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Lyapunov-type inequalities and applications to boundary value problems

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ABSTRACT

LYAPUNOV-TYPE INEQUALITIES AND APPLICATIONS TO BOUNDARY VALUE PROBLEMS

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In this dissertation, we derive Lyapunov-type inequalities for integer and fractional order differential equations and use them to study the nonexistence, uniqueness, and existence-uniqueness criteria for several classes of boundary value problems.

First, we consider third-order half-linear differential equations of the form

$$\left(\phi_{\alpha_2}\left(\left(\phi_{\alpha_1}\left(x'\right)\right)'\right)\right)' + q\left(t\right)\phi_{\alpha_1\alpha_2}\left(x\right) = 0,$$

where $\phi_p(x) = |x|^{p-1}x$, and $\alpha_1, \alpha_2 > 0$. We obtain Lyapunov-type inequalities which utilize integrals of both $q_+(t)$ and $q_-(t)$ rather than those of $|q(t)|$ as in most papers in the literature. Furthermore, by combining these inequalities with the "uniqueness implies existence" theorems by many authors, we establish the uniqueness and hence existence-uniqueness for several classes of boundary value problems for third-order linear equations. This is the first time for Lyapunov-type inequalities to be used to deal with the existence-uniqueness of boundary value problems. These inequalities are further extended to higher order half-linear differential equations. Our results cover and improve many results in the literature when the equations become linear.

For the third-order linear differential equation $x''' + q(t)x = 0$, using the Green's function method in a subtle way, we obtain the sharpest Lyapunov-type inequalities in the literature. We further extend these inequalities to more general third-order and higher order linear differential equations. We also discuss their applications to the existence-uniqueness of boundary value problems.

Then we investigate boundary value problems for Riemann-Liouville fractional differential equations with certain fractional integral boundary conditions. Such boundary conditions are different from the widely considered pointwise conditions in the sense that they allow solutions to have singularities. We derive Lyapunov-type inequalities for linear fractional differential equations with order $\alpha \in (1, 2]$ and $\alpha \in (2, 3]$, respectively. Our results are good in the sense that they are consistent with the existing ones for the second-order and third-order problems when $\alpha = 2, 3$.

Finally, we establish some Lyapunov-type inequalities for Riemann-Liouville fractional differential equations with order $\alpha \in (2, 3]$ and certain pointwise or mixed boundary conditions. Results are first given for univariate case, and then extended to multivariate case. All the results are new and one of them extends and improves substantially the one in the literature for third-order multivariate boundary value problems.

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SOUGATA DHAR

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DEDICATION

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CHAPTER 1

INTRODUCTION

1.1 Background

For the second-order linear differential equation

$$x'' + q(t)x = 0, \tag{1.1.1}$$

where $q \in C([a,b], \mathbb{R})$, the following result is known as the Lyapunov inequality, see [58] and [6].

Theorem 1.1.1. *Assume $x(t)$ is a solution of Eq. (1.1.1) such that $x(a) = x(b) = 0$ and $x(t) \neq 0$ for $t \in (a,b)$. Then*

$$\int_a^b |q(t)|dt > \frac{4}{b-a}. \tag{1.1.2}$$

This result plays an important role in the study of various problems of Eq. (1.1.1) such as oscillation, disconjugacy, eigenvalue problems, and many other areas of differential equations. Due to its importance, the Lyapunov inequality has been improved and generalized in many forms. It was first noticed by Wintner [80] and later by several other authors that inequality (1.1.2) can be improved by replacing $|q(t)|$ by $q_+(t) := \max\{q(t), 0\}$, the nonnegative part of $q(t)$, to become

$$\int_a^b q_+(t)dt > \frac{4}{b-a}. \tag{1.1.3}$$

The Lyapunov inequality was extended by Hartman [37, Chap. XI] to the more general equation

$$(r(t)x')' + q(t)x = 0, \tag{1.1.4}$$

where $q, r \in C([a, b], \mathbb{R})$ such that $r(t) > 0$ for $t \in [a, b]$, as follows:

Theorem 1.1.2. *Assume $x(t)$ is a solution of Eq. (1.1.4) such that $x(a) = x(b) = 0$ and $x(t) \neq 0$ for $t \in (a, b)$. Then*

$$\int_a^b q_+(t)dt > \frac{4}{\int_a^b r^{-1}(t)dt}.$$

The above inequalities have been further improved by replacing $\int_a^b q_+(t)\,dt$ by some integrals of $q(t)$ on parts of or the whole interval $[a, b]$. In fact, under the assumptions of Theorem 1.1.1, Harris and Kong [36, Theorem 2.3] showed that there exist two disjoint subintervals I_1 and I_2 such that

$$\int_{I_1 \cup I_2} q(t)dt > \frac{4}{b-a} \quad \text{and} \quad \int_{[a,b]\setminus(I_1 \cup I_2)} q(t)dt \leq 0,$$

and Brown and Hinton [7] showed that

$$\Big| \int_a^b q(t)dt \Big| > \frac{4}{b-a}.$$

We note that the number 4 in the above inequalities is the best in the sense that if it is replaced by any larger number, then the inequalities fail to hold, see [37, p. 345] and [57] for examples.

Lyapunov-type inequalities have also been extended to half-linear differential equations by many authors, see [22, 77, 84]. In fact, Yang [84] obtained the following result for a Lyapunov-type inequality for the general second order half-linear equation

$$(r(t)\phi_p(x'))' + q(t)\phi_p(x) = 0, \tag{1.1.5}$$

where $q, r \in C([a,b],\mathbb{R})$ such that $r(t) > 0$ for $t \in [a,b]$, and $\phi_p(x) = |x|^{p-1}x$ for $p > 0$.

Theorem 1.1.3. *Assume Eq. (1.1.5) has a solution $x(t)$ such that $x(a) = x(b) = 0$ and $x(t) \neq 0$ for $t \in (a,b)$. Then*

$$\int_a^b q_+(t)dt > \frac{2^{p+1}}{(\int_a^b r^{-\frac{1}{p}}(t)dt)^p}.$$

In recent years, Lyapunov-type inequalities have also been developed for higher order linear and half-linear differential equations. In particular, Parhi and Panigrahi [65] established Lyapunov-type inequalities for the third-order linear differential equation

$$x''' + q(t)x = 0, \tag{1.1.6}$$

where $-\infty < a < b < c < \infty$ and $q \in C([a,c],\mathbb{R})$.

Theorem 1.1.4. *(a) Assume Eq.* (1.1.6) *has a solution $x(t)$ such that $x(a) = x(b) = 0$ and $x(t) \neq 0$ for $t \in (a,b)$. Suppose there exists a $\xi \in [a,b]$ such that $x''(\xi) = 0$. Then*

$$\int_a^b |q(t)|dt > \frac{4}{(b-a)^2}. \tag{1.1.7}$$

(b) Assume Eq. (1.1.6) *has a solution $x(t)$ with $x(a) = x(b) = x(c) = 0$ and $x(t) \neq 0$ for $t \in (a,b) \cup (b,c)$. Then*

$$\int_a^c |q(t)|dt > \frac{4}{(c-a)^2}. \tag{1.1.8}$$

Kisel'ak first established a Lyapunov-type inequality for the general third-order half-linear equation. In [53], he considered the following equation

$$(r_2(t)\phi_{\alpha_2}((r_1(t)\phi_{\alpha_1}(x'))'))' + q(t)\phi_{\alpha_1\alpha_2}(x(t)) = 0, \tag{1.1.9}$$

where $r_1, r_2, q \in C([a,b],\mathbb{R})$ such that $r_1(t), r_2(t) > 0$ for $t \in \mathbb{R}$, $\phi_p(x) = |x|^{p-1}x$, and $\alpha_1, \alpha_2 > 0$.

By changing the equation to a system of two equations, he obtained a Lyapunov-type inequality under the stronger assumption that $r_k \in C^{3-k}\left([a,b], (0,\infty)\right), k = 1, 2$. However, the obtained inequalities in [53] involves $|q(t)|$. We summarize the main results in [53] below.

Theorem 1.1.5. *Assume Eq. (1.1.9) has a solution $x(t)$ satisfying one of the following conditions:*

(a) $x(a) = x(b) = 0$, and there exists a $\xi \in [a,b]$ such that $\left(r_1^{-1}\phi_{\alpha_1}(x')\right)'(\xi) = 0$;

(b) $x(a) = x(c) = x(b) = 0$, $x(t) \neq 0$ for $t \in (a,c) \cup (c,b)$, and $\left(r_1^{-1}\phi_{\alpha_1}(x')\right)'(t) \neq 0$ for $t \in [a,c]$.

Then

$$2\left(\int_a^b |q(t)|\,dt\right)^{\frac{1}{\alpha_1\alpha_2}} > \min_{d\in[a,b]} h(d), \tag{1.1.10}$$

where

$$h(d) = \frac{1}{\left(\int_a^d r_1^{-\frac{1}{\alpha_1}}(t)\,dt\right)\left(\int_a^d r_2^{-\frac{1}{\alpha_2}}(t)\,dt\right)^{\frac{1}{\alpha_1}}} + \frac{1}{\left(\int_d^b r_1^{-\frac{1}{\alpha_1}}(t)\,dt\right)\left(\int_d^b r_2^{-\frac{1}{\alpha_2}}(t)\,dt\right)^{\frac{1}{\alpha_1}}}.$$

We note that inequality (1.1.10) is not only complicated, but also uses $|q(t)|$ rather than $q_\pm(t)$. In Chapter 2, we will improve the inequality to a sharper form using $q_\pm(t)$.

Yang [83] generalized the Lyapunov-type inequalities to the higher order linear equation

$$x^{(m)} + q(t)x = 0 \tag{1.1.11}$$

with $m \in \mathbb{N}$, $m \geq 2$ and $q \in C([a,b],\mathbb{R})$, respectively.

Theorem 1.1.6. *Let $m = 2n+1$ for $n \in \mathbb{N}$. Assume Eq. (1.1.11) has a solution $x(t)$ satisfying*

$$x^{(i)}(a) = x^{(i)}(b) = 0, \quad i = 0, 1, ..., n-1,$$

and $x(t) \neq 0$ on (a,b). Suppose there exists a $d \in (a,b)$ such that $x^{(2n)}(d) = 0$. Then

$$\int_a^b |q(t)|dt > \frac{n!2^{n+1}}{(b-a)^{2n}}.$$

Theorem 1.1.7. *Let $m = 2n$ for $n \in \mathbb{N}$. Assume Eq. (1.1.11) has a solution $x(t)$ satisfying*

$$x^{(i)}(a) = x^{(i)}(b) = 0, \quad i = 0, 1, ..., n-1.$$

and $x(t) \neq 0$ on (a,b). Then

$$\int_a^b |q(t)|dt > \frac{4^{2n-1}(2n-1)[(n-1)!]^2}{(b-a)^{2n-1}}.$$

Theorem 1.1.8. *Assume Eq. (1.1.11) has a solution $x(t)$ satisfying*

$$x(a) = x(t_2) = ... = x(t_{n-1}) = x(b) = 0,$$

where $a = t_1 < t_2 < ... < t_{n-1} < t_n = b$ *and* $x(t) \neq 0$ *on* $t \in (t_k, t_{k+1}),\ k = 1, 2, ..., n-1$. *Then*

$$\int_a^b |q(t)|dt > \frac{n^n(n-2)!}{(n-1)^{n-2}(b-a)^{n-1}}.$$

Theorem 1.1.9. *Let* $m = 2n$ *for* $n \in \mathbb{N}$. *Assume Eq. (1.1.11) has a solution* $x(t)$ *satisfying*

$$x^{(2i)}(a) = x^{(2i)}(b) = 0, \quad i = 0, 1, ..., n-1.$$

and $x(t) \neq 0$ *on* (a, b). *Then*

$$\int_a^b |q(t)|dt > \frac{2^n}{(b-a)^n}.$$

Later on Cakmak [11] improved the conclusions of Theorems 1.1.8 and 1.1.9. For more on higher order Lyapunov-type inequalities, we refer the reader to Aktas [1], Aktas et al. [3, 4], Cakmak [9, 10], Cakmak et al. [13], Guseinov and Kaymakcalan [34], Guseinov and Zafer [35], He and Tang [39], Jiang and Zhou [52], Ji and Fan [46], Pachpatte [62, 63], Panigrahi [64], Parhi and Panigrahi [66], Tiryaki et al. [72, 73, 74] , Unal et al. [75], Unal and Cakmak [76], Watanabe [78], Watanabe et al. [79], Yang et al. [81], Yang and Lo [82], Zhang and He [86], and the references cited therein. Also, Pinasco [68] provided an excellent survey on various Lyapunov-type inequalities.

Although Lyapunov-type inequalities have been developed in many directions for the integer-order differential equations, there are only a few known results for the fractional differential equations. Recall that, for $\gamma > 0$ and $t > a$,

$$\left(I_{a^+}^{\gamma} x\right)(t) := \frac{1}{\Gamma(\gamma)} \int_a^t (t-s)^{\gamma-1} x(s) ds$$

is the γth-order left-sided Riemann-Liouville fractional integral of $x(t)$ at a, and

$$\left(D_{a^+}^{\gamma}x\right)(t) := \frac{d^n}{dt^n}\left(I_{a^+}^{n-\gamma}x\right)(t) = \frac{1}{\Gamma(n-\gamma)}\frac{d^n}{dt^n}\int_a^t (t-s)^{n-\gamma-1}x(s)ds, \tag{1.1.12}$$

is the γth-order left-sided Riemann-Liouville fractional derivative of $x(t)$ at a, where $n = \lfloor\gamma\rfloor + 1$ with $\lfloor\gamma\rfloor$ the integer part of γ and $\Gamma(\gamma) = \int_0^\infty t^{\gamma-1}e^{-t}dt$ is the gamma function. In particular, when $\gamma = i \in \mathbb{N}_0$, then

$$\left(D_{a^+}^{i}u\right)(t) = u^{(i)}(t). \tag{1.1.13}$$

BVPs for fractional differential equations are important in applications and have been studied extensively by many authors, see [8, 20, 30, 31, 32, 33, 52, 55, 56, 59, 69, 71, 85] and the references cited therein.

In [27], Ferreira obtained Lyapunov-type inequalities for a Riemann-Liouville fractional boundary value problem (BVP) consisting of the equation

$$\left(D_{a^+}^{\alpha}x\right)(t) + q(t)x = 0, \quad 1 < \alpha \leq 2, \tag{1.1.14}$$

where $q \in C([a,b],\mathbb{R})$.

Theorem 1.1.10. *Assume Eq. (1.1.14) has a solution $x(t)$ satisfying*

$$x(a) = x(b) = 0 \tag{1.1.15}$$

and $x(t) \neq 0$ for $t \in (a,b)$. Then

$$\int_a^b |q(t)|dt > \Gamma(\alpha)\Big(\frac{4}{b-a}\Big)^{\alpha-1}.$$

With a simple modification in the theorem, we can easily obtain a variation of Theorem 1.1.10.

Theorem 1.1.11. *Assume Eq. (1.1.14) has a solution* $x(t)$ *satisfying* $x(a) = x(b) = 0$ *and* $x(t) \neq 0$ *for* $t \in (a,b)$*. Then*

$$\int_a^b q_+(t)dt > \Gamma(\alpha)\Big(\frac{4}{b-a}\Big)^{\alpha-1}.$$

These results are derived using the Green's function

$$G(t,s) = \frac{1}{\Gamma(\alpha)}\begin{cases} \frac{(t-a)^{\alpha-1}(b-s)^{\alpha-1}}{(b-a)^{\alpha-1}} - (t-s)^{\alpha-1}, & a \leq s \leq t \leq b \\ \frac{(t-a)^{\alpha-1}(b-s)^{\alpha-1}}{(b-a)^{\alpha-1}}, & a \leq t \leq s \leq b \end{cases} \tag{1.1.16}$$

for BVP (1.1.14), (1.1.15) obtained in [27], which is an extension of the one given in [8] for the case that $a = 0$ and $b = 1$.

In [61], the authors considered the following higher order fractional differential equation to obtain Lyapunov-type inequalities.

$$\Big(D_{a+}^{\alpha}x\Big)(t) + q(t)x = 0, \quad 3 < \alpha \leq 4, \tag{1.1.17}$$

where $q \in C([a,b],\mathbb{R})$.

Theorem 1.1.12. *Assume Eq. (1.1.17) has a solution* $x(t)$ *satisfying*

$$x(a) = x'(a) = x''(a) = x''(b) = 0.$$

and $x(t) \neq 0$ *for* $t \in (a,b)$*. Then*

$$\int_a^b |q(t)|dt > \frac{\Gamma(\alpha)(\alpha-2)^{\alpha-2}}{2(\alpha-3)^{\alpha-3}(b-a)^{\alpha-1}}.$$

Recently, Lyapunov-type inequalities has also been extended to fractional BVPs involving the Caputo fractional derivatives. In [26], Ferreira obtained such inequalities for the following equation:

$$\left({}^{C}D^{\alpha}_{a^+}x\right)(t)+q(t)x=0, \quad 1<\alpha\le 2, \tag{1.1.18}$$

where $q\in C([a,b],\mathbb{R})$.

Theorem 1.1.13. *Assume Eq. (1.1.18) has a nontrivial solution $x(t)$ satisfying*

$$x(a)=x(b)=0$$

and $x(t)\neq 0$ for $t\in(a,b)$. Then

$$\int_a^b |q(t)|dt > \frac{\Gamma(\alpha)\alpha^{\alpha}}{[(\alpha-1)(b-a)]^{\alpha-1}}.$$

Jleli and Samet studied the BVP consisting of Eq. (1.1.18) and a Sturm-Liouville boundary condition (BC). In [48], they obtained the following

Theorem 1.1.14. *Assume Eq. (1.1.18) has a solution $x(t)$ satisfying*

$$px(a)-rx'(a)=x(b)=0,$$

where $p>0$, $\frac{r}{p}>\frac{b-a}{\alpha-1}$ and $x(t)\neq 0$ for $t\in(a,b)$. Then

$$\int_a^b |q(t)|dt > \left(1+\frac{p}{r}(b-a)\right)\frac{\Gamma(\alpha)}{(b-a)^{\alpha-1}}.$$

Jleli, Ragoub and Samet studied the BVP consisting of Eq. (1.1.18) and a Robin BC. In [51], they obtained the following

Theorem 1.1.15. *Assume Eq. (1.1.18) has a solution $x(t)$ satisfying*

$$x(a) - x'(a) = x(b) + x'(b) = 0$$

and $x(t) \neq 0$ for $t \in (a, b)$. Then

$$\int_a^b (b-t)^{\alpha-2}(b-t+\alpha-1)|q(t)|dt > \frac{(b-a+2)\Gamma(\alpha)}{\max\{b-a+1, ((2-\alpha)/(\alpha-1))(b-a)-1\}}.$$

Lyapunov-type inequalities of integer and fractional order have also been obtained on multivariate domains or for partial differential equations by many authors. To name a few, we refer the reader to [5, 38, 60, 50]. In particular, Anastassiou [5] obtained Lyapunov-type inequalities for third-order multivariate equations on special domains in $\mathbb{R}^n$, as shown below: For $N \geq 2$ denote

$$B(0, R) := \{x \in \mathbb{R}^N : |x| < R\} \text{ for } R > 0,$$

and let A be an open spherical shell in $\mathbb{R}^N$ centered at the origin, i.e., $A := B(0, b) \setminus \overline{B(0, a)}$ for $0 < a < b$. Consider the following equation

$$\frac{\partial^3 y(x)}{\partial r^3} + q(x)y(x) = 0 \tag{1.1.19}$$

with the BC

$$y(\partial B(0, a)) = y(\partial B(0, b)) = 0 \text{ and } \frac{\partial^2 y(\partial B(0, \xi))}{\partial r^2} = 0 \text{ for } \xi \in [a, b], \tag{1.1.20}$$

where $q \in (\overline{A})$ and the derivatives with respect to r are directional derivatives in the radial direction.

Theorem 1.1.16. *Assume Eq. (1.1.19) has a solution $y(x)$ satisfying (1.1.20) and $y(x) \neq 0$ on A. Then*

$$\int_A |q(x)| dx > \frac{8\pi^{N/2} a^{N-1}}{\Gamma(N/2)(b-a)^2}. \tag{1.1.21}$$

Note that inequality (1.1.21) holds only when the domain A is radially symmetric and contains a circular hole of radius $a > 0$ inside. Moreover, it becomes less sharp when a is small and provides no information when $a = 0$, i.e., when the hole shrinks to a point.

1.2 Outline of Research

In this dissertation, we will establish various Lyapunov-type inequalities for several types of BVPs. While the Lyapunov inequality and its generalizations have been used as a great tool in oscillation, disconjugacy, eigenvalue problems, and many other areas of differential equations, we have not seen any applications of Lyapunov-type inequalities to the existence and uniqueness of solutions of BVPs. Here we will use the obtained inequalities to study the nonexistence, existence, uniqueness, and relations of solutions for several classes of linear and nonlinear BVPs.

In Chapter 2, we will consider the third-order half-linear differential equations of the form

$$(\phi_{\alpha_2}((\phi_{\alpha_1}(x'))'))' + q(t)\phi_{\alpha_1\alpha_2}(x) = 0,$$

where $q \in C([a,c], \mathbb{R})$, $\phi_p(x) = |x|^{p-1}x$, and $\alpha_1, \alpha_2 > 0$, with one of the following boundary conditions

$$x(a) = x(b) = 0 \text{ and } (\phi_{\alpha_1}(x'))'(\xi) = 0 \text{ for } \xi \in [a,b];$$

or

$$x(a) = x(b) = x(c) = 0.$$

We will obtain Lyapunov-type inequalities for each of the BVPs. We point out that our inequalities will utilize integrals of both $q_+(t)$ and $q_-(t)$, the positive and negative parts of $q(t)$; which is different from most existing results for higher-order cases, where integrals of $|q(t)|$ are involved. As a special case, our work improves the result in Theorem 1.1.4 for the linear case.

In Section 2.2, we will apply the obtained Lyapunov-type inequalities to show the nonexistence of nontrivial solutions of third-order half-linear BVPs and the uniqueness of solutions of third-order linear BVPs. Furthermore, by employing the "uniqueness implies existence" theorems given by Jackson [43, 44] and by Jackson and Schrader [45], we establish an existence-uniqueness result for several classes of third-order linear BVPs. We believe that these results are new and hard to obtain by the traditional methods for BVPs. We expect that the Lyapunov-type inequality approach can be further applied to study general higher-order BVPs. For detailed discussion on the "uniqueness implies existence" criteria for general BVPs, the reader is referred to [12, 23, 24, 25, 29, 40, 41, 42].

In Chapter 3, we will consider the third-order linear differential equations of the form

$$x''' + q(t)x = 0$$

with $q \in C([a, c], \mathbb{R})$ and BC

$$x(a) = x(b) = x(c) = 0.$$

In a different approach, by using the Green's function $G(t, s)$ for a corresponding Dirichlét problem defined on $[t_1, t_2] \subset (a, c)$, Aktas, Cakmak, and Tiryaki [2] tried to improve the constant 4 on the right-hand side of (1.1.8). More specifically, they claim that the following inequality holds:

$$\int_a^c |q(t)|dt \geq \frac{16}{(c-a)^2}. \tag{1.2.1}$$

Unfortunately, there is an error in their proof. In fact, they mistakenly assumed $G(t,s) \geq 0$ for all $(t,s) \in [a,c] \times [t_1,t_2]$ which is actually true only when $(t,s) \in [t_1,t_2] \times [t_1,t_2]$. In fact, (2.7) in [2] does not hold unless G is replaced by $|G|$. However, with G replaced by $|G|$, (2.11) in [2] no longer holds. As a result, Lemma 2.1 in [2] fails to hold. Therefore, (1.2.1) in the above is unjustified and is probably too good to be true.

On the other hand, their idea of applying Green's functions for second-order BVPs to deal with third-order linear equations are novel and feasible. Motivated by this idea, we will derive new Lyapunov-type inequalities for the third-order linear equation. In Section 3.2, we will extend the work to more general third-order linear equations. Finally, in the last section, we will apply the obtained Lyapunov-type inequalities to derive conditions for the nonexistence of nontrivial solutions of third-order homogeneous linear BVPs and the uniqueness of solutions of third-order nonhomogeneous linear BVPs. Furthermore, by employing the "uniqueness implies existence" theorems given by Jackson [43, 44] and by Jackson and Schrader [48], we establish existence-uniqueness results for several classes of third-order linear BVPs.

In Chapter 4, we will consider higher order half-linear differential equations in the form

$$\left(x^{[N]}\right)' + q(t)\phi_{\alpha[1,N]}(x) = 0,$$

where

(i) $N \in \mathbb{N}$;

(ii) $x^{[i]} = \phi_{\alpha_i}\left[\left(x^{[i-1]}\right)'\right]$ for $i = 1, ..., N$ with $x^{[0]} = x$, and $\phi_p(x) = |x|^{p-1}x$ for $p > 0$;

(iii) $q \in C([a,b], \mathbb{R})$ with $-\infty < a < b < \infty$;

(iv) $\alpha[i,j] = \alpha_i...\alpha_j$ for $1 \leq i < j \leq N$ and $\alpha[i,i] = \alpha_i$ with $\alpha_i > 0$ for $i = 1, ..., N$;

with one of the following BC

$$x^{[2i]}(a) = x^{[2i]}(b) = 0, \ i = 0, ..., n-1;$$

$$x^{[i]}(a) = x^{[i]}(b) = 0, \ i = 0, ..., n-1;$$

$$x^{[2i]}(a) = x^{[2i]}(b) = 0, \ i = 0, ..., n-1 \ and \ x^{[2n]}(\xi) = 0;$$

$$x^{[i]}(a) = x^{[i]}(b) = 0, \ i = 0, ..., n-1; \ and \ x^{[2n]}(\xi) = 0.$$

To the best of our knowledge, Lyapunov-type inequalities have not been well-developed for higher order half-linear differential equations. This is due to the nonlinear feature of the equation and the complexity caused by the multiple p-Laplacian operators in the equation. In this chapter, by subtle applications of forward and backward inductions, we establish Lyapunov-type inequalities for each types of even and odd order half-linear BVPs mentioned above. As shown in Corollary 4.1.3, our results reduce to the existing results by Cakmak [11], Yang [83], and Yang and Lo [82] for the linear case. We also apply the obtained Lyapunov-type inequalities to some related half-linear eigenvalue problems to estimate the range of their eigenvalues.

In Chapter 5, we will extend the results from Chapter 3 to derive Lyapunov-type inequalities for odd order equations. These inequalities will also improve those from Chapter 4 when the equations become linear. More specifically, we will consider the odd order linear differential equations in the form

$$x^{(2n+1)} + (-1)^{n-1} q(t) x = 0$$

with $n \in \mathbb{N}$, $q \in C([a,b],\mathbb{R})$ and BCs

$$x^{(i+1)}(a) = x^{(i+1)}(b) = 0, \quad i = 0, 1, ..., n-1$$

and $x(c) = 0$ for $c \in [a,b]$.

We will use the Green's function for even order linear BVPs to obtain Lyapunov-type inequalities for certain types of BVPs associated with odd order linear equations. Furthermore, by utilizing the Fredholm Alternative theorem, we obtain a criterion for the existence and uniqueness of solutions of corresponding nonhomogeneous linear BVPs for odd order equations.

In Chapter 6, we will consider the Riemann-Liouville fractional BVP consisting of the fractional linear differential equation

$$\left(D_{a^+}^{\alpha} x\right)(t) + q(t)x = 0, \quad 1 < \alpha \leq 2,$$

and the integral BC

$$\left(D_{a^+}^{\alpha-2} x\right)(a^+) = \left(D_{a^+}^{\alpha-2} x\right)(b) = 0.$$

Here we point out that the BCs considered in this chapter are different from the general integral BCs with a singular kernel in the sense that they originate from the fractional initial conditions with which the existence-uniqueness results are derived. Problems with such BCs can be investigated in many approaches based on results on initial value problems. For instance, using the Fredholm alternative method to study the existence and uniqueness of boundary value problems, as shown in our Theorems 6.4.1 and 6.4.2. Such approaches are not allowed for general integral BCs. Lyapunov-type inequalities are derived and used to establish the existence and uniqueness for solutions of this BVP. Parallel results are also obtained for certain sequential fractional BVPs.

In Chapter 7, we will extend the results from Chapter 6 for the case when $2 < \alpha \leq 3$. In particular, we will consider the following fractional linear differential equation

$$\left(D_{a^+}^{\alpha} x\right)(t) + q(t)x = 0, \quad 2 < \alpha \leq 3$$

with one of the following boundary conditions:

$$\left(D_{a^+}^{\alpha-2} x\right)(a^+) = \left(D_{a^+}^{\alpha-2} x\right)(b) = 0 \text{ and } \left(D_{a^+}^{\alpha-3} x\right)(c) = 0, \quad a \leq c \leq b;$$

$$\left(D_{a^+}^{\alpha-3} x\right)(a^+) = \left(D_{a^+}^{\alpha-3} x\right)(b) = 0 \text{ and } \left(D_{a^+}^{\alpha-2} x\right)(a^+) = 0, \quad a < b;$$

$$\left(D_{a^+}^{\alpha-3} x\right)(a^+) = \left(D_{a^+}^{\alpha-3} x\right)(b) = 0 \text{ and } \left(D_{a^+}^{\alpha-1} x\right)(a) = 0, \quad a < b.$$

We will derive Lyapunov-type inequalities for each of the BVPs and utilize them to establish the existence and uniqueness for solutions of related homogeneous and nonhomogeneous linear BVPs. In special cases, our work covers and improves some existing results for the third-order linear boundary value problems.

In Chapter 8, we will first establish Lyapunov-type inequalities for a univariate Riemann-Liouville fractional differential equation of order $\alpha \in (2, 3]$ together with several pointwise or mixed BCs. In particular, we will consider the following fractional linear differential equation

$$\left(D_{a^+}^{\alpha} x\right)(t) + q(t)x = 0, \quad 2 < \alpha \leq 3$$

with one of the following boundary conditions:

$$x(a) = 0 \quad \text{and} \quad x'(a) = x'(b) = 0;$$

$$x(a) = x(b) = 0 \quad \text{and} \quad x'(a) = 0;$$

$$x(a) = 0 \quad \text{and} \quad \left(D_{a^+}^{\alpha-2} x\right)(a) = \left(D_{a^+}^{\alpha-2} x\right)(b) = 0;$$

$$x(a) = x(b) = 0 \quad \text{and} \quad \left(D_{a^+}^{\alpha-2} x\right)(a) = 0;$$

$$x(a) = x(b) = 0 \quad \text{and} \quad \left(D_{a^+}^{\alpha-1} x\right)(\xi) = 0, \quad \xi \in [a, b].$$

Based on the results, we will further develop Lyapunov-type inequalities for multivariate fractional BVPs on domains in $\mathbb{R}^N$ which are not necessarily radially symmetric. We point out that when $\alpha = 3$, our result improves Theorem 1.1.16 even for the special domain A, see Remark 8.3.1, Part (ii). Moreover, our result provides estimates for general simply connected regions as long as they contain the origin inside, see Remark 8.3.1, Part (iii).

CHAPTER 2

THIRD-ORDER HALF-LINEAR EQUATIONS

2.1 Lyapunov-Type Inequalities

In this chapter we let $-\infty < a < b < c < \infty$ and consider third-order half-linear differential equations of the form

$$\left(\phi_{\alpha_2}\left(\left(\phi_{\alpha_1}\left(x'\right)\right)'\right)\right)' + q\left(t\right)\phi_{\alpha_1\alpha_2}\left(x\right) = 0, \tag{2.1.1}$$

where $q \in C\left(\mathbb{R}, \mathbb{R}\right)$, $\phi_p\left(x\right) = \left|x\right|^{p-1}x$, and $\alpha_1, \alpha_2 > 0$. For simplicity, we denote $\alpha = \left(\alpha_1 + 1\right)\alpha_2$.

