + q(t)y = \lambda \sum_{i=1}^{m} w_i(t)y, \ t \in (a,b) \tag{3.2.1}$$

and the two-point BC

$$\begin{aligned} &\cos\alpha \ y(a) - \sin\alpha \ y'(a) = 0, \quad \alpha \in [0,\pi), \\ &(py')(b) = 0. \end{aligned} \tag{3.2.2}$$

It is well known that any eigenfunction associated with λ_n has n simple zeros in (a,b); see [65, Theorem 4.3.2].

For each $i \in \{1, ..., m\}$, let $F_i(y) = \int_0^y f_i(\xi)\, d\xi$ for $y \in \mathbb{R}$ and denote

$$H(t,y) := \frac{1}{2}(q^* - q(t))y^2 + \sum_{i=1}^{m} w_i(t)F_i(y) \quad \text{and} \quad k_0 = \int_a^b l(t)dt.$$

By (H2), each F_i is strictly increasing on $[0,\infty)$ and strictly decreasing on $(-\infty, 0]$. Thus, for any fixed $t \in [a,b]$, H is strictly increasing on $t \times [0,\infty)$ and strictly decreasing on $t \times (-\infty, 0]$. Let H_+^{-1} and H_-^{-1} be the inverses of H on $t \times [0,\infty)$ and $t \times (-\infty, 0]$, respectively, and define

$$H_{\max}^{-1} = \max\{H_+^{-1}, -H_-^{-1}\} \ \text{ and } \ H_{\min}^{-1} = \min\{H_+^{-1}, -H_-^{-1}\}. \tag{3.2.3}$$

Clearly, when all f_i are odd functions on $\mathbb{R}$, i.e., $f_i(-y) = -f_i(y)$ for $y \in \mathbb{R}$ and $i \in \{1...m\}$, then

$$H_{\max}^{-1} = H_{+}^{-1} = -H_{-}^{-1} = H_{\min}^{-1}.$$

It is useful to note the special case of $H(t, y)$ when $m = 1$ and $q \equiv 0$. In particular, if $c = H(t_0, y)$ for some $c > 0$ and $t_0 \in [a, b]$, then

$$c = H(t_0, y) = w(t_0)F(y).$$

As above, we take F_{+}^{-1} and F_{-}^{-1} to be the inverses of F on $t \times [0, \infty)$ and $t \times (-\infty, 0]$, respectively. Similarly, we define $F_{\max}^{-1}$ and $F_{\min}^{-1}$ as above. Hence, $y(t_0) = F_{+}^{-1}\Big(\dfrac{c}{w(t_0)}\Big)$ if $y \in [0, \infty)$ and $y(t_0) = F_{-}^{-1}\Big(\dfrac{c}{w(t_0)}\Big)$ if $y \in (-\infty, 0]$.

We now present our main results on the existence and nonexistence of nodal solutions of BVP (3.0.1), (1.4.2) with the proofs given in a later section after several technical lemmas are derived. The first theorem concerns the existence of certain types of nodal solutions.

3.2.1 Main Theorems

Theorem 3.2.1. Assume that for some $n \in \mathbb{N}_0$ and all $t \in [a, b]$, either

$$(i) \sum_{i=1}^{m} w_i(t)(f_i)_0 < \lambda_n \sum_{i=1}^{m} w_i(t) \quad \text{and} \quad \lambda_{n+1} \sum_{i=1}^{m} w_i(t) < \sum_{i=1}^{m} w_i(t)(f_i)_\infty \tag{3.2.4}$$

or

$$(ii) \sum_{i=1}^{m} w_i(t)(f_i)_\infty < \lambda_n \sum_{i=1}^{m} w_i(t) \quad \text{and} \quad \lambda_{n+1} \sum_{i=1}^{m} w_i(t) < \sum_{i=1}^{m} w_i(t)(f_i)_0. \tag{3.2.5}$$

Suppose that for any $c > 0$

$$\int_a^b H_{\max}^{-1}\left(s, ce^{k_0}\right) d\left(\xi_1(s) + \xi_2(s)\right) < H_{\min}^{-1}(b, c). \tag{3.2.6}$$

Then BVP (3.0.1), (1.4.2) has two solutions $y_n^\gamma \in \mathcal{T}_n^\gamma$ for $\gamma \in \{+, -\}$.

As consequences of Theorem 3.2.1 we have the corollaries below.

Corollary 3.2.1. Assume (3.2.6) holds and $q \equiv 0$ and $m = 1$. Then

(i) (3.2.6) is replaced with

$$\int_a^b F_{\max}^{-1}\Big(\frac{ce^{k_0}}{w(s)}\Big) d(\xi_1(s) + \xi_2(s)) < F_{\min}^{-1}\Big(\frac{c}{w(b)}\Big);$$

(ii) if $w \equiv w^* > 0$ on $[a, b]$, f is odd, and $p \equiv 1$, then (3.2.6) reduces to

$$\int_a^b d(\xi_1(s) + \xi_2(s)) < 1.$$

Remark 3.2.1. As in Corollary 6.3, assume (3.2.6) holds and $q \equiv 0$ and $m = 1$. If the second condition of BC (1.4.2) is replaced with the multipoint BC (1.3.10), then set $\xi(s) = \sum_{j=1}^d k_i \chi(s - x_j)$. Thus $\xi(s) = \xi_1(s) - \xi_2(s)$ with

$$\xi_1(s) = \sum_{j=1}^d (k_j)_+ \chi(s - x_j) \;\text{ and }\; \xi_2(s) = \sum_{j=1}^m (k_j)_- \chi(s - x_j),$$

where $(k_j)_\pm = \max\{\pm k_j, 0\}$. Hence $\xi_1(s) + \xi_2(s) = \sum_{j=1}^d |k_j| \chi(s - x_j)$. Then

(i) (3.2.6) reduces to

$$\sum_{j=1}^d |k_j| F_{\max}^{-1}\Big(\frac{ce^{k_0}}{w(x_j)}\Big) < F_{\min}^{-1}\Big(\frac{c}{w(b)}\Big);$$

(ii) if $w \equiv w^* > 0$ on $[a, b]$, f is odd, and $p \equiv 1$, then (3.2.6) reduces to

$$\sum_{j=1}^{d} |k_j| < 1;$$

(iii) if $f(y) = |y|^{r-1}y$ for $r > 0$, then (3.2.6) reduces to

$$\sum_{j=1}^{d} |k_j| \Big(\frac{w(b)e^{k_0}}{w(x_j)}\Big)^{1/(r+1)} < 1.$$

Corollary 3.2.2. Assume (3.2.6) holds and either (3.2.4) or (3.2.5) holds with $n = 0$. Then BVP (3.0.1), (1.4.2);

(i) has positive and negative solutions in $\mathcal{T}_0^\gamma$ for $\gamma \in \{+, -\}$ if $\xi(s)$ is increasing on $[a, b]$;

(ii) has solutions in $\mathcal{T}_0^\gamma$ for $\gamma \in \{+, -\}$ with exactly one zero in (a, b) if $\xi(s)$ is decreasing on $[a, b]$ such that $\int_a^{b-\epsilon} d\xi(s) < 0$ for some $\epsilon > 0$.