Theorem 2.1.1. *Assume $x(t)$ is a solution of Eq. (2.1.1) with $x\left(a\right) = x\left(b\right) = 0$ and $x\left(t\right) \neq 0$ for all $t \in (a, b)$. Suppose that there is a $\xi \in [a, b]$ such that $\left(\phi_{\alpha_1}\left(x'\right)\right)'\left(\xi\right) = 0$. Then*

$$\int_a^\xi q_-(s)ds + \int_\xi^b q_+(s)ds > \left(\frac{2}{b-a}\right)^\alpha. \tag{2.1.2}$$

Proof. Without loss of generality, we may assume $x(t) > 0$ on (a, b). Then there exists a $d \in [a, b]$ such that $m := x(d) = \max_{t\in[a,b]} x(t)$. It follows that

$$m = \int_a^d x'(t)dt \leq \int_a^d \left|x'(t)\right| dt$$

and

$$m = -\int_d^b x'(t)dt \leq \int_d^b \left|x'(t)\right| dt.$$

Therefore,

$$2m \le \int_a^b |x'(t)|\, dt. \tag{2.1.3}$$

Applying Holder's inequality

$$\int_a^b |f(t)g(t)|\, dt \le \left(\int_a^b |f(t)|^p\, dt\right)^{\frac{1}{p}} \left(\int_a^b |g(t)|^q\, dt\right)^{\frac{1}{q}} \tag{2.1.4}$$

with $f(t) = x'(t)$, $g(t) = 1$, $p = \alpha_1 + 1$ and $q = 1 + \frac{1}{\alpha_1}$, we obtain that

$$\int_a^b |x'(t)|\, dt \le (b-a)^{\alpha_1/(\alpha_1+1)} \left(\int_a^b |x'(t)|^{\alpha_1+1} dt\right)^{1/(\alpha_1+1)}.$$

Hence by (2.1.3) we have

$$(2m)^{\alpha_1+1} \le (b-a)^{\alpha_1} \int_a^b |x'(t)|^{\alpha_1+1}\, dt = (b-a)^{\alpha_1} \int_a^b x'(t)\phi_{\alpha_1}(x'(t))\, dt.$$

Note that $x(a) = x(b) = 0$. Then integration by parts leads to

$$\frac{(2m)^{\alpha_1+1}}{(b-a)^{\alpha_1}} \le \int_a^b x(t)\left(-\phi_{\alpha_1}(x'(t))\right)' dt. \tag{2.1.5}$$

For $t \in [a,b]$, by integrating (2.1.1) from ξ to t and using the fact that $(\phi_{\alpha_1}(x'))'(\xi) = 0$, we see that

$$\phi_{\alpha_2}\left(\left(-\phi_{\alpha_1}(x')\right)'\right)(t) = \int_\xi^t q(s)\phi_{\alpha_1\alpha_2}(x(s))\, ds. \tag{2.1.6}$$

Since $-q_-(t) \le q(t) \le q_+(t)$, we have

$$\int_\xi^t q(s)\phi_{\alpha_1\alpha_2}(x(s))\, ds \le \begin{cases} \int_\xi^t q_+(s)\phi_{\alpha_1\alpha_2}(x(s))\, ds, & t \ge \xi; \\ \int_\xi^t -q_-(s)\phi_{\alpha_1\alpha_2}(x(s))\, ds, & t < \xi. \end{cases} \tag{2.1.7}$$

Define

$$q^{*}(s)=\begin{cases} q_{+}(s), & \xi \leq s \leq b; \\ -q_{-}(s), & a \leq s < \xi. \end{cases} \tag{2.1.8}$$

From (2.1.7) we see that for $t \in [a,b]$

$$\int_{\xi}^{t} q(s)\phi_{\alpha_1\alpha_2}(x(s))\,ds \leq \int_{\xi}^{t} q^{*}(s)\phi_{\alpha_1\alpha_2}(x(s))\,ds.$$

Then from (2.1.6),

$$\phi_{\alpha_2}\left((-\phi_{\alpha_1}(x'))'\right)(t) \leq \int_{\xi}^{t} q^{*}(s)\phi_{\alpha_1\alpha_2}(x(s))\,ds.$$

It follows that

$$(-\phi_{\alpha_1}(x'))'(t) \leq \phi_{\alpha_2}^{-1}\left(\int_{\xi}^{t} q^{*}(s)\phi_{\alpha_1\alpha_2}(x(s))\,ds\right). \tag{2.1.9}$$

Combining (2.1.5) and (2.1.9) and using the facts that $0 \leq x(t) \leq m$, $x(t) \not\equiv m$ on $[a,b]$, and $\int_{\xi}^{t} q^{*}(s)ds \geq 0$ for all $t \in [a,b]$, we have

$$\begin{aligned} \frac{(2m)^{\alpha_1+1}}{(b-a)^{\alpha_1}} &\leq \int_a^b x(t)\phi_{\alpha_2}^{-1}\left(\int_{\xi}^{t} q^{*}(s)\phi_{\alpha_1\alpha_2}(x(s))\,ds\right)dt \\ &< m^{\alpha_1+1}\int_a^b \phi_{\alpha_2}^{-1}\left(\int_{\xi}^{t} q^{*}(s)ds\right)dt \\ &= m^{\alpha_1+1}\left\{\int_a^{\xi} \phi_{\alpha_2}^{-1}\left(\int_{\xi}^{t} q^{*}(s)ds\right)dt + \int_{\xi}^{b} \phi_{\alpha_2}^{-1}\left(\int_{\xi}^{t} q^{*}(s)ds\right)dt\right\}. \end{aligned}$$

This together with (2.1.8) yields that

$$\begin{aligned}\frac{2^{\alpha_1+1}}{(b-a)^{\alpha_1}} &< \int_a^\xi \phi_{\alpha_2}^{-1}\left(\int_\xi^t -q_-(s)ds\right)dt + \int_\xi^b \phi_{\alpha_2}^{-1}\left(\int_\xi^t q_+(s)ds\right)dt \\ &= \int_a^\xi \phi_{\alpha_2}^{-1}\left(\int_t^\xi q_-(s)ds\right)dt + \int_\xi^b \phi_{\alpha_2}^{-1}\left(\int_\xi^t q_+(s)ds\right)dt \\ &\le \int_a^\xi \phi_{\alpha_2}^{-1}\left(\int_a^\xi q_-(s)ds\right)dt + \int_\xi^b \phi_{\alpha_2}^{-1}\left(\int_\xi^b q_+(s)ds\right)dt.\end{aligned}$$

Therefore,

$$\frac{2^{\alpha_1+1}}{(b-a)^{\alpha_1}} < (\xi-a)\,\phi_{\alpha_2}^{-1}\left(\int_a^\xi q_-(s)ds\right) + (b-\xi)\,\phi_{\alpha_2}^{-1}\left(\int_\xi^b q_+(s)ds\right). \tag{2.1.10}$$

We discuss the two cases with $\alpha_2 \ge 1$ and $0 < \alpha_2 < 1$ separately.

Case I: $\alpha_2 \ge 1$. In this case, ϕ_{α_2} is a concave-up function on $[0,\infty)$. Hence for any $x_1, x_2 \in [0,\infty)$ and $t \in [0,1]$, we have

$$\phi_{\alpha_2}(tx_1 + (1-t)x_2) \le t\phi_{\alpha_2}(x_1) + (1-t)\,\phi_{\alpha_2}(x_2). \tag{2.1.11}$$

Now dividing both sides of (2.1.10) by $b-a$ and applying ϕ_{α_2} we get that

$$\begin{aligned}\left(\frac{2}{b-a}\right)^{(\alpha_1+1)\alpha_2} &= \phi_{\alpha_2}\left[\left(\frac{2}{b-a}\right)^{\alpha_1+1}\right] \\ < \phi_{\alpha_2}&\left[\left(\frac{\xi-a}{b-a}\right)\phi_{\alpha_2}^{-1}\left(\int_a^\xi q_-(s)ds\right) + \left(\frac{b-\xi}{b-a}\right)\phi_{\alpha_2}^{-1}\left(\int_\xi^b q_+(s)ds\right)\right].\end{aligned}$$

Note that $\frac{\xi-a}{b-a} + \frac{b-\xi}{b-a} = 1$. Then using (2.1.11) with $t = \frac{\xi-a}{b-a}$ we obtain

$$\left(\frac{2}{b-a}\right)^{(\alpha_1+1)\alpha_2} < \left(\frac{\xi-a}{b-a}\right)\int_a^\xi q_-(s)ds + \left(\frac{b-\xi}{b-a}\right)\int_\xi^b q_+(s)ds.$$

Hence

$$\int_a^\xi q_-(s)ds + \int_\xi^b q_+(s)ds > \left(\frac{2}{b-a}\right)^\alpha,$$

i.e., (2.1.2) holds.

Case II: $0 < \alpha_2 < 1$. In this case, ϕ_{α_2} is a concave-down function on $[0,\infty)$. Hence for $x_1, x_2 \in [0,\infty)$, we have

$$\phi_{\alpha_2}(x_1 + x_2) \leq \phi_{\alpha_2}(x_1) + \phi_{\alpha_2}(x_2). \tag{2.1.12}$$

It follows from (2.1.10) that

$$\frac{2^{\alpha_1+1}}{(b-a)^{\alpha_1}} < (b-a)\left[\phi_{\alpha_2}^{-1}\left(\int_a^\xi q_-(s)ds\right) + \phi_{\alpha_2}^{-1}\left(\int_\xi^b q_+(s)ds\right)\right].$$

Hence

$$\left(\frac{2}{b-a}\right)^{\alpha_1+1} < \phi_{\alpha_2}^{-1}\left(\int_a^\xi q_-(s)ds\right) + \phi_{\alpha_2}^{-1}\left(\int_\xi^b q_+(s)ds\right). \tag{2.1.13}$$

Applying ϕ_{α_2} to both sides of (2.1.13) and using (2.1.12) we have

$$\int_a^\xi q_-(s)ds + \int_\xi^b q_+(s)ds > \left(\frac{2}{b-a}\right)^{(\alpha_1+1)\alpha_2},$$

i.e., (2.1.2) holds. □

Theorem 2.1.2. *Assume $x(t)$ is a solution of Eq. (2.1.1) with $x(a) = x(b) = x(c) = 0$ and $x(t) \neq 0$ for $t \in (a,b) \cup (b,c)$. Then either*

$$\max_{\xi\in[a,b]}\left\{\int_a^\xi q_-(s)ds + \int_\xi^b q_+(s)ds\right\} > \left(\frac{2}{b-a}\right)^\alpha \tag{2.1.14}$$

or

$$\max_{\xi\in[b,c]}\left\{\int_b^{\xi} q_-(s)ds + \int_{\xi}^{c} q_+(s)ds\right\} > \left(\frac{2}{c-b}\right)^{\alpha}. \tag{2.1.15}$$

As a result,

$$\max_{\xi\in[a,c]}\left\{\int_a^{\xi} q_-(s)ds + \int_{\xi}^{c} q_+(s)ds\right\} > \left(\frac{2}{c-a}\right)^{\alpha}. \tag{2.1.16}$$

Proof. By Rolle's Theorem, there exists a $\xi_1 \in (a,b)$ and $\xi_2 \in (b,c)$ such that $\phi_{\alpha_1}(x')(\xi_1) = 0$ and $\phi_{\alpha_1}(x')(\xi_2) = 0$. A further application of Rolle's Theorem yields that there exists a $\xi \in (\xi_1, \xi_2)$ such that $(\phi_{\alpha_1}(x'))'(\xi) = 0$. Clearly, either $\xi \in (a,b]$ or $\xi \in [b,c)$.

If $\xi \in (a,b]$, then by applying Theorem 2.1.1 to the interval $[a,b]$, we obtain that

$$\int_a^{\xi} q_-(s)ds + \int_{\xi}^{b} q_+(s)ds > \left(\frac{2}{b-a}\right)^{\alpha},$$

and hence (2.1.14) holds. If $\xi \in [b,c)$, a similar argument leads to (2.1.15). It is easy to see that both (2.1.14) and (2.1.15) lead to (2.1.16). □

We note that $q_-(t), q_+(t) \le |q(t)|$ for $t \in [a,b]$. Then the results below follow from Theorems 2.1.1 and 2.1.2 directly.

Corollary 2.1.1. *(a) Assume $x(t)$ is a solution of Eq. (2.1.1) with $x(a) = x(b) = 0$ and $x(t) \neq 0$ for $t \in (a,b)$. Suppose there is a $\xi \in [a,b]$ such that $(\phi_{\alpha_1}(x'))'(\xi) = 0$. Then*

$$\int_a^b |q(s)|\, ds > \left(\frac{2}{b-a}\right)^{\alpha}. \tag{2.1.17}$$

(b) Assume $x(t)$ is a solution of Eq. (2.1.1) with $x(a) = x(b) = x(c) = 0$ and $x(t) \neq 0$ for all $t \in (a,b) \cup (b,c)$. Then

$$\text{either} \quad \int_a^b |q(s)|\, ds > \left(\frac{2}{b-a}\right)^{\alpha} \quad \text{or} \quad \int_b^c |q(s)|\, ds > \left(\frac{2}{c-b}\right)^{\alpha}.$$

As a result,

$$\int_a^c |q(s)|\,ds > \left(\frac{2}{c-a}\right)^\alpha .$$

Remark 2.1.1. Although the inequalities in corollary 2.1.1 are simpler, they are not as sharp as those in Theorems 2.1.1 and 2.1.2 due to the fact that $|q(s)| = q_+(s) + q_-(s)$. For instance, if $a = \xi$, then (2.1.2) becomes

$$\int_a^b q_+(s)ds > \left(\frac{2}{b-a}\right)^\alpha ;$$

and if $\xi = b$, then (2.1.2) becomes

$$\int_a^b q_-(s)ds > \left(\frac{2}{b-a}\right)^\alpha .$$

Both are sharper than (2.1.17). Furthermore, under the assumptions of Theorem 2.1.1, we never expect that

$$q(s) \begin{cases} \geq 0, & s \in [a,\xi] \\ \leq 0, & s \in (\xi, b] \end{cases}$$

could happen. However, this cannot be observed from corollary 2.1.1.

In the following, as direct applications of Theorem 2.1.2, we discuss the zero count and distances between consecutive zeros of a non-trivial solution of Eq. (2.1.1) on a given interval. The first result gives us an estimate for the number of zeros.

Theorem 2.1.3. *Let $x(t)$ be a nontrivial solution of Eq. (2.1.1). Let $\{t_k\}_{k=1}^{2N+1}$, $N \geq 1$, be an increasing sequence of zeros of $x(t)$ in a compact interval I with length l. Then*

$$N < \left(\frac{l}{2}\right)^{\frac{\alpha}{\alpha+1}} \left[\sum_{k=1}^{N} \max_{\xi_k \in [t_{2k-1}, t_{2k+1}]} \left\{ \int_{t_{2k-1}}^{\xi_k} q_-(t)dt + \int_{\xi_k}^{t_{2k+1}} q_+(t)dt \right\}\right]^{\frac{1}{\alpha+1}} . \qquad (2.1.18)$$

Proof. For $k = 1, 2, ..., N$, we apply Theorem 2.1.2 to the interval $[t_{2k-1}, t_{2k+1}] \subseteq I$. It follows from (2.1.16) that

$$\max_{\xi_k \in [t_{2k-1}, t_{2k+1}]} \left\{ \int_{t_{2k-1}}^{\xi_k} q_-(t)dt + \int_{\xi_k}^{t_{2k+1}} q_+(t)dt \right\} > \left(\frac{2}{t_{2k+1} - t_{2k-1}} \right)^{\alpha}.$$

Taking the sum on both sides for k from 1 to N, we get

$$\sum_{k=1}^{N} \max_{\xi_k \in [t_{2k-1}, t_{2k+1}]} \left\{ \int_{t_{2k-1}}^{\xi_k} q_-(t)dt + \int_{\xi_k}^{t_{2k+1}} q_+(t)dt \right\} > 2^{\alpha} \sum_{k=1}^{N} \left(t_{2k+1} - t_{2k-1} \right)^{-\alpha}. \tag{2.1.19}$$

Note that $a_k = t_{2k+1} - t_{2k-1} > 0$ for $1 \leq k \leq N$. Then $\psi(x) := x^{-\alpha}$ is a concave-up function for $\alpha > 0$ on $(0, \infty)$. Thus

$$\frac{1}{N} \sum_{k=1}^{N} \psi(a_k) \geq \psi\left(\frac{1}{N} \sum_{k=1}^{N} a_k \right)$$

to the right-hand side of (2.1.19), we obtain

$$\begin{aligned} &\sum_{k=1}^{N} \max_{\xi_k \in [t_{2k-1}, t_{2k+1}]} \left\{ \int_{t_{2k-1}}^{\xi_k} q_-(t)dt + \int_{\xi_k}^{t_{2k+1}} q_+(t)dt \right\} \\ &> 2^{\alpha} N \left(\frac{1}{N} \sum_{k=1}^{N} (t_{2k+1} - t_{2k-1}) \right)^{-\alpha} \\ &= 2^{\alpha} N^{1+\alpha} \left(t_{2N+1} - t_1 \right)^{-\alpha} \geq \left(\frac{2}{l} \right)^{\alpha} N^{1+\alpha}. \end{aligned}$$

Then (2.1.18) follows. □

Next, we show how the distance between consecutive zeros of an oscillatory solution of Eq. (2.1.1) may change.

Theorem 2.1.4. *Let $x(t)$ be an oscillatory solution of Eq. (2.1.1) with $\{t_n\}_{n=1}^{\infty}$ its increasing sequence of zeros in $[0,\infty)$. Assume there exists a $\sigma \geq 1$, such that for any $M > 0$, we have*

$$\int_t^{t+M} |q(s)|^{\sigma} \, ds \to 0 \text{ as } t \to \infty. \tag{2.1.20}$$

Then $t_{n+2} - t_n \to \infty$ as $n \to \infty$.

Proof. It suffices to prove for the case when $\sigma = 1$. In fact, if (2.1.20) holds for a $\sigma > 1$, then by Holder's inequality (2.1.4) with $p = \sigma$ and $q = \frac{\sigma}{\sigma-1}$, we have

$$\int_t^{t+M} |q(s)| \, ds \leq \left(\int_t^{t+M} |q(s)|^{\sigma} \, ds \right)^{\frac{1}{\sigma}} M^{\frac{\sigma-1}{\sigma}} \to 0 \text{ as } t \to \infty,$$

and hence (2.1.20) holds for $\sigma = 1$.

Assume the contrary. Then there exists $M > 0$ and a subsequence $\{t_{n_k}\}_{k=1}^{\infty}$ of $\{t_n\}_{n=1}^{\infty}$ such that $t_{n_k+2} - t_{n_k} \leq M$ for all large k. By the assumption,

$$\int_{t_{n_k}}^{t_{n_k+2}} |q(t)| \, dt \leq \int_{t_{n_k}}^{t_{n_k}+M} |q(t)| \, dt \to 0 \text{ as } k \to \infty.$$

Applying Corollary 2.1.1 to the interval $[t_{n_k}, t_{n_k+2}]$ we have

$$\int_{t_{n_k}}^{t_{n_k+2}} |q(t)| \, dt > \left(\frac{2}{t_{n_k+2} - t_{n_k}} \right)^{\alpha},$$

i.e.,

$$\begin{aligned} 1 &< 2^{-\alpha} \left(t_{n_k+2} - t_{n_k}\right)^{\alpha} \int_{t_{n_k}}^{t_{n_k+2}} |q(t)| \, dt \\ &\leq 2^{-\alpha} M^{\alpha} \int_{t_{n_k}}^{t_{n_k+2}} |q(t)| \, dt \to 0 \text{ as } k \to \infty. \end{aligned}$$

We have reached a contradiction. □

Remark 2.1.2. (a) We observe that (2.1.20) is satisfied if $\lim_{t\to\infty} q(t) = 0$ or $q \in L^{\sigma}([0,\infty],\mathbb{R})$ for some $\sigma \geq 1$. However, the converse is not true in general.

(b) In [65, Theorem 5], Parhi and Panigrahi derived a similar result to Theorem 2.1.4 for the linear case under the stronger assumption that $q \in L^{\sigma}([0,\infty),\mathbb{R})$. Obviously, their result has been improved by Theorem 2.1.4, even for the linear case.

2.2 Generalizations

We now extend the results in Section 2.1 to equations in the more general form

$$\left(r_2(t)\,\phi_{\alpha_2}\left((r_1(t)\,\phi_{\alpha_1}(x'))'\right)\right)' + q(t)\,\phi_{\alpha_1\alpha_2}(x) = 0, \tag{2.2.1}$$

where $r_1, r_2, q \in C(\mathbb{R},\mathbb{R})$ such that $r_1(t), r_2(t) > 0$, $\phi_p(x) = |x|^{p-1}x$, and $\alpha_1, \alpha_2 > 0$. As before, we denote $\alpha = (\alpha_1 + 1)\alpha_2$.

Theorem 2.2.1. *Assume $x(t)$ is a solution of Eq. (2.2.1) with $x(a) = x(b) = 0$ and $x(t) \neq 0$ for $t \in (a,b)$. Suppose that there is a $\xi \in [a,b]$ such that $(r_1\phi_{\alpha_1}(x'))'(\xi) = 0$. Then*

$$\int_a^{\xi} q_-(s)ds + \int_{\xi}^{b} q_+(s)ds > \frac{2^{\alpha}}{\left(\int_a^b r_1^{-\frac{1}{\alpha_1}}(t)dt\right)^{\alpha_1\alpha_2}\left(\int_a^b r_2^{-\frac{1}{\alpha_2}}(t)dt\right)^{\alpha_2}}.$$

Proof. This proof is essentially in the same way as that of Theorem 2.1.1. We only give an outline of the proof here. Without loss of generality, we may assume $x(t) > 0$ in (a,b). Then there exists a $d \in [a,b]$ such that $m := x(d) = \max_{t\in[a,b]} x(t)$. As in the proof of Theorem 2.1.1, we have

$$2m \leq \int_a^b |x'(t)|\,dt = \int_a^b r_1^{-\frac{1}{\alpha_1+1}}(t)\left(r_1^{\frac{1}{\alpha_1+1}}(t)\,|x'(t)|\right)dt.$$

Applying Holder's Inequality (2.1.4) to the right-hand side of the inequality with $p = \alpha_1 + 1$ and $q = 1 + \frac{1}{\alpha_1}$, we get

$$\begin{aligned}(2m)^{\alpha_1+1} &\leq \left(\int_a^b r_1^{-\frac{1}{\alpha_1}}(t)dt\right)^{\alpha_1} \int_a^b r_1(t)\left|x'(t)\right|^{\alpha_1+1} dt \\ &= \left(\int_a^b r_1^{-\frac{1}{\alpha_1}}(t)dt\right)^{\alpha_1} \int_a^b x'(t)\left[r_1(t)\,\phi_{\alpha_1}(x'(t))\right] dt.\end{aligned}$$

Using integration by parts to the second integral on the right-hand side we have

$$\frac{(2m)^{\alpha_1+1}}{\left(\int_a^b r_1^{-\frac{1}{\alpha_1}}(t)dt\right)^{\alpha_1}} \leq \int_a^b x(t)\left(-r_1(t)\phi_{\alpha_1}(x'(t))\right)' dt.$$

The rest of the proof is similar to that of Theorem 2.1.1 and hence is omitted. □

Theorem 2.2.2. *Assume $x(t)$ is a solution of Eq. (2.2.1) with $x(a) = x(b) = x(c) = 0$ and $x(t) \neq 0$ for all $t \in (a, b) \cup (b, c)$. Then either*

$$\max_{\xi \in [a,b]} \left\{\int_a^\xi q_-(s)ds + \int_\xi^b q_+(s)ds\right\} > \frac{2^\alpha}{\left(\int_a^b r_1^{-\frac{1}{\alpha_1}}(t)dt\right)^{\alpha_1\alpha_2}\left(\int_a^b r_2^{-\frac{1}{\alpha_2}}(t)dt\right)^{\alpha_2}} \tag{2.2.2}$$

or

$$\max_{\xi \in [b,c]} \left\{\int_b^\xi q_-(s)ds + \int_\xi^c q_+(s)ds\right\} > \frac{2^\alpha}{\left(\int_b^c r_1^{-\frac{1}{\alpha_1}}(t)dt\right)^{\alpha_1\alpha_2}\left(\int_b^c r_2^{-\frac{1}{\alpha_2}}(t)dt\right)^{\alpha_2}}. \tag{2.2.3}$$

As a result,

$$\max_{\xi \in [a,c]} \left\{\int_a^\xi q_-(s)ds + \int_\xi^c q_+(s)ds\right\} > \frac{2^\alpha}{\left(\int_a^c r_1^{-\frac{1}{\alpha_1}}(t)dt\right)^{\alpha_1\alpha_2}\left(\int_a^c r_2^{-\frac{1}{\alpha_2}}(t)dt\right)^{\alpha_2}}. \tag{2.2.4}$$

Proof. The proof is similar to that of Theorem 2.1.2. By Rolle's Theorem, there exists a $\xi_1 \in (a,b)$ and $\xi_2 \in (b,c)$ such that $(r_1\phi_{\alpha_1}(x'))(\xi_1) = 0$ and $(r_1\phi_{\alpha_1}(x'))(\xi_2) = 0$. Again by Rolle's Theorem, there is $\xi \in (\xi_1,\xi_2)$ such that $(r_1\phi_{\alpha_1}(x'))'(\xi) = 0$. Clearly $\xi \in (a,b]$ or $\xi \in [b,c)$.

If $\xi \in (a,b]$, then applying Theorem 2.2.1 to the interval $(a,b]$, we obtain that

$$\int_a^\xi q_-(s)ds + \int_\xi^b q_+(s)ds > \frac{2^\alpha}{\left(\int_a^b r_1^{-\frac{1}{\alpha_1}}(t)dt\right)^{\alpha_1\alpha_2}\left(\int_a^b r_2^{-\frac{1}{\alpha_2}}(t)dt\right)^{\alpha_2}}.$$

It follows that

$$\max_{\xi\in[a,b]}\left\{\int_a^\xi q_-(s)ds + \int_\xi^b q_+(s)ds\right\} > \frac{2^\alpha}{\left(\int_a^b r_1^{-\frac{1}{\alpha_1}}(t)dt\right)^{\alpha_1\alpha_2}\left(\int_a^b r_2^{-\frac{1}{\alpha_2}}(t)dt\right)^{\alpha_2}},$$

i.e. (2.2.2) holds. Similarly, if $\xi \in [b,c)$, then (2.2.3) holds. It is easy to see that both (2.2.2) and (2.2.3) lead to (2.2.4). □

Recently, J. Kisel'ak [53] considered Eq. (2.2.1) under the stronger assumption that $r_k \in C^{3-k}([a,b],(0,\infty)), k = 1,2$. By changing the equation to a system of two equations, he obtained a Lyapunov-type inequality. For comparison purposes, we summarize the main results in [53], using our notation, in the next proposition.

Proposition 2.2.3. *Assume $x(t)$ is a nontrivial solution of Eq. (2.2.1) satisfying one of the following conditions:*

(a) $x(a) = x(b) = 0$, and there exists a $\xi \in [a,b]$ such that $\left(r_1^{-1}\phi_{\alpha_1}(x')\right)'(\xi) = 0$;

(b) $x(a) = x(b) = x(c) = 0$, $x(t) \neq 0$ for $t \in (a,b)\cup(b,c)$, and $\left(r_1^{-1}\phi_{\alpha_1}(x')\right)'(t) \neq 0$ for $t \in [a,b]$.

Then

$$2\left(\int_a^c |q(t)|\,dt\right)^{\frac{1}{\alpha_1\alpha_2}} > \min_{d\in[a,c]} h(d), \tag{2.2.5}$$

where

$$h(d) = \frac{1}{\left(\int_a^d r_1^{-\frac{1}{\alpha_1}}(t)\,dt\right)\left(\int_a^d r_2^{-\frac{1}{\alpha_2}}(t)\,dt\right)^{\frac{1}{\alpha_1}}} + \frac{1}{\left(\int_d^c r_1^{-\frac{1}{\alpha_1}}(t)\,dt\right)\left(\int_d^c r_2^{-\frac{1}{\alpha_2}}(t)\,dt\right)^{\frac{1}{\alpha_1}}}.$$

We note that inequality (2.2.5) is much more complicated than its counterparts (2.1.2) and (2.1.16) given in Theorems 2.1.1 and 2.1.2, respectively; and is not comparable in form with second-order case. Moreover, since the inequality in (2.2.5) uses $|q(t)|$ rather than $q_{\pm}(t)$, it is not as sharp as those in (2.1.2) and (2.1.16).

In [53, Theorem 3.2], the author also gave an estimate for the number of zeros of an oscillatory solution of Eq. (2.1.1) for the case with $\alpha \geq 1$. However, the case with $0 < \alpha < 1$ was left as an open problem there. In Theorem 2.1.3, we solve the problem for the general case including the case with $0 < \alpha < 1$, which improves the estimate for the number of zeros even for the case with $\alpha \geq 1$.

2.3 The Linear Case

Finally, for the convenience of the reader, we summarize the results for the linear case extracted from Section 2.1. Consider the third-order linear differential equation

$$x''' + q(t)x = 0. \tag{2.3.1}$$

Note that $\alpha_1 = \alpha_2 = 1$ and $\alpha = 2$ in this case. By Theorem 2.1.1, we obtain the following result.

Corollary 2.3.1. *Assume $x(t)$ is a solution of Eq. (2.3.1) with $x(a) = x(b) = 0$ and $x(t) \neq 0$ for $t \in (a,b)$. Suppose that there is a $\xi \in [a,b]$ such that $x''(\xi) = 0$. Then*

$$\int_a^{\xi} q_-(s)ds + \int_{\xi}^{b} q_+(s)ds > \frac{4}{(b-a)^2}.$$

The next result is from Theorem 2.1.2.

Corollary 2.3.2. *Assume $x(t)$ is a solution of Eq. (2.3.1) with $x(a) = x(b) = x(c) = 0$ and $x(t) \neq 0$ for all $t \in (a,b) \cup (b,c)$. Then either*

$$\max_{\xi \in [a,b]} \left\{ \int_a^{\xi} q_-(s)ds + \int_{\xi}^{b} q_+(s)ds \right\} > \frac{4}{(b-a)^2}$$

or

$$\max_{\xi \in [b,c]} \left\{ \int_b^{\xi} q_-(s)ds + \int_{\xi}^{c} q_+(s)ds \right\} > \frac{4}{(c-b)^2}.$$

As a result,

$$\max_{\xi \in [a,c]} \left\{ \int_a^{\xi} q_-(s)ds + \int_{\xi}^{c} q_+(s)ds \right\} > \frac{4}{(c-a)^2}.$$

It is easy to see that these coraollaries provide sharper results for the linear case than those by Parhi and Panigrahi [65] as summarized in Theorem 1.1.4.

2.4 Applications to boundary value problems

In the last section, we apply the results on the Lyapunov-type inequalities obtained in Section 2.1 to study the nonexistence, uniqueness, and existence-uniqueness for solutions of certain third-order BVPs. Consider the BVPs consisting of Eq. (2.1.1) and one of the boundary conditions

$$x(a) = 0, \quad x(b) = 0, \quad (\phi_{\alpha_1}(x'))'(\xi) = 0, \tag{2.4.1}$$

where $-\infty < a \leq \xi \leq b < \infty$; and

$$x(a) = 0, \quad x(b) = 0, \quad x(c) = 0, \tag{2.4.2}$$

where $-\infty < a < b < c < \infty$.

The first result is on the nonexistence of solutions of these BVPs.

Theorem 2.4.1. *(a) Assume*

$$\int_a^\xi q_-(s)ds + \int_\xi^b q_+(s)ds \leq \left(\frac{2}{b-a}\right)^\alpha. \tag{2.4.3}$$

Then BVP (2.1.1), (2.4.1) has no nontrivial solution.

(b) Assume

$$\max_{\xi\in[a,b]}\left\{\int_a^\xi q_-(s)ds + \int_\xi^b q_+(s)ds\right\} \leq \left(\frac{2}{b-a}\right)^\alpha \tag{2.4.4}$$

and

$$\max_{\xi\in[b,c]}\left\{\int_b^\xi q_-(s)ds + \int_\xi^c q_+(s)ds\right\} \leq \left(\frac{2}{c-b}\right)^\alpha. \tag{2.4.5}$$

Then BVP (2.1.1), (2.4.2) has no nontrivial solution.

Proof. (a) Assume the contrary, i.e., BVP (2.1.1), (2.4.1) has a nontrivial solution $x(t)$. Then there exist consecutive zeros t_1, t_2 of $x(t)$ such that $a \le t_1 < t_2 \le b$ and $\xi \in [t_1, t_2]$. By Theorem 2.1.1 we have

$$\int_{t_1}^{\xi} q_-(s)ds + \int_{\xi}^{t_2} q_+(s)ds > \left(\frac{2}{t_2 - t_1}\right)^{\alpha}.$$

It follows that

$$\int_{a}^{\xi} q_-(s)ds + \int_{\xi}^{b} q_+(s)ds > \left(\frac{2}{b - a}\right)^{\alpha}.$$

This contradicts assumption (2.4.3).

(b) Assume the contrary, i.e., BVP (2.1.1), (2.4.2) has a nontrivial solution $x(t)$. Then there exist three consecutive zeros t_1, t_2, t_3 of $x(t)$ such that $a \le t_1 < t_2 < t_3 \le b$. By Theorem 2.1.2 we have either

$$\max_{\xi \in [t_1, t_2]} \left\{ \int_{t_1}^{\xi} q_-(s)ds + \int_{\xi}^{t_2} q_+(s)ds \right\} > \left(\frac{2}{t_2 - t_1}\right)^{\alpha}$$

or

$$\max_{\xi \in [t_2, t_3]} \left\{ \int_{t_2}^{\xi} q_-(s)ds + \int_{\xi}^{t_3} q_+(s)ds \right\} > \left(\frac{2}{t_3 - t_2}\right)^{\alpha}.$$

It follows that either

$$\max_{\xi \in [a, b]} \left\{ \int_{a}^{\xi} q_-(s)ds + \int_{\xi}^{b} q_+(s)ds \right\} > \left(\frac{2}{b - a}\right)^{\alpha}$$

or

$$\max_{\xi \in [b, c]} \left\{ \int_{b}^{\xi} q_-(s)ds + \int_{\xi}^{c} q_+(s)ds \right\} > \left(\frac{2}{c - b}\right)^{\alpha}.$$

This contradicts assumptions (2.4.4) and (2.4.5). □

As a direct consequence of Theorem 2.4.1, Part (a), we have the following corollary.

Corollary 2.4.1. *Assume* $q(t)\begin{cases}\geq 0, & t\in[a,\xi)\\ \leq 0, & t\in[\xi,b]\end{cases}$. *Then BVP (2.1.1), (2.4.1) has no nontrivial solution. In particular, if either*

$$a=\xi \text{ and } q(t)\leq 0 \text{ for } t\in[a,b]$$

or

$$\xi=b \text{ and } q(t)\geq 0 \text{ for } t\in[a,b],$$

then BVP (2.1.1), (2.4.1) has no nontrivial solution.

Next we consider the third-order nonhomogeneous linear BVPs consisting of the equation

$$x'''+q(t)x=h(t) \text{ on } (A,B)\,, \tag{2.4.6}$$

where $-\infty<A<B<\infty$ and $q,h\in C\left((A,B)\,,\mathbb{R}\right)$, and one of the following boundary conditions

$$x\left(t_1\right)=k_1,\quad x\left(t_2\right)=k_2,\quad x'\left(t_2\right)=k_3; \tag{2.4.7}$$

$$x\left(t_1\right)=k_1,\quad x'\left(t_1\right)=k_2,\quad x\left(t_2\right)=k_3; \tag{2.4.8}$$

$$x\left(t_1\right)=k_1,\quad x\left(t_2\right)=k_2,\quad x\left(t_3\right)=k_3; \tag{2.4.9}$$

where

$$A<t_1<t_2<t_3<B \text{ and } k_1,k_2,k_3\in\mathbb{R}. \tag{2.4.10}$$

Now, we present an existence and uniqueness criterion for a nontrivial solution of these BVPs.

Theorem 2.4.2. *Assume*

$$\max_{\xi\in[A,B]}\left\{\int_A^{\xi} q_-(t)dt+\int_{\xi}^{B} q_+(t)dt\right\}\leq\frac{4}{(B-A)^2}. \tag{2.4.11}$$

Then each of the BVPs (2.4.6), (2.4.7); (2.4.6), (2.4.8); and (2.4.6), (2.4.9) has a unique solution on (A,B) *for any* t_1,t_2,t_3 *and* k_1,k_2,k_3 *satisfying (2.4.10).*

To prove Theorem 2.4.2, we first introduce the following results by Jackson and Schrader [45] and Jackson [44] on BVPs associated with the general third-order equation

$$x'''=f\left(t,x,x',x''\right), \tag{2.4.12}$$

where

(a) $f\in C\left((A,B)\times\mathbb{R}^3,\mathbb{R}\right)$, and

(b) any initial value problem associated with Eq. (2.4.12) has a unique solution which exists on the whole interval (A,B).

Lemma 2.4.1. *Assume (a) and (b) are satisfied and BVP (2.4.12), (2.4.9) has at most one solution on* (A,B). *Then each of the BVPs (2.4.12), (2.4.7); (2.4.12), (2.4.8); and (2.4.12),(2.4.9) has a unique solution on* (A,B).

Under the assumptions (a) and (b), Jackson and Schrader [45, Theorem 2, 3] showed that if the solution of BVP (2.4.12), (2.4.9), when exists, is unique on (A,B), then each of the BVPs (2.4.12), (2.4.7); (2.4.12), (2.4.8); and (2.4.12),(2.4.9) has at least one solution on (A,B). Then, Jackson [44, Theorem 1, 2] further proved that the uniqueness of the solution of BVP (2.4.12), (2.4.9) guarantees the uniqueness of the solutions of BVPs (2.4.12), (2.4.7) and (2.4.12), (2.4.8).

Based on Lemma 2.4.1, we give the proof for Theorem 2.4.2 below.

Proof. of Theorem 2.2.2. Note that assumptions (a) and (b) are satisfied by Eq. (2.4.6) since it is linear. First we show that BVP (2.4.6), (2.4.9) has at most one solution for any t_1, t_2, t_3 and k_1, k_2, k_3 satisfying (2.4.10). Assume the contrary, i.e., BVP (2.4.6), (2.4.9) has two solutions $x_1(t)$ and $x_2(t)$ in (A, B). Define $x(t) = x_1(t) - x_2(t)$. Then $x(t)$ is a solution of the third-order homogeneous linear BVP

$$x''' + q(t)x = 0, \tag{2.4.13}$$

$$x(t_1) = 0, \quad x(t_2) = 0, \quad x(t_3) = 0. \tag{2.4.14}$$

On the other hand, from (2.4.11) we have

$$\max_{\xi \in [t_1, t_2]} \left\{ \int_{t_1}^{\xi} q_-(s)ds + \int_{\xi}^{t_2} q_+(s)ds \right\} \leq \frac{4}{(t_2 - t_1)^2}$$

and

$$\max_{\xi \in [t_2, t_3]} \left\{ \int_{t_2}^{\xi} q_-(s)ds + \int_{\xi}^{t_3} q_+(s)ds \right\} \leq \frac{4}{(t_3 - t_2)^2}.$$

Then by applying Theorem 2.4.1, Part (b) with $\alpha = 2$ we see that BVP (2.4.13), (2.4.14) has no nontrivial solution in (A, B). Hence $x(t) \equiv 0$ which follows that $x_1(t) \equiv x_2(t)$. Thus, the uniqueness of the solution of BVP (2.4.6), (2.4.9) is established. By Lemma 3.1, we conclude that each of the BVPs (2.4.6), (2.4.7); (2.4.6), (2.4.8); and (2.4.6),(2.4.9) has a unique solution on (A, B). $\square$

CHAPTER 3

THIRD-ORDER LINEAR EQUATIONS

3.1 Lyapunov-type inequalities

In this section, we let $-\infty < a < b < c < \infty$ and consider the third-order linear differential equation

$$x''' + q(t)x = 0, \tag{3.1.1}$$

where $q \in C([a, c], \mathbb{R})$.

Theorem 3.1.1. *Assume Eq. (3.1.1) has a solution $x(t)$ satisfying*

$$x(a) = x(b) = x(c) = 0 \quad \textit{and} \quad x(t) \neq 0 \textit{ for } t \in (a, b) \cup (b, c). \tag{3.1.2}$$

Then one of the following holds:

(a) $\displaystyle\int_a^c (t-a)(c-t)q_-(t)dt > 2,$

(b) $\displaystyle\int_a^c (t-a)(c-t)q_+(t)dt > 2,$

(c) $\displaystyle\int_a^b (t-a)(c-t)q_-(t)dt + \int_b^c (t-a)(c-t)q_+(t)dt > 2.$

Proof. By the continuity of $x(t)$ on $[a, c]$, there exist $t_1 \in (a, b)$ and $t_2 \in (b, c)$ such that $|x(t_1)| = \max\{|x(t)| : \ t \in [a, b]\}$ and $|x(t_2)| = \max\{|x(t)| : \ t \in [b, c]\}$. As a result,

$$x'(t_1) = x'(t_2) = 0.$$

Let

$$G(t,s)=\frac{1}{t_2-t_1}\begin{cases}(s-t_1)(t_2-t), & t_1\le s\le t\le t_2;\\ (t-t_1)(t_2-s), & t_1\le t\le s\le t_2.\end{cases}$$

Then $G(t,s)$ is the Green's function for the BVP

$$-y''=r(t),\quad y(t_1)=y(t_2)=0. \tag{3.1.3}$$

Hence the solution $y(t)$ satisfies

$$y(t)=\int_{t_1}^{t_2}G(t,s)r(s)ds. \tag{3.1.4}$$

We observe that for the solution $x(t)$ of Eq. (3.1.1), $y(t):=x'(t)$ satisfies (3.1.3) with $r(t)=q(t)x(t)$. By (3.1.4)

$$x'(t)=\int_{t_1}^{t_2}G(t,s)q(s)x(s)ds.$$

It follows that

$$x(t)=\int_b^t\int_{t_1}^{t_2}G(\tau,s)q(s)x(s)dsd\tau=\int_{t_1}^{t_2}\Big(\int_b^t G(\tau,s)d\tau\Big)q(s)x(s)ds. \tag{3.1.5}$$

Note that for $s\in[t_1,t_2]$

$$\begin{aligned}\int_{t_1}^{t_2}G(t,s)dt&=\int_{t_1}^{s}G(t,s)dt+\int_{s}^{t_2}G(t,s)dt\\&=\frac{1}{t_2-t_1}\Big[(t_2-s)\int_{t_1}^{s}(t-t_1)dt+(s-t_1)\int_{s}^{t_2}(t_2-t)dt\Big]\\&=\frac{1}{t_2-t_1}\Big[\frac{(t_2-s)(s-t_1)^2}{2}+\frac{(t_2-s)^2(s-t_1)}{2}\Big]\\&=\frac{1}{2}(s-t_1)(t_2-s),\end{aligned}$$

and hence

$$\int_{t_1}^{t_2} G(t,s)dt \le \frac{1}{2}(s-a)(c-s). \tag{3.1.6}$$

Without loss of generality, we may assume $x(t)$ satisfies one of following cases:

I. $x(t) > 0$ on $(a,b) \cup (b,c)$ and $x(t_1) \ge x(t_2)$;

II. $x(t) > 0$ on $(a,b) \cup (b,c)$ and $x(t_1) < x(t_2)$;

III. $x(t) > 0$ on (a,b), $x(t) < 0$ on (b,c), and $x(t_1) \ge -x(t_2)$;

IV. $x(t) > 0$ on (a,b), $x(t) < 0$ on (b,c), and $x(t_1) < -x(t_2)$.

In the sequel, we denote $m = \max\{|x(t_1)|, |x(t_2)|\}$.

Case I: In this case, $m = x(t_1)$. Then (3.1.5) with $t = t_1$ shows that

$$m = \int_{t_1}^{t_2} \Big(\int_{t_1}^{b} G(t,s)dt\Big)(-q(s))x(s)ds.$$

Using the facts that $G(t,s) \ge 0$ on $[t_1,t_2] \times [t_1,t_2]$, $0 \le x(t) \le m$ and $x(t) \not\equiv m$ on $[t_1,t_2]$, and $-q(t) \le q_-(t)$, we have

$$m < m\int_{t_1}^{t_2} \Big(\int_{t_1}^{t_2} G(t,s)dt\Big)q_-(s)ds.$$

Cancelling m from both sides and using (3.1.6) we obtain

$$1 < \frac{1}{2}\int_{t_1}^{t_2} (s-a)(c-s)q_-(s)ds \le \frac{1}{2}\int_{a}^{c} (s-a)(c-s)q_-(s)ds,$$

i.e., conclusion (a) holds.

Case II: In this case, $m = x(t_2)$. Then (3.1.5) with $t = t_2$ shows that

$$m = \int_{t_1}^{t_2} \Big(\int_{b}^{t_2} G(t,s)dt\Big)q(s)x(s)ds.$$

Using the facts that $G(t,s) \geq 0$ on $[t_1,t_2] \times [t_1,t_2]$, $0 \leq x(t) \leq m$ and $x(t) \not\equiv m$ on $[t_1,t_2]$, and $q(t) \leq q_+(t)$, we have

$$m < m \int_{t_1}^{t_2} \Big(\int_{t_1}^{t_2} G(t,s)dt \Big) q_+(s)ds.$$

Cancelling m from both sides and using (3.1.6) we obtain

$$1 < \frac{1}{2} \int_{t_1}^{t_2} (s-a)(c-s)q_+(s)ds \leq \frac{1}{2} \int_a^c (s-a)(c-s)q_+(s)ds,$$

i.e., conclusion (b) holds.

Case III: In this case, $m = x(t_1)$. Then (3.1.5) with $t = t_1$ shows that

$$\begin{aligned} m &= \int_{t_1}^{t_2} \Big(\int_{t_1}^{b} G(t,s)dt \Big) (-q(s))x(s)ds \\ &= \int_{t_1}^{b} \Big[\int_{t_1}^{b} G(t,s)(-q(s))x(s)ds + \int_{b}^{t_2} G(t,s)q(s)(-x(s))ds \Big] dt. \end{aligned}$$

Note that $x(t) > 0$ on (a,b) and $x(t) < 0$ on (b,c). Then by a similar argument to Cases I and II, we see that

$$1 < \frac{1}{2} \Big[\int_a^b (s-a)(c-s)q_-(s)ds + \int_b^c (s-a)(c-s)q_+(s)ds \Big],$$

i.e., conclusion (c) holds.

Case IV: The same argument as in Case III also leads to conclusion (c). We omit the detail. □

From the fact that $4\alpha\beta \leq (\alpha+\beta)^2$ for any $\alpha, \beta \in \mathbb{R}$, we see that $4(s-a)(c-s) \leq (c-a)^2$. Then the following corollary is a direct consequence of Theorem 3.1.1.

Corollary 3.1.1. *Assume Eq. (3.1.1) has a solution* $x(t)$ *satisfying (3.1.2). Then one of the following holds:*

(a) $\displaystyle\int_a^c q_-(t)dt > \frac{8}{(c-a)^2}$,

(b) $\displaystyle\int_a^c q_+(t)dt > \frac{8}{(c-a)^2}$,

(c) $\displaystyle\int_a^b q_-(t)dt + \int_b^c q_+(t)dt > \frac{8}{(c-a)^2}$.

As a result,

$$\int_a^c |q(t)|dt > \frac{8}{(c-a)^2}.$$

Remark 3.1.1. It is clear that the inequalities obtained in this section are sharper than those in (1.1.8) and (2.1.16) and hence is the best in the literature. In particular, by Corollary 3.1.1, the constant 4 on both the right-hand sides is improved to a larger constant 8, the integral of $|q(t)|$ on the left-hand side of (1.1.8) is replaced by integrals of $q_+(t)$ and $q_-(t)$, and the maximum for $\xi \in [a, c]$ on the left-hand side of (2.1.16) is replaced by the maximum over three values $\xi = a, b, c$.

3.2 Generalizations

In this section, we extend the Lyapunov-type inequalities obtained in Section 3.1 to several more general third-order linear equations. All the results obtained in this section are new in the literature. Here we let $-\infty < a < b < c < \infty$.

We first consider the differential equation

$$(p(t)x'')' + q(t)x = 0, \tag{3.2.1}$$

where $p, q \in C([a, c], \mathbb{R})$ and $p(t) > 0$ for $t \in [a, c]$. We denote $P = \max\{p(t) : t \in [a, c]\}$.

Theorem 3.2.1. *Assume Eq. (3.2.1) has a solution $x(t)$ satisfying (3.1.2). Then one of the following holds:*

(a) $\displaystyle\int_a^c \Big(\int_a^t \frac{d\tau}{p(\tau)}\Big)\Big(\int_t^c \frac{d\tau}{p(\tau)}\Big) q_-(t)dt > \frac{2}{P},$

(b) $\displaystyle\int_a^c \Big(\int_a^t \frac{d\tau}{p(\tau)}\Big)\Big(\int_t^c \frac{d\tau}{p(\tau)}\Big) q_+(t)dt > \frac{2}{P},$

(c) $\displaystyle\int_a^b \Big(\int_a^t \frac{d\tau}{p(\tau)}\Big)\Big(\int_t^c \frac{d\tau}{p(\tau)}\Big) q_-(t)dt + \int_b^c \Big(\int_a^t \frac{d\tau}{p(\tau)}\Big)\Big(\int_t^c \frac{d\tau}{p(\tau)}\Big) q_+(t)dt > \frac{2}{P}.$

Proof. We first consider the case where $0 < p(t) \le 1$ for $t \in [a, c]$. Let t_1 and t_2 be defined as in the proof of Theorem 3.1.1 and let

$$G(t,s) = \frac{1}{\int_{t_1}^{t_2} \frac{d\tau}{p(\tau)}} \begin{cases} \int_{t_1}^{s} \frac{d\tau}{p(\tau)} \int_{t}^{t_2} \frac{d\tau}{p(\tau)}, & t_1 \le s \le t \le t_2; \\ \int_{t_1}^{t} \frac{d\tau}{p(\tau)} \int_{s}^{t_2} \frac{d\tau}{p(\tau)}, & t_1 \le t \le s \le t_2. \end{cases} \tag{3.2.2}$$

Then $G(t, s)$ is the Green's function for the BVP

$$-(p(t)y')' = r(t), \quad y(t_1) = y(t_2) = 0. \tag{3.2.3}$$

As shown in the proof of Theorem 3.1.1, (3.1.5) holds with $G(t, s)$ defined by (3.2.2), i.e.,

$$x(t) = \int_{t_1}^{t_2} \Big(\int_b^t G(\tau, s)d\tau\Big) q(s)x(s)ds. \tag{3.2.4}$$

Note that for $s \in [t_1, t_2]$

$$\begin{aligned} \int_{t_1}^{t_2} G(t,s)dt &= \int_{t_1}^{s} G(t,s)dt + \int_{s}^{t_2} G(t,s)dt \\ &= \frac{1}{\int_{t_1}^{t_2} \frac{d\tau}{p(\tau)}} \Big[\int_s^{t_2} \frac{d\tau}{p(\tau)} \int_{t_1}^{s} \int_{t_1}^{t} \frac{d\tau}{p(\tau)}\, dt + \int_{t_1}^{s} \frac{d\tau}{p(\tau)} \int_s^{t_2} \int_t^{t_2} \frac{d\tau}{p(\tau)}\, dt\Big]. \end{aligned} \tag{3.2.5}$$

We claim that for $s \in [t_1, t_2]$

$$H(s) := \int_{t_1}^{s} \int_{t_1}^{t} \frac{d\tau}{p(\tau)}\, dt - \frac{1}{2}\Big(\int_{t_1}^{s} \frac{d\tau}{p(\tau)}\Big)^2 \le 0$$

and hence

$$\int_{t_1}^{s} \int_{t_1}^{t} \frac{d\tau}{p(\tau)}\, dt \le \frac{1}{2}\Big(\int_{t_1}^{s} \frac{d\tau}{p(\tau)}\Big)^2. \tag{3.2.6}$$

This is because $H(t_1) = 0$ and

$$H'(s) = \int_{t_1}^{s} \frac{d\tau}{p(\tau)} - \frac{1}{p(s)} \int_{t_1}^{s} \frac{d\tau}{p(\tau)} = \Big(1 - \frac{1}{p(s)}\Big) \int_{t_1}^{s} \frac{d\tau}{p(\tau)} \le 0$$

due to the assumption that $0 < p(t) \le 1$. Similarly, for $s \in [t_1, t_2]$

$$\int_{s}^{t_2} \int_{t}^{t_2} \frac{d\tau}{p(\tau)}\, dt \le \frac{1}{2}\Big(\int_{s}^{t_2} \frac{d\tau}{p(\tau)}\Big)^2. \tag{3.2.7}$$

Substituting (3.2.6) and (3.2.7) into (3.2.5) we see that for $s \in [t_1, t_2]$

$$\begin{aligned}
\int_{t_1}^{t_2} G(t,s)dt &\le \frac{1}{2\int_{t_1}^{t_2} \frac{d\tau}{p(\tau)}} \Big[\Big(\int_{t_1}^{s} \frac{d\tau}{p(\tau)}\Big)^2 \int_{s}^{t_2} \frac{d\tau}{p(\tau)} + \Big(\int_{s}^{t_2} \frac{d\tau}{p(\tau)}\Big)^2 \int_{t_1}^{s} \frac{d\tau}{p(\tau)}\Big] \\
&= \frac{1}{2\int_{t_1}^{t_2} \frac{d\tau}{p(\tau)}} \int_{t_1}^{s} \frac{d\tau}{p(\tau)} \int_{s}^{t_2} \frac{d\tau}{p(\tau)} \Big[\int_{t_1}^{s} \frac{d\tau}{p(\tau)} + \int_{s}^{t_2} \frac{d\tau}{p(\tau)}\Big] \\
&= \frac{1}{2} \int_{t_1}^{s} \frac{d\tau}{p(\tau)} \int_{s}^{t_2} \frac{d\tau}{p(\tau)} \le \frac{1}{2} \int_{a}^{s} \frac{d\tau}{p(\tau)} \int_{s}^{c} \frac{d\tau}{p(\tau)}.
\end{aligned}$$

Then a similar argument to that in the proof of Theorem 3.1.1 shows that one of the following holds:

(i) $\int_a^c \Big(\int_a^t \frac{d\tau}{p(\tau)}\Big)\Big(\int_t^c \frac{d\tau}{p(\tau)}\Big) q_-(t)dt > 2,$

(ii) $\int_a^c \Big(\int_a^t \frac{d\tau}{p(\tau)}\Big)\Big(\int_t^c \frac{d\tau}{p(\tau)}\Big) q_+(t)dt > 2,$

(iii) $\int_a^b \Big(\int_a^t \frac{d\tau}{p(\tau)}\Big)\Big(\int_t^c \frac{d\tau}{p(\tau)}\Big) q_-(t)dt + \int_b^c \Big(\int_a^t \frac{d\tau}{p(\tau)}\Big)\Big(\int_t^c \frac{d\tau}{p(\tau)}\Big) q_+(t)dt > 2.$

Now, we consider the general case with $p(t) > 0$. Dividing both sides of Eq. (3.2.1) by P we obtain

$$\Big(\frac{p(t)}{P}x''\Big)' + \frac{q(t)}{P}x = 0. \tag{3.2.8}$$

Note that Eq. (3.2.8) is in the form of Eq. (3.2.1) with $p(t)$ and $q(t)$ replaced by $p(t)/P$ and $q(t)/P$, respectively; and $0 < p(t)/P \le 1$. Then the conclusions (a)-(c) follow from (i)-(iii) above with $p(t)$ and $q(t)$ replaced by $p(t)/P$ and $q(t)/P$, respectively. □

Again, from the fact that $4\alpha\beta \le (\alpha+\beta)^2$ for any $\alpha, \beta \in \mathbb{R}$ we see that

$$4\int_a^t \frac{d\tau}{p(\tau)} \int_t^c \frac{d\tau}{p(\tau)} \le \Big(\int_a^t \frac{d\tau}{p(\tau)} + \int_t^c \frac{d\tau}{p(\tau)}\Big)^2 = \Big(\int_a^c \frac{d\tau}{p(\tau)}\Big)^2.$$

Then the following corollary is a direct consequence of Theorem 3.2.1.

Corollary 3.2.1. *Assume Eq. (3.2.1) has a solution $x(t)$ satisfying (3.1.2). Then one of the following holds:*

(a) $\int_a^c q_-(t)dt > \frac{8}{P(\int_a^c p^{-1}(\tau)\, d\tau)^2},$

(b) $\int_a^c q_+(t)dt > \frac{8}{P(\int_a^c p^{-1}(\tau)\, d\tau)^2},$

(c) $\int_a^b q_-(t)dt + \int_b^c q_+(t)dt > \frac{8}{P(\int_a^c p^{-1}(\tau)\, d\tau)^2}.$

As a result,

$$\int_a^c |q(t)|dt > \frac{8}{P(\int_a^c p^{-1}(\tau)\, d\tau)^2}.$$

The result below is provided by the anonymous referee:

Corollary 3.2.2. *Assume Eq. (3.2.1) has a solution $x(t)$ satisfying (3.1.2). Then one of the following holds:*

(a) $\displaystyle\int_a^c q_-(t)dt > \frac{4}{(c-a)\int_a^c p^{-1}(\tau)\,d\tau},$

(b) $\displaystyle\int_a^c q_+(t)dt > \frac{4}{(c-a)\int_a^c p^{-1}(\tau)\,d\tau},$

(c) $\displaystyle\int_a^b q_-(t)dt + \int_b^c q_+(t)dt > \frac{4}{(c-a)\int_a^c p^{-1}(\tau)\,d\tau}.$

As a result,

$$\int_a^c |q(t)|dt > \frac{4}{(c-a)\int_a^c p^{-1}(\tau)\,d\tau}.$$

Proof. The proof is a modification of that of Theorem 3.2.1. We only provide the outline here. For the function $G(t,s)$ given in (3.2.2) we have

$$G(s,s) = \frac{1}{\int_{t_1}^{t_2} \frac{d\tau}{p(\tau)}} \int_{t_1}^{s} \frac{d\tau}{p(\tau)} \int_{s}^{t_2} \frac{d\tau}{p(\tau)}.$$

Again, using the inequality $4\alpha\beta \le (\alpha+\beta)^2$, we have

$$G(s,s) \le \frac{1}{4\int_{t_1}^{t_2} \frac{d\tau}{p(\tau)}} \Big(\int_{t_1}^{s} \frac{d\tau}{p(\tau)} + \int_{s}^{t_2} \frac{d\tau}{p(\tau)} \Big)^2 = \frac{1}{4} \int_{t_1}^{t_2} \frac{d\tau}{p(\tau)}.$$

Note that $\max_{t_1 \le t \le t_2} G(t,s) = G(s,s)$, we have

$$\int_{t_1}^{t_2} G(t,s)\,dt \le \int_{t_1}^{t_2} G(s,s)\,dt \le \frac{t_2 - t_1}{4} \int_{t_1}^{t_2} \frac{d\tau}{p(\tau)} \le \frac{c-a}{4} \int_a^c \frac{d\tau}{p(\tau)}.$$

Then the rest of the proof is essentially in the same way as that of Theorem 3.1.1 and hence is omitted. □

Remark 3.2.1. Corollary 3.2.2 is a variation of Corollary 3.2.1. It is not intended to replace Corollary 3.2.1; however, it provides a better result for the case when $p(t)$ has a large maximum but not a large average. More precisely, the inequalities in Corollary 3.2.2 are sharper than those in Corollary 3.2.1 when $P > 2(c-a)/\int_a^c p^{-1}(\tau)\,d\tau$.

We next consider the equation

$$x''' + f(t)x'' + h(t)x = 0, \tag{3.2.9}$$

where $f, h \in C([a,c], \mathbb{R})$. We denote

$$p(t) = \exp\Big(\int_a^t f(s)ds\Big) \text{ and } P = \exp\Big(\max\Big\{\int_a^t f(s)ds : t \in [a,c]\Big\}\Big).$$

Theorem 3.2.2. *Assume Eq. (3.2.9) has a solution $x(t)$ satisfying (3.1.2). Then one of the following holds:*

(a) $\displaystyle\int_a^c \Big(\int_a^t \frac{d\tau}{p(\tau)}\Big)\Big(\int_t^c \frac{d\tau}{p(\tau)}\Big)p(t)h_-(t)dt > \frac{2}{P},$

(b) $\displaystyle\int_a^c \Big(\int_a^t \frac{d\tau}{p(\tau)}\Big)\Big(\int_t^c \frac{d\tau}{p(\tau)}\Big)p(t)h_+(t)dt > \frac{2}{P},$

(c) $$\int_a^b \Big(\int_a^t \frac{d\tau}{p(\tau)}\Big)\Big(\int_t^c \frac{d\tau}{p(\tau)}\Big)p(t)h_-(t)dt + \int_b^c \Big(\int_a^t \frac{d\tau}{p(\tau)}\Big)\Big(\int_t^c \frac{d\tau}{p(\tau)}\Big)p(t)h_+(t)dt > \frac{2}{P}.$$

Proof. Let $q(t) = h(t)\exp(\int_a^t f(s)ds)$. Multiplying both sides of (3.2.9) by the integral factor $p(t)$, we obtain Eq. (3.2.1). Note that $p(t) > 0$ for $t \in [a,c]$. Then the result follows from Theorem 3.2.1 immediately. □

The following corollary is an immediate consequence of Theorem 3.2.2.

Corollary 3.2.3. *Assume Eq. (3.2.9) has a solution $x(t)$ satisfying (3.1.2). Then one of the following holds:*

(a) $\displaystyle\int_a^c p(t)h_-(t)dt > \frac{8}{P\big(\int_a^c p^{-1}(\tau)d\tau\big)^2}$,

(b) $\displaystyle\int_a^c p(t)h_+(t)dt > \frac{8}{P\big(\int_a^c p^{-1}(\tau)d\tau\big)^2}$,

(c) $\displaystyle\int_a^b p(t)h_-(t)dt + \int_b^c p(t)h_+(t)dt > \frac{8}{P\big(\int_a^c p^{-1}(\tau)d\tau\big)^2}$.

As a result,

$$\int_a^c |h(t)|dt > \frac{8}{P\big(\int_a^c p^{-1}(\tau)d\tau\big)^2}.$$

Finally, we consider the general third-order linear equation

$$x''' + f(t)x'' + g(t)x' + h(t)x = 0, \tag{3.2.10}$$

where $f, g, h \in C([a, c], \mathbb{R})$. We denote $p(t) = \exp(\int_a^t f(s)ds)$.

Theorem 3.2.3. *Assume Eq. (3.2.10) has a solution $x(t)$ satisfying (3.1.2). Then*

$$\int_a^c \Big(|g(t)| + (c-a)|h(t)|\Big)p(t)dt > \frac{4}{\int_a^c p^{-1}(t)dt}. \tag{3.2.11}$$

Proof. Multiplying both sides of Eq. (3.2.10) by the integral factor $p(t)$ we obtain the equation

$$-(p(t)x'')' = r(t), \tag{3.2.12}$$

where $r(t) = p(t)(g(t)x'(t) + h(t)x(t))$. Let t_1 and t_2 be defined as in the proof of Theorem 3.1.1. Let $G(t, s)$ be given by (3.2.2). By (3.2.12),

$$\begin{aligned} x'(t) &= \int_{t_1}^{t_2} G(t,s)r(s)ds \\ &= \int_{t_1}^{t_2} G(t,s)p(s)\Big(g(s)x'(s) + h(s)x(s)\Big)ds \\ &= \int_{t_1}^{t_2} G(t,s)p(s)\Big(g(s)x'(s) + h(s)\int_b^s x'(\tau)d\tau\Big)ds. \end{aligned}$$

Taking absolute value on both sides and considering that $G(t,s) \geq 0$ on $[t_1, t_2] \times [t_1, t_2]$ and $p(t) > 0$, we have

$$|x'(t)| \leq \int_{t_1}^{t_2} G(t,s)p(s)\Big(|g(s)||x'(s)| + |h(s)|\Big|\int_b^s x'(\tau)d\tau\Big|\Big)ds. \tag{3.2.13}$$

Note $G(t,s) \leq G(s,s)$ and $4\alpha\beta \leq (\alpha+\beta)^2$. Then

$$G(t,s) \leq \frac{1}{\int_{t_1}^{t_2} \frac{d\tau}{p(\tau)}} \int_{t_1}^{s} \frac{d\tau}{p(\tau)} \int_{s}^{t_2} \frac{d\tau}{p(\tau)} \leq \frac{1}{4}\int_{t_1}^{t_2} \frac{d\tau}{p(\tau)} \leq \frac{1}{4}\int_{a}^{c} \frac{d\tau}{p(\tau)}. \tag{3.2.14}$$

Let $m_1 = \max\{|x'(t)| : t \in [t_1, t_2]\}$. Taking the maximum of $|x'(t)|$ on (3.2.13) and using the fact that $0 \leq |x'(t)| \leq m_1$ and $|x'(t)| \not\equiv m_1$ on $[t_1, t_2]$, we have

$$\begin{aligned} m_1 &= m_1 \int_{t_1}^{t_2} G(s,s)p(s)\Big(|g(s)| + |s-b||h(s)|\Big)ds \\ &\leq m_1 \int_a^c G(s,s)p(s)\Big(|g(s)| + (c-a)|h(s)|\Big). \end{aligned}$$

By substituting (3.2.14) in the above and cancelling m_1 from both sides, we obtain (3.2.11).

□

If $f(t) \equiv 0$, then Eq. (3.2.10) becomes

$$x''' + g(t)x' + h(t)x = 0; \tag{3.2.15}$$

and if $h(t) \equiv 0$, then (3.2.10) becomes

$$x''' + f(t)x'' + g(t)x' = 0. \tag{3.2.16}$$

We observe that $p(t) \equiv 1$ for the former case. As direct consequences of Theorem 3.2.3, we have the following results.