Corollary 3.2.3. Assume the special condition that $q(t) \equiv 0$, (3.2.6) holds, $\alpha \in [0, \pi/2)$, and either

$$(f_i)_0 = 0 \text{ for all } i \in \{1, 2, ..., m\} \text{ and } (f_j)_\infty = \infty \text{ for some } j \in \{1, 2, ..., m\}$$

or

$$(f_i)_\infty = 0 \text{ for all } i \in \{1, 2, ..., m\} \text{ and } (f_j)_0 = \infty \text{ for some } j \in \{1, 2, ..., m\}.$$

Then for all $n \geq 0$ and $\gamma \in \{+,-\}$, BVP (3.0.1), (1.4.2) has two solutions $y_n^\gamma \in \mathcal{T}_n^\gamma$.

The next theorem is about the nonexistence of certain types of nodal solutions.

Theorem 3.2.2. (i) Assume

$$\sum_{i=1}^{m} \frac{f_i(y)w_i(t)}{y} \leq \lambda_n \sum_{i=1}^{m} w_i(t)$$

for some $n \in \mathbb{N}_0$ and all $y \neq 0$. Then BVP (3.0.1), (1.4.2) has no solution in $\mathcal{T}_j^\gamma$ for all $j \geq n$ and $\gamma \in \{+,-\}$.

(ii) Assume

$$\sum_{i=1}^{m} \frac{f_i(y)w_i(t)}{y} \geq \lambda_n \sum_{i=1}^{m} w_i(t)$$

for some $n \in \mathbb{N}_0$ and all $y \neq 0$. Then BVP (3.0.1), (1.4.2) has no solution in $\mathcal{T}_j^\gamma$ for all $j \leq n$ and $\gamma \in \{+,-\}$.

3.2.2 Proofs

To prove Theorem 3.2.1, we need some preliminaries. We present three lemmas that are simple generalizations of results obtained for only one term, $w(t)f(y)$, in Equation (3.0.1). In each lemma, we consider multiple terms, $\sum_{i=1}^{m} w_i(t)f_i(y)$, on the right-hand side of Equation (3.0.1). The first lemma is a generalization of a combination of Propositions 3.1 and 3.2 and Corollary 3.1 in [28].

Lemma 3.2.1. Any initial value problem associated with Eq. (3.0.1) has a unique solution which exists on the whole interval $[a, b]$. Consequently, the solution depends continuously on the initial condition.

As a consequence we have the following:

Corollary 3.2.4. For any nontrivial solution y of Eq. (3.0.1), y and py' have only simple zeros in $[a, b]$.

Proof of Corollary 3.2.4. Assume y has a nonsimple zero $t_0 \in [a, b]$, i.e., $y(t_0) = y'(t_0) = 0$. Then from Lemma 3.2.1, $y(t) \equiv 0$ on $[a, b]$, a contradiction. Assume py' has nonsimple zero $t_0 \in [a, b]$; i.e., $py'(t_0) = (py')'(t_0) = 0$. For $y(t_0) \neq 0$, from Eq. (3.0.1), $\sum_{i=1}^{m} w_i(t_0) \frac{f_i(y(t_0))}{y(t_0)} - q(t_0) = 0$. By (H4), this implies that $y(t_0) = 0$. Then as above, $y(t) \equiv 0$ on $[a, b]$, a contradiction. □

For $\gamma \in \{+, -\}$, let $y(t, \rho)$ be the solution of Eq. (3.0.1) satisfying

$$y(a) = \gamma\rho \sin\alpha \quad \text{and} \quad (py')(a) = \gamma\rho\cos\alpha, \tag{3.2.7}$$

where $\rho > 0$ is a parameter. Let $\theta(t, \rho)$ be the Prüfer angle of $y(t, \rho)$; i.e., $\theta(\cdot, \rho)$ is a continuous function on $[a, b]$ such that

$$\tan\theta(t, \rho) = y(t, \rho)/(py')(t, \rho) \quad \text{and} \quad \theta(a, \rho) = \alpha.$$

By Lemma 3.2.1, $\theta(t, \rho)$ is continuous in ρ on $(0, \infty)$ for any $t \in [a, b]$. The following results are from generalizations of Lemmas 4.1, 4.2, 4.4, and 4.5 in [28].

Lemma 3.2.2. (i) Assume

$$\sum_{i=1}^{m} w_i(t)(f_i)_0 < \lambda_n \sum_{i=1}^{m} w_i(t)$$

for some $n \in \mathbb{N}_0$. Then there exists $\rho_* > 0$ such that $\theta(b, \rho) < n\pi + \pi/2$ for all $\rho \in (0, \rho_*)$.

(ii) Assume

$$\lambda_n \sum_{i=1}^{m} w_i(t) < \sum_{i=1}^{m} w_i(t)(f_i)_\infty$$

for some $n \in \mathbb{N}_0$. Then there exists $\rho^* > 0$ such that $\theta(b, \rho) > n\pi + \pi/2$ for all $\rho \in (\rho^*, \infty)$.

Lemma 3.2.3. (i) Assume

$$\sum_{i=1}^{m} w_i(t)(f_i)_\infty < \lambda_n \sum_{i=1}^{m} w_i(t)$$

for some $n \in \mathbb{N}_0$. Then there exists $\rho^* > 0$ such that $\theta(b, \rho) < n\pi + \pi/2$ for all $\rho \in (\rho^*, \infty)$.

(ii) Assume

$$\lambda_n \sum_{i=1}^{m} w_i(t) < \sum_{i=1}^{m} w_i(t)(f_i)_0$$

for some $n \in \mathbb{N}_0$. Then there exists $\rho_* > 0$ such that $\theta(b, \rho) > n\pi + \pi/2$ for all $\rho \in (0, \rho_*)$.

Proof of Theorem 3.2.1. We first prove it for the case where $\sum_{i=1}^{m} w_i(t)(f_i)_0 < \lambda_n \sum_{i=1}^{m} w_i(t)$ and $\lambda_{n+1} \sum_{i=1}^{m} w_i(t) < \sum_{i=1}^{m} w_i(t)(f_i)_\infty$. Without loss of generality we assume $\gamma = +$. The case with $\gamma = -$ can be proved in the same way. Let $y(t, \rho)$ be the solution of Eq. (3.0.1) satisfying (3.2.6) with $\gamma = +$ and $\theta(t,\rho)$ its Prüfer angle. By Lemma 3.2.2, there exist $0 < \rho_* < \rho^* < \infty$ such that

$$\theta(b,\rho) < n\pi + \pi/2 \quad \text{for all } \rho \in (0,\rho_*)$$

and

$$\theta(b,\rho) > (n+1)\pi + \pi/2 \quad \text{for all } \rho \in (\rho^*,\infty).$$

By the continuity of $\theta(t,\rho)$ in ρ, there exist $\rho_* \le \rho_n < \rho_{n+1} \le \rho^*$ such that

$$\theta(b,\rho_n) = n\pi + \pi/2 \quad \text{and} \quad \theta(b,\rho_{n+1}) = (n+1)\pi + \pi/2 \tag{3.2.8}$$

and

$$n\pi + \pi/2 < \theta(b,\rho) < (n+1)\pi + \pi/2 \quad \text{for } \rho_n < \rho < \rho_{n+1}. \tag{3.2.9}$$