Corollary 3.2.4. *(a) Assume Eq. (3.2.15) has a solution $x(t)$ satisfying (3.1.2). Then*

$$\int_a^c \Big(|g(t)| + (c-a)|h(t)|\Big)dt > \frac{4}{c-a}.$$

(b) Assume Eq. (3.2.16) has a solution $x(t)$ satisfying (3.1.2). Then

$$\int_a^c p(t)|g(t)|dt > \frac{4}{\int_a^c p^{-1}(t)dt}.$$

Remark 3.2.2. We note that Eq. (3.2.10) also includes Eqs. (3.1.1) and (3.2.9) as special cases. However, Theorem 3.2.3 does not cover the results in Theorems 3.1.1 and 3.2.2. This is due to the complexity caused by the appearance of the term $g(t)x'$, and hence the result has to be derived from an inequality for x' rather than from one for x as in the proofs of Theorems 3.1.1 and 3.2.2. Moreover, since the zeros of x' may have accumulation points in $[a, c]$, we cannot locate any specific subintervals of $[a, c]$ where x' has a fixed sign. Therefore, the Lyapunov-type inequalities given in Theorem 3.2.3 and Corollary 3.2.4 can only involve $|g|$ and $|h|$ instead of $g_\pm$ and $h_\pm$.

3.3 Applications to boundary value problems

In the last section, we apply the results on the Lyapunov-type inequalities obtained in Sections 3.1 and 3.2 to study the nonexistence, uniqueness, and existence-uniqueness for solutions of certain third-order BVPs.

We first consider the BVP consisting of Eq. (3.1.1) and the following boundary condition (BC)

$$x(a) = x(b) = x(c) = 0, \tag{3.3.1}$$

where $-\infty < a < b < c < \infty$ and $q \in C([a,c],\mathbb{R})$.

The result below is on the nonexistence of nontrivial solutions of the three point BVP (3.1.1), (3.3.1).

Theorem 3.3.1. *Assume all the following three conditions are satisfied:*

(a) $\displaystyle\int_a^c q_-(t)dt \le \frac{8}{(c-a)^2}$,

(b) $\displaystyle\int_a^c q_+(t)dt \le \frac{8}{(c-a)^2}$,

(c) $\displaystyle\int_a^b q_-(t)dt + \int_b^c q_+(t)dt \le \frac{8}{(c-a)^2}$.

Then BVP (3.1.1), (3.3.1) has only the trivial solution.

Proof. Assume the contrary, i.e., BVP (3.1.1), (3.3.1) has a nontrivial solution $x(t)$. Then there exist $a_1 \in [a,b)$ and $c_1 \in (b,c]$ such that $x(a_1) = x(c_1) = 0$ and $x(t) \neq 0$ for $t \in (a_1,b) \cup (b,c_1)$. It follows that one of the conclusions (a)-(c) in Corollary 3.1.1 holds with a and c replaced by a_1 and c_1, respectively. Without loss of generality, we assume (a) holds. Hence,

$$\int_a^c q_-(t)dt \ge \int_{a_1}^{c_1} q_-(t)dt > \frac{8}{(c_1-a_1)^2} \ge \frac{8}{(c-a)^2}$$

which contradicts condition (a) in the theorem. □

Next we consider the third-order nonhomogeneous linear BVPs consisting of the equation

$$x''' + q(t)x = w(t) \text{ on } (A,B), \tag{3.3.2}$$

where $-\infty < A < B < \infty$ and $q, w \in C((A,B),\mathbb{R})$; and one of the following BCs

$$x(t_1) = k_1, \quad x(t_2) = k_2, \quad x'(t_2) = k_3, \tag{3.3.3}$$

$$x(t_1) = k_1, \quad x'(t_1) = k_2, \quad x(t_2) = k_3, \tag{3.3.4}$$

and

$$x(t_1) = k_1, \quad x(t_2) = k_2, \quad x(t_3) = k_3, \tag{3.3.5}$$

where

$$A < t_1 < t_2 < t_3 < B \text{ and } k_1, k_2, k_3 \in \mathbb{R}. \tag{3.3.6}$$

Here we present a criterion for the above BVPs to have a unique solution.

Theorem 3.3.2. *Assume*

$$\max_{\xi \in [A,B]} \left\{ \int_A^\xi q_-(t)dt + \int_\xi^B q_+(t)dt \right\} \leq \frac{8}{(B-A)^2}. \tag{3.3.7}$$

Then each of the BVPs (3.3.2), (3.3.3); (3.3.2), (3.3.4); and (3.3.2), (3.3.5) has a unique solution on (A, B) *for any* t_1, t_2, t_3 *and* k_1, k_2, k_3 *satisfying (3.3.6).*

To prove Theorem 3.3.2, we first introduce the following results by Jackson and Schrader [48] and Jackson [44] on BVPs associated with the general third-order equation

$$x''' = F(t, x, x', x''), \tag{3.3.8}$$

where

(i) $F \in C((A, B) \times \mathbb{R}^3, \mathbb{R})$, and

(ii) any initial value problem associated with Eq. (3.3.8) has a unique solution which exists on the whole interval.

Lemma 3.3.1. *Assume (i) and (ii) are satisfied and BVP (3.3.8), (3.3.5) has at most one solution on* (A, B)*. Then each of the BVPs (3.3.8), (3.3.3); (3.3.8), (3.3.4); and (3.3.8),(3.3.5) has a unique solution on* (A, B)*.*

Under assumptions (i) and (ii), Jackson and Schrader [48, Theorem 2, 3] showed that if the solution of BVP (3.3.8), (3.3.5), when exists, is unique on (A, B), then each of the BVPs (3.3.8), (3.3.3); (3.3.8), (3.3.4); and (3.3.8),(3.3.5) has at least one solution on (A, B). Furthermore, Jackson [44, Theorem 1, 2] proved that the uniqueness of the solution of BVP (3.3.8), (3.3.5) guarantees the uniqueness of the solution of BVPs (3.3.8), (3.3.3); (3.3.8), (3.3.4); and (3.3.8),(3.3.5).

Using Lemma 3.3.1, we prove Theorem 3.3.2 below.

Proof of Theorem 3.3.2. Note that assumptions (i) and (ii) are satisfied by Eq. (3.3.2) since it is linear. First we show that BVP (3.3.2), (3.3.5) has at most one solution for any t_1, t_2, t_3 and k_1, k_2, k_3 satisfying (3.3.6). For otherwise, it has two solutions $x_1(t)$ and $x_2(t)$ in (A, B). Define $x(t) = x_1(t) - x_2(t)$. Then $x(t)$ is a solution of the BVP consisting of Eq. (3.1.1) and the BC

$$x(t_1) = 0, \quad x(t_2) = 0, \quad x(t_3) = 0. \tag{3.3.9}$$

From (3.3.7) we have for any $\xi \in [t_1, t_3]$

$$\max_{\xi \in [t_1, t_3]} \left\{ \int_{t_1}^{\xi} q_-(t)dt + \int_{\xi}^{t_3} q_+(t)dt \right\} \leq \frac{8}{(t_3 - t_1)^2}.$$

In particular, for $\xi = t_3$, t_1, t_2, respectively, we see that conditions (a)-(c) in Theorem 3.3.1 are satisfied with $a = t_1, b = t_2, c = t_3$. Then by Theorem 3.3.1, $x(t) \equiv 0$, i.e., $x_1(t) \equiv x_2(t)$. This shows the uniqueness of the solution of BVP (3.3.2), (3.3.5). By Lemma 3.3.1 each of the BVPs (3.3.2), (3.3.3); (3.3.2), (3.3.4); and (3.3.2), (3.3.5) has a unique solution on (A, B). $\square$

Now we consider the BVPs consisting of one of the equations

$$x''' + f(t)x'' + h(t)x = r(t) \tag{3.3.10}$$

and

$$x''' + f(t)x'' + g(t)x' + h(t)x = r(t), \tag{3.3.11}$$

where $f, g, h, r \in C((A, B), \mathbb{R})$; and one of the BCs (3.3.3), (3.3.4), (3.3.5). In the following, we state criteria for these BVPs to have a unique solution. As before, we denote

$$p(t) = \exp\Big(\int_A^t f(s)ds\Big) \quad \text{and} \quad P = \exp\Big(\max\Big\{\int_A^t f(s)ds : t \in [a, c]\Big\}\Big).$$

Theorem 3.3.3. *Assume*

$$\max_{\xi \in [A,B]} \Big\{\int_A^\xi p(t)h_-(t)dt + \int_\xi^B p(t)h_+(t)dt\Big\} \le \frac{8}{P(\int_A^B p^{-1}(\tau)d\tau)^2}.$$

Then each of the BVPs (3.3.10), (3.3.3); (3.3.10), (3.3.4); and (3.3.10), (3.3.5) has a unique solution on (A, B) *for any* t_1, t_2, t_3 *and* k_1, k_2, k_3 *satisfying (3.3.6).*

Theorem 3.3.4. *Assume*

$$\int_A^B \Big(|g(t)| + (B - A)|h(t)|\Big)p(t)dt \le \frac{4}{\big(\int_A^B p^{-1}(\tau)d\tau\big)^2}.$$

Then each of the BVPs (3.3.11), (3.3.3); (3.3.11), (3.3.4); and (3.3.11), (3.3.5) has a unique solution on (A, B) *for any* t_1, t_2, t_3 *and* k_1, k_2, k_3 *satisfying (3.3.6).*

The proofs of Theorems 3.3.3 and 3.3.4 are similar to that of Theorem 3.3.2. We omit the details.

CHAPTER 4

HIGHER ORDER HALF-LINEAR EQUATIONS

4.1 Lyapunov-type inequalities

We consider higher order half-linear differential equations in the form

$$\left(x^{[N]}\right)' + q(t)\phi_{\alpha[1,N]}(x) = 0, \tag{4.1.1}$$

where

(i) $N \in \mathbb{N}$;

(ii) $x^{[i]} = \phi_{\alpha_i}\left[\left(x^{[i-1]}\right)'\right]$ for $i = 1, ..., N$ with $x^{[0]} = x$, and $\phi_p(x) = |x|^{p-1}x$ for $p > 0$;

(iii) $q \in C([a,b], \mathbb{R})$ with $-\infty < a < b < \infty$;

(iv) $\alpha[i,j] = \alpha_i ... \alpha_j$ for $1 \leq i < j \leq N$ and $\alpha[i,i] = \alpha_i$ with $\alpha_i > 0$ for $i = 1, ..., N$.

Note that, for $N = 1$ and $N = 2$, Eq. (4.1.1) becomes the second and third order half-linear differential equations

$$(\phi_{\alpha_1}(x'))' + q(t)\phi_{\alpha_1}(x) = 0$$

and

$$\left(\phi_{\alpha_2}(\phi_{\alpha_1}(x'))'\right)' + q(t)\phi_{\alpha_1\alpha_2}(x) = 0,$$

respectively.

As is conventional, we denote

$$\sum_{i=j}^{k} a_i = 0 \text{ and } \prod_{i=j}^{k} a_i = 1 \text{ for } j > k. \tag{4.1.2}$$

We also denote by ϕ_p^{-1} the inverse function of ϕ_p for any $p > 0$. Clearly, $\phi_p^{-1} = \phi_{\frac{1}{p}}$. When $p = 1$, we have $\phi_p = \phi_p^{-1} = I$, the identity operator. We also observe that

$$\frac{\alpha\,[i,k]}{\alpha\,[i,j]} = \alpha\,[j+1,k] \text{ for } 1 \le i \le j < k \le N. \tag{4.1.3}$$

To simplify the statements and proofs of the main results, we will use the notation

$$\beta_N = \sum_{i=1}^{N} \alpha[i,N]. \tag{4.1.4}$$

Clearly,

$$(\beta_j + 1)\,\alpha_{j+1} = \beta_{j+1} \text{ for } 1 \le j \le N. \tag{4.1.5}$$

If $N = 2n-1$ for some $n \in \mathbb{N}$, then Eq. (4.1.1) becomes the even order equation

$$\left(x^{[2n-1]}\right)' + q(t)\phi_{\alpha[1,2n-1]}\,(x) = 0; \tag{4.1.6}$$

and if $N = 2n$, then Eq. (4.1.1) becomes the odd order equation

$$\left(x^{[2n]}\right)' + q(t)\phi_{\alpha[1,2n]}\,(x) = 0. \tag{4.1.7}$$

First, we present results on Lyapunov-type inequalities for the even order equation (4.1.6) with certain BCs.

Theorem 4.1.1. *Assume Eq. (4.1.6) has a nontrivial solution $x(t)$ satisfying the BCs*

$$x^{[2i]}(a) = x^{[2i]}(b) = 0, \ \ i = 0, ..., n-1. \tag{4.1.8}$$

Then

$$\int_a^b |q(t)|\, dt > \frac{2^{\beta_{2n-1}+1}}{(b-a)^{\beta_{2n-1}}}. \tag{4.1.9}$$

Theorem 4.1.2. *Assume Eq. (4.1.6) has a nontrivial solution $x(t)$ satisfying the BCs*

$$x^{[i]}(a) = x^{[i]}(b) = 0, \ \ i = 0, ..., n-1. \tag{4.1.10}$$

Then

$$\int_a^b |q(t)|\, dt > \prod_{j=1}^{n-1} \left[\sum_{k=j-1}^{n-1} \frac{1}{\alpha\,[j,k]} \right]^{\alpha[j,2n-1]} \frac{2^{\sum_{k=1}^{n+1} \alpha[k,2n-1]}}{(b-a)^{\beta_{2n-1}}}. \tag{4.1.11}$$

Then we present Lyapunov-type inequalities for the odd order equation (4.1.7) with certain BCs.

Theorem 4.1.3. *Let $\xi \in [a,b]$. Assume Eq. (4.1.7) has a nontrivial solution $x(t)$ satisfying BCs*

$$x^{[2i]}(a) = x^{[2i]}(b) = 0, \ \ i = 0, ..., n-1; \ \ \text{and} \ x^{[2n]}(\xi) = 0. \tag{4.1.12}$$

Then

$$\int_a^b |q(t)|\, dt > \left(\frac{2}{b-a} \right)^{\beta_{2n}}. \tag{4.1.13}$$

Theorem 4.1.4. *Let $\xi \in [a,b]$. Assume Eq. (4.1.7) has a nontrivial solution $x(t)$ satisfying BCs*

$$x^{[i]}(a) = x^{[i]}(b) = 0, \ \ i = 0, ..., n-1; \ \ \text{and} \ x^{[2n]}(\xi) = 0. \tag{4.1.14}$$

Then

$$\int_a^b |q(t)|\,dt > \prod_{j=1}^{n-1}\left[\sum_{k=j-1}^{n-1}\frac{1}{\alpha\,[j,k]}\right]^{\alpha[j,2n]}\frac{2^{\sum_{k=1}^{n+1}\alpha[k,2n]}}{(b-a)^{\beta_{2n}}}. \tag{4.1.15}$$

As immediate consequences of Theorems 4.1.1-4.1.4, the following corollary gives sufficient conditions for the nonexistence of nontrivial solutions of the above BVPs.

Corollary 4.1.1. *(a) Assume*

$$\int_a^b |q(t)|\,dt \le \frac{2^{\beta_{2n-1}+1}}{(b-a)^{\beta_{2n-1}}}. \tag{4.1.16}$$

Then BVP (4.1.6), (4.1.8) has no nontrivial solution.

(b) Assume

$$\int_a^b |q(t)|\,dt \le \prod_{j=1}^{n-1}\left[\sum_{k=j-1}^{n-1}\frac{1}{\alpha\,[j,k]}\right]^{\alpha[j,2n-1]}\frac{2^{\sum_{k=1}^{n+1}\alpha[k,2n-1]}}{(b-a)^{\beta_{2n-1}}}.$$

Then BVP (4.1.6), (4.1.10) has no nontrivial solution.

(c) Assume

$$\int_a^b |q(t)|\,dt \le \left(\frac{2}{b-a}\right)^{\beta_{2n}}.$$

Then BVP (4.1.7), (4.1.12) has no nontrivial solution.

(d) Assume

$$\int_a^b |q(t)|\,dt \le \prod_{j=1}^{n-1}\left[\sum_{k=j-1}^{n-1}\frac{1}{\alpha\,[j,k]}\right]^{\alpha[j,2n]}\frac{2^{\sum_{k=1}^{n+1}\alpha[k,2n]}}{(b-a)^{\beta_{2n}}}.$$

Then BVP (4.1.7), (4.1.14) has no nontrivial solution.

In the following, we apply the obtained Lyapunov-type inequalities to the eigenvalue problems associated with the half-linear equations

$$\left(x^{[2n-1]}\right)' + \lambda q(t)\phi_{\alpha[1,2n-1]}(x) = 0 \tag{4.1.17}$$

and

$$\left(x^{[2n]}\right)' + \lambda q(t)\phi_{\alpha[1,2n]}(x) = 0, \tag{4.1.18}$$

where $\lambda \in \mathbb{R}$ is an eigenvalue parameter. We say that $\lambda = \lambda^*$ is an eigenvalue of a BVP associated with Eq. (4.1.17) if the BVP with $\lambda = \lambda^*$ has a nontrivial solution. As direct consequences of Theorems 4.1.1 and 4.1.2, we obtain the following results.

Corollary 4.1.2. *(a) Assume λ is an eigenvalue of BVP (4.1.17), (4.1.8). Then*

$$|\lambda| > \frac{2^{\beta_{2n-1}+1}}{(b-a)^{\beta_{2n-1}} \int_a^b |q(t)|\,dt}.$$

(b) Assume λ is an eigenvalue of BVP (4.1.17), (4.1.10). Then

$$|\lambda| > \prod_{j=1}^{n-1} \left[\sum_{k=j-1}^{n-1} \frac{1}{\alpha\,[j,k]}\right]^{\alpha[j,2n-1]} \frac{2^{\sum_{k=1}^{n+1} \alpha[k,2n-1]}}{(b-a)^{\beta_{2n-1}} \int_a^b |q(t)|\,dt}.$$

Following Theorems 4.1.3 and 4.1.4, we can obtain parallel results for eigenvalue problems associated with Eq. (4.1.18) under BC (4.1.12) or (4.1.14). We omit the detail.

Finally, for the convenience of the reader, we interpret the results in Theorems 4.1.1-4.1.4 to the linear case. In this case, $\alpha_i = 1$ for $1 \le i \le m$ and hence Eq. (4.1.6) becomes the even order linear equation

$$x^{(2n)} + q(t)x = 0 \tag{4.1.19}$$

and Eq. (4.1.7) becomes the odd order linear equation

$$x^{(2n+1)} + q(t)x = 0. \tag{4.1.20}$$

Here we note that $\alpha\,[j, k] = 1$ for any $1 \leq j \leq k$. Hence from (4.1.4)

$$\beta_m = \sum_{i=1}^{m} \alpha[i, m] = m.$$

Corollary 4.1.3. *(a) Assume Eq. (5.1.20) has a nontrivial solution $x(t)$ satisfying*

$$x^{(2i)}\,(a) = x^{(2i)}\,(b) = 0,\ \ i = 0, ..., n-1. \tag{4.1.21}$$

Then

$$\int_a^b |q(t)|\, dt > \frac{2^{2n}}{(b-a)^{2n-1}}. \tag{4.1.22}$$

(b) Assume Eq. (5.1.20) has a nontrivial solution $x(t)$ satisfying

$$x^{(i)}\,(a) = x^{(i)}\,(b) = 0,\ \ i = 0, ..., n-1. \tag{4.1.23}$$

Then

$$\int_a^b |q(t)|\, dt > \frac{n!2^{n+1}}{(b-a)^{2n-1}}. \tag{4.1.24}$$

(c) Let $\xi \in [a, b]$. Assume Eq. (4.1.20) has a nontrivial solution $x(t)$ satisfying

$$x^{(2i)}\,(a) = x^{(2i)}\,(b) = 0,\ \ i = 0, ..., n-1;\ \ \textit{and}\ x^{(2n)}\,(\xi) = 0. \tag{4.1.25}$$

Then

$$\int_a^b |q(t)|\,dt > \left(\frac{2}{b-a}\right)^{2n}. \tag{4.1.26}$$

(d) Let $\xi \in [a,b]$. Assume Eq. (4.1.20) has a nontrivial solution $x(t)$ satisfying

$$x^{(i)}(a) = x^{(i)}(b) = 0,\ i = 0, ..., n-1;\ \textit{and}\ x^{(2n)}(\xi) = 0. \tag{4.1.27}$$

Then

$$\int_a^b |q(t)|\,dt > \frac{n!2^{n+1}}{(b-a)^{2n}}. \tag{4.1.28}$$

It is easy to see that parts (a), (b) and (d) of Corollary 4.1.3 are the same as those in Cakmak [11, Theorem 2], Yang [83, Theorem 1], and Yang and Lo [82, Corollary 3], respectively. To the best of our knowledge, part (c) is new for odd order equations. We point out that for the even order linear equations, sharper results than (4.1.24) have been obtained by the Green's function method; see for example, [14]. However, the Green's function method fails to work for half-linear equations due to their nonlinear nature.

4.2 Proofs

The following version of Jensen's inequality will be used in the proofs of the theorems in Section 4.1.

Lemma 4.2.1. *Assume $\psi(u)$ is a concave-up function on $(0,\infty)$ and $u_1, u_2 \in (0,\infty)$. Then*

$$\psi(u_1) + \psi(u_2) \geq 2\psi\left(\frac{u_1+u_2}{2}\right).$$

We first give a proof for Theorem 4.1.1.

Proof of Theorem 4.1.1. Let $m := \max_{t\in[a,b]} |x(t)|$. Using the Rolle's Theorem and (4.1.8), we see that for $i = 0, ..., n-1$, there exists a $\xi_i \in (a,b)$ such that $\left|x^{[2i]}(\xi_i)\right| = \max_{t\in[a,b]} \left|x^{[2i]}(t)\right|$. It follows that $\left(x^{[2i]}\right)'(\xi_i) = 0$, i.e., $x^{[2i+1]}(\xi_i) = 0$. Recall that $\left(x^{[2i+1]}(t)\right)' = \phi^{-1}_{\alpha_{2i+2}}\left(x^{[2i+2]}(t)\right)$. Then for $i = 0, ..., n-1$

$$\left|x^{[2i+1]}(t)\right| = \left|\int_{\xi_i}^{t} \phi^{-1}_{\alpha_{2i+2}}\left(x^{[2i+2]}(s)\right)\right| ds \le \int_{\xi_i}^{b} \phi^{-1}_{\alpha_{2i+2}}\left(\left|x^{[2i+2]}(s)\right|\right) ds,\ t \in [\xi_i, b). \tag{4.2.1}$$

Note that $x^{[2i]}(b) = 0$. Then

$$x^{[2i]}(\xi_i) = -\int_{\xi_i}^{b} \left(x^{[2i]}(t)\right)' dt = -\int_{\xi_i}^{b} \phi^{-1}_{\alpha_{2i+1}}\left(x^{[2i+1]}(t)\right) dt.$$

Then

$$\left|x^{[2i]}(\xi_i)\right| \le \int_{\xi_i}^{b} \phi^{-1}_{\alpha_{2i+1}}\left(\left|x^{[2i+1]}(t)\right|\right) dt. \tag{4.2.2}$$

The combination of (4.2.1) and (4.2.2) leads to

$$\begin{aligned}\left|x^{[2i]}(\xi_i)\right| &\le \int_{\xi_i}^{b} \phi^{-1}_{\alpha_{2i+1}}\left(\int_{\xi_i}^{b} \phi^{-1}_{\alpha_{2i+2}}\left(\left|x^{[2i+2]}(s)\right|\right) ds\right) dt \\ &= (b-\xi_i)\, \phi^{-1}_{\alpha_{2i+1}}\left(\int_{\xi_i}^{b} \phi^{-1}_{\alpha_{2i+2}}\left(\left|x^{[2i+2]}(s)\right|\right) ds\right).\end{aligned}$$

Let $t \in [a,b]$. By the definition of ξ_i and (4.2.2) we see that

$$\left|x^{[2i]}(t)\right| \le \left|x^{[2i]}(\xi_i)\right| \le (b-\xi_i)\, \phi^{-1}_{\alpha_{2i+1}}\left(\int_{\xi_i}^{b} \phi^{-1}_{\alpha_{2i+2}}\left(\left|x^{[2i+2]}(s)\right|\right) ds\right). \tag{4.2.3}$$

Dividing both sides of (4.2.3) by $(b-\xi_i)$ and applying $\phi_{\alpha_{2i+1}}$ we get

$$\frac{1}{(b-\xi_i)^{\alpha_{2i+1}}} \phi_{\alpha_{2i+1}}\left(\left|x^{[2i]}(t)\right|\right) \le \int_{\xi_i}^{b} \phi^{-1}_{\alpha_{2i+2}}\left(\left|x^{[2i+2]}(s)\right|\right) ds. \tag{4.2.4}$$

With a similar argument as above we obtain that

$$\frac{1}{(\xi_i - a)^{\alpha_{2i+1}}}\phi_{\alpha_{2i+1}}\left(\left|x^{[2i]}\,(t)\right|\right) \leq \int_a^{\xi_i} \phi^{-1}_{\alpha_{2i+2}}\left(\left|x^{[2i+2]}(s)\right|\right) ds. \tag{4.2.5}$$

Adding (4.2.4) and (4.2.5) we have

$$\left[\frac{1}{(b-\xi_i)^{\alpha_{2i+1}}} + \frac{1}{(\xi_i - a)^{\alpha_{2i+1}}}\right]\phi_{\alpha_{2i+1}}\left(\left|x^{[2i]}(t)\right|\right) \leq \int_a^b \phi^{-1}_{\alpha_{2i+2}}\left(\left|x^{[2i+2]}(s)\right|\right) ds. \tag{4.2.6}$$

It is easy to see that $\psi(u) := u^{-\alpha_{2i+1}}$ is a concave-up function on $(0, \infty)$. By Lemma 4.2.1 with $u_1 = b - \xi_i$ and $u_2 = \xi_i - a$, we have

$$\left[\frac{1}{(b-\xi_i)^{\alpha_{2i+1}}} + \frac{1}{(\xi_i - a)^{\alpha_{2i+1}}}\right] \geq \frac{2}{\left(\frac{b-a}{2}\right)^{\alpha_{2i+1}}} = \frac{2^{\alpha_{2i+1}+1}}{(b-a)^{\alpha_{2i+1}}}.$$

Then from (4.2.6)

$$\frac{2^{\alpha_{2i+1}+1}}{(b-a)^{\alpha_{2i+1}}}\phi_{\alpha_{2i+1}}\left(\left|x^{[2i]}(t)\right|\right) \leq \int_a^b \phi^{-1}_{\alpha_{2i+2}}\left(\left|x^{[2i+2]}(s)\right|\right) ds. \tag{4.2.7}$$

It follows that

$$\left|x^{[2i]}(t)\right| \leq \frac{b-a}{2}\phi^{-1}_{\alpha_{2i+1}}\left[\frac{1}{2}\int_a^b \phi^{-1}_{\alpha_{2i+2}}\left(\left|x^{[2i+2]}(s)\right|\right) ds\right]. \tag{4.2.8}$$

Now we prove by induction that for $i = 0, ..., n-1$,

$$\int_a^b \phi^{-1}_{\alpha_{2i+2}}\left(\left|x^{[2i+2]}(s)\right|\right) ds \geq K_i := \frac{2^{\beta_{2i+1}+1}m^{\alpha[1,2i+1]}}{(b-a)^{\beta_{2i+1}}}, \tag{4.2.9}$$

where $\beta_{2i+1} = \sum_{j=1}^{2i+1} \alpha[j, 2i+1]$, as given by (4.1.4). Note that $\beta_1 = \alpha_1$. For the case when $n = 1$, (4.2.9) follows from (4.2.7) with $i = 0$. Assume $n \geq 2$ and (4.2.9) holds for some $i \in \{0, ..., n-2\}$. It follows from (4.2.9) and (4.2.8) that

$$\begin{aligned} K_i &\leq \int_a^b \phi_{\alpha_{2i+2}}^{-1} \left[\frac{b-a}{2} \phi_{\alpha_{2i+3}}^{-1} \left\{ \frac{1}{2} \int_a^b \phi_{\alpha_{2i+4}}^{-1} \left(\left| x^{[2i+4]}(s) \right| \right) ds \right\} \right] dt \\ &= (b-a) \phi_{\alpha_{2i+2}}^{-1} \left[\frac{b-a}{2} \phi_{\alpha_{2i+3}}^{-1} \left\{ \frac{1}{2} \int_a^b \phi_{\alpha_{2i+4}}^{-1} \left(\left| x^{[2i+4]}(s) \right| \right) ds \right\} \right]. \end{aligned}$$

Thus

$$\frac{b-a}{2} \phi_{\alpha_{2i+3}}^{-1} \left\{ \frac{1}{2} \int_a^b \phi_{\alpha_{2i+4}}^{-1} \left(\left| x^{[2i+4]}(s) \right| \right) ds \right\} \geq \phi_{\alpha_{2i+2}} \left(\frac{K_i}{b-a} \right),$$

and hence

$$\int_a^b \phi_{\alpha_{2i+4}}^{-1} \left(\left| x^{[2i+4]}(s) \right| \right) ds \geq 2\phi_{\alpha_{2i+3}} \left\{ \frac{2}{b-a} \phi_{\alpha_{2i+2}} \left(\frac{K_i}{b-a} \right) \right\}. \tag{4.2.10}$$

From (4.2.9) and (4.1.5) we have

$$2\phi_{\alpha_{2i+3}} \left\{ \frac{2}{b-a} \phi_{\alpha_{2i+2}} \left(\frac{K_i}{b-a} \right) \right\} = K_{i+1}. \tag{4.2.11}$$

Hence (4.2.9) holds for $i+1$.

By (4.2.9) with $i = n-1$ we have

$$\int_a^b \phi_{\alpha_{2n}}^{-1} \left(\left| x^{[2n]}(s) \right| \right) ds \geq K_{n-1} = \frac{2^{\beta_{2n-1}+1} m^{\alpha[1,2n-1]}}{(b-a)^{\beta_{2n-1}}}.$$

Note that $\phi_{\alpha_{2n}}^{-1} \left(\left| x^{[2n]}(s) \right| \right) = \left| \left(x^{[2n-1]}(s) \right)' \right|$. From Eq. (4.1.6) and using the fact that $|x(t)| \leq m$ and $x(t) \not\equiv m$ on $[a, b]$, we have

$$\frac{2^{\beta_{2n-1}+1} m^{\alpha[1,2n-1]}}{(b-a)^{\beta_{2n-1}}} < \int_a^b |q(s)| \, \phi_{\alpha[1,2n-1]}(|x|) \, ds < m^{\alpha[1,2n-1]} \int_a^b |q(s)| \, ds.$$

Hence

$$\int_a^b |q(t)|\,dt > \frac{2^{\beta_{2n-1}+1}}{(b-a)^{\beta_{2n-1}}},$$

i.e., (4.1.9) holds. □

Now we give a proof for Theorem 4.1.2.

Proof of Theorem 4.1.2. Let $m := |x(c)| = \max_{t\in[a,b]} |x(t)|$ for some $c \in (a,b)$. Recall that $\left(x^{[i]}(t)\right)' = \phi^{-1}_{\alpha_{i+1}}\left(x^{[i+1]}(t)\right)$. From (4.1.10) it follows that for $i = 0, ..., n-1$

$$\left|x^{[i]}(t)\right| = \left|\int_a^t \phi^{-1}_{\alpha_{i+1}}\left(x^{[i+1]}(s)\right) ds\right| \le \int_a^t \phi^{-1}_{\alpha_{i+1}}\left(\left|x^{[i+1]}(s)\right|\right) ds,\ t \in [a,b]. \qquad (4.2.12)$$

Using (4.1.10) and applying Rolle's Theorem repeatedly, we see that $x^{[i]}(t)$ has i zeros in (a,b) with at least one of them in $(a,c]$ for $i = 1, ..., n$. In particular, $x^{[n]}(t)$ has a zero $t_n \in (a,c]$. Hence

$$\left|x^{[n]}(t)\right| = \left|\int_{t_n}^t \phi^{-1}_{\alpha_{n+1}}\left(x^{[n+1]}(s)\right) ds\right| \le \int_a^c \phi^{-1}_{\alpha_{n+1}}\left(\left|x^{[n+1]}(s)\right|\right) ds,\ t \in [a,c]. \qquad (4.2.13)$$

Further applications of Rolle's Theorem imply that $x^{[i]}(t)$ has $2n-i$ zeros in (a,b) for $i = n+1, ..., 2n-1$. For $i = n+1, ..., 2n-1$ let $t_i \in (a,b)$ be a zero of $x^{[i]}(t)$. Then

$$\left|x^{[i]}(t)\right| = \left|\int_{t_i}^t \phi^{-1}_{\alpha_{i+1}}\left(\left|x^{[i+1]}(s)\right|\right) ds\right| \le \int_a^b \phi^{-1}_{\alpha_{i+1}}\left(\left|x^{[i+1]}(s)\right|\right) ds,\ t \in [a,b]. \qquad (4.2.14)$$

Now we use a backward induction to show that for $i = 2n-1, ..., n+1$,

$$\left|x^{[i]}(t)\right| < M_i,\ t \in [a,b], \qquad (4.2.15)$$

where

$$M_i = m^{\alpha[1,i]} \phi^{-1}_{\alpha[i+1,2n-1]}\left(\int_a^b |q(s)|\,ds\right)(b-a)^{\sum_{k=i}^{2n-2} \frac{1}{\alpha[i+1,k]}} \qquad (4.2.16)$$

with the convention as defined in (4.1.2). Note that $\phi^{-1}_{\alpha_{2n}}\left(\left|x^{[2n]}(s)\right|\right)=\left|\left(x^{[2n-1]}(s)\right)'\right|$. Then (4.2.14) with $i=2n-1$ together with (4.1.6) implies that for $t\in[a,b]$

$$\left|x^{[2n-1]}(t)\right|\le\int_a^b\left|\left(x^{[2n-1]}(s)\right)'\right|ds=\int_a^b|q(s)|\,\phi_{\alpha[1,2n-1]}\left(|x(s)|\right)ds. \tag{4.2.17}$$

From (4.2.17) and using the facts that $|x(t)|\le m$ and $x(t)\not\equiv m$ on $[a,b]$, we have

$$\left|x^{[2n-1]}(t)\right|<m^{\alpha[1,2n-1]}\int_a^b|q(s)|\,ds=M_{2n-1}.$$

This shows that (4.2.15) holds for $i=2n-1$. Assume (4.2.15) holds for some $i\in\{2n-1,...,n+2\}$. We observe from (4.2.16) that $(b-a)\phi^{-1}_{\alpha_i}(M_i)=M_{i-1}$. Then from (4.2.14) and (4.2.15) we have

$$\left|x^{[i-1]}(t)\right|\le\int_a^b\phi^{-1}_{\alpha_i}\left(\left|x^{[i]}(s)\right|\right)ds<\int_a^b\phi^{-1}_{\alpha_i}(M_i)\,ds=(b-a)\phi^{-1}_{\alpha_i}(M_i)=M_{i-1}.$$

Hence (4.2.15) holds for $i-1$.

Using (4.2.15) with $i=n+1$, we have

$$\left|x^{[n+1]}(t)\right|<M_{n+1},\ t\in[a,b]. \tag{4.2.18}$$

By (4.2.13)

$$\left|x^{[n]}(t)\right|<\int_a^c\phi^{-1}_{\alpha_{n+1}}(M_{n+1})\,ds=(c-a)\phi^{-1}_{\alpha_{n+1}}(M_{n+1}),\ t\in[a,c]. \tag{4.2.19}$$

Again we prove by backward induction that for $i=n,...,0$

$$\left|x^{[i]}(t)\right|<(t-a)^{\sum_{k=i}^{n-1}\frac{1}{\alpha[i+1,k]}}N_i,\ t\in(a,c], \tag{4.2.20}$$

where

$$N_i = \frac{(c-a)^{\frac{1}{\alpha[i+1,n]}}}{\prod_{j=i+1}^{n-1}\left[\sum_{k=j-1}^{n-1}\frac{1}{\alpha[j,k]}\right]^{\frac{1}{\alpha[i+1,j-1]}}}\phi^{-1}_{\alpha[i+1,n+1]}\left(M_{n+1}\right), \tag{4.2.21}$$

where the convention (4.1.2) applies. In fact, (4.2.20) with $i = n$ follows from (4.2.19) immediately. We observe that

$$\phi^{-1}_{\alpha_i}\left(N_i\right) = N_{i-1}\sum_{k=i-1}^{n-1}\frac{1}{\alpha[i,k]}.$$

It is easy to see that

$$1+\sum_{k=i}^{n-1}\frac{1}{\alpha\left[i,k\right]} = \sum_{k=i-1}^{n-1}\frac{1}{\alpha\left[i,k\right]} \quad \text{and} \quad \frac{1}{\alpha_i}\sum_{k=i}^{n-1}\frac{1}{\alpha[i+1,k]} = \sum_{k=i}^{n-1}\frac{1}{\alpha[i,k]}.$$

Then

$$\begin{aligned}\int_a^t \phi^{-1}_{\alpha_i}\left((s-a)^{\sum_{k=i}^{n-1}\frac{1}{\alpha[i+1,k]}}N_i\right)ds &= \phi^{-1}_{\alpha_i}\left(N_i\right)\int_a^t (s-a)^{\sum_{k=i}^{n-1}\frac{1}{\alpha[i,k]}}\,ds \\ &= (t-a)^{\sum_{k=i-1}^{n-1}\frac{1}{\alpha[i,k]}}N_{i-1}.\end{aligned}$$

Assume (4.2.20) holds for some $i \in \{n, ..., 1\}$. From (4.2.12) and (4.2.20) we have that for $t \in (a, c]$

$$\begin{aligned}\left|x^{[i-1]}\left(t\right)\right| &\le \int_a^t \phi^{-1}_{\alpha_i}\left(\left|x^{[i]}\left(s\right)\right|\right)ds \\ &< \int_a^t \phi^{-1}_{\alpha_i}\left((s-a)^{\sum_{k=i}^{n-1}\frac{1}{\alpha[i+1,k]}}N_i\right)ds = (t-a)^{\sum_{k=i-1}^{n-1}\frac{1}{\alpha[i,k]}}N_{i-1}.\end{aligned}$$

Hence (4.2.20) holds for $i-1$.

Using (4.2.20) with $i = 0$ we have

$$|x(t)| = \left|x^{[0]}\left(t\right)\right| < (t-a)^{\sum_{k=0}^{n-1}\frac{1}{\alpha[1,k]}}N_0,\ t \in (a,c], \tag{4.2.22}$$

where as in (4.2.21)

$$N_0 = \frac{(c-a)^{\frac{1}{\alpha[1,n]}}}{\prod_{j=1}^{n-1}\left[\sum_{k=j-1}^{n-1}\frac{1}{\alpha[j,k]}\right]^{\frac{1}{\alpha[1,j-1]}}}\phi^{-1}_{\alpha[1,n+1]}\left(M_{n+1}\right).$$

Recall that $m = |x(c)| = \max_{t\in[a,b]}|x(t)|$. Then letting $t = c$ in (4.2.22) we have

$$m < \frac{(c-a)^{\sum_{k=0}^{n}\frac{1}{\alpha[1,k]}}}{\prod_{j=1}^{n-1}\left[\sum_{k=j-1}^{n-1}\frac{1}{\alpha[j,k]}\right]^{\frac{1}{\alpha[1,j-1]}}}\phi^{-1}_{\alpha[1,n+1]}\left(M_{n+1}\right).$$

Taking $\phi_{\alpha[1,2n-1]}$ on both sides and using (4.1.3) we obtain

$$m^{\alpha[1,2n-1]} < \frac{(c-a)^{\sum_{k=1}^{n+1}\alpha[k,2n-1]}}{\prod_{j=1}^{n-1}\left[\sum_{k=j-1}^{n-1}\frac{1}{\alpha[j,k]}\right]^{\alpha[j,2n-1]}}\phi_{\alpha[n+2,2n-1]}\left(M_{n+1}\right). \tag{4.2.23}$$

From (4.2.16) with $i = n+1$,

$$M_{n+1} = m^{\alpha[1,n+1]}\phi^{-1}_{\alpha[n+2,2n-1]}\left(\int_a^b |q(t)|\,dt\right)(b-a)^{\sum_{k=n+1}^{2n-2}\frac{1}{\alpha[n+2,k]}}.$$

Taking $\phi_{\alpha[n+2,2n-1]}$ on both sides and using (4.1.3) we obtain

$$\phi_{\alpha[n+2,2n-1]}\left(M_{n+1}\right) = m^{\alpha[1,2n-1]}\left(\int_a^b |q(t)|\,dt\right)(b-a)^{\sum_{k=n+1}^{2n-2}\frac{1}{\alpha[n+2,k]}}. \tag{4.2.24}$$

Substituting (4.2.24) to (4.2.23) and canceling $m^{\alpha[1,2n-1]}$ from both sides, we have

$$1 < \frac{(c-a)^{\sum_{k=1}^{n+1}\alpha[k,2n-1]}}{\prod_{j=1}^{n-1}\left[\sum_{k=j-1}^{n-1}\frac{1}{\alpha[j,k]}\right]^{\alpha[j,2n-1]}}\left(\int_a^b |q(t)|\,dt\right)(b-a)^{\sum_{k=n+2}^{2n-1}\alpha[k,2n-1]}.$$

This leads to

$$\int_a^b |q(t)|\,dt > \frac{\prod_{j=1}^{n-1}\left[\sum_{k=j-1}^{n-1}\frac{1}{\alpha[j,k]}\right]^{\alpha[j,2n-1]}}{(c-a)^{\sum_{k=1}^{n+1}\alpha[k,2n-1]}(b-a)^{\sum_{k=n+2}^{2n-1}\alpha[k,2n-1]}}. \tag{4.2.25}$$

With a similar discussion as above, we can show that

$$\int_a^b |q(t)|\,dt > \frac{\prod_{j=1}^{n-1}\left[\sum_{k=j-1}^{n-1}\frac{1}{\alpha[j,k]}\right]^{\alpha[j,2n-1]}}{(b-c)^{\sum_{k=1}^{n+1}\alpha[k,2n-1]}(b-a)^{\sum_{k=n+2}^{2n-1}\alpha[k,2n-1]}}. \tag{4.2.26}$$

Let $\gamma = \sum_{k=1}^{n+1}\alpha[k,2n-1]$. Adding (4.2.25) and (4.2.26) we have

$$2\int_a^b |q(t)|\,dt > \frac{\prod_{j=1}^{n-1}\left[\sum_{k=j-1}^{n-1}\frac{1}{\alpha[j,k]}\right]^{\alpha[j,2n-1]}}{(b-a)^{\sum_{k=n+2}^{2n-1}\alpha[k,2n-1]}}\left(\frac{1}{(c-a)^{\gamma}}+\frac{1}{(b-c)^{\gamma}}\right). \tag{4.2.27}$$

It is easy to see that $\psi(u) := u^{-\gamma}$ is a concave-up function on $(0,\infty)$. By Lemma 4.2.1 with $u_1 = c-a$ and $u_2 = b-c$, we have

$$\frac{1}{(c-a)^{\gamma}}+\frac{1}{(b-c)^{\gamma}} \geq \frac{2}{\left(\frac{b-a}{2}\right)^{\gamma}} = \frac{2^{\gamma+1}}{(b-a)^{\gamma}}.$$

This together with (4.2.27) yields

$$\int_a^b |q(t)|\,dt > \frac{\prod_{j=1}^{n-1}\left[\sum_{k=j-1}^{n-1}\frac{1}{\alpha[j,k]}\right]^{\alpha[j,2n-1]}}{(b-a)^{\sum_{k=n+2}^{2n-1}\alpha[k,2n-1]}}\left(\frac{2}{b-a}\right)^{\gamma}. \tag{4.2.28}$$

Note that

$$\gamma + \sum_{k=n+2}^{2n-1}\alpha[k,2n-1] = \sum_{k=1}^{n+1}\alpha[k,2n-1] + \sum_{k=n+2}^{2n-1}\alpha[k,2n-1] = \sum_{k=1}^{2n-1}\alpha[k,2n-1] = \beta_{2n-1}.$$

Then (4.2.28) becomes

$$\int_a^b |q(t)|\, dt > \prod_{j=1}^{n-1}\left[\sum_{k=j-1}^{n-1}\frac{1}{\alpha\,[j,k]}\right]^{\alpha[j,2n-1]}\frac{2^{\sum_{k=1}^{n+1}\alpha[k,2n-1]}}{(b-a)^{\beta_{2n-1}}},$$

i.e., (4.1.11) holds. □

The proofs of Theorems 4.1.3 and 4.1.4 are similar to those of Theorems 4.1.1 and 4.1.2, respectively. We only provide outlines of the proofs here.

Proof of Theorem 4.1.3. Let $m := \max_{t\in[a,b]}|x(t)|$. Then as shown in the proof of Theorem 4.1.1, (4.2.9) holds for $i = 0, ..., n-1$. By (4.2.9) with $i = n-1$,

$$\int_a^b \phi_{\alpha_{2n}}^{-1}\left(\left|x^{[2n]}(t)\right|\right) dt > K_{n-1} = \frac{2^{\beta_{2n-1}+1}m^{\alpha[1,2n-1]}}{(b-a)^{\beta_{2n-1}}}. \tag{4.2.29}$$

Note that $x^{[2n]}(\xi) = 0$ with $\xi \in [a,b]$. By integrating Eq. (4.1.7) from ξ to t we get

$$x^{[2n]}(t) = -\int_\xi^t q(s)\phi_{\alpha[1,2n]}(x(s))\, ds,\ t \in [a,b].$$

Using the facts $|x(t)| \le m$ and $x(t) \not\equiv m$ on $[a,b]$, we have

$$\begin{aligned}\left|x^{[2n]}(t)\right| &= \left|-\int_\xi^t q(s)\phi_{\alpha[1,2n]}(x(s))\, ds\right| \\ &\le \int_a^b |q(s)|\,\phi_{\alpha[1,2n]}(|x(s)|)\, ds < m^{\alpha[1,2n]}\int_a^b |q(s)|\, ds.\end{aligned}$$

Hence it follows from (4.2.29) that

$$\begin{aligned}\frac{2^{\beta_{2n-1}+1}m^{\alpha[1,2n-1]}}{(b-a)^{\beta_{2n-1}}} &< \int_a^b \phi_{2n}^{-1}\left(m^{\alpha[1,2n]}\int_a^b |q(t)|\, dt\right) ds \\ &= (b-a)m^{\alpha[1,2n-1]}\phi_{\alpha_{2n}}^{-1}\left(\int_a^b |q(t)|\, dt\right).\end{aligned}$$

Dividing both sides by $(b-a)$ and canceling $m^{\alpha[1,2n-1]}$, we obtain

$$\phi_{\alpha_{2n}}^{-1}\left(\int_a^b |q(t)|\,dt\right) > \left(\frac{2}{b-a}\right)^{\beta_{2n-1}+1}.$$

This together with (4.1.5) leads to

$$\int_a^b |q(t)|\,dt > \left(\frac{2}{b-a}\right)^{\beta_{2n}},$$

i.e., (4.1.13) holds. □

Proof of Theorem 4.1.4. Let $m := |x(c)| = \max_{t\in[a,b]} |x(t)|$ for some $c \in (a,b)$. Then as shown in the proof of Theorem 4.1.2, (4.2.12)-(4.2.14) hold. Note that $x^{[2n]}(\xi) = 0$ with $\xi \in [a,b]$. Then integrating Eq. (4.1.7) from ξ to t we get

$$x^{[2n]}(t) = -\int_\xi^t q(s)\phi_{\alpha[1,2n]}(x(s))\,ds,\ \ t\in[a,b].$$

Using the facts $|x(t)| \le m$ and $x(t) \not\equiv m$ on $[a,b]$, it follows that

$$\left|x^{[2n]}(t)\right| \le \int_a^b |q(s)|\,\phi_{\alpha[1,2n]}(|x(s)|)\,ds < m^{\alpha[1,2n]}\int_a^b |q(s)|\,ds. \tag{4.2.30}$$

With a similar argument to the proof of Theorem 4.1.2, we have that for $i = 2n, ..., n+1$,

$$\left|x^{[i]}(t)\right| < M_i,\ \ t\in[a,b] \tag{4.2.31}$$

with

$$M_i = m^{\alpha[1,i]}\phi_{\alpha[i+1,2n]}^{-1}\left(\int_a^b |q(t)|\,dt\right)(b-a)^{\sum_{k=i}^{2n-1}\frac{1}{\alpha[i+1,k]}}, \tag{4.2.32}$$

where the convention (4.1.2) applies.

The rest of the proof is essentially the same as that of Theorem 4.1.2 and hence is omitted.

□

CHAPTER 5

ODD ORDER LINEAR EQUATIONS

5.1 Lyapunov-type inequalities

We let $-\infty < a < b < \infty$ and consider the odd order linear differential equation

$$x^{(2n+1)} + (-1)^{n-1} q(t) x = 0 \tag{5.1.1}$$

with $n \in \mathbb{N}$ and $q \in C([a,b], \mathbb{R})$. To simplify the notation, in the following, we denote

$$S_n = \sum_{j=0}^{n-1} \sum_{k=0}^{j} 2^{2k-2j} \binom{n-1+j}{j} \binom{j}{k} B(n+1, n+k-j), \tag{5.1.2}$$

where $B(\alpha, \beta) = \displaystyle\int_0^1 z^{\alpha-1}(1-z)^{\beta-1} dz$ is the Beta function for $\alpha, \beta > 0$.

Theorem 5.1.1. *Assume Eq. (5.1.1) has a nontrivial solution $x(t)$ satisfying*

$$x^{(i+1)}(a) = x^{(i+1)}(b) = 0, \quad i = 0, 1, ..., n-1$$

and $x(c) = 0$ for $c \in [a, b]$. Then

$$\int_a^b |q(t)| dt > \frac{2^{2n}(2n-1)!}{(b-a)^{2n} S_n}. \tag{5.1.3}$$

Proof. As shown in [14], the Green's function for the BVP

$$\begin{aligned} y^{(2n)} + (-1)^{n-1} h(t) = 0, \\ y^{(i)}(a) = y^{(i)}(b) = 0, \quad i = 0, 1, ..., n-1 \end{aligned} \tag{5.1.4}$$

is given by

$$G(t,s) = \begin{cases} \dfrac{1}{(2n-1)!}\left(\dfrac{(t-a)(b-s)}{b-a}\right)^n \displaystyle\sum_{j=0}^{n-1} \binom{n+j-1}{j} (s-t)^{n-j-1} \left(\dfrac{(b-t)(s-a)}{b-a}\right)^j, \\ \qquad\qquad a \le t \le s \le b; \\ \dfrac{1}{(2n-1)!}\left(\dfrac{(s-a)(b-t)}{b-a}\right)^n \displaystyle\sum_{j=0}^{n-1} \binom{n+j-1}{j} (t-s)^{n-j-1} \left(\dfrac{(t-a)(b-s)}{b-a}\right)^j, \\ \qquad\qquad a \le s \le t \le b. \end{cases} \tag{5.1.5}$$

Hence the solution $y(t)$ of BVP (5.1.4) satisfies

$$y(t) = \int_a^b G(t,s) h(s) ds. \tag{5.1.6}$$

We notice that for the solution $x(t)$ of Eq. (5.1.1), $y(t) := x'(t)$ satisfies (5.1.4) with $h(t) = q(t)x(t)$. By (5.1.6)

$$x'(t) = \int_a^b G(t,s) q(s) x(s) ds. \tag{5.1.7}$$

Integrating (5.1.7) from c to t and noting that $x(c) = 0$, we have

$$x(t) = \int_c^t \int_a^b G(\tau,s) q(s) x(s) ds d\tau = \int_a^b \Big(\int_c^t G(\tau,s) d\tau \Big) q(s) x(s) ds. \tag{5.1.8}$$

It is easy to see that $G(t,s)\geq 0$ on $[a,b]\times[a,b]$. It follows that

$$|x(t)|=\Big|\int_a^b\Big(\int_c^t G(\tau,s)d\tau\Big)q(s)x(s)ds\Big|\leq\int_a^b\Big(\int_a^b G(\tau,s)d\tau\Big)|q(s)||x(s)|ds. \tag{5.1.9}$$

We first show that for $s\in[a,b]$

$$\int_a^b G(\tau,s)d\tau\leq\frac{(b-a)^{2n}S_n}{2^{2n}(2n-1)!} \tag{5.1.10}$$

where S_n is defined in (5.1.2). In fact,

$$\int_a^b G(\tau,s)d\tau=\int_a^s G(\tau,s)d\tau+\int_s^b G(\tau,s)d\tau. \tag{5.1.11}$$

We consider each integral separately. To ease the notation, we denote

$$A_j(s)=\frac{1}{(2n-1)!}\begin{pmatrix} n+j-1\\ j\end{pmatrix}\frac{(b-s)^n(s-a)^j}{(b-a)^{n+j}}. \tag{5.1.12}$$

Then from (5.1.5)

$$\begin{aligned}\int_a^s G(\tau,s)d\tau &= \int_a^s(\tau-a)^n\sum_{j=0}^{n-1}A_j(s)(s-\tau)^{n-j-1}(b-\tau)^j d\tau\\ &= \sum_{j=0}^{n-1}A_j(s)\int_a^s(\tau-a)^n(s-\tau)^{n-j-1}(b-\tau)^j d\tau. \end{aligned}\tag{5.1.13}$$

We write

$$(b-\tau)^j=(b-s+s-\tau)^j=\sum_{k=0}^{j}\begin{pmatrix} j\\ k\end{pmatrix}(b-s)^{j-k}(s-\tau)^k.$$

Substituting it into (5.1.13), we obtain

$$\begin{aligned}\int_a^s G(\tau,s)d\tau &= \sum_{j=0}^{n-1} A_j(s)\int_a^s (\tau-a)^n(s-\tau)^{n-j-1}\sum_{k=0}^{j}\left(\begin{array}{c} j \\ k \end{array}\right)(b-s)^{j-k}(s-\tau)^k d\tau \\ &= \sum_{j=0}^{n-1} A_j(s)\sum_{k=0}^{j}\left(\begin{array}{c} j \\ k \end{array}\right)(b-s)^{j-k}\int_a^s (\tau-a)^n(s-\tau)^{n-j+k-1}d\tau. \quad (5.1.14)\end{aligned}$$

To evaluate the integral in (5.1.14), we use the transformation $u=(\tau-a)/(s-a)$ which implies $1-u=(s-\tau)/(s-a)$. Hence

$$\begin{aligned}\int_a^s (\tau-a)^n(s-\tau)^{n-j+k-1}d\tau &= (s-a)^{2n-j+k}\int_0^1 u^n(1-u)^{n-j+k-1}du \\ &= (s-a)^{2n-j+k}B(n+1,n-j+k).\end{aligned}$$

Then by (5.1.14),

$$\int_a^s G(\tau,s)d\tau = \sum_{j=0}^{n-1} A_j(s)\sum_{k=0}^{j}\left(\begin{array}{c} j \\ k \end{array}\right)(b-s)^{j-k}(s-a)^{2n-j+k}B(n+1,n-j+k). \quad (5.1.15)$$

Using the expression for $A_j(s)$ in (5.1.15) and rearranging terms we obtain

$$\begin{aligned}\int_a^s G(\tau,s)d\tau &= \frac{1}{(2n-1)!(b-a)^n}\sum_{j=0}^{n-1}\sum_{k=0}^{j}\left(\begin{array}{c} n+j-1 \\ j \end{array}\right)\left(\begin{array}{c} j \\ k \end{array}\right)B(n+1,n-j+k) \\ &\quad \times\frac{(b-s)^{n+j-k}(s-a)^{2n+k}}{(b-a)^j}. \quad (5.1.16)\end{aligned}$$

Using the same technique, we also have

$$\int_s^b G(\tau,s)d\tau = \frac{1}{(2n-1)!(b-a)^n}\sum_{j=0}^{n-1}\sum_{k=0}^{j}\binom{n+j-1}{j}\binom{j}{k}B(n+1,n-j+k) \times\frac{(b-s)^{2n+k}(s-a)^{n+j-k}}{(b-a)^j}. \tag{5.1.17}$$

Substituting (5.1.16) and (5.1.17) into (5.1.11) we obtain

$$\int_a^b G(\tau,s)d\tau = \frac{1}{(2n-1)!(b-a)^n}\sum_{j=0}^{n-1}\sum_{k=0}^{j}\binom{n+j-1}{j}\binom{j}{k}B(n+1,n-j+k) \times\left\{\frac{(b-s)^{n+j-k}(s-a)^{2n+k}}{(b-a)^j}+\frac{(b-s)^{2n+k}(s-a)^{n+j-k}}{(b-a)^j}\right\}. \tag{5.1.18}$$

Note that $\alpha\beta \le (\alpha+\beta)^2/4$ and $\alpha^l+\beta^l \le (\alpha+\beta)^l$ for $\alpha,\beta>0$ and $l\in\mathbb{N}$. Letting $\alpha = b-s,\ \beta = s-a$ and $l = n-j+2k$ we have

$$\begin{aligned} &\frac{(b-s)^{n+j-k}(s-a)^{2n+k}}{(b-a)^j}+\frac{(b-s)^{2n+k}(s-a)^{n+j-k}}{(b-a)^j} \\ = &\frac{(b-s)^{n+j-k}(s-a)^{n+j-k}}{(b-a)^j}\Big[(s-a)^{n-j+2k}+(b-s)^{n-j+2k}\Big] \\ \le &\frac{(b-a)^{2n+2j-2k}}{2^{2n+2j-2k}(b-a)^j}(b-a)^{n-j+2k} = \frac{(b-a)^{3n}}{2^{2n+2j-2k}}. \end{aligned}$$

Then (5.1.10) follows from (5.1.18).

We then show that (5.1.3) holds. Define $m := \max\{|x(t)| : t\in[a,b]\}$. Then taking maximum of $|x(t)|$ in (5.1.9) and using the fact that $x(t)\not\equiv m$ on $[a,b]$, we have

$$m < m\int_a^b\Big(\int_a^b G(\tau,s)d\tau\Big)|q(s)|ds.$$

Cancelling m from both sides and using (5.1.10), we obtain (5.1.3). □

If, in addition to the assumptions of Theorem 5.1.1, we assume $x(t) \neq 0$ for $t \in (a,c) \cup (c,b)$, then stronger Lyapunov-type inequalities can be derived. We present the results in the next Theorem.

Theorem 5.1.2. *Assume Eq. (5.1.1) has a solution $x(t)$ satisfying*

$$x^{(i+1)}(a) = x^{(i+1)}(b) = 0, \quad i = 0, 1, ..., n-1.$$

(a) Suppose $x(c) = 0$ for $c \in (a,b)$ and $x(t) \neq 0$ for $t \in [a,c) \cup (c,b]$. Then one of the following holds:

(i) $\displaystyle\int_a^b q_-(t)dt > \frac{2^{2n}(2n-1)!}{(b-a)^{2n}S_n}$,

(ii) $\displaystyle\int_a^b q_+(t)dt > \frac{2^{2n}(2n-1)!}{(b-a)^{2n}S_n}$,

(iii) $\displaystyle\int_a^c q_-(t)dt + \int_c^b q_+(t)dt > \frac{2^{2n}(2n-1)!}{(b-a)^{2n}S_n}$.

(b) Suppose $x(a) = 0$ and $x(t) \neq 0$ for $t \in (a,b]$. Then

$$\int_a^b q_+(t)dt > \frac{2^{2n}(2n-1)!}{(b-a)^{2n}S_n}.$$

(c) Suppose $x(b) = 0$ and $x(t) \neq 0$ for $t \in [a,b)$. Then

$$\int_a^b q_-(t)dt > \frac{2^{2n}(2n-1)!}{(b-a)^{2n}S_n}.$$

Proof. As in the proof of Theorem 5.1.1, we see that (5.1.8) and (5.1.10) holds.

(a) Since $x(t)$ is continuous and $x(c) = 0$ for $c \in (a,b)$, there exist $t_1 \in (a,c)$ and $t_2 \in (c,b)$ such that $|x(t_1)| = \max\{|x(t)| : t \in [a,c]\}$ and $|x(t_2)| = \max\{|x(t)| : t \in [c,b]\}$. Without loss of generality, we may assume $x(t)$ satisfies one of the following cases:

I. $x(t) > 0$ on $(a,c) \cup (c,b)$ and $x(t_1) \geq x(t_2)$;

II. $x(t) > 0$ on $(a,c) \cup (c,b)$ and $x(t_1) < x(t_2)$;

III. $x(t) > 0$ on (a, c) and $x(t) < 0$ on (c, b), and $x(t_1) \geq -x(t_2)$;

IV. $x(t) > 0$ on (a, c) and $x(t) < 0$ on (c, b), and $x(t_1) < -x(t_2)$.

In the sequel, we denote $m = \max\{|x(t_1)|, |x(t_2)|\}$.

Case I: In this case, $m = x(t_1)$. Then (5.1.8) with $t = t_1$ shows that

$$m = \int_a^b \Big(\int_{t_1}^c G(\tau, s) d\tau \Big) (-q(s)) x(s) ds$$

Using the facts that $0 \leq x(t) \leq m$ and $x(t) \not\equiv m$, and $-q(t) \leq q_-(t)$, we have

$$m < m \int_a^b \Big(\int_a^b G(\tau, s) d\tau \Big) q_-(s) ds.$$

Cancelling m from both sides and using (5.1.10) we obtain

$$\int_a^b q_-(s) ds > \frac{2^{2n}(2n-1)!}{(b-a)^{2n} S_n}.$$

i.e., conclusion (i) in Part (a) holds.

Case II: In this case, $m = x(t_2)$. Then (5.1.8) with $t = t_2$ shows that

$$m = \int_a^b \Big(\int_c^{t_2} G(\tau, s) d\tau \Big) q(s) x(s) ds.$$

Again using the facts that $0 \leq x(t) \leq m$ and $x(t) \not\equiv m$, and $q(t) \leq q_+(t)$, we have

$$m < m \int_a^b \Big(\int_a^b G(\tau, s) d\tau \Big) q_+(s) ds.$$

Cancelling m from both sides and using (5.1.10) we obtain

$$\int_a^b q_+(s) ds > \frac{2^{2n}(2n-1)!}{(b-a)^{2n} S_n}.$$

i.e., conclusion (ii) in Part (a) holds.

Case III: In this case, $m = x(t_1)$. Then (5.1.8) with $t = t_1$ shows that

$$\begin{aligned} m &= \int_a^b \Big(\int_{t_1}^c G(\tau,s)d\tau\Big)(-q(s))x(s)ds \\ &= \int_a^c \Big(\int_{t_1}^c G(\tau,s)d\tau\Big)(-q(s))x(s)ds + \int_c^b \Big(\int_{t_1}^c G(\tau,s)d\tau\Big)q(s)(-x(s))ds. \end{aligned}$$

Note that $x(t) > 0$ on $[a,c)$ and $x(t) < 0$ on $(c,b]$. Then by a similar argument to Cases I and II, we see that

$$\int_a^c q_-(s)ds + \int_c^b q_+(s)ds > \frac{2^{2n}(2n-1)!}{(b-a)^{2n}S_n}.$$

i.e., conclusion (iii) in Part (a) holds.

Case IV: The same argument as in Case III shows that conclusion (iii) in Part (a) holds. We omit the detail.

(b) Note that $x(a) = 0$. Then it follows from (5.1.8) that

$$x(t) = \int_a^b \Big(\int_a^t G(\tau,s)d\tau\Big)q(s)x(s)ds. \tag{5.1.19}$$

Without loss of generality, we may assume $x(t) > 0$ in $(a,b]$. Then there exists $t_2 \in (a,b]$ such that $m = x(t_2) = \max\{x(t) : t \in [a,b]\}$. Using $t = t_2$ in (5.1.19) we obtain

$$m = \int_a^b \Big(\int_a^{t_2} G(\tau,s)d\tau\Big)q(s)x(s)ds \le \int_a^b \Big(\int_a^b G(\tau,s)d\tau\Big)q_+(s)x(s)ds.$$

Using (5.1.10) and a similar technique as before, we see that conclusion in Part (b) holds.

(c) In this case, a similar argument as Part (b) holds with $m = x(t_1) = \max\{|x(t)| : t \in [a,b]\}|$. We omit the detail.

□

Now we interpret the results in Theorems 5.1.1 and 5.1.2 to the special case with $n = 1$, i.e., the third-order linear differential equation

$$x''' + q(t)x = 0. \tag{5.1.20}$$

From (5.1.2),

$$S_1 = \sum_{j=0}^{0}\sum_{k=0}^{j} 2^{2k-2j}\binom{j}{j}\binom{j}{k}B(2, 1+k-j) = B(2,1) = \frac{1}{2}.$$

Corollary 5.1.1. *Assume Eq. (5.1.20) has a nontrivial solution $x(t)$ satisfying*

$$x'(a) = x'(b) = 0$$

and $x(c) = 0$ for $c \in [a, b]$. Then

$$\int_a^b |q(t)|dt > \frac{8}{(b-a)^2}.$$

Corollary 5.1.2. *Assume Eq. (5.1.20) has a solution $x(t)$ satisfying*

$$x'(a) = x'(b) = 0.$$

(a) Suppose $x(c) = 0$ for $c \in (a, b)$ and $x(t) \neq 0$ for $t \in [a, c) \cup (c, b]$. Then one of the following holds:

(i) $\displaystyle\int_a^b q_-(t)dt > \frac{8}{(b-a)^2}$,

(ii) $\displaystyle\int_a^b q_+(t)dt > \frac{8}{(b-a)^2}$,

(iii) $\int_a^c q_-(t)dt + \int_c^b q_+(t)dt > \frac{8}{(b-a)^2}$.