Then, for all $t \in [a,b]$ and all $\rho > 0$, we define an energy function $E(t,\rho)$ for $y(t,\rho)$ by

$$E(t,\rho) = \frac{1}{2p(t)}\big(p(t)y'(t,\rho)\big)^2 + \frac{1}{2}\big(q^* - q(t)\big)y^2(t,\rho) + \sum_{i=1}^{m} w_i(t)F_i\big(y(t,\rho)\big), \tag{3.2.10}$$

where $F_i(y) = \int_0^y f_i(s)ds$ for each $i = 1..m$. By (H1) and (H2), $F_i(y) \ge 0$ on $\mathbb{R}$ for each $i = 1..m$, yielding $E(t,\rho) \ge 0$ on $[a,b]$. For ease of notation, in the following, we use $p = p(t)$, $q = q(t)$, $w_i = w_i(t)$, $l = l(t)$, and $y = y(t,\rho)$, $E = E(t,\rho)$. Then, by (3.0.1) and (H1), we find that

$$\begin{aligned}
E' &= -\frac{p'}{2p^2}(py')^2 - \frac{1}{2}q'y^2 + q^*yy' + \sum_i^m w_i'F_i(y) \\
&\geq -\frac{p'}{2p^2}(py')^2 - \frac{1}{2}q'y^2 - \frac{q^*}{2}\big(y^2 + y'^2\big) + \sum_{i=1}^m w_i'F_i(y) \\
&= -\frac{(p'+q^*)}{2p^2}(py')^2 - \frac{1}{2}(q'+q^*)y^2 + \sum_{i=1}^m w_i'F_i(y) \\
&\geq -\Big(\frac{p'+q^*}{p}\Big)_+\Big(\frac{1}{2p}(py')^2\Big) - l\Big(\frac{1}{2}(q^*-q)y^2\Big) - \sum_{i=1}^m \frac{w'_{i_-}}{w_i}w_iF_i(y) \\
&\geq -l\frac{(py')^2}{2p} - l\Big(\frac{1}{2}(q^*-q)y^2\Big) - l\sum_{i=1}^m w_iF_i(y) \\
&= -lE(t,\rho).
\end{aligned}$$

Thus, $E'(t,\rho) + lE(t,\rho) \geq 0$ for all $t \in [a,b]$ and $\rho > 0$. By solving this inequality, we obtain

$$E(s,\rho) \leq E(b,\rho)e^{\int_s^b l(\tau)d\tau} \leq E(b,\rho)e^{k_0}, \quad s \in [a,b]. \tag{3.2.11}$$

We observe that for $\rho = \rho_n$ and $\rho = \rho_{n+1}$,

$$E(s,\rho) \geq \frac{1}{2}(q^* - q(s))y(s,\rho)^2 + \sum_{i=1}^m w_i(s)F_i(y(s,\rho)), \quad s \in [a,b], \tag{3.2.12}$$

and

$$E(b,\rho) = \frac{1}{2}(q^* - q(b))y(b,\rho)^2 + \sum_{i=1}^m w_i(b)F_i(y(b,\rho)). \tag{3.2.13}$$

We note that H_+^{-1} is increasing and H_-^{-1} is decreasing. Thus, from (3.2.12) we see that for $\rho = \rho_n$ and $\rho = \rho_{n+1}$ and $s \in [a,b]$,

$$y(s,\rho) \leq H_+^{-1}(s, E(s,\rho)) \quad \text{if } y(s,\rho) \geq 0$$

and

$$-y(s,\rho) \le -H_-^{-1}(s, E(s,\rho)) \quad \text{if } y(s,\rho) \le 0.$$

Therefore, by (3.2.3),

$$|y(s,\rho)| \le H_{\max}^{-1}(s, E(s,\rho)). \tag{3.2.14}$$

Define

$$\Gamma(\rho) = y(b,\rho) - \int_a^b y(s,\rho)d(\xi(s)). \tag{3.2.15}$$

Let $n = 2k$ with $k \in \mathbb{N}_0$. Since $y(b,\rho_{2k}) > 0$ and $y(b,\rho_{2k+1}) < 0$, by (3.2.11), (3.2.12), and (3.2.6),

$$\begin{aligned}
\Gamma(\rho_{2k}) &= y(b,\rho_{2k}) - \int_a^b y(s,\rho_{2k})d(\xi(s)) \\
&\ge y(b,\rho_{2k}) - \int_a^b |y(s,\rho_{2k})|d(\xi_1(s)+\xi_2(s)) \\
&\ge H_+^{-1}(b, E(b,\rho_{2k})) - \int_a^b H_{\max}^{-1}(s, E(s,\rho_{2k}))d(\xi_1(s)+\xi_2(s)) \\
&\ge H_+^{-1}(b, E(b,\rho_{2k})) - \int_a^b H_{\max}^{-1}(s, E(b,\rho_{2k})e^{k_0})d(\xi_1(s)+\xi_2(s)) > 0 \\
&> H_+^{-1}(b, E(b,\rho_{2k})) - H_{\min}^{-1}(b, E(b,\rho_{2k})) \ge 0
\end{aligned} \tag{3.2.16}$$

and

$$\begin{aligned} & \Gamma(\rho_{2k+1}) = y(b,\rho_{2k+1}) - \int_a^b y(s,\rho_{2k+1})d(\xi(s)) \\ \leq\ & y(b,\rho_{2k+1}) + \int_a^b |y(s,\rho_{2k+1})|d(\xi_1(s)+\xi_2(s)) \\ \leq\ & H_-^{-1}(b,E(b,\rho_{2k+1})) + \int_a^b H_{\max}^{-1}(s,E(s,\rho_{2k+1}))d(\xi_1(s)+\xi_2(s)) \\ \leq\ & H_-^{-1}(b,E(b,\rho_{2k+1})) + \int_a^b H_{\max}^{-1}(s,E(b,\rho_{2k+1})e^{k_0})d(\xi_1(s)+\xi_2(s)) \\ <\ & H_-^{-1}(b,E(b,\rho_{2k+1})) + H_{\min}^{-1}(b,E(b,\rho_{2k+1})) \leq 0. \end{aligned} \tag{3.2.17}$$

By the continuity of $\Gamma(\rho)$, there exists $\bar{\rho} \in (\rho_{2k}, \rho_{2k+1})$ such that $\Gamma(\bar{\rho}) = 0$. Similarly, for $n = 2k+1$ with $k \in \mathbb{N}_0$, there exists $\bar{\rho} \in (\rho_{2k+1}, \rho_{2k+2})$ such that $\Gamma(\bar{\rho}) = 0$. In both cases, from (3.2.9),

$$n\pi + \pi/2 < \theta(b,\bar{\rho}) < (n+1)\pi + \pi/2.$$

Since

$$\theta'(t,\rho) = \frac{1}{p(t)}\cos^2\theta(t,\rho) + \sum_{i=1}^m w_i(t)\frac{f_i(y(t,\rho))y(t,\rho)}{r^2(t,\rho)} - q(t)\sin^2\theta(t,\rho), \tag{3.2.18}$$

where $r = (y^2 + (py')^2)^{1/2}$, by (H4), we have that $\theta(\cdot,\rho)$ is strictly increasing on $[a,b]$. We note that $y(t) = 0$ if and only if $\theta(t,\rho) = 0 \pmod{\pi}$ and $(py')(t) = 0$ if and only if $\theta(t,\rho) = \pi/2 \pmod{\pi}$. Thus, py' has exactly $n+1$ zeros in (a,b) if $\alpha \in [0,\pi/2)$ and n zeros in (a,b) if $\alpha \in [\pi/2,\pi)$, and y has exactly one zero strictly between any two consecutive zeros of py'. Initial condition (3.2.7) implies that $y(t,\bar{\rho}) > 0$ in a right-neighborhood of a. Therefore, $y(t,\bar{\rho}) \in \mathcal{T}_n^+$.