(b) Suppose $x(a) = 0$ *and* $x(t) \neq 0$ *for* $t \in (a, b]$. *Then*

$$\int_a^b q_+(t)dt > \frac{8}{(b-a)^2}.$$

(c) Suppose $x(b) = 0$ *and* $x(t) \neq 0$ *for* $t \in [a, b)$. *Then*

$$\int_a^b q_-(t)dt > \frac{8}{(b-a)^2}.$$

We observe that the inequalities in Corollaries 5.1.1 and 5.1.2 are supplements to [17, Corollary 2.1] by Dhar and Kong for different boundary conditions.

5.2 Application to boundary value problems

In the final section, we apply the results on the Lyapunov-type Inequalities obtained in Section 5.1 to study the nonexistence, uniqueness, and existence-uniqueness for solutions of certain BVPs. Consider the BVP consisting of Eq. (5.1.1) and the following boundary conditions

$$\begin{gathered} x^{(i+1)}(a) = x^{(i+1)}(b) = 0,\ i = 0, 1, ..., n-1; \\ x(c) = 0,\ c \in [a, b]. \end{gathered} \tag{5.2.1}$$

In the following, we let S_n be defined by (5.1.2). The first result is on the nonexistence of solutions of the BVP (5.1.1), (5.2.1).

Theorem 5.2.1. *Assume*

$$\int_a^b |q(t)|dt \leq \frac{2^{2n}(2n-1)!}{(b-a)^{2n} S_n}. \tag{5.2.2}$$

Then BVP (5.1.1),(5.2.1) has no nontrivial solution for any $c \in [a, b]$.

Proof. Assume the contrary, i.e., BVP (5.1.1), (5.2.1) has a nontrivial solution $x(t)$. Then by Theorem 5.1.1, (5.1.3) holds. This contradicts assumption (5.2.2). □

As a direct application of Theorem 5.1.2, we present the following result.

Theorem 5.2.2. *Assume*

$$\max_{\xi\in[a,b]}\left\{\int_a^\xi q_-(t)dt+\int_\xi^b q_+(t)dt\right\}\le\frac{2^{2n}(2n-1)!}{(b-a)^{2n}S_n}. \tag{5.2.3}$$

Then every nontrivial solution of BVP (5.1.1), (5.2.1) *has at least two zeros in* $[a,b]$*.*

Proof. Assume the contrary, i.e., BVP (5.1.1), (5.2.1) has a nontrivial solution $x(t)$ with only one zero c_1 in $[a,b]$. Then $c_1=c$ and $x(t)\neq 0$ for $t\in[a,c)\cup(c,b]$. It follows that one of the conclusions in Part (a) of Theorem 5.1.2 holds. This contradicts (5.2.3).

□

Next we consider the odd order nonhomogeneous linear BVPs consisting of the equation

$$x^{(2n+1)}+(-1)^{n-1}q(t)x=f(t)\text{ on }(A,B) \tag{5.2.4}$$

with $-\infty<A<B<\infty$ and $q,f\in C((A,B),\mathbb{R})$; and boundary condition

$$\begin{aligned}&x^{(i+1)}(a)=k_{i1},\,x^{(i+1)}(b)=k_{i2},\quad i=0,1,...,n-1\\&x(c)=k_{i3},\,c\in[a,b]\end{aligned} \tag{5.2.5}$$

with

$$A<a<b<B\text{ and }k_{i1},k_{i2},k_{i3}\in\mathbb{R}. \tag{5.2.6}$$

Based on Theorem 5.1.1, we obtain a criterion for BVP (5.2.4), (5.2.5) to have a unique solution.

Theorem 5.2.3. *Assume*

$$\int_a^b|q(t)|dt\le\frac{2^{2n}(2n-1)!}{(B-A)^{2n}S_n}.$$

Then BVP (5.2.4), (5.2.5) has a unique solution on (A, B) *for any* $a, b \in (A, B)$*, and* $c \in [a, b]$*, and* k_{i1}, k_{i2}, k_{i3} *satisfying* (5.2.6).

Proof. We first show that BVP (5.2.4), (5.2.5) has at most one solution for any a, b and k_1, k_2, k_3 satisfying (5.2.6). Assuming the contrary, it has two solutions $x_1(t)$ and $x_2(t)$ in (A, B). Define $x(t) = x_1(t) - x_2(t)$. Then $x(t)$ is a solution of the BVP (5.1.1), (5.2.1). Then by Theorem 5.2.1, $x(t) \equiv 0$, i.e., $x_1(t) \equiv x_2(t)$. This shows the uniqueness of solution of BVP (5.2.4), (5.2.5).

Since the homogeneous linear BVP (5.1.1), (5.2.1) only has the zero solution, then by the Fredholm alternative theorem [28], we conclude that the nonhomogeneous linear BVP (5.2.4), (5.2.5) has a unique solution.

□

CHAPTER 6

FRACTIONAL DIFFERENTIAL EQUATIONS I

6.1 Fractional integral boundary conditions

A lot of work has been done on fractional BVPs consisting of a fractional differential equation in the form

$$\left(D_{a^+}^{\alpha} x\right)(t) = f(t, x) \quad \text{on } (a, b) \tag{6.1.1}$$

with $\alpha > 0$, and a pointwise BC at the end points; in particular, the Dirichlet BC

$$x(a) = x(b) = 0 \tag{6.1.2}$$

for $1 < \alpha \leq 2$. Here $\left(D_{a^+}^{\alpha} x\right)(t)$ denotes the αth-order left-sided Riemann-Liouville fractional derivative of $x(t)$ at a as defined in (1.1.12). In the following, for the consistence of notations for BCs, we also denote $\left(D_{a^+}^{-\alpha} x\right)(t) := \left(I_{a^+}^{\alpha} x\right)(t)$ for $0 < \alpha < 1$.

We note that with Riemann-Liouville fractional derivative involved, any solution of Eq. (6.1.1) with a pointwise BC such as (6.1.2), if it exists, must be bounded on $[a, b]$. However, unlike integer-order differential equations, the majority of the solutions of Eq. (6.1.1) is unbounded at the left endpoint a no matter how good the right-hand function $f(t, x)$ is. This can be seen from [54, (2.1.39)] that every solution of Eq. (6.1.1) satisfies

$$x(t) = \left(I_{a^+}^{\alpha} D_{a^+}^{\alpha} x\right)(t) + \sum_{j=1}^{n} \frac{c_j}{\Gamma(\alpha - j + 1)} (t - a)^{\alpha - j}, \tag{6.1.3}$$

which shows that either $x(a) = 0$ or $x(t)$ is unbounded at a. Consequently, we should not expect that any BVP consisting Eq. (6.1.1) and a pointwise BC to have any solution unless the BC includes or implies the condition $x(a) = 0$. This is the reason why fractional BVPs have been studied mainly with the Dirichlet BC at a so far. More specifically, any pointwise BC including one of the following is ill-posed:

(i) $x(a) = c$ for some $c \neq 0$,

(ii) $x(a) + cx'(a) = 0$ for some $c \neq 0$,

(iii) $x(a) + cx^{(i)}(b) = 0$ for some $c \neq 0, i = 0, 1$.

In fact, the BC in (i) violates (6.1.3); and the BCs in (ii) and (iii) are each equivalent to one of the two sets of conditions: $x(a) = x'(a) = 0$ and $x(a) = x^{(i)}(b) = 0$, and hence does not agree with the number requirement for well-posed BCs.

In this chapter, we use fractional integral BCs to allow and "smoothen" the singularity of solutions at a. This idea is motivated by the initial conditions for Cauchy problems associated with Eq. (6.1.1) given in [54, (3.1.2)]:

$$\left(D_{a^+}^{\alpha-k}x\right)(a^+) = b_k, \quad b_k \in \mathbb{R}, \quad k = 1, 2, ..., n, \tag{6.1.4}$$

where $n = \lfloor \alpha \rfloor + 1$ for $\alpha \notin \mathbb{N}$ and $n = \alpha$ for $\alpha \in \mathbb{N}$, and

$$\left(D_{a^+}^{\alpha-k}x\right)(a^+) := \lim_{t \to a^+} \left(D_{a^+}^{\alpha-k}x\right)(t)$$

except that

$$\left(D_{a^+}^{\alpha-n}x\right)(a^+) = \lim_{t \to a^+} \left(D_{a^+}^{\alpha-n}x\right)(t) = \lim_{t \to a^+} \left(I_{a^+}^{n-\alpha}x\right)(t) \quad \text{for } \alpha \notin \mathbb{N}.$$

Note that $\alpha - n = 0$ for $\alpha \in \mathbb{N}$. By (1.1.13)

$$\left(D^0_{a^+}x\right)(a^+) = \lim_{t \to a^+} \left(D^0_{a^+}x\right)(t) = x(a). \tag{6.1.5}$$

We notice that the existence and uniqueness of solutions have been established for the Cauchy problem (6.1.1), (6.1.4) with any $b_k \in \mathbb{R}$. From this point of view, a more reasonable BC should involve $\left(D^{\alpha-k}_{a^+}x\right)(a^+)$ rather than $x^{(k)}(a)$ for $k = 1, 2, ..., n$. In particular, for Eq. (6.1.1) with $1 < \alpha \leq 2$, we may assign a homogeneous linear separated BC as

$$\begin{cases} c_{11}\left(D^{\alpha-2}_{a^+}x\right)(a^+) + c_{12}\left(D^{\alpha-2}_{a^+}x\right)(a^+) = 0, \\ c_{21}\left(D^{\alpha-2}_{a^+}x\right)(b) \quad + c_{22}\left(D^{\alpha-2}_{a^+}x\right)(b) \quad = 0; \end{cases} \tag{6.1.6}$$

and a coupled BC as

$$\begin{bmatrix} D^{\alpha-2}_{a^+}x \\ D^{\alpha-1}_{a^+}x \end{bmatrix}(b) = K \begin{bmatrix} D^{\alpha-2}_{a^+}x \\ D^{\alpha-1}_{a^+}x \end{bmatrix}(a^+); \tag{6.1.7}$$

where $c_{ij} \in \mathbb{R}$ and $K \in \mathbb{R}^{2\times 2}$ such that $\det K \neq 0$. Nonhomogeneous BCs can be defined accordingly. Such BCs permit solutions unbounded at a and hence are more general than pointwise BCs. It is easy to see from (6.1.5) that when $\alpha = 2$, BCs (6.1.6) and (6.1.7) reduce to the two point BCs

$$\begin{cases} c_{11}x(a) + c_{12}x(a) = 0, \\ c_{21}x(b) + c_{22}x(b) = 0; \end{cases}$$

and

$$\begin{bmatrix} x \\ x' \end{bmatrix}(b) = K \begin{bmatrix} x \\ x' \end{bmatrix}(a);$$

respectively. Therefore, (6.1.6) and (6.1.7) are natural extensions of the self-adjoint BCs for second-order linear differential equations to fractional differential equations with $1 < \alpha \leq 2$.

We point out that the BCs considered in this chapter are different from the general integral BCs with a singular kernel in the sense that they originate from the fractional initial conditions with which the existence-uniqueness results are derived. Problems with such BCs can be investigated in many approaches based on results on initial value problems. For instance, using the Fredholm alternative method to study the existence and uniqueness of BVPs, as shown in our Theorems 6.4.1 and 6.4.2. Such approaches are not allowed for general integral BCs.

In this chapter, we consider the fractional BVP consisting of the linear equation

$$\left(D_{a^+}^{\alpha} x\right)(t) + q(t)x = 0, \quad 1 < \alpha \leq 2, \tag{6.1.8}$$

and the BC

$$\left(D_{a^+}^{\alpha-2} x\right)(a^+) = \left(D_{a^+}^{\alpha-2} x\right)(b) = 0. \tag{6.1.9}$$

Lyapunov-type inequalities are derived and used to establish the existence and uniqueness for solutions of this BVP. Parallel results are also obtained for certain sequential fractional BVPs.

This chapter is organized as follows: After this section, we derive new Lyapunov-type inequalities for fractional differential equations and sequential fractional differential equations, respectively in Sections 6.2 and 6.3. Finally in Section 6.4, we apply the obtained Lyapunov-type inequalities to establish the existence and uniqueness for solutions of some fractional BVPs. We also give an example to show how computer programs and numerical algorithms can be used to verify the conditions and to apply the results.

6.2 Fractional Lyapunov-type inequalities

In this section, we let $-\infty < a < b < \infty$ and consider fractional differential equation

$$\left(D_{a^+}^{\alpha} x\right)(t) + q(t)x = 0 \quad \text{on } (a, b), \tag{6.2.1}$$

where $1 < \alpha \leq 2$ and $q \in L(a, b)$. To present our main results, we need the concept of γ-th right-sided Riemann-Liouville fractional derivative of a function $u(t)$ at b defined as

$$\left(D_{b^-}^{\gamma} u\right)(t) = \frac{-1}{\Gamma(n-\gamma)} \frac{d^n}{dt^n} \int_t^b (s-t)^{n-\gamma-1} u(s) ds \quad \text{for } t < b, \tag{6.2.2}$$

where $\gamma \geq 0$ and $n = \lfloor \gamma \rfloor + 1$. In particular, when $0 \leq \gamma < 1$, (6.2.2) reduces to

$$\left(D_{b^-}^{\gamma} u\right)(t) = \frac{-1}{\Gamma(1-\gamma)} \frac{d}{dt} \int_t^b (s-t)^{-\gamma} u(s) ds \quad \text{for } t < b. \tag{6.2.3}$$

More specifically, $\left(D_{b^-}^{0} u\right)(t) = u(t)$. With the left-sided and right-sided fractional derivatives given in (1.1.12) and (6.2.3), we have the following fractional integration by parts formula, see [71, (2.64)]:

$$\int_a^b \phi(s) D_{a^+}^{\gamma} \psi(s) ds = \int_a^b \psi(s) D_{b^-}^{\gamma} \phi(s) ds \quad \text{for } 0 \leq \gamma < 1, \tag{6.2.4}$$

where $\phi \in L_p(a, b)$ and $\psi \in L_r(a, b)$ such that $p^{-1} + r^{-1} \leq 1 + \gamma$.

In the following we define

$$G(t, s) := \frac{1}{b-a} \begin{cases} (s-a)(b-t), & a \leq s \leq t \leq b, \\ (t-a)(b-s), & a \leq t \leq s \leq b; \end{cases} \tag{6.2.5}$$

and let $D_{b^-}^{2-\alpha}[G(t,s)q(s)]$ be the right-sided fractional derivative of $G(t,s)q(s)$ with respect to s and $\left[D_{b^-}^{2-\alpha}[G(t,s)q(s)]\right]_+$ be the positive part of $D_{b^-}^{2-\alpha}[G(t,s)q(s)]$. Now we present our main result on fractional Lyapunov-type inequalities.

Theorem 6.2.1. *(a) Assume Eq. (6.2.1) has a nontrivial solution $x(t)$ satisfying*

$$\left(D_{a^+}^{\alpha-2}x\right)(a^+) = \left(D_{a^+}^{\alpha-2}x\right)(b) = 0. \tag{6.2.6}$$

Then

$$\max_{t\in[a,b]}\left\{\int_a^b \left|D_{b^-}^{2-\alpha}[G(t,s)q(s)]\right| ds\right\} > 1. \tag{6.2.7}$$

(b) Assume Eq. (6.2.1) has a solution $x(t)$ satisfying

$$\left(D_{a^+}^{\alpha-2}x\right)(a^+) = \left(D_{a^+}^{\alpha-2}x\right)(b) = 0 \text{ and } \left(D_{a^+}^{\alpha-2}x\right)(t) \neq 0 \text{ on } (a,b). \tag{6.2.8}$$

Then

$$\max_{t\in[a,b]}\left\{\int_a^b \left[D_{b^-}^{2-\alpha}[G(t,s)q(s)]\right]_+ ds\right\} > 1. \tag{6.2.9}$$

Proof. (a) Let $y(t) = \left(D_{a^+}^{\alpha-2}x\right)(t)$ for $a < t \le b$ and $y(a) = \left(D_{a^+}^{\alpha-2}x\right)(a^+)$. Then $y(t)$ is continuous on $[a,b]$. Note that $x(t) = (D_{a^+}^{2-\alpha}y)(t)$. We claim that

$$\left(D_{a^+}^{\alpha-2}x\right)(t) = y''(t) \quad \text{for } 1 < \alpha \le 2. \tag{6.2.10}$$

In fact, for $1 < \alpha < 2$, from (1.1.12) we have

$$\left(D_{a^+}^{\alpha}x\right)(t) = \frac{d^2}{dt^2}\left(I_{a^+}^{2-\alpha}x\right)(t) = \frac{d^2}{dt^2}\left(D_{a^+}^{\alpha-2}x\right)(t) = y''(t);$$

and (6.2.10) holds clearly when $\alpha = 2$ since $y(t) = x(t)$. Then it follows that BVP (6.2.1), (6.2.6) becomes the second-order linear BVP

$$-y'' = q(t)x, \quad y(a) = y(b) = 0. \tag{6.2.11}$$

Hence the solution $y(t)$ satisfies

$$y(t) = \int_a^b G(t,s)q(s)x(s)ds = \int_a^b G(t,s)q(s)D_{a^+}^{2-\alpha}y(s)ds, \tag{6.2.12}$$

where $G(t,s)$, given in (6.2.5), is the Green's function for BVP (6.2.11). For a fixed $t \in [a,b]$, applying (6.2.4) with $\phi(s) = G(t,s)q(s) \in L(a,b)$ and $\psi(s) = y(s) \in L_\gamma(a,b)$ with $\gamma = 2-\alpha$, we obtain

$$y(t) = \int_a^b G(t,s)q(s)D_{a^+}^{2-\alpha}y(s)ds = \int_a^b y(s)D_{b^-}^{2-\alpha}[G(t,s)q(s)]ds. \tag{6.2.13}$$

By taking the absolute value on both sides we have

$$|y(t)| = \left|\int_a^b y(s)D_{b^-}^{2-\alpha}[G(t,s)q(s)]ds\right| \le \int_a^b |y(s)|\left|D_{b^-}^{2-\alpha}[G(t,s)q(s)]\right| ds.$$

Let $m = \max_{t\in[a,b]}|y(t)|$. By taking the maximum of $|y(t)|$ on both sides we obtain

$$m \le \max_{t\in[a,b]}\int_a^b |y(s)|\left|D_{b^-}^{2-\alpha}[G(t,s)q(s)]\right| ds. \tag{6.2.14}$$

If $|y(t)| < m$ a.e. on $[a,b]$, then

$$m < m \max_{t\in[a,b]}\int_a^b \left|D_{b^-}^{2-\alpha}[G(t,s)q(s)]\right| ds$$

which leads to (6.2.7). Otherwise, there exists $J = \cup_{i=1}^{k} [a_i, b_i] \subset [a, b]$ for $1 \leq k \leq \infty$ with $a_i < b_i$ such that $y(t) \equiv m$ on J and $y(t) < m$ a.e. on $[a, b] \backslash J$. Then for $t \in J$, $y''(t) = 0$ and

$$x(t) = (D_{a^+}^{2-\alpha} y)(t) = (D_{a^+}^{2-\alpha} m) = \frac{m(t-a)^{\alpha-2}}{\Gamma(\alpha-1)} \neq 0.$$

From (6.2.11), $q(t) \equiv 0$ on J. This implies that for any $t \in [a, b]$, $D_{b^-}^{2-\alpha}[G(t, s)q(s)] = 0$ for $s \in J$. Hence it follows from (6.2.14) that

$$\begin{aligned} m &\leq \max_{t\in[a,b]} \int_{[a,b]\backslash J} |y(s)| \left| D_{b^-}^{3-\alpha}[G(t,s)q(s)] \right| ds \\ &< m \max_{t\in[a,b]} \int_{[a,b]\backslash J} \left| D_{b^-}^{3-\alpha}[G(t,s)q(s)] \right| ds \\ &= m \max_{t\in[a,b]} \int_a^b \left| D_{b^-}^{3-\alpha}[G(t,s)q(s)] \right| ds \end{aligned}$$

which also leads to (6.2.7).

(b) From the proof of Part (a) we see that (6.2.13) holds. By the assumption, $y(t) \neq 0$ on (a, b). Without loss of generality, we assume that $y(t) > 0$ on (a, b). Then it follows that

$$y(t) \leq \int_a^b y(s) \left[D_{b^-}^{2-\alpha}[G(t,s)q(s)] \right]_+ ds. \tag{6.2.15}$$

Now, a similar argument as in Part (a) leads to (6.2.9). □

The corollary below is a special case of Theorem 6.2.1.

Corollary 6.2.1. *Assume $D_{b^-}^{2-\alpha}[G(t, s)q(s)] \geq 0$ for $t, s \in [a, b]$ and Eq. (6.2.1) has a nontrivial solution $x(t)$ satisfying (6.2.6). Then*

$$\int_a^b q_+(t)dt > \frac{\alpha^\alpha \Gamma(\alpha-1)}{(\alpha-1)^{\alpha-1}(b-a)^{\alpha-1}}. \tag{6.2.16}$$

Proof. By Theorem 6.2.1 we see that (6.2.7) holds. By the assumption and the definition of $D_{b^-}^{2-\alpha}[G(t,s)q(s)]$ given in (6.2.3) we have

$$\begin{aligned}\int_a^b \left|D_{b^-}^{2-\alpha}[G(t,s)q(s)]\right| ds &= \int_a^b D_{b^-}^{2-\alpha}[G(t,s)q(s)]ds \\ &= \frac{-1}{\Gamma(\alpha-1)}\int_a^b \Big(\int_s^b (\tau-s)^{\alpha-2}G(t,\tau)q(\tau)d\tau\Big)' ds \\ &= \frac{1}{\Gamma(\alpha-1)}\int_a^b (\tau-a)^{\alpha-2}G(t,\tau)q(\tau)d\tau.\end{aligned}$$

Hence (6.2.9) becomes

$$\max_{t\in[a,b]}\int_a^b (\tau-a)^{\alpha-2}G(t,\tau)q(\tau)d\tau > \Gamma(\alpha-1).$$

Using the facts that $q(t) \le q_+(t)$, $G(t,s) \ge 0$ on $[a,b]\times[a,b]$, and

$$\max_{t\in[a,b]} G(t,\tau) = G(\tau,\tau) = \frac{(\tau-a)(b-\tau)}{b-a}, \quad \tau\in[a,b],$$

we see that

$$\begin{aligned}\Gamma(\alpha-1) &< \max_{t\in[a,b]}\int_a^b (\tau-a)^{\alpha-2}G(t,\tau)q_+(\tau)d\tau \\ &\le \frac{1}{b-a}\int_a^b (\tau-a)^{\alpha-1}(b-\tau)q_+(\tau)d\tau. \qquad (6.2.17)\end{aligned}$$

Denote $g(\tau) = (\tau-a)^{\alpha-1}(b-\tau)$. From the fact that $1<\alpha\le 2$, we see $g(\tau)$ is continuous on $[a,b]$, $g(a) = g(b) = 0$, and $g(\tau) > 0$ on (a,b). Thus there exists a $c\in(a,b)$ such that $g(c) = \max_{\tau\in[a,b]} g(\tau)$. Now a simple calculation shows that $c = [(\alpha-1)b+a]/\alpha$ and hence

$$g(\tau) \le g(c) = \frac{(\alpha-1)^{\alpha-1}(b-a)^{\alpha}}{\alpha^{\alpha}}. \qquad (6.2.18)$$

Substituting (6.2.18) in (6.2.17) we see that (6.2.16) holds. □

Remark 6.2.1. By (1.1.13) we have $(D^2_{a^+}x)(t) = x''(t)$ and $(D^0_{a^+}x)(t) = x(t)$. Hence for $\alpha = 2$, Eq. (6.2.1) with condition (6.2.8) becomes the second-order equation with pointwise condition

$$x'' + q(t)x = 0, \quad x(a) = x(b) = 0 \text{ and } x(t) \neq 0 \text{ on } (a,b). \tag{6.2.19}$$

Since $G(t,s) \geq 0$ on $[a,b] \times [a,b]$, it follows from Theorem 6.2.1, Part (b) with $\alpha = 2$ that

$$\begin{aligned} 1 &< \max_{t\in[a,b]} \left\{ \int_a^b \Big[D^{2-\alpha}_{b^-}[G(t,s)q(s)] \Big]_+ ds \right\} \\ &= \max_{t\in[a,b]} \left\{ \int_a^b \Big[G(t,s)q(s) \Big]_+ ds \right\} \\ &= \max_{t\in[a,b]} \int_a^b G(t,s)q_+(s)ds. \end{aligned} \tag{6.2.20}$$

Note that

$$\max_{t\in[a,b]} G(t,s) = G(s,s) = \frac{(s-a)(b-s)}{b-a} \leq \frac{b-a}{4}.$$

Hence (6.2.20) reduces to

$$\int_a^b q_+(t)dt > \frac{4}{b-a},$$

which becomes the Lyapunov inequality for the second-order equations.

6.3 Sequential fractional Lyapunov-type inequalities

Here we let $-\infty < a < b < \infty$ and consider the sequential fractional differential equation

$$\Big[\Big(D^{\beta}_{a+}(D^{\alpha}_{a+}x) \Big) \Big](t) + q(t)x = 0 \quad \text{on } (a,b), \tag{6.3.1}$$

where $q \in L([a,b],\mathbb{R})$, and $0 < \alpha, \beta \leq 1$. In the following, we define

$$G(t,s) := \frac{1}{\Gamma(\beta+1)} \begin{cases} \frac{(t-a)^\beta (b-s)^\beta}{(b-a)^\beta} - (t-s)^\beta, & a \leq s \leq t \leq b, \\ \frac{(t-a)^\beta (b-s)^\beta}{(b-a)^\beta}, & a \leq t \leq s \leq b; \end{cases} \tag{6.3.2}$$

and let $D_{b^-}^{1-\alpha}[G(t,s)q(s)]$ and $\left[D_{b^-}^{1-\alpha}[G(t,s)q(s)]\right]_+$ be defined in the same way as in Section 6.2. Now we present Lyapunov-type inequalities for Eq. (6.3.1).

Theorem 6.3.1. *(a) Assume Eq. (6.3.1) has a nontrivial solution $x(t)$ satisfying*

$$\left(D_{a^+}^{\alpha-1} x\right)(a^+) = \left(D_{a^+}^{\alpha-1} x\right)(b) = 0. \tag{6.3.3}$$

Then

$$\max_{t \in [a,b]} \left\{ \int_a^b \left| D_{b^-}^{1-\alpha}[G(t,s)q(s)] \right| ds \right\} > 1. \tag{6.3.4}$$

(b) Assume Eq. (6.3.1) has a solution $x(t)$ satisfying

$$\left(D_{a^+}^{\alpha-1} x\right)(a^+) = \left(D_{a^+}^{\alpha-1} x\right)(b) = 0 \text{ and } \left(D_{a^+}^{\alpha-1} x\right)(t) \neq 0 \text{ on } (a,b). \tag{6.3.5}$$

Then

$$\max_{t \in [a,b]} \left\{ \int_a^b \left[D_{b^-}^{1-\alpha}[G(t,s)q(s)] \right]_+ ds \right\} > 1, \tag{6.3.6}$$

Proof. Let $y(t) = \left(D_{a^+}^{\alpha-1} x\right)(t)$ for $a < t \leq b$ and $y(a) = \left(D_{a^+}^{\alpha-1} x\right)(a^+)$. Then $y(t)$ is continuous on $[a,b]$. Note that $x(t) = (D_{a^+}^{1-\alpha} y)(t)$. As shown in the proof of Theorem 6.2.1, we have $\left(D_{a^+}^{\alpha} x\right)(t) = y'(t)$. It follows that BVP (6.3.1), (6.3.3) becomes the fractional BVP

$$-(D_{a+}^{\beta} y')(t) = q(t)x, \quad y(a) = y(b) = 0. \tag{6.3.7}$$

We claim that $\left(D_{a^+}^{\beta+1}y\right)(t) = \left(D_{a+}^{\beta}y'\right)(t)$. In fact, from the relation [54, (2.1.28)] we have

$$\left(D_{a+}^{\beta}y\right)(t) = \frac{1}{\Gamma(1-\beta)}\left[\frac{y(a)}{(t-a)^{\beta}} + \int_a^t \frac{y'(s)}{(t-s)^{\beta}}ds\right].$$

Using the fact that $y(a) = 0$ and differentiating both sides with respect to t we have

$$\left(D_{a+}^{\beta+1}y\right)(t) = \frac{d}{dt}\left(D_{a+}^{\beta}y\right)(t) = \frac{1}{\Gamma(1-\beta)}\frac{d}{dt}\int_a^t \frac{y'(s)}{(t-s)^{\beta}}ds = (D_{a+}^{\beta}y')(t).$$

Thus BVP (6.3.7) becomes

$$-(D_{a+}^{\beta+1}y)(t) = q(t)x, \quad y(a) = y(b) = 0. \tag{6.3.8}$$

Note that BVP (6.3.8) is in the form of BVP (6.1.8), (6.1.2) with α replaced by $\beta+1$, and the Green's function $G(t,s)$ in (1.1.16) for BVP (6.1.8), (6.1.2) becomes the one in (6.3.2). Then the solution $y(t)$ satisfies

$$y(t) = \int_a^b G(t,s)q(s)x(s)ds = \int_a^b G(t,s)q(s)D_{a^+}^{1-\alpha}y(s)ds.$$

The rest of the proof is essentially the same as the proof of Theorem 6.2.1. We omit the detail. □

The following corollary is a special case of Theorem 6.3.1.

Corollary 6.3.1. *Assume* $D_{b^-}^{1-\alpha}[G(t,s)q(s)] \geq 0$ *for* $t, s \in [a,b]$ *and* $1 < \alpha+\beta \leq 2$*. Suppose Eq. (6.3.1) has a nontrivial solution* $x(t)$ *satisfying (6.3.3). Then*

$$\int_a^b q_+(t)dt > \frac{(\alpha+2\beta-1)^{\alpha+2\beta-1}\Gamma(\alpha)\Gamma(\beta+1)}{(\alpha+\beta-1)^{\alpha+\beta-1}\beta^{\beta}(b-a)^{\alpha+\beta-1}}. \tag{6.3.9}$$

Proof. The proof is similar to that of Corollary 6.2.1. By Theorem 6.3.1 we see that (6.3.4) holds. From the assumption and the definition of $D_{b^-}^{1-\alpha}[G(t,s)q(s)]$ given in (6.2.3) we have

$$\begin{aligned}
\int_a^b \Big| D_{b^-}^{1-\alpha}[G(t,s)q(s)] \Big| ds &= \int_a^b D_{b^-}^{1-\alpha}[G(t,s)q(s)] ds \\
&= \frac{-1}{\Gamma(\alpha)} \int_a^b \Big(\int_s^b (\tau - s)^{\alpha-1} G(t,\tau) q(\tau) d\tau \Big)' ds \\
&= \frac{1}{\Gamma(\alpha)} \int_a^b (\tau - a)^{\alpha-1} G(t,\tau) q(\tau) d\tau.
\end{aligned}$$

Hence (6.3.4) becomes

$$\max_{t\in[a,b]} \int_a^b (\tau - a)^{\alpha-1} G(t,\tau) q(\tau) d\tau > \Gamma(\alpha).$$

Using the facts that $q(t) \le q_+(t)$, $G(t,s) \ge 0$ on $[a,b] \times [a,b]$, and

$$\max_{t\in[a,b]} G(t,\tau) = G(\tau,\tau) = \frac{(\tau-a)^\beta (b-\tau)^\beta}{(b-a)^\beta \Gamma(\beta+1)}, \quad \tau \in [a,b],$$

we see that

$$\begin{aligned}
\Gamma(\alpha) &< \max_{t\in[a,b]} \int_a^b (\tau - a)^{\alpha-1} G(t,\tau) q_+(\tau) d\tau \\
&\le \frac{1}{(b-a)^\beta \Gamma(\beta+1)} \int_a^b (\tau-a)^{\alpha+\beta-1} (b-\tau)^\beta q_+(\tau) d\tau. \qquad (6.3.10)
\end{aligned}$$

Denote $h(\tau) = (\tau - a)^{\alpha+\beta-1}(b-\tau)^\beta$. From the fact that $0 < \beta \le 1$ and $1 < \alpha + \beta \le 2$, we see $h(\tau)$ is continuous on $[a,b]$, $h(a) = h(b) = 0$ and $h(\tau) > 0$ on (a,b). Thus there exists a $d \in (a,b)$ such that $h(d) = \max_{\tau\in[a,b]} h(\tau)$. Now a simple calculation shows that $d = [(\alpha+\beta-1)b + \beta a]/(\alpha + 2\beta - 1)$ and hence

$$h(\tau) \le h(d) = \frac{(\alpha+\beta-1)^{\alpha+\beta-1}\beta^\beta}{(\alpha+2\beta-1)^{\alpha+2\beta-1}} (b-a)^{\alpha+2\beta-1}. \qquad (6.3.11)$$

Substituting (6.3.11) in (6.3.10) we see that (6.3.9) holds. □

Remark 6.3.1. By (1.1.13) we have $(D^2_{a^+}x)(t) = x''(t)$ and $(D^0_{a^+}x)(t) = x(t)$. Hence for $\alpha = \beta = 1$, Eq. (6.3.1) and condition (6.3.5) becomes (6.2.19). Letting $\beta = 1$ in (6.3.2), we see that $G(t,s)$ becomes the same as the one in (6.2.5). Since $G(t,s) \geq 0$ on $[a,b] \times [a,b]$, it follows from Theorem 6.3.1, Part (b) with $\alpha = 1$ that

$$\begin{aligned} 1 &< \max_{t\in[a,b]} \left\{ \int_a^b \Big[D_{b^-}^{1-\alpha}[G(t,s)q(s)] \Big]_+ ds \right\} \\ &= \max_{t\in[a,b]} \left\{ \int_a^b \Big[G(t,s)q(s) \Big]_+ ds \right\} \\ &= \max_{t\in[a,b]} \int_a^b G(t,s)q_+(s)ds. \end{aligned} \tag{6.3.12}$$

Note that

$$\max_{t\in[a,b]} G(t,s) = G(s,s) = \frac{(s-a)(b-s)}{b-a} \leq \frac{b-a}{4}.$$

Hence (6.3.12) reduces to

$$\int_a^b q_+(t)dt > \frac{4}{b-a},$$

which becomes the Lyapunov inequality for the second-order equations.

6.4 Applications to boundary value problems

In the last section, we apply the results on the Lyapunov-type Inequalities obtained in Sections 6.2 and 6.3 to study the nonexistence, uniqueness, and existence-uniqueness of solutions of related fractional-order linear BVPs. We first consider the BVP consisting of the equation

$$\left(D^\alpha_{a^+}x\right)(t) + q(t)x = 0, \quad 1 < \alpha \leq 2, \tag{6.4.1}$$

and the BC

$$\left(D_{a^+}^{\alpha-2}x\right)(a^+) = \left(D_{a^+}^{\alpha-2}x\right)(b) = 0. \tag{6.4.2}$$

Definition 6.4.1. *A solution $x(t)$ of Eq.* (6.4.1) *is said to be an I-positive solution if* $\left(I_{a^+}^{n-\alpha}x\right)(t) > 0$ *on (a,b), where $n = \lfloor\alpha\rfloor + 1$.*

The following result is on the nonexistence of solutions of BVP (6.4.1), (6.4.2).

Lemma 6.4.1. *(a) Assume*

$$\max_{t\in[a,b]}\left\{\int_a^b \left|D_{b^-}^{2-\alpha}[G(t,s)q(s)]\right| ds\right\} \le 1. \tag{6.4.3}$$

Then BVP (6.4.1), (6.4.2) has no nontrivial solution.

(b) Assume

$$\max_{t\in[a,b]}\left\{\int_a^b \left[D_{b^-}^{2-\alpha}[G(t,s)q(s)]\right]_+ ds\right\} \le 1. \tag{6.4.4}$$

Then BVP (6.4.1), (6.4.2) has no I-positive solution.

Proof. (a) Assume the contrary, i.e., BVP (6.4.1), (6.4.2) has a nontrivial solution $x(t)$. Then by Theorem 6.2.1 Part (a), (6.2.7) holds. This contradicts assumption (6.4.3).

(b) The proof is similar to Part (a) and hence is omitted. □

Next we consider the fractional-order nonhomogeneous linear BVP consisting of the equation

$$\left(D_{a^+}^{\alpha}x\right)(t) + q(t)x = w(t) \quad \text{on } (a,b) \tag{6.4.5}$$

with $1 < \alpha \le 2$ and $q, w \in L((a,b),\mathbb{R})$; and the BC

$$\left(D_{a^+}^{\alpha-2}x\right)(a^+) = k_1, \quad \left(D_{a^+}^{\alpha-2}x\right)(b) = k_2, \tag{6.4.6}$$

where $k_1, k_2 \in \mathbb{R}$. Based on Theorem 6.2.1, we obtain a criterion for BVP (6.4.5), (6.4.6) to have a unique solution and a relation among solutions if the problem has more than one solution.

Theorem 6.4.1. *(a) Assume*

$$\max_{t\in[a,b]} \left\{ \int_a^b \left| D_{b^-}^{2-\alpha}[G(t,s)q(s)] \right| ds \right\} \leq 1.$$

Then BVP (6.4.5), (6.4.6) has a unique solution on (a,b) for any $k_1, k_2 \in \mathbb{R}$.

(b) Assume

$$\max_{t\in[a,b]} \left\{ \int_a^b \left[D_{b^-}^{2-\alpha}[G(t,s)q(s)] \right]_+ ds \right\} \leq 1 < \max_{t\in[a,b]} \left\{ \int_a^b \left| D_{b^-}^{2-\alpha}[G(t,s)q(s)] \right| ds \right\}.$$

If BVP (6.4.5), (6.4.6) has two solutions $x_1(t)$ and $x_2(t)$, then there exists a $c \in (a,b)$ such that $\left(I_{a^+}^{2-\alpha} x_1\right)(c) = \left(I_{a^+}^{2-\alpha} x_2\right)(c)$.

Proof. (a) We first show that BVP (6.4.5), (6.4.6) has at most one solution for any $k_1, k_2 \in \mathbb{R}$. Assume the contrary, i.e., it has two solutions $x_1(t)$ and $x_2(t)$ in (a,b). Let $x(t) = x_1(t) - x_2(t)$. Then $x(t)$ is a solution of BVP (6.4.1), (6.4.2). By Lemma 6.4.1, Part (a), we have $x(t) \equiv 0$, i.e., $x_1(t) \equiv x_2(t)$. This shows the uniqueness of the solution of BVP (6.4.5), (6.4.6).

Since BVP (6.2.1), (6.4.2) has only the zero solution, then by the Fredholm alternative theorem [28], we conclude that BVP (6.4.5), (6.4.6) has a unique solution.

(b) Let $x(t) = x_1(t) - x_2(t)$. Then $x(t)$ is a solution of the BVP (6.4.1), (6.4.2). By Lemma 6.4.1 Part (b), $x(t)$ is not an I-positive on $[a,b]$. Then there exists a $c \in (a,b)$ such that $\left(I_{a^+}^{2-\alpha} x\right)(c) = 0$, i.e., $\left(I_{a^+}^{2-\alpha} x_1\right)(c) = \left(I_{a^+}^{2-\alpha} x_2\right)(c)$. □

With a similar argument, from Corollary 6.2.1 we obtain the result below.

Corollary 6.4.1. *Assume* $D_{b^-}^{2-\alpha}[G(t,s)q(s)] \geq 0$ *in* $[a,b]$ *and*

$$\int_a^b q_+(t)dt \leq \frac{\alpha^\alpha \Gamma(\alpha-1)}{(\alpha-1)^{\alpha-1}(b-a)^{\alpha-1}}. \tag{6.4.7}$$

Then BVP (6.4.5), (6.4.6) has a unique solution on (a,b) *for any* $k_1, k_2 \in \mathbb{R}$.

Remark 6.4.1. We note from Section 6.1 that the BVP consisting of Eq. (6.4.5) and the pointwise BC

$$x(a) = k_1, \quad x(b) = k_2 \tag{6.4.8}$$

does not have a solution unless $k_1 = 0$. Even for the case with $k_1 = 0$, the existence and uniqueness of solutions of BVP (6.4.5), (6.4.8) cannot be established by the Fredholm alternative method. This is due to the fact that Eq. (6.4.5) with a pointwise initial condition may not have a unique solution.

For the case with $k_1 = k_2 = 0$ and $w(t) \equiv 0$, from Theorem 1.1.10, we can easily derive the following result: Assume

$$\int_a^b |q(t)|dt \leq \Gamma(\alpha)\Big(\frac{4}{b-a}\Big)^{\alpha-1}. \tag{6.4.9}$$

Then BVP (6.4.5), (6.4.8) has only the zero solution.

We observe that this result agrees with Corollary 6.4.1 for $\alpha = 2$ since BVPs (6.4.5), (6.4.6) and (6.4.5), (6.4.8) become the same second-order homogeneous linear problem. When $1 < \alpha < 2$, we compare the two results by comparing the right-hand numbers of (6.4.7) and (6.4.9) (under the assumption that $D_{b^-}^{2-\alpha}[G(t,s)q(s)] \geq 0$ for BVP (6.4.5), (6.4.6)). We claim that

$$H(\alpha) := \frac{\alpha^\alpha \Gamma(\alpha-1)}{(\alpha-1)^{\alpha-1}(b-a)^{\alpha-1}} - \Gamma(\alpha)\Big(\frac{4}{b-a}\Big)^{\alpha-1} > 0 \tag{6.4.10}$$

and $H(\alpha) \to \infty$ as $\alpha \to 1^+$. In fact,

$$H(\alpha) = \frac{(\alpha-1)\Gamma(\alpha-1)}{(b-a)^{\alpha-1}} \left(\left(\frac{\alpha}{\alpha-1} \right)^{\alpha} - \left(4^{1-\frac{1}{\alpha}}\right)^{\alpha} \right).$$

Then (6.4.10) follows from the fact that $\alpha/(\alpha-1) > 4^{1-1/\alpha}$ for $1 < \alpha < 2$. This shows that condition (6.4.7) is weaker than condition (6.4.9), and much weaker when α is close to 1; which is reasonable since BC (6.4.6) allows the solution $x(t)$ to have a singularity at a, while BC (6.4.8) requires the solution to be bounded.

Now, we state the results for the sequential fractional BVPs which are parallel to Theorem 6.4.1 and Corollary 6.4.1. We omit the proofs since they are essentially in the same way. Consider the BVP consisting of the equation

$$\left[\left(D^{\beta}_{a+}(D^{\alpha}_{a+}x)\right)\right](t) + q(t)x = 0, \quad 0 < \alpha, \beta \le 1, \tag{6.4.11}$$

and the BC

$$\left(D^{\alpha-1}_{a^+}x\right)(a^+) = \left(D^{\alpha-1}_{a^+}x\right)(b) = 0. \tag{6.4.12}$$

The following result is on the nonexistence of solutions of BVP (6.4.11), (6.4.12).

Lemma 6.4.2. *(a) Assume*

$$\max_{t\in[a,b]} \left\{ \int_a^b \left| D^{1-\alpha}_{b^-}[G(t,s)q(s)] \right| ds \right\} \le 1.$$

Then BVP (6.4.11), (6.4.12) has only the trivial solution.

(b) Assume

$$\max_{t\in[a,b]} \left\{ \int_a^b \left[D^{1-\alpha}_{b^-}[G(t,s)q(s)] \right]_+ ds \right\} \le 1.$$

Then BVP (6.4.11), (6.4.12) has no I-positive solution.

Next we consider the sequential nonhomogeneous linear BVPs consisting of the equation

$$(D^{\beta}_{a+}(D^{\alpha}_{a+}x))(t) + q(t)x = w(t) \text{ on } (a,b), \tag{6.4.13}$$

where $0 < \alpha, \beta \leq 1$ and $q, w \in L((a,b)), \mathbb{R})$, and the BC

$$\left(D^{\alpha-1}_{a^+}x\right)(a^+) = k_1, \quad \left(D^{\alpha-1}_{a^+}x\right)(b) = k_2, \tag{6.4.14}$$

where $k_1, k_2 \in \mathbb{R}$. Now we present a criterion for BVP (6.4.13), (6.4.14) to have a unique solution and a relation among the solutions if the problem has more than one solution.

Theorem 6.4.2. *(a) Assume*

$$\max_{t\in[a,b]}\left\{\int_a^b \left|D^{1-\alpha}_{b^-}[G(t,s)q(s)]\right| ds\right\} \leq 1.$$

Then BVP (6.4.13), (6.4.14) have a unique solution on (a,b) for any $k_1, k_2 \in \mathbb{R}$.

(b) Assume

$$\max_{t\in[a,b]}\left\{\int_a^b \left[D^{1-\alpha}_{b^-}[G(t,s)q(s)]\right]_+ ds\right\} \leq 1 < \max_{t\in[a,b]}\left\{\int_a^b \left|D^{1-\alpha}_{b^-}[G(t,s)q(s)]\right| ds\right\}.$$

If BVP (6.4.13), (6.4.14) has two solutions $x_1(t)$ and $x_2(t)$, then there exists a $c \in (a,b)$ such that $\left(I^{1-\alpha}_{a^+}x_1\right)(c) = \left(I^{1-\alpha}_{a^+}x_2\right)(c)$.

As before, we have the following corollary from Corollary 6.3.1.

Corollary 6.4.2. *Let $1 < \alpha + \beta \leq 2$. Assume $D^{1-\alpha}_{b^-}[G(t,s)q(s)] \geq 0$ in $[a,b]$ and*

$$\int_a^b q_+(t)dt \leq \frac{(\alpha+2\beta-1)^{\alpha+2\beta-1}\Gamma(\alpha)\Gamma(\beta+1)}{(\alpha+\beta-1)^{\alpha+\beta-1}\beta^{\beta}(b-a)^{\alpha+\beta-1}}.$$

Then BVP (6.4.13), (6.4.14) have a unique solution on (a, b) for any $k_1, k_2 \in \mathbb{R}$.

Finally, we point out that the applications of the results in this chapter involve evaluations of fractional derivatives of functions. However, conditions involving fractional derivatives and integrals are hard to check analytically, even with pointwise BCs. So computer programs and numerical algorithms are the main tools for applications. We refer the reader to [33] for numerical algorithms for computing fractional derivatives. Here, we give an example to illustrate the application of Theorem 6.4.1. A similar example for Theorem 6.4.2 can be easily elaborated and hence is left to the interested reader.

Example 1. We consider the BVP

$$\left(D_{0^+}^{\alpha} x\right)(t) + k(\sin t)x = w(t), \quad \left(D_{0^+}^{\alpha-2} x\right)(0^+) = k_1, \quad \left(D_{0^+}^{\alpha-2} x\right)(2\pi) = k_2, \tag{6.4.15}$$

where $1 < \alpha \leq 2$, $w \in L((0, 2\pi)), \mathbb{R})$, and $k, k_1, k_2 \in \mathbb{R}$. Using Mathematica, we sketch the graphs of the integrals $\int_0^{2\pi} \left|D_{2\pi^-}^{2-\alpha}[G(t,s)\sin s]\right| ds$ and $\int_0^{2\pi} \left[D_{2\pi^-}^{2-\alpha}[G(t,s)\sin s]\right]_+ ds$ as functions of t. From them we find that

$$\max_{t\in[0,2\pi]} \int_0^{2\pi} \left|D_{2\pi^-}^{2-\alpha}[G(t,s)\sin s]\right| ds = 3.29$$

and

$$\max_{t\in[0,2\pi]} \int_0^{2\pi} \left[D_{2\pi^-}^{2-\alpha}[G(t,s)\sin s]\right]_+ ds = 1.81,$$

see Figures 6.1 and 6.2 respectively. Hence, applying Theorem 6.4.1, we observe the following:

(a) For $k \leq 0.3$, BVP (6.4.15) has a unique solution on $(0, 2\pi)$;

(b) for $0.3 < k \leq 0.55$, if BVP (6.4.15) has two solutions $x_1(t)$ and $x_2(t)$, then there exists a $c \in (0, 2\pi)$ such that $\left(I_{0^+}^{2-\alpha} x_1\right)(c) = \left(I_{0^+}^{2-\alpha} x_2\right)(c)$.

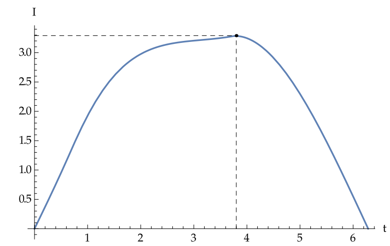

Figure 6.1: $\int_0^{2\pi} \left| D_{2\pi^-}^{2-\alpha}[G(t,s)\sin s] \right| ds$.

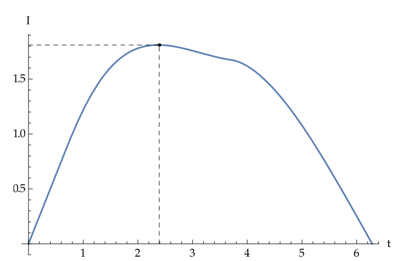

Figure 6.2: $\int_0^{2\pi} \left[D_{2\pi^-}^{2-\alpha}[G(t,s)\sin s] \right]_+ ds$

CHAPTER 7

FRACTIONAL DIFFERENTIAL EQUATIONS II

7.1 Fractional Lyapunov-type inequalities

We consider the α-th order fractional linear differential equation

$$\left(D_{a^+}^{\alpha} x\right)(t) + q(t)x = 0, \quad 2 < \alpha \leq 3. \tag{7.1.1}$$

Here $\left(D_{a^+}^{\alpha} x\right)(t)$ denotes the αth-order left-sided Riemann-Liouville fractional derivative of $x(t)$ at a as defined in (1.1.12). In the following, we denote

$$\left(D_{a^+}^{\alpha-k} x\right)(a^+) := \lim_{t\to a^+} \left(D_{a^+}^{\alpha-k} x\right)(t) \quad \text{for } k = 1, 2, 3$$

with

$$\left(D_{a^+}^{\alpha-3} x\right)(a^+) := \lim_{t\to a^+} \left(I_{a^+}^{3-\alpha} x\right)(t) \quad \text{for } 2 < \alpha < 3.$$

In this chapter, we derive Lyapunov-type inequalities for the BVPs consisting of Eq. (7.1.1) and one of the following BCs:

$$\left(D_{a^+}^{\alpha-2} x\right)(a^+) = \left(D_{a^+}^{\alpha-2} x\right)(b) = 0 \text{ and } \left(D_{a^+}^{\alpha-3} x\right)(c) = 0, \quad a \leq c \leq b; \tag{7.1.2}$$

$$\left(D_{a^+}^{\alpha-3} x\right)(a^+) = \left(D_{a^+}^{\alpha-3} x\right)(b) = 0 \text{ and } \left(D_{a^+}^{\alpha-2} x\right)(a^+) = 0, \quad a < b; \tag{7.1.3}$$

$$\left(D_{a^+}^{\alpha-3}x\right)(a^+) = \left(D_{a^+}^{\alpha-3}x\right)(b) = 0 \text{ and } \left(D_{a^+}^{\alpha-1}x\right)(a) = 0, \quad a < b. \tag{7.1.4}$$

Also we utilize them to establish the existence and uniqueness for solutions of related homogeneous and nonhomogeneous linear BVPs.

In this chapter, we let $-\infty < a < b < \infty$ and assume $q \in L([a,b],\mathbb{R})$. To prove our results, we will need the following fractional integration by parts formula, see [71, (2.64)]:

$$\int_a^b \phi(s)\left(D_{a^+}^{\gamma}\psi\right)(s)ds = \int_a^b \psi(s)\left(D_{b^-}^{\gamma}\phi\right)(s)ds, \quad 0 \le \gamma < 1, \tag{7.1.5}$$

for any $\phi \in L_p(a,b)$, $\psi \in L_r(a,b)$ such that $p^{-1} + r^{-1} \le 1 + \gamma$, where

$$\left(D_{b^-}^{\gamma}\phi\right)(s) := \frac{-1}{\Gamma(1-\gamma)}\frac{d}{ds}\int_s^b (\tau - s)^{-\gamma}\phi(\tau)d\tau.$$

In the following, we denote

$$G(t,s) := \frac{1}{b-a}\begin{cases}(s-a)(b-t), & a \le s \le t \le b\\ (t-a)(b-s), & a \le t \le s \le b\end{cases} \tag{7.1.6}$$

is the Green's function for the BVP

$$-u'' = h(t), \quad u(a) = u(b) = 0 \tag{7.1.7}$$

with $h \in L([a,b)],\mathbb{R})$; Moreover, similar to the notations in Chapter 6, we will denote by $D_{b^-}^{3-\alpha}[G(t,s)q(s)]$ the $(3-\alpha)$-th order right-sided fractional derivative of $G(t,s)q(s)$ with respect to s at b for fixed $t \in [a,b]$ and by $\left[D_{b^-}^{3-\alpha}[G(t,s)q(s)]\right]_{\pm}$ the positive and negative parts of $D_{b^-}^{3-\alpha}[G(t,s)q(s)]$, respectively. First we present the Lyapunov-type Inequalities for BVP (7.1.1), (7.1.2).

Theorem 7.1.1. *Assume Eq. (7.1.1) has a nontrivial solution $x(t)$ satisfying BC (7.1.2). Then*

$$\int_a^b \int_a^b \left| D_{b^-}^{3-\alpha}[G(t,s)q(s)] \right| ds dt > 1. \tag{7.1.8}$$

Proof. Let

$$y(t) := \left(D_{a^+}^{\alpha-3} x \right)(t) \text{ for } a < t \le b \text{ and } y(a) = \left(D_{a^+}^{\alpha-3} x \right)(a^+). \tag{7.1.9}$$

Then $y(t)$ is continuous on $[a,b]$. Note that $x(t) = (D_{a^+}^{3-\alpha} y)(t)$. We claim that

$$\left(D_{a^+}^{\alpha} x \right)(t) = y'''(t). \tag{7.1.10}$$

In fact, for $2 < \alpha < 3$, from (1.1.12) we have

$$\left(D_{a^+}^{\alpha} x \right)(t) = \frac{d^3}{dt^3} \left(I_{a^+}^{3-\alpha} x \right)(t) = \frac{d^3}{dt^3} \left(D_{a^+}^{\alpha-3} x \right)(t) = y'''(t);$$

and (7.1.10) holds clearly when $\alpha = 3$ since $y(t) = x(t)$. Also,

$$\left(D_{a^+}^{\alpha-2} x \right)(t) = \frac{d}{dt} \left(D_{a^+}^{\alpha-3} x \right)(t) = y'(t).$$

Then BVP (7.1.1), (7.1.2) becomes the third-order linear BVP

$$-y''' = q(t)x, \quad y'(a) = y'(b) = 0 \text{ and } y(c) = 0 \quad \text{for } c \in [a,b]. \tag{7.1.11}$$

We denote $z(t) = y'(t)$ and rewrite BVP (7.1.11) as

$$-z'' = q(t)x, \quad z(a) = z(b) = 0.$$

Using the Green's function $G(t,s)$ defined in (7.1.6) for BVP (7.1.7) we have

$$z(t) = \int_a^b G(t,s)q(s)x(s)ds = \int_a^b G(t,s)q(s)(D_{a^+}^{3-\alpha}y)(s)ds.$$

For a fixed $t \in [a,b]$ and $2 < \alpha \leq 3$, applying (7.1.5) with $\phi(s) = G(t,s)q(s)$, $\psi(s) = y(s)$, and $\gamma = 3 - \alpha$, we obtain

$$z(t) = \int_a^b G(t,s)q(s)(D_{a^+}^{3-\alpha}y)(s)ds = \int_a^b y(s)D_{b^-}^{3-\alpha}[G(t,s)q(s)]ds. \tag{7.1.12}$$

Replacing $z(t)$ by $y'(t)$, and then integrating both sides from c to t and using the fact that $y(c) = 0$, we see that

$$y(t) = \int_c^t \int_a^b y(s)D_{b^-}^{3-\alpha}[G(\tau,s)q(s)]dsd\tau. \tag{7.1.13}$$

Hence

$$|y(t)| = \Big| \int_c^t \int_a^b y(s)D_{b^-}^{3-\alpha}[G(\tau,s)q(s)]dsd\tau \Big| \leq \int_a^b \int_a^b |y(s)| \left| D_{b^-}^{3-\alpha}[G(\tau,s)q(s)] \right| dsd\tau. \tag{7.1.14}$$

Let $m := \max_{t\in[a,b]} |y(t)|$. Then

$$m \leq m \int_a^b \int_a^b \left| D_{b^-}^{3-\alpha}[G(\tau,s)q(s)] \right| dsd\tau$$

from which it follows that

$$1 \leq \int_a^b \int_a^b \left| D_{b^-}^{3-\alpha}[G(\tau,s)q(s)] \right| dsd\tau.$$

Using the same argument as given in the proof of Theorem 3.1 in [16], we come to the conclusion that

$$1 < \int_a^b \int_a^b \left| D_{b^-}^{3-\alpha}[G(\tau,s)q(s)] \right| dsd\tau.$$

We omit the details.

□

In the following, we say that a function $u(t)$ does not change sign on an interval J if $u(t) \geq 0$ on J or $u(t) \leq 0$ on J. Under the assumptions that $\left(D_{a^+}^{\alpha-3}x\right)(t)$ does not change sign on (a,c) and on (c,b), we derive sharper Lyapunov-type inequalities than (7.1.8).

Theorem 7.1.2. *Assume Eq. (7.1.1) has a nontrivial solution $x(t)$ satisfying BC (7.1.2) with $c \in (a,b)$. Furthermore, assume $\left(D_{a^+}^{\alpha-3}x\right)(t)$ does not change sign on (a,c) and on (c,b). Then one of the following holds:*

(a) $\displaystyle\int_a^c \int_a^b \left[D_{b^-}^{3-\alpha}[G(t,s)q(s)]\right]_- dsdt > 1,$

(b) $\displaystyle\int_c^b \int_a^b \left[D_{b^-}^{3-\alpha}[G(t,s)q(s)]\right]_+ dsdt > 1,$

(c) $\displaystyle\int_a^c \int_a^c \left[D_{b^-}^{3-\alpha}[G(t,s)q(s)]\right]_- dsdt + \int_a^c \int_c^b \left[D_{b^-}^{3-\alpha}[G(t,s)q(s)]\right]_+ dsdt > 1.$

(d) $\displaystyle\int_c^b \int_a^c \left[D_{b^-}^{3-\alpha}[G(t,s)q(s)]\right]_- dsdt + \int_c^b \int_c^b \left[D_{b^-}^{3-\alpha}[G(t,s)q(s)]\right]_+ dsdt > 1.$

Proof. Let $y(t)$ be defined by (7.1.9). As shown in the proof of Theorem 7.1.1, (7.1.13) holds. Since $y(t)$ is continuous on $[a,b]$ and $y(c)=0$, there exist $t_1 \in [a,c)$ and $t_2 \in (c,b]$ such that $|y(t_1)| = \max\{|y(t)| : t \in [a,c]\}$ and $|y(t_2)| = \max\{|y(t)| : t \in [c,b]\}$. Without loss of generality, we may assume $y(t)$ satisfies one of the following cases:

I. $y(t) \geq 0$ on $(a,c) \cup (c,b)$ and $y(t_1) \geq y(t_2)$;

II. $y(t) \geq 0$ on $(a,c) \cup (c,b)$ and $y(t_1) < y(t_2)$;

III. $y(t) \geq 0$ on (a,c) and $y(t) \leq 0$ on (c,b), and $y(t_1) \geq -y(t_2)$;

IV. $y(t) \geq 0$ on (a, c) and $y(t) \leq 0$ on (c, b), and $y(t_1) < -y(t_2)$.

In the sequel, we denote $m = \max\{|y(t_1)|, |y(t_2)|\}$.

Case I: In this case, $m = y(t_1)$. Then (7.1.13) with $t = t_1$ shows that

$$m = \int_{t_1}^{c} \int_{a}^{b} y(s)\Big[-D_{b^-}^{3-\alpha}[G(\tau, s)q(s)]\Big] ds d\tau \leq m \int_{a}^{c} \int_{a}^{b} \Big[D_{b^-}^{3-\alpha}[G(\tau, s)q(s)]\Big]_- ds d\tau.$$

Similar to the proof of Theorem 7.1.1, we have

$$1 < \int_{a}^{c} \int_{a}^{b} \Big[D_{b^-}^{3-\alpha}[G(\tau, s)q(s)]\Big]_- ds d\tau$$

which shows that conclusion (a) holds.

Case II: In this case, $m = y(t_2)$. Then (7.1.13) with $t = t_2$ shows that

$$m = \int_{c}^{t_2} \int_{a}^{b} y(s) D_{b^-}^{3-\alpha}[G(\tau, s)q(s)] ds d\tau \leq m \int_{c}^{b} \int_{a}^{b} \Big[D_{b^-}^{3-\alpha}[G(\tau, s)q(s)]\Big]_+ ds d\tau.$$

Again, this leads to

$$1 < \int_{c}^{b} \int_{a}^{b} \Big[D_{b^-}^{3-\alpha}[G(\tau, s)q(s)]\Big]_+ ds d\tau$$

which shows that conclusion (b) holds.

Case III: In this case, $m = y(t_1)$. Then (7.1.13) with $t = t_1$ shows that

$$\begin{aligned} m &= \int_{t_1}^{c} \int_{a}^{b} y(s)\Big[-D_{b^-}^{3-\alpha}[G(\tau, s)q(s)]\Big] ds d\tau \\ &= \int_{t_1}^{c} \int_{a}^{c} y(s)\Big[-D_{b^-}^{3-\alpha}[G(\tau, s)q(s)]\Big] ds d\tau + \int_{t_1}^{c} \int_{c}^{b} (-y(s))\Big[D_{b^-}^{3-\alpha}[G(\tau, s)q(s)]\Big] ds d\tau \\ &\leq m \int_{t_1}^{c} \int_{a}^{c} \Big[-D_{b^-}^{3-\alpha}[G(\tau, s)q(s)]\Big] ds d\tau + m \int_{t_1}^{c} \int_{c}^{b} \Big[D_{b^-}^{3-\alpha}[G(\tau, s)q(s)]\Big] ds d\tau. \end{aligned}$$

Once again, this shows that conclusion (c) holds.

Case IV: The same argument as in Case III shows that conclusion (d) holds. We omit the detail. □

As a consequence of Theorem 7.1.2, we have the following corollary.

Corollary 7.1.1. *Assume Eq. (7.1.1) has a nontrivial solution $x(t)$ satisfying BC (7.1.2). Furthermore, assume $\left(D_{a^+}^{\alpha-3}x\right)(t)$ does not change sign on (a,c) and on (c,b).*

(a) Suppose $D_{b^-}^{3-\alpha}[G(t,s)q(s)] \le 0$ for $(s,t) \in [a,b]\times[a,b]$. Then

$$\int_a^b (s-a)^{\alpha-2}(b-s)q_-(s)ds > 2\Gamma(\alpha-2). \tag{7.1.15}$$

(b) Suppose $D_{b^-}^{3-\alpha}[G(t,s)q(s)] \ge 0$ for $(s,t) \in [a,b]\times[a,b]$. Then

$$\int_a^b (s-a)^{\alpha-2}(b-s)q_+(s)ds > 2\Gamma(\alpha-2). \tag{7.1.16}$$

(c) Suppose $D_{b^-}^{3-\alpha}[G(t,s)q(s)] \le 0$ for $(s,t) \in [a,c]\times[a,b]$ and $D_{b^-}^{3-\alpha}[G(t,s)q(s)] \ge 0$ for $(s,t) \in [c,b]\times[a,b]$. Then

$$\int_a^c (s-a)^{\alpha-2}(b-s)q_-(s)ds + \int_c^b (s-a)^{\alpha-2}(b-s)q_+(s)ds > 2\Gamma(\alpha-2). \tag{7.1.17}$$

Proof. (a) By the assumption we have $\left[D_{b^-}^{3-\alpha}[G(t,s)q(s)]\right]_- = -D_{b^-}^{3-\alpha}[G(t,s)q(s)]$ and $\left[D_{b^-}^{3-\alpha}[G(t,s)q(s)]\right]_+ = 0$ for $(s,t) \in [a,b]\times[a,b]$. It is easy to see that in this case, all feasible inequalities in (a)-(d) of Theorem 7.1.2 lead to

$$-\int_a^b \int_a^b D_{b^-}^{3-\alpha}[G(t,s)q(s)]dsdt > 1. \tag{7.1.18}$$

By the definition of $D_{b^-}^{3-\alpha}[G(t,s)q(s)]$ given in we have

$$-\int_a^b \int_a^b D_{b^-}^{3-\alpha}[G(t,s)q(s)]dsdt = \frac{1}{\Gamma(\alpha-2)} \int_a^b \int_a^b \Big(\int_s^b (\tau - s)^{\alpha-3} G(t,\tau) q(\tau) d\tau \Big)' dsdt$$
$$= \frac{-1}{\Gamma(\alpha-2)} \int_a^b \int_a^b (\tau - a)^{\alpha-3} G(t,\tau) q(\tau) d\tau dt$$
$$= \frac{-1}{\Gamma(\alpha-2)} \int_a^b \Big(\int_a^b G(t,\tau)dt \Big) (\tau - a)^{\alpha-3} q(\tau) d\tau.$$

Hence (7.1.18) becomes

$$\int_a^b \Big(\int_a^b G(t,\tau)dt \Big) (\tau - a)^{\alpha-3} (-q(\tau)) d\tau > \Gamma(\alpha-2).$$

Using the facts that $-q(t) \leq q_-(t)$, $G(t,\tau) \geq 0$ on $[a,b] \times [a,b]$ we have

$$\int_a^b \Big(\int_a^b G(t,\tau)dt \Big) (\tau - a)^{\alpha-3} q_-(\tau) d\tau > \Gamma(\alpha-2). \tag{7.1.19}$$

Note that for $\tau \in [a,b]$

$$\int_a^b G(t,\tau)dt = \frac{1}{2}(\tau - a)(b - \tau).$$

Therefore, (7.1.19) leads to (7.1.15).