The proof for the case where $\sum_{i=1}^{m} w_i(t)(f_i)_\infty < \lambda_n \sum_{i=1}^{m} w_i(t)$ together with $\lambda_{n+1} \sum_{i=1}^{m} w_i(t) < \sum_{i=1}^{m} w_i(t)(f_i)_0$ is essentially the same as above except that the discussion is based on Lemma 3.2.3 instead of Lemma 3.2.2. □

Proof of Corollary 3.2.2. Without loss of generality we let $\gamma = +$. The case for $\gamma = -$ can be proved in the same way. Let $y(t, \rho)$ be the solution of Eq. (3.0.1) satisfying (3.2.6) with $\gamma = +$ and $\theta(t, \rho)$ its Prüfer angle. Let ρ_0 and ρ_1 be given in (3.2.8) with $n = 0$. Then for the function $\Gamma(\rho)$ defined by (3.2.15), from (3.2.16) and (3.2.17) we have $\Gamma(\rho_0) > 0$ and $\Gamma(\rho_1) < 0$.

(i) Assume $\xi(s)$ is increasing on $[a, b]$. Then $\xi(s) = \xi_1(s) - \xi_2(s)$ implies that we may take $\xi_2(s) \equiv 0$, and hence $d(\xi(s)) \geq 0$ on $[a, b]$. By the mean value theorem for the Riemann-Stieltjes integral and the continuity of solution y, there exists $t_0 \in [a, b]$ such that $\int_a^b y(s)d(\xi(s)) = y(t_0) \int_a^b d(\xi(s))$. By the continuity of $\theta(t, \rho)$ in ρ, there exists $\tilde{\rho} \in (\rho_0, \rho_1)$ such that $\theta(b, \tilde{\rho}) = \pi$ and $\theta(b, \rho) < \pi$ for $\rho \in (\rho_0, \tilde{\rho})$. Since $\theta(s, \tilde{\rho}) < \pi$ for $s \in [a, b]$, we see that

$$\Gamma(\tilde{\rho}) = -\int_a^b y(s, \tilde{\rho})d(\xi(s)) = -y(t_0, \tilde{\rho}) \int_a^b d(\xi(s)) \leq 0.$$

Therefore, there exists $\bar{\rho} \in (\rho_0, \tilde{\rho}]$ such that $\Gamma(\bar{\rho}) = 0$. This means that $y(t, \bar{\rho}) \in \mathcal{T}_0^+$ and is a positive solution.

(ii) Assume $\xi(s)$ is decreasing on $[a, b]$ such that $\int_a^{b-\epsilon} d\xi(s) < 0$ for some $\epsilon > 0$. Since $\xi(s)$ is decreasing, we may take $\xi_1(s) \equiv 0$. As in (i), there exists $\tilde{\rho} \in (\rho_0, \rho_1)$ such that $\theta(b, \tilde{\rho}) = \pi$ and $\theta(b, \rho) > \pi$ for $\rho \in (\tilde{\rho}, \rho_1)$. Furthermore, for some $t_0 \in [a, b]$,

$$\Gamma(\tilde{\rho}) = -\int_a^b y(s, \tilde{\rho})d(\xi(s)) = -y(t_0, \tilde{\rho}) \int_a^b d(\xi(s)) \geq -y(t_0, \tilde{\rho}) \int_a^{b-\epsilon} d(\xi(s)) > 0.$$

Therefore, there exists $\bar{\rho} \in (\tilde{\rho}, \rho_1)$ such that $\Gamma(\bar{\rho}) = 0$. This means that $\theta(t, \bar{\rho}) \in (\pi, 3\pi/2)$ and hence $y(t, \bar{\rho}) \in \mathcal{T}_0^+$ and has exactly one zero in (a, b). □

Proof of Corollary 3.2.3. It is easy to see that $\lambda_0 = 0$ is the first eigenvalue of the BVP consisting of Eq. (3.1.1) and the BC

$$y'(a) = y'(b) = 0,$$

i.e., the BC (3.1.2) with $\alpha = \pi/2$. In fact, $y_0(t) \equiv 1$ is an associated eigenfunction. From Theorem 4.2 in [31] we see that λ_0 as a function of α is strictly decreasing. This shows that $\lambda_0 > 0$ for $\alpha \in [0, \pi/2)$. By Theorem 3.2.1 we see that BVP (3.0.1), (1.4.2) has a solution $y_n^\gamma \in \mathcal{T}_n^\gamma$ for all $n \geq 0$ and $\gamma \in \{+, -\}$. □

Proof of Theorem 3.2.2. (i) Assume to the contrary that BVP (3.0.1), (1.4.2) has a solution $y \in \mathcal{T}_j^\gamma$ for some $j \geq n$ and $\gamma \in \{+, -\}$. Let $\tilde{w}(t) = \sum_{i=1}^m w_i(t) f_i(y(t))/y(t)$. Then $\tilde{w}(t)$ is continuous on $[a, b]$ by the continuous extension since $(f_i)_0 < \infty$ for each $i = 1 : m$. Let $\theta(t)$ be the Prüfer angle of $y(t)$ with $\theta(a) = \alpha$. Then $\theta(t)$ satisfies Eq. (3.2.13) and hence is strictly increasing on $[a, b]$. Note from the assumption that $\tilde{w}(t) < \lambda_j \sum_{i=1}^m w_i(t)$ on $[a, b]$, we have

$$\theta'(t) < \cos^2 \theta(t) + \lambda_j w(t) \sin^2 \theta(t).$$

Let $u(t)$ be an eigenfunction of SLP (3.1.1), (3.1.2) associated with the eigenvalue λ_j and $\phi(t)$ its Prüfer angle with $\phi(a) = \alpha$. Then

$$\phi'(t) = \cos^2 \phi(t) + \lambda_j w(t) \sin^2 \phi(t)$$

and $\phi(b) = j\pi + \pi/2$. By the theory of differential inequalities we find that $\theta(b) < \phi(b)$. This contradicts the assumption that $y \in \mathcal{T}_j^\gamma$ since condition (ii) of the definition of $\mathcal{T}_j^\gamma$ is violated.

(ii) It is similar to (i) and hence is omitted. □

3.3 Dependence of Nodal Solutions on the Problem

In this section, we investigate the changes of the existence of different types of nodal solutions of BVP (3.0.1), (1.4.3) as the problem changes. Our work is based on the following lemma for the dependence of the nth eigenvalue of SLP (3.1.1), (3.1.2) on the problem which can be excerpted from [30, Theorems 2.2 and 2.3], [31, Theorem 4.2], and [29, Lemma 3.32]. We assume throughout the remainder of the this paper, and without further mention, that $w(t) = \sum_{i=1}^{m} w_i(t)$. Clearly, $w \in C^1[a,b]$ with $w(t) > 0$ on $[a,b]$.

Lemma 3.3.1. For any $n \in \mathbb{N}_0$, we have the following conclusions:

(a) Consider the nth eigenvalue of SLP (3.1.1), (3.1.2) as a function of b for $b \in (a,\infty)$, denoted by $\lambda_n(b)$. Then $\lambda_n(b) \to \infty$ as $b \to a^+$.

(b) Consider the nth eigenvalue of SLP (3.1.1), (3.1.2) as a function of w for $w \in C^1[a,b]$, denoted by $\lambda_n(w)$. Then $\lambda_n(w)$ is decreasing as long as it is positive; i.e., for $w_1 = \sum_i w_i, w_2 = \sum_j w_j \in C^1[a,b]$ such that $w_1(t) \le w_2(t)$ for $t \in [a,b]$, we have $\lambda_n(w_1) \ge \lambda_n(w_2)$ as long as $\min\{\lambda_n(w_1), \lambda_n(w_2)\} \ge 0$.

(c) Consider the nth eigenvalue of SLP (3.1.1), (3.1.2) as a function of q for $q \in C^1[a,b]$, denoted by $\lambda_n(q)$. Then $\lambda_n(q)$ is increasing; i.e., for $q_1, q_2 \in C^1[a,b]$ such that $q_1(t) \le q_2(t)$ for $t \in [a,b]$, we have $\lambda_n(q_1) \le \lambda_n(q_2)$.

(d) Consider the nth eigenvalue of SLP (3.1.1), (3.1.2) as a function of $1/p$ for $1/p \in C^1[a,b]$, denoted by $\lambda_n(1/p)$. Then $\lambda_n(1/p)$ is decreasing; i.e., for $1/p_1, 1/p_2 \in C^1[a,b]$ such that $1/p_1(t) \leq 1/p_2(t)$ for $t \in [a,b]$, we have $\lambda_n(1/p_1) \geq \lambda_n(1/p_2)$.

(e) Consider the nth eigenvalue of SLP (3.1.1), (3.1.2) as a function of the boundary condition angle α, denoted by $\lambda_n(\alpha)$. Then $\lambda_n(\alpha)$ is a continuous and decreasing function on $[0,\pi)$. Furthermore,

$$\lim_{\alpha \to \pi^-} \lambda_0(\alpha) = -\infty \quad \text{and} \quad \lim_{\alpha \to \pi^-} \lambda_{n+1}(\alpha) = \lambda_n(0) \;\; \text{for} \;\; n \geq 1.$$

The first result is about the changes as the interval $[a,b]$ shrinks, more precisely, as $b \to a^+$. We discuss both the cases when one of f_0 and f_∞ is infinite and when both of them are finite.

Theorem 3.3.1. Let Eq. (3.0.1) and BC (1.4.3) be fixed and let (3.1.3) hold.

(i) Assume either $(f_i)_0 < \infty$ for all $i \in \{1,2,...,m\}$ and $(f_j)_\infty = \infty$ for some $j \in \{1,2,...,m\}$ or $(f_i)_\infty < \infty$ for all $i \in \{1,2,...,m\}$ and $(f_j)_0 = \infty$ for some $j \in \{1,2,...,m\}$. Then for any $n \in \mathbb{N}_0$, there exists $b_n > a$ such that for any $b \in (a,b_n)$ and for any $k \geq n$, BVP (3.0.1), (1.4.3) has a solution $y_k^\gamma \in \mathcal{S}_{k+1}^\gamma$ for $\gamma \in \{+,-\}$.

(ii) Assume $(f_i)_0 < \infty$ and $(f_i)_\infty < \infty$ for all $i \in \{1,2,...,m\}$. Then for any $n \in \mathbb{N}_0$, there exists $b_n > a$ such that for any $b \in (a,b_n)$ and for any $k \geq n+1$, BVP (3.0.1), (1.4.3) has no solutions in $\mathcal{S}_k^\gamma$ for $\gamma \in \{+,-\}$.

Proof of Theorem 3.3.1. (i) Without loss of generality, assume $(f_i)_0 < \infty$ for all $i \in \{1,2,...,m\}$ and $(f_j)_\infty = \infty$ for some $j \in \{1,2,...,m\}$. Let $\lambda_n(b)$ be defined as in Lemma 3.3.1 (a). By Lemma 3.3.1 (a), for any $n \in \mathbb{N}_0$, there exists $b_n > a$ such that for any $b \in (a, b_n)$, we have

$$\sum_{i=1}^{m} w_i(t)(f_i)_0 < \lambda_n(b) \sum_{i=1}^{m} w_i(t) < \sum_{i=1}^{m} w_i(t)(f_i)_\infty$$

and hence

$$\sum_{i=1}^{m} w_i(t)(f_i)_0 < \lambda_k(b) \sum_{i=1}^{m} w_i(t) < \sum_{i=1}^{m} w_i(t)(f_i)_\infty$$

for all $k \geq n$. Then the conclusion follows from Theorem 3.1.1.

(ii) By Lemma 3.3.1 (a), for any $n \in \mathbb{N}$, there exists $b_n > a$ such that for any $b \in (a, b_n)$, we have that

$$\lambda_n(b) > f^* := \sup\{f_i(y)/y : i \in \{1, 2, ..., m\} \text{ and } y \in (0, \infty)\}.$$

Then the conclusion follows from Theorem 3.1.2 (i). □

We then present a result on the nonexistence of certain types of nodal solutions of BVP (3.0.1), (1.4.3) as the function w increases in a given direction. More precisely, let $s \geq 0$ and $h \in C^1[a, b]$ such that $h(t) > 0$ on $[a, b]$, and consider the equation

$$-(p(t)y')' + q(t)y = \sum_{i=1}^{m} \left(w_i(t) + s\frac{h(t)}{m} \right) f_i(y). \tag{3.3.1}$$

Theorem 3.3.2. Let the interval $[a, b]$ and BC (1.4.3) be fixed and let (3.1.3) hold. Assume $f_i(y)/y \geq f_* > 0$ for all $y \neq 0$ and $i \in \{1, 2, ..., m\}$. Then for any $n \in \mathbb{N}_0$, there exists $s_n \geq 0$ such that for any $s > s_n$ and for any $k \leq n$, BVP (3.3.1), (1.4.3) has no solution in $\mathcal{S}_k^\gamma$ for $\gamma \in \{+, -\}$.