(b) The proof is similar to case (a) and hence is omitted.

(c) It is easy to see that Theorem 7.1.2, conclusions (a)-(d) leads to

$$\int_a^b \int_a^c \Big[D_{b^-}^{3-\alpha}[G(t,s)q(s)] \Big]_- dsdt + \int_a^b \int_c^b \Big[D_{b^-}^{3-\alpha}[G(t,s)q(s)] \Big]_+ dsdt > 1.$$

Then the proof is similar to case (a) and hence is omitted.

□

Remark 7.1.1. Let $g(s) := (s-a)^{\alpha-1}(b-s)$ for $2 < \alpha \leq 3$. It is easy to see that the maximum of $g(s)$ occurs at $d = [(\alpha-2)b + a]/(\alpha-1)$. Hence for $s \in [a,b]$,

$$g(s) \leq g(d) = \frac{(\alpha-2)^{\alpha-2}(b-a)^{\alpha-1}}{(\alpha-1)^{\alpha-1}}.$$

Therefore, (7.1.15)-(7.1.17) become respectively the following

(i) $\displaystyle\int_a^b q_-(s)ds > \frac{2(\alpha-1)^{\alpha-1}\Gamma(\alpha-2)}{(\alpha-2)^{\alpha-2}(b-a)^{\alpha-1}}.$

(ii) $\displaystyle\int_a^b q_+(s)ds > \frac{2(\alpha-1)^{\alpha-1}\Gamma(\alpha-2)}{(\alpha-2)^{\alpha-2}(b-a)^{\alpha-1}}.$

(iii) $\displaystyle\int_a^c q_-(s)ds + \int_c^b q_+(s)ds > \frac{2(\alpha-1)^{\alpha-1}\Gamma(\alpha-2)}{(\alpha-2)^{\alpha-2}(b-a)^{\alpha-1}}.$

The following provides supplements to Theorem 7.1.2.

Theorem 7.1.3. *(a) Assume Eq. (7.1.1) has a nontrivial solution $x(t)$ satisfying*

$$\left(D_{a^+}^{\alpha-2}x\right)(a^+) = \left(D_{a^+}^{\alpha-2}x\right)(b) \text{ and } \left(D_{a^+}^{\alpha-3}x\right)(a) = 0. \tag{7.1.20}$$

Then

$$\int_a^b \int_a^b \left| D_{b^-}^{3-\alpha}[G(t,s)q(s) \right| dsdt > 1. \tag{7.1.21}$$

Furthermore, assume $\left(D_{a^+}^{\alpha-3}x\right)(t)$ does not change sign on (a,b). Then

$$\int_a^b \int_a^b \left[D_{b^-}^{3-\alpha}[G(t,s)q(s)] \right]_+ dsdt > 1. \tag{7.1.22}$$

(b) Assume Eq. (7.1.1) has a nontrivial solution $x(t)$ satisfying

$$\left(D_{a^+}^{\alpha-2}x\right)(a^+) = \left(D_{a^+}^{\alpha-2}x\right)(b) \text{ and } \left(D_{a^+}^{\alpha-3}x\right)(b) = 0.$$

Then

$$\int_a^b \int_a^b \Big| D_{b^-}^{3-\alpha}[G(t,s)q(s)\Big| dsdt > 1.$$

Furthermore, assume $\left(D_{a^+}^{\alpha-3}x\right)(t)$ *does not change sign on* (a,b)*. Then*

$$\int_a^b \int_a^b \Big[D_{b^-}^{3-\alpha}[G(t,s)q(s)]\Big]_- dsdt > 1.$$

Proof. (a) As in the proof of Theorem 7.1.1, we see that (7.1.13) holds with $c = a$, i.e.,

$$y(t) = \int_a^t \int_a^b y(s) D_{b^-}^{3-\alpha}[G(\tau,s)q(s)]dsd\tau. \tag{7.1.23}$$

Hence

$$|y(t)| = \Big| \int_a^t \int_a^b y(s) D_{b^-}^{3-\alpha}[G(\tau,s)q(s)]dsd\tau \Big| \leq \int_a^b \int_a^b |y(s)| \Big| D_{b^-}^{3-\alpha}[G(\tau,s)q(s)]\Big| dsd\tau.$$

Let $m = \max_{t\in[a,b]} |y(t)|$. Then as shown in the proof of Theorem 7.1.1, this leads to (7.1.21).

Furthermore, assume $y(t) = \left(D_{a^+}^{\alpha-3}x\right)(t)$ does not change sign on (a,b). Without loss of generality, we may assume $y(t) \geq 0$ in $(a,b]$. Then there exists $t_2 \in (a,b]$ such that $m = y(t_2) = \max\{y(t) : t \in [a,b]\}$. Letting $t = t_2$ in (7.1.23) we obtain

$$m = \int_a^{t_2} \int_a^b y(s) D_{b^-}^{3-\alpha}[G(\tau,s)q(s)]dsd\tau \leq m \int_a^b \int_a^b \Big[D_{b^-}^{3-\alpha}[G(\tau,s)q(s)]\Big]_+ dsd\tau.$$

As shown in the proof of Theorem 7.1.1, this leads to (7.1.22).

(b) The proof is similar to Part (a) and hence is omitted. □

Remark 7.1.2. Now, we remark on the special case of Theorems 7.1.1-7.1.3 with $\alpha = 3$, where BVP (7.1.1), (7.1.2) becomes the third-order linear BVP

$$x''' + q(t)x = 0, \quad x'(a) = x'(b) = 0 \text{ and } x(c) = 0 \text{ for } c \in [a, b]. \tag{7.1.24}$$

In this case, conclusion (7.1.8) in Theorem 7.1.1 becomes

$$\int_a^b \int_a^b |G(t, s)q(s)| ds dt > 1. \tag{7.1.25}$$

Note that $G(t, s) \geq 0$ for $(s, t) \in [a, b] \times [a, b]$. Hence

$$\int_a^b \int_a^b |G(t, s)q(s)| ds dt = \int_a^b \int_a^b G(t, s)|q(s)| ds dt = \int_a^b \Big(\int_a^b G(t, s) dt\Big)|q(s)| ds.$$

With a simple calculation we have

$$\int_a^b G(t, s) dt = \frac{1}{2}(b - s)(s - a) \leq \frac{(b - a)^2}{8}$$

and hence (7.1.25) leads to

$$\int_a^b |q(s)| ds > \frac{8}{(b - a)^2}.$$

Similarly, conclusions (a)-(d) in Theorem 7.1.2 lead to

(i) $\displaystyle\int_a^b q_-(t) dt > \frac{8}{(b - a)^2}$,

(ii) $\displaystyle\int_a^b q_+(t) dt > \frac{8}{(b - a)^2}$,

(iii) $\displaystyle\int_a^c q_-(t) dt + \int_c^b q_+(t) dt > \frac{8}{(b - a)^2}$.

The same remark applies to the case when $c = a$ or b given in Theorem 7.1.3. It is easy to see that the condition $x(t)$ does not change sign on (a, c) and on (c, b) in Theorem 7.1.2

and its parallel conditions in Theorem 7.1.3 are not essential for the integer-order differential equations. Therefore, these results agree with Theorem 3.1.1.

Next we derive the Lyapunov-type inequalities for BVP (7.1.1), (7.1.3). Note from (7.1.6) that the Green's function of BVP

$$-u'' = h(t), \quad u(a) = u(\eta) = 0$$

is given by

$$G_\eta(t,s) := \frac{1}{\eta - a}\begin{cases}(s-a)(\eta - t), & a \le s \le t \le \eta,\\ (t-a)(\eta - s), & a \le t \le s \le \eta\end{cases}$$

for any $\eta \in (a,b)$. Then the following results for BVP (7.1.1), (7.1.3) are derived from Theorems 7.1.1 and 7.1.3 for BVP (7.1.1), (7.1.2). Here, we will use $D_{\eta^-}^{3-\alpha}[G_\eta(t,s)q(s)]$ to denote the $(3-\alpha)$-th order right-sided fractional derivative of $G_\eta(t,s)q(s)$ with respect to s at η for fixed $t \in [a,b]$, i.e.,

$$D_{\eta^-}^{3-\alpha}[G(t,s)q(s)] := \frac{-1}{\Gamma(-2+\alpha)}\frac{d}{ds}\int_s^\eta (\tau - s)^{-3+\alpha} G_\eta(t,\tau)q(\tau)d\tau.$$

Theorem 7.1.4. *(a) Assume Eq. (7.1.1) has a nontrivial solution $x(t)$ satisfying BC (7.1.3). Then*

$$\sup_{\eta\in(a,b)} \int_a^\eta \int_a^\eta \left| D_{\eta^-}^{3-\alpha}[G_\eta(t,s)q(s)]\right| dsdt > 1. \tag{7.1.26}$$

(b) Assume Eq. (7.1.1) has a nontrivial solution $x(t)$ satisfying BC (7.1.3) and $\left(D_{a^+}^{\alpha-3}x\right)(t)$ does not change sign on (a,b). Then

$$\sup_{\eta\in(a,b)} \int_a^\eta \int_a^\eta \left[D_{\eta^-}^{3-\alpha}[G_\eta(t,s)q(s)]\right]_+ dsdt > 1. \tag{7.1.27}$$

Proof. (a) Since $\left(D_{a^+}^{\alpha-3}x\right)(a^+) = \left(D_{a^+}^{\alpha-3}x\right)(b) = 0$, by Rolle's Theorem there exists a $\eta \in (a,b)$ such that $\left(D_{a^+}^{\alpha-2}x\right)(\eta) = \left(D_{a^+}^{\alpha-3}x\right)'(\eta) = 0$. Hence it satisfies

$$\left(D_{a^+}^{\alpha-2}x\right)(a^+) = \left(D_{a^+}^{\alpha-2}x\right)(\eta) = 0 \text{ and } \left(D_{a^+}^{\alpha-3}x\right)(a^+) = 0. \tag{7.1.28}$$

Applying Theorem 7.1.1 to BVP (7.1.1), (7.1.28) we have

$$\int_a^\eta \int_a^\eta \left| D_{\eta^-}^{3-\alpha}[G_\eta(t,s)q(s)] \right| dsdt > 1.$$

Then (7.1.26) follows.

(b) From the proof of Part (a) we see that there exists a $\eta \in (a,b)$ such that $\left(D_{a^+}^{\alpha-2}x\right)(\eta) = 0$. Hence $x(t)$ satisfies (7.1.28). By the assumption, $\left(D_{a^+}^{\alpha-3}x\right)(t)$ does not change sign on (a,η). Applying the second part of Theorem 7.1.3, Part (a) to BVP (7.1.1), (7.1.28) we have

$$\int_a^\eta \int_a^\eta \left[D_{\eta^-}^{3-\alpha}[G_\eta(t,s)q(s)] \right]_+ dsdt > 1.$$

Then (7.1.27) follows. □

With the same argument as in Corollary 7.1.1, we obtain the corollary below from Theorem 7.1.4.

Corollary 7.1.2. *Assume Eq. (7.1.1) has a nontrivial solution $x(t)$ satisfying BC (7.1.3). Suppose $D_{\eta^-}^{3-\alpha}[G_\eta(\tau,s)q(s)] \geq 0$ on $[a,\eta] \times [a,\eta]$ for every $\eta \in (a,b)$. Then*

$$\int_a^b q_+(s)ds > \frac{2(\alpha-1)^{\alpha-1}\Gamma(\alpha-2)}{(\alpha-2)^{\alpha-2}(b-a)^{\alpha-1}}.$$

In the last part of this section, we derive the Lyapunov-type inequalities for BVP (7.1.1), (7.1.4).

Theorem 7.1.5. *(a) Assume Eq. (7.1.1) has a nontrivial solution $x(t)$ satisfying BC (7.1.4). Then*

$$\max_{t\in[a,b]} \int_a^b \int_a^s G(t,s) \left| \left(D_{s^-}^{3-\alpha} q \right)(\tau) \right| d\tau ds > 1. \tag{7.1.29}$$

(b) Assume Eq. (7.1.1) has a nontrivial solution $x(t)$ satisfying BC (7.1.4) and $\left(D_{a^+}^{\alpha-3}x\right)(t)$ does not change sign on (a,b). Then

$$\max_{t\in[a,b]} \int_a^b \int_a^s G(t,s) \left[\left(D_{s^-}^{3-\alpha} q \right)(\tau) \right]_+ d\tau ds > 1. \tag{7.1.30}$$

Proof. Let $y(t)$ be defined by (7.1.9). As shown in the proof of Theorem 7.1.1, BVP (7.1.1), (7.1.4) becomes

$$-y''' = q(t)x, \quad y(a) = y(b) = 0 \text{ and } y''(a) = 0.$$

It follows that

$$-y'' = \int_a^t q(\tau)x(\tau)d\tau = \int_a^t q(\tau) \left(D_{a^+}^{3-\alpha} y\right)(\tau)d\tau.$$

For any $t \in (a,b]$, applying (7.1.5) with $\phi(\tau) = q(\tau)$, $\psi(\tau) = \left(D_{a^+}^{3-\alpha}y\right)(\tau)$, $\gamma = 3-\alpha$, and b replaced by t, we obtain

$$-y'' = \int_a^t y(\tau) \left(D_{t^-}^{3-\alpha} q\right)(\tau)d\tau.$$

Using the Green's function $G(t,s)$ given by (7.1.6) for BVP (7.1.7) we see

$$y(t) = \int_a^b \int_a^s G(t,s)y(\tau) \left(D_{s^-}^{3-\alpha} q\right)(\tau)d\tau ds. \tag{7.1.31}$$

Hence

$$|y(t)| = \left| \int_a^b \int_a^s G(t,s)y(\tau) \left(D_{s^-}^{3-\alpha} q\right)(\tau)d\tau ds \right| \le \int_a^b \int_a^s G(t,s)|y(\tau)| \left| \left(D_{s^-}^{3-\alpha} q\right)(\tau) \right| d\tau ds.$$

Let $m = \max_{t\in[a,b]} |y(t)|$. Then

$$m \leq m \max_{t\in[a,b]} \int_a^b \int_a^s G(t,s) \left| \left(D_{s^-}^{3-\alpha} q\right)(\tau) \right| d\tau ds$$

from which it follows that

$$1 \leq \max_{t\in[a,b]} \int_a^b \int_a^s G(t,s) \left| \left(D_{s^-}^{3-\alpha} q\right)(\tau) \right| d\tau ds.$$

This, together with the argument in the proof of Theorem 7.1.1, leads to (7.1.29).

(b) From the proof of Part (a) we see that (7.1.31) holds. Without loss of generality, assume that $y(t) \geq 0$ on (a,b). Let $m = \max_{t\in[a,b]} y(t)$. Then from (7.1.31) we see that

$$m \leq \max_{t\in[a,b]} \int_a^b \int_a^s G(t,s) y(\tau) \left[\left(D_{s^-}^{3-\alpha} q\right)(\tau) \right]_+ d\tau ds.$$

After this the same argument as in the proof of Theorem 7.1.1 leads us to (7.1.30). □

With the same argument as in Corollary 7.1.1, we obtain the corollary below from Theorem 7.1.5.

Corollary 7.1.3. *Assume Eq. (7.1.1) has a nontrivial solution $x(t)$ satisfying BC (7.1.4). Suppose $\left(D_{s^-}^{3-\alpha} q\right)(\tau) \geq 0$ for $s \in [a,b]$. Then*

$$\int_a^b (t-a)^{\alpha-3} q_+(t) dt > \frac{8\Gamma(\alpha-2)}{(b-a)^2}.$$

Remark 7.1.3. Here we remark on the special case of Theorem 7.1.5 with $\alpha = 3$ where BVP (7.1.1),(7.1.4) becomes the third-order linear BVP

$$x''' + q(t)x = 0, \quad x(a) = x(b) = 0 \text{ and } x''(a) = 0. \tag{7.1.32}$$

Note that

$$\max_{t\in[a,b]}\int_a^b\int_a^s G(t,s)|q(\tau)|d\tau ds \le \max_{t\in[a,b]}\int_a^b\int_a^b G(t,s)|q(\tau)|d\tau ds$$
$$\le \int_a^b\left(\max_{t\in[a,b]}\int_a^b G(t,s)ds\right)|q(\tau)|d\tau$$

and by Remark 7.1.2

$$\max_{t\in[a,b]}\int_a^b G(t,s)ds = \frac{(b-a)^2}{8}.$$

Hence conclusion (7.1.29) in Theorem 7.1.5 leads to

$$\int_a^b |q(\tau)|d\tau > \frac{8}{(b-a)^2}.$$

Similarly, conclusion (7.1.30) in Theorem 7.1.5 leads to

$$\int_a^b q_+(\tau)d\tau > \frac{8}{(b-a)^2}.$$

These inequalities improve those in Theorem 2.1.1 when $\xi = a$.

7.2 Applications to boundary value problems

In the last section, we apply the results on the Lyapunov-type Inequalities obtained in Section 7.1 to study the nonexistence, uniqueness, and existence-uniqueness of solutions of certain fractional order linear BVPs.

Definition 7.2.1. *A nontrivial solution $x(t)$ of Eq. (7.1.1) is said to be I-positive if $\left(I_{a^+}^{n-\alpha}x\right)(t) \ge 0$ on $[a,b]$, where $n = \lfloor\alpha\rfloor + 1$.*

The following result is on the nonexistence of certain solutions of BVP (7.1.1), (7.1.2).

Theorem 7.2.1. *(a) Assume*

$$\int_a^b \int_a^b \left| D_{b^-}^{3-\alpha}[G(t,s)q(s)] \right| dsdt \leq 1. \tag{7.2.1}$$

Then BVP (7.1.1), (7.1.2) has no nontrivial solution.

(b) Assume

$$\int_a^c \int_a^b \Big[D_{b^-}^{3-\alpha}[G(t,s)q(s)] \Big]_- dsdt \leq 1 \tag{7.2.2}$$

and

$$\int_c^b \int_a^b \Big[D_{b^-}^{3-\alpha}[G(t,s)q(s)] \Big]_+ dsdt \leq 1 \tag{7.2.3}$$

Then BVP (7.1.1), (7.1.2) has no I-positive solution.

Proof. (a) Assume the contrary, i.e., BVP (7.1.1), (7.1.2) has a nontrivial solution $x(t)$. Then by Theorem 7.1.1, (7.1.8) holds. This contradicts assumption (7.2.1).

(b) Assume the contrary, i.e., BVP (7.1.1), (7.1.2) has an I-positive solution $x(t)$. Then from the proof of Theorem 7.1.2, we see that only Cases I and II in the proof are feasible. Hence either

$$\int_a^c \int_a^b \Big[D_{b^-}^{3-\alpha}[G(t,s)q(s)] \Big]_- dsdt > 1$$

or

$$\int_c^b \int_a^b \Big[D_{b^-}^{3-\alpha}[G(t,s)q(s)] \Big]_+ dsdt > 1.$$

This contradicts the assumptions. □

Next we apply the results of Theorem 7.2.1 to study the nonhomogeneous linear BVP consisting of the equation

$$\Big(D_{a^+}^{\alpha} x \Big)(t) + q(t)x = w(t), \quad \text{on } (a,b) \tag{7.2.4}$$

and the BC

$$\left(D_{a^+}^{\alpha-2}x\right)(a^+) = k_1,\ \left(D_{a^+}^{\alpha-2}x\right)(b) = k_2,\ \text{and}\ \left(D_{a^+}^{\alpha-3}x\right)(c) = k_3, \tag{7.2.5}$$

where $q, w \in L((a,b),\mathbb{R})$, $2 < \alpha \leq 3$, and $k_1, k_2, k_3 \in \mathbb{R}$. Based on Theorem 7.2.1, we obtain a criterion for BVP (7.2.4), (7.2.5) to have a unique solution and reveal a relation among the solutions if the problem has more than one solution.

Theorem 7.2.2. *(a) Assume*

$$\int_a^b \int_a^b \left| D_{b^-}^{3-\alpha}[G(t,s)q(s)] \right| dsdt \leq 1. \tag{7.2.6}$$

Then BVP (7.2.4), (7.2.5) has a unique solution on (a,b) *for any* $k_1, k_2, k_3 \in \mathbb{R}$.

(b) Assume

$$\int_a^b \int_a^b \left[D_{b^-}^{3-\alpha}[G(t,s)q(s)] \right]_{\pm} dsdt \leq 1 < \int_a^b \int_a^b \left| D_{b^-}^{3-\alpha}[G(t,s)q(s)] \right| dsdt.$$

If BVP (7.2.4), (7.2.5) has two solutions $x_1(t)$ *and* $x_2(t)$*, then there exists a* $d \in (a,b)$ *such that* $\left(I_{a^+}^{3-\alpha}x_1\right)(d) = \left(I_{a^+}^{3-\alpha}x_2\right)(d)$.

Proof. (a) By Theorem 7.2.1, Part (a), BVP (7.1.1), (7.1.2) has only the zero solution. Then by the Fredholm alternative theorem [28], we conclude that BVP (7.2.4), (7.2.5) has a unique solution.

(b) The conclusion is clearly true when $x_1(t) \equiv x_2(t)$ on $[a,b]$. Assume $x_1(t) \not\equiv x_2(t)$ on $[a,b]$ and let $x(t) = x_1(t) - x_2(t)$. Then $x(t)$ is a nontrivial solution of BVP (7.1.1), (7.1.2). By Theorem 7.2.1, Part (b), $x(t)$ is not I-positive on $[a,b]$. With the same reason, $-x(t)$ is not an I-positive solution on [a,b] either. Then there exists a $d \in (a,b)$ such that $\left(I_{a^+}^{3-\alpha}x\right)(d) = 0$, i.e., $\left(I_{a^+}^{3-\alpha}x_1\right)(d) = \left(I_{a^+}^{3-\alpha}x_2\right)(d)$. □

The results in this section can be easily extended to the homogeneous linear BVPs (7.1.1), (7.1.3) and (7.1.1), (7.1.4), and their corresponding nonhomogeneous linear BVPs. We left the details to the interested reader.

CHAPTER 8

FRACTIONAL DIFFERENTIAL EQUATIONS III

8.1 Fractional Lyapunov-type inequalities

We consider the α-th order fractional linear differential equation

$$\left(D_{a^+}^{\alpha}x\right)(t)+q(t)x=0,\quad 2<\alpha\leq 3, \tag{8.1.1}$$

where $q\in L([a,b],\mathbb{R})$ and $\left(D_{a^+}^{\alpha}x\right)(t)$ denotes the αth-order left-sided Riemann-Liouville fractional derivative of $x(t)$ at a as defined in (1.1.12). We will derive Lyapunov-type inequalities for the BVPs consisting of Eq. (8.1.1) and one of the following BCs:

$$x(a)=0\quad\text{and}\quad x'(a)=x'(b)=0; \tag{8.1.2}$$

$$x(a)=x(b)=0\quad\text{and}\quad x'(a)=0; \tag{8.1.3}$$

$$x(a)=0\quad\text{and}\quad\left(D_{a^+}^{\alpha-2}x\right)(a)=\left(D_{a^+}^{\alpha-2}x\right)(b)=0; \tag{8.1.4}$$

$$x(a)=x(b)=0\quad\text{and}\quad\left(D_{a^+}^{\alpha-2}x\right)(a)=0; \tag{8.1.5}$$

$$x(a)=x(b)=0\quad\text{and}\quad\left(D_{a^+}^{\alpha-1}x\right)(\xi)=0,\quad \xi\in[a,b]. \tag{8.1.6}$$

In the following, we say that a solution $x(t)$ of Eq. (8.1.1) does not change sign on $[a,b]$ if $x(t)\geq 0$ on $[a,b]$ or $x(t)\leq 0$ on $[a,b]$. The first result is for Eq. (8.1.1) with BC (8.1.2) or BC (8.1.3).

Theorem 8.1.1. *Assume Eq. (8.1.1) has a nontrivial solution $x(t)$ satisfying either (8.1.2) or (8.1.3) and $x(t)$ does not change sign on $[a,b]$. Then*

$$\int_a^b q_+(t)dt > \frac{(\alpha-1)^{\alpha-1}\Gamma(\alpha)}{(\alpha-2)^{\alpha-2}(b-a)^{\alpha-1}}. \tag{8.1.7}$$

The next result is for Eq. (8.1.1) with BC (8.1.4) or BC (8.1.5).

Theorem 8.1.2. *Assume Eq. (8.1.1) has a nontrivial solution $x(t)$ satisfying either (8.1.4) or (8.1.5) and $x(t)$ does not change sign on $[a,b]$. Then*

$$\int_a^b q_+(t)dt > \frac{4\Gamma(\alpha-1)}{(b-a)^{\alpha-1}}. \tag{8.1.8}$$

The last result is for Eq. (8.1.1) with BC (8.1.6).

Theorem 8.1.3. *Assume Eq. (8.1.1) has a nontrivial solution $x(t)$ satisfying (8.1.6) and $x(t)$ does not change sign on $[a,b]$. Then*

$$\int_a^\xi q_-(t)dt + \int_\xi^b q_+(t)dt > \frac{(\alpha-1)^{\alpha-1}\Gamma(\alpha)}{(\alpha-2)^{\alpha-2}(b-a)^{\alpha-1}}. \tag{8.1.9}$$

Consequently,

$$\int_a^b |q(t)|dt > \frac{(\alpha-1)^{\alpha-1}\Gamma(\alpha)}{(\alpha-2)^{\alpha-2}(b-a)^{\alpha-1}}. \tag{8.1.10}$$

Remark 8.1.1. As special cases, when $a=\xi$, then (8.1.9) becomes

$$\int_a^b q_+(t)dt > \frac{(\alpha-1)^{\alpha-1}\Gamma(\alpha)}{(\alpha-2)^{\alpha-2}(b-a)^{\alpha-1}};$$

and when $\xi=b$, then (8.1.9) becomes

$$\int_a^b q_-(t)dt > \frac{(\alpha-1)^{\alpha-1}\Gamma(\alpha)}{(\alpha-2)^{\alpha-2}(b-a)^{\alpha-1}}.$$

In general, (8.1.9) is sharper than (8.1.10). Furthermore, (8.1.9) shows that under the assumptions of Theorem 8.1.3, we never expect that

$$q(t)\begin{cases}\geq 0, & t\in[a,\xi]\\ \leq 0, & t\in(\xi,b]\end{cases}$$

would happen. However, this cannot be observed from (8.1.10).

8.2 Proofs

In order to prove Theorems 8.1.1-8.1.3, we will use the following lemmas. The first one is from [27] which extends the one given in [8] for the case that $a=0$ and $b=1$.

Lemma 8.2.1. *Consider the BVP*

$$-\left(D_{a^+}^{\beta}u\right)(t)=h(t),\quad u(a)=u(b)=0 \tag{8.2.1}$$

with $1<\beta\leq 2$ *and* $h\in C([a,b],\mathbb{R})$*. Then* $u(t)$ *is a solution of BVP (8.2.1) if and only if* $u(t)$ *satisfies the following integral equation*

$$u(t)=\int_a^b G_\beta(t,s)h(s)ds, \tag{8.2.2}$$

where $G_\beta(t,s)$ *is referred as the Green's function for BVP (8.2.1) and is given by*

$$G_\beta(t,s)=\frac{1}{\Gamma(\beta)}\begin{cases}\frac{(t-a)^{\beta-1}(b-s)^{\beta-1}}{(b-a)^{\beta-1}}-(t-s)^{\beta-1}, & a\leq s\leq t\leq b\\ \frac{(t-a)^{\beta-1}(b-s)^{\beta-1}}{(b-a)^{\beta-1}}, & a\leq t\leq s\leq b\end{cases}. \tag{8.2.3}$$

Moreover, $G_\beta(t,s) \ge 0$.

The next lemma provides an estimate for the integral of $G_\beta(t,s)$ in t.

Lemma 8.2.2. *For* $s \in [a,b]$ *we have*

$$\int_a^b G_\beta(t,s)dt \le \frac{(\beta-1)^{\beta-1}(b-a)^\beta}{\beta^\beta\Gamma(\beta+1)}. \tag{8.2.4}$$

Proof. For $s \in [a,b]$,

$$\begin{aligned}
&\int_a^b G_\beta(t,s)dt = \int_a^s G_\beta(t,s)dt + \int_s^b G_\beta(t,s)dt \\
=&\frac{1}{\Gamma(\beta)}\left[\left(\frac{b-s}{b-a}\right)^{\beta-1}\int_a^s (t-a)^{\beta-1}\,dt + \left(\frac{b-s}{b-a}\right)^{\beta-1}\int_s^b (t-a)^{\beta-1}\,dt - \int_s^b (t-s)^{\beta-1}dt\right] \\
=&\frac{1}{\Gamma(\beta)}\left[\left(\frac{b-s}{b-a}\right)^{\beta-1}\int_a^b (t-a)^{\beta-1}\,dt - \int_s^b (t-s)^{\beta-1}dt\right] \\
=&\frac{1}{\beta\Gamma(\beta)}\left[\left(\frac{b-s}{b-a}\right)^{\beta-1}(b-a)^\beta - (b-s)^\beta\right] \\
=&\frac{1}{\Gamma(\beta+1)}(s-a)(b-s)^{\beta-1}.
\end{aligned}$$

Let $g(s) := (s-a)(b-s)^{\beta-1}$. It is easy to see that the maximum of $g(s)$ occurs at $d = [b+(\beta-1)a]/\beta$. Hence

$$g(s) \le g(d) = \frac{(\beta-1)^{\beta-1}(b-a)^\beta}{\beta^\beta}.$$

This shows that (8.2.4) holds. □

The last lemma provides an extension of Rolle's theorem to Riemann-Liouville fractional derivatives.

Lemma 8.2.3. *Assume a function* $u(t)$ *satisfies* $u(a) = u(b) = 0$ *and for* $0 < \gamma \le 1$, $\left(D^\gamma_{a^+}u\right)(t)$ *defined by* (1.1.12) *exists on* (a,b). *Then there exists a* $c \in (a,b)$ *such that* $\left(D^\gamma_{a^+}u\right)(c) = 0$.

Proof. The case for $\gamma = 1$ follows immediately from Rolle's theorem. Now, let $0 < \gamma < 1$. Assume the contrary and without loss of generality, let $\left(D^{\gamma}_{a^+}u\right)(t) > 0$ for $t \in (a, b)$. By integration we have

$$u(t) = I^{\gamma}_{a^+}\left(D^{\gamma}_{a^+}u(t)\right) + C(t-a)^{\gamma-1}$$

with $C \in \mathbb{R}$. Now $u(a) = 0$ implies $C = 0$. Hence

$$u(t) = I^{\gamma}_{a^+}\left(D^{\gamma}_{a^+}u(t)\right) = \frac{1}{\Gamma(\gamma)}\int_a^t (t-s)^{\gamma-1}\left(D^{\gamma}_{a^+}u\right)(s)ds.$$

Then

$$u(b) = \frac{1}{\Gamma(\gamma)}\int_a^b (b-s)^{\gamma-1}\left(D^{\gamma}_{a^+}u\right)(s)ds > 0.$$

We have reached a contradiction. $\square$

Now we prove Theorems 8.1.1-8.1.2.

Proof of Theorem 8.1.1. (i) We assume $x(t)$ satisfies (8.1.2) and without loss of generality, let that $x(t) \geq 0$ for $t \in [a, b]$. We claim that $\Big(D^{\alpha}_{a^+}x\Big)(t) = \Big(D^{\alpha-1}_{a^+}x'\Big)(t)$ for $x(a) = 0$. Clearly the claim holds for $\alpha = 3$. Now let $2 < \alpha < 3$ which implies $0 < \alpha - 2 < 1$. By [54, (2.1.28)] and from the fact that $x(a) = 0$, we have

$$\Big(D^{\alpha-2}_{a+}x\Big)(t) = \frac{1}{\Gamma(3-\alpha)}\left[\frac