Proof of Theorem 3.3.2. For $s \geq 0$ and $k \in \mathbb{N}_0$, we denote by $\lambda_k(s)$ the kth eigenvalue of the SLP consisting of the equation

$$-(p(t)y')' + q(t)y = \lambda\big[\sum_{i=1}^{m}\big(w_i(t) + s\frac{h(t)}{m}\big)\big]y = \lambda\big[w(t) + sh(t)\big]y$$

and BC (3.1.2). Let $h_* = \min\{h(t)/w(t) : t \in [a,b]\}$, and denote by $\mu_k(s)$ the kth eigenvalue of the SLP consisting of the equation

$$-(p(t)y')' + q(t)y = \mu(1 + sh_*)w(t)y$$

and BC (3.1.2). Since

$$w(t) + sh(t) \geq (1 + sh_*)w(t) \quad \text{for } s \geq 0,$$

by Lemma 3.3.1 (b),

$$\lambda_k(s) \leq \mu_k(s) \ \text{ for all } s \geq 0 \text{ and } k \geq 0, \text{ whenever } \lambda_k(s) \geq 0. \tag{3.3.2}$$

Note that for $k \geq 0$, $\mu_k(s)(1 + sh_*) = \mu_k(0)$, we have

$$\mu_k(s) = \frac{\mu_k(0)}{1 + sh_*} \to 0 \ \text{ as } s \to \infty.$$

This together with (3.3.2) implies that $\lambda_k(s) < f_*$ as $s \to \infty$. Then, for any $n \in \mathbb{N}_0$, there exists $s_n \geq 0$ such that $\lambda_n(s) < f_*$ for $s > s_n$. Therefore, the conclusion follows from Theorem 3.3.2 (ii). □

The next result is on the nonexistence and existence of certain types of nodal solutions of BVP (3.0.1), (1.4.3) as the function q changes in a given direction. More precisely, let $s \in \mathbb{R}$ and $h \in C^1[a,b]$ such that $h(t) > 0$ on $[a,b]$, and consider the equation

$$-(p(t)y')' + \big[q(t) + sh(t)\big]y = \sum_{i=1}^{m} w_i(t) f_i(y). \tag{3.3.3}$$

Theorem 3.3.3. Let the interval $[a,b]$ and BC (1.4.3) be fixed and let (3.1.3) hold.
(i) For any $n \in \mathbb{N}_0$, there exists $s_n \le 0$ such that for any $s < s_n$ and for any $k \le n$, BVP (3.3.3), (1.4.3) has no solutions in $\mathcal{S}_k^\gamma$ for $\gamma \in \{+,-\}$.
(ii) Assume either

$(f_i)_0 < \infty$ for all $i \in \{1, 2, ..., m\}$ and $(f_j)_\infty = \infty$ for some $j \in \{1, 2, ..., m\}$

or

$(f_i)_\infty < \infty$ for all $i \in \{1, 2, ..., m\}$ and $(f_j)_0 = \infty$ for some $j \in \{1, 2, ..., m\}$.

Then for any $n \in \mathbb{N}_0$, there exists $s_n \ge 0$ such that for any $s > s_n$ and for any $k \ge n$, BVP (3.3.3), (1.4.3) has two solutions $y_{k,\gamma} \in \mathcal{S}_{k+1}^\gamma$ for $\gamma \in \{+,-\}$.

(iii) Assume $(f_i)_0 < \infty$ and $(f_i)_\infty < \infty$ for all $i \in \{1, 2, ..., m\}$. Then for any $n \in \mathbb{N}_0$, there exists $s_* \ge 0$ such that for any $s > s_*$, BVP (3.3.3), (1.4.3) has no solution in $\mathcal{S}_k^\gamma$ for all $k \ge n+1$ and $\gamma \in \{+,-\}$.

Proof of Theorem 3.3.3. For $s \in \mathbb{R}$ and $i \in \mathbb{N}_0$, we denote by $\lambda_k(s)$ the kth eigenvalue of the SLP consisting of the equation

$$-(p(t)y')' + \big[q(t) + sh(t)\big]y = \lambda \sum_{i=1}^{m} w_i(t) y = \lambda w(t) y$$

and BC (3.1.2). Let $h_* = \min\{h(t)/w(t) : t \in [a,b]\}$, and denote by $\mu_k(s)$ the kth eigenvalue of the SLP consisting of the equation

$$-(p(t)y')' + [q(t) + sh_*(t)w(t)]y = \mu w(t)y \tag{3.3.4}$$

and BC (3.1.2).

(i) Since for $s \leq 0$,

$$q(t) + sh(t) \leq q(t) + sh_* w(t),$$

by Lemma 3.3.1 (c), $\lambda_k(s) \leq \mu_k(s)$ for all $s \leq 0$ and $k \geq 0$. Note that Eq. (3.3.4) yields

$$-(p(t)y')' + q(t)y = (\mu - sh_*)w(t)y.$$

Thus, for $s \leq 0$ and $k \geq 0$, $\mu_k(0) = \mu_k(s) - sh_*$, which implies that

$$\mu_k(s) = \mu_k(0) + sh_* \to -\infty \quad \text{as } s \to -\infty,$$

and hence $\lambda_k(s) \to -\infty$ as $s \to -\infty$ for all $k \geq 0$. Then, for any $n \in \mathbb{N}_0$ there exists $s_n \leq 0$ such that $\lambda_n < 0$ for all $s < s_n$. Therefore, the conclusion follows from Theorem 3.1.2 (ii).

(ii) Without loss of generality, assume $(f_i)_0 < \infty$ for all $i \in \{1, 2, ..., m\}$ and $(f_j)_\infty = \infty$ for some $j \in \{1, 2, ..., m\}$. Similar to the argument in (i), we have $\lambda_k(s) \to \infty$ as $s \to \infty$ for all $k \geq 0$. Then for any $n \in \mathbb{N}_0$ there exists $s_n \geq 0$ such that for any $s > s_n$ we have

$$\sum_{i=1}^{m} w_i(t)(f_i)_0 < \lambda_n(s) \sum_{i=1}^{m} w_i(t) < \sum_{i=1}^{m} w_i(t)(f_i)_\infty$$

and hence

$$\sum_{i=1}^{m} w_i(t)(f_i)_0 < \lambda_k(s) \sum_{i=1}^{m} w_i(t) < \sum_{i=1}^{m} w_i(t)(f_i)_\infty$$

for all $k \geq n$. Therefore, the conclusion follows from Theorem 3.1.1.

(iii) As we can see from Part (ii), for any $n \in \mathbb{N}_0$, there exists $s_* \geq 0$ such that for all $s > s_*$ we have $\lambda_n(s) > f^* := \sup\{f_i(y)/y : y \neq 0 \text{ and } i = 1, 2, ..., m\}$. Thus, the conclusion follows from Theorem 3.1.2 (i). □

Similar to Theorem 3.3.2, we show a result on the nonexistence of certain types of nodal solutions of BVP (3.0.1), (1.4.3) as the function $1/p(t)$ increases in a certain direction. More precisely, let $s \geq 0$ and $h \in C[a, b]$ such that $h(t) > 0$ on $[a, b]$, and consider the equation

$$-\left(\frac{1}{1/p(t) + sh(t)} y'\right)' + q(t)y = \sum_{i=1}^{m} w_i(t) f_i(y). \qquad (3.3.5)$$

Theorem 3.3.4. Let the interval $[a, b]$ and BC (1.4.3) be fixed and let (1.3.4) hold. Define $\hat{q} := \max\{q(t)/w(t) : t \in [a, b]\}$ and assume $f_i(y)/y \geq f_* > \hat{q}$ for all $i \in \{1, 2, ..., m\}$ and $y \neq 0$. Then for any $n \in \mathbb{N}_0$, there exists $s_n \geq 0$ such that for any $s > s_n$, BVP (3.3.5), (1.4.3) has no solution in $\mathcal{S}_k^\gamma$ for all $k \leq n$ and $\gamma \in \{+, -\}$.

Proof of Theorem 3.3.4. For $s \geq 0$ and $k \in \mathbb{N}_0$, we denote by $\lambda_k(s)$ the kth eigenvalue of the SLP consisting of the equation

$$-\left(\frac{1}{1/p(t) + sh(t)} y'\right)' + q(t)y = \lambda \sum_{i=1}^{m} w_i(t) y = \lambda w(t) y$$

and BC (3.1.2) with an eigenfunction $u_k(t,s)$. Let $\theta_k(t,s)$ be the Prüfer angle of $u_k(t,s)$ satisfying $\theta_k(a,s) = \alpha$. Then

$$\theta_k'(t,s) = \left[\frac{1}{p(t)} + sh(t)\right]\cos^2\theta_k(t,s) + [\lambda_k w(t) - q(t)]\sin^2\theta_k(t,s). \tag{3.3.6}$$

By Lemma 3.3.1 (d), $\lambda_k(s)$ is decreasing and hence

$$\lim_{s\to\infty}\lambda_k(s) = \lambda_k^* \in [-\infty, \infty).$$

We show that $\lambda_k^* < f_*$ and then the conclusion follows from Theorem 3.1.2 (ii). Assume the contrary, i.e., $\lambda_k^* \geq f_*$. Let $w_* = \min\{w(t) : t \in [a,b]\}$. By (3.3.6),

$$\begin{aligned}
\theta_k'(t,s) &\geq \left[\frac{1}{p(t)} + sh(t)\right]\cos^2\theta_k(t,s) + [\lambda_k^* w(t) - q(t)]\sin^2\theta_k(t,s) \\
&= \left[\frac{1}{p(t)} + sh(t)\right]\cos^2\theta_k(t,s) + [\lambda_k^* - q(t)/w(t)]w(t)\sin^2\theta_k(t,s) \\
&\geq \left[\frac{1}{p(t)} + sh(t)\right]\cos^2\theta_k(t,s) + [f_* - \hat{q}]w_*\sin^2\theta_k(t,s).
\end{aligned}$$

Let $\phi(t,s)$ be the solution of the equation

$$\phi'(t,s) = \left[\frac{1}{p(t)} + sh(t)\right]\cos^2\phi(t,s) + [f_* - \hat{q}]w_*\sin^2\phi(t,s) \tag{3.3.7}$$

satisfying $\phi(a,s) = \alpha$. By the theory of differential inequalities, we have $\phi(t,s) \leq \theta_k(t,s)$. In particular,

$$\phi(b,s) \leq \theta_k(b,s) = (k+1)\pi. \tag{3.3.8}$$

We observe from (3.3.7) that $\phi(t,s)$ is strictly increasing in t and s, and $0 < \phi(t,s) \leq (k+1)\pi$ for $t \in [a,b]$ and $s \geq 0$. Let $\phi^*(t) = \lim_{s\to\infty} \phi(t,s)$. Then $0 < \phi^*(t) \leq (k+1)\pi$ for $t \in [a,b]$. We claim that

$$\phi^*(t) \not\equiv j\pi + \frac{\pi}{2} \quad \text{on} \quad (a,b] \tag{3.3.9}$$

for any $0 \leq j \leq k$. If not, for any $a_1 \in (a,b]$ and $\epsilon > 0$, there exists $s^* > 0$ such that for $s \geq s^*$,

$$\phi(a_1, s) \in (j\pi + \pi/2 - \epsilon, j\pi + \pi/2),$$

which yields that

$$\phi(t,s) \in (k\pi + \pi/2 - \epsilon, k\pi + \pi/2) \quad \text{for} \quad t \in [a_1, b].$$

This implies that

$$0 < \phi(b,s) - \phi(a_1,s) < \epsilon. \tag{3.3.10}$$

However, from (3.3.7), we see that for s sufficiently large,

$$\phi'(t,s) \geq \frac{1}{2}(f_* - \hat{q})w_* \quad \text{for} \quad t \in [a_1, b].$$

This contradicts (3.3.10) and hence verifies (3.3.9).

It is easy to see that $\phi(t,s) \to \phi^*(t)$ uniformly on $[a_1, b]$ as $s \to \infty$. Thus, $\phi^*(t)$ is continuous on $[a_1, b]$. From (3.3.9), we can find a nontrivial closed interval $[c,d] \subset [a,b]$ such that $\cos^2 \phi^*(t) \geq \nu > 0$ for $t \in [c,d]$. Then from (3.3.7),

$$\phi'(t,s) \geq \left[\frac{1}{p(t)} + sh(t)\right]\nu \to \infty \quad \text{uniformly for} \quad t \in [c,d] \quad \text{as} \quad s \to \infty.$$

Therefore,

$$\begin{aligned}\phi(b,s) &\geq \phi(d,s) \geq \phi(c,s) + \int_c^d \left[\frac{1}{p(t)} + sh(t)\right] \nu dt \\ &\geq \int_c^d \left[\frac{1}{p(t)} + sh(t)\right] \nu dt \to \infty \quad \text{as} \quad s \to \infty.\end{aligned}$$

This contradicts (3.3.8) and hence completes the proof. □

The last result is on the existence of certain types of nodal solutions of BVP (3.0.1), (1.4.3) as the boundary condition angle α changes.

Theorem 3.3.5. Let Eq. (3.0.1) and the interval $[a, b]$ be fixed and let (3.1.3) hold. Assume either

$$(f_i)_0 = 0 \text{ for all } i \in \{1, 2, ..., m\} \text{ and } (f_j)_\infty = \infty \text{ for some } j \in \{1, 2, ..., m\}$$

or

$$(f_i)_\infty = 0 \text{ for all } i \in \{1, 2, ..., m\} \text{ and } (f_j)_0 = \infty \text{ for some } j \in \{1, 2, ..., m\}.$$

For $n \in \mathbb{N}_0$ denote $\lambda_n(\alpha)$ the nth eigenvalue of the SLP (3.1.1), (3.1.2). Suppose k is the first nonnegative integer such that $\lambda_k(\alpha^*) > 0$ for some $\alpha^* \in (0, \pi)$. Then

(i) for $\alpha \in [0, \alpha^*)$, BVP (3.0.1), (1.4.3) has a solution $y_n^\gamma \in \mathcal{S}_{n+1}^\gamma$ for all $n \geq k$ and $\gamma \in \{+, -\}$;

(ii) for $\alpha \in [\alpha^*, \pi)$, BVP (3.0.1), (1.4.3) has a solution $y_n^\gamma \in \mathcal{S}_{n+1}^\gamma$ for all $n \geq k+1$ and $\gamma \in \{+, -\}$.

Proof of Theorem 3.3.5. By assumption, $\lambda_k(\alpha^*) > 0$. Then Lemma 3.3.1 (e) shows that $\lambda_k(\alpha) > 0$ for $\alpha \in [0, \alpha^*]$, and for $\alpha \in (\alpha^*, \pi)$, $\lambda_{k+1}(\alpha) > 0$. Therefore, the conclusion follows from Theorem 6.1. □

Remark 3.3.1. Theorems 3.3.1-3.3.5 show that we can "create" or "eliminate" certain types of nodal solutions by changing the interval $[a, b]$, the coefficient functions q, p, w, and the boundary condition angle α. Since the eigenvalues of SLP (3.1.1), (3.1.2) can be easily computed using computer software such as that in [4], we are able to construct specific BVPs (3.0.1), (1.4.3) which have or do not have nodal solutions in $\mathcal{S}_n^\gamma$ for a prescribed $n \in \mathbb{N}_0$.

CHAPTER 4

PLAN FOR FURTHER STUDY

4.1 Statement of the Problems

Many possible scenarios present themselves for future study. For example, we would like to consider the BVPs consisting of the differential equation

$$-(p(t)y')' = w(t)f(y), \quad t \in (a,b), \tag{4.1.1}$$

and the BCs:

$$\begin{aligned} y(a) - \textstyle\int_a^b y(s)d\eta(s) = 0, \\ y(b) - \textstyle\int_a^b y(s)d\xi(s) = 0, \end{aligned} \tag{4.1.2}$$

$$\begin{aligned} (py')(a) - \textstyle\int_a^b (py')(s)d\eta(s) = 0, \\ (py')(b) - \textstyle\int_a^b (py')(s)d\xi(s) = 0. \end{aligned} \tag{4.1.3}$$

We would like to readily extend our results from Theorem 2.1.1 and Theorem 2.2.1 to similar theorems concerning the BVPs (4.1.1), (4.1.3) and (4.1.1), (4.1.2) respectively. In what follows, we assume the results of Thereom 2.1.1 hold for the BVP consisting of (4.1.1) and BCs:

$$\begin{aligned} &\cos\alpha \ y(a) - \sin\alpha \ (py')(a) = 0, \quad \alpha \in [0,\pi), \\ &(py')(b) - \textstyle\int_a^b (py')(s)d\xi(s) = 0. \end{aligned} \tag{4.1.4}$$

Similarly, we assume the results of Theorem 2.2.1 hold for the BVP consisting of (4.1.1) and the BCs:

$$\begin{aligned} &\cos\alpha \; y(a) - \sin\alpha \; (py')(a) = 0, \quad \alpha \in [0, \pi), \\ &y(b) - \textstyle\int_a^b y(s) d\xi(s) = 0. \end{aligned} \tag{4.1.5}$$

As in the previous chapters, the eigenvalues of the associated linear BVPs with separated BCs are used for comparison with f_0 and f_∞.

4.2 Ideas for Approaches

We show an outline of a strategy for the BVP (4.1.1), (4.1.3). The outline for the BVP (4.1.1), (4.1.2) would be similarly outlined and is omitted. The associated linear BVP for BVP (4.1.1), (4.1.3) is equation (4.1.1) and BC

$$\begin{aligned} &\cos\alpha \; y(a) - \sin\alpha \; (py')(a) = 0, \quad \alpha \in [0, \pi), \\ &y(b) = 0. \end{aligned} \tag{4.2.1}$$

By Theorem 4.1 in [31], the eigenvalues are differentiable functions of the parameters of the problem. Specifically, we have

$$\lambda'(a) = \frac{1}{p(a)} |py'(a)|^2 - |y(a)|^2 [(q(a) - \lambda(a) w(a)].$$

Since $q \equiv 0$, we have that λ is an increasing function with respect to the left endpoint, a. Thus, if we replace a with $c \in [a, b)$ in the first part of BC (4.2.1) we have $\lambda_n(c) > \lambda_n(a)$ for any $n \in \mathbb{N}_2$ and $c \in [a, b)$. Hence, if $f_0 < \lambda_n(a)$ and $f_\infty = \infty$, then $f_0 < \lambda_n(c)$ and $\lambda_{n+1}(c) < f_\infty$ for any $c \in [a, b)$.

Now for any $c \in [a,b)$, our strategy is to break up BC (4.1.4) into two separate BCs:

$$\begin{aligned} &\cos\alpha\ y(c) - \sin\alpha\ (py')(c) = 0, \quad \alpha \in [0,\pi), \\ &(py')(b) - \textstyle\int_a^b (py')(s) d\xi(s) = 0, \end{aligned} \tag{4.2.2}$$

$$\begin{aligned} &\cos\alpha\ y(c) - \sin\alpha\ (py')(c) = 0, \quad \alpha \in [0,\pi), \\ &(py')(a) - \textstyle\int_a^b (py')(s) d\eta(s) = 0. \end{aligned} \tag{4.2.3}$$

It is left for future work to verify that Theorem 2.1.1 applies to these two BVPs (4.1.1), (4.2.2) and (4.1.1), (4.2.3). As in Thereom 2.1.1, we want to use a generalized energy function and the shooting method. For the shooting method, we parameterize the initial condition at $c \in [a,b)$ with a parameter ρ. For $\gamma \in \{+,-\}$, let $y(t,\rho)$ be the solution of Eq. (4.1.1) satisfying

$$y(c) = \gamma\rho\sin\alpha \quad \text{and} \quad (py')(c) = \gamma\rho\cos\alpha, \tag{4.2.4}$$

where $\rho > 0$ is a parameter. We denote $\rho_b(c)$, the value of the ρ when b is fixed, and we choose the left endpoint $c \in [a,b)$; similarly, $\rho_a(c)$, the value of ρ when a is fixed, and we choose the right endpoint $c \in (c,b]$. It is left as future work to show that $\limsup_{c\to b} \rho_b(c) = \infty$ and $\limsup_{c\to a} \rho_a(c) = \infty$. If we can show this, then there exists c_* near a and c^* near b such that

$$\begin{aligned} &\rho_b(c) > \rho_a(c) \quad \text{for any} \quad c \in [c^*, b), \\ &\rho_a(c) > \rho_b(c) \quad \text{for any} \quad c \in (a, c_*]. \end{aligned}$$

It then follows that there is $\bar{c} \in (c_*, c^*)$ such that $\rho_a(\bar{c}) = \rho_b(\bar{c})$. Thus, as c assumes values from a to b, there will be at least one value $c = \bar{c}$ where the solutions of the two separate BVPs match.

In summary, if we can

- show Theorem 2.1.1 holds on BVPs (4.1.1), (4.2.2) and (4.1.1), (4.2.3) for any $c \in [a, b)$ and
- show $\limsup_{c \to b} \rho_b(c) = \infty$ and $\limsup_{c \to a} \rho_a(c) = \infty$,

then, using the fact that $f_0 < \lambda_n(c)$ and $\lambda_{n+1}(c) < f_\infty$ for any $c \in [a, b)$, we may proceed in similar fashion to prove a theorem similar to Theorem 2.1.1.

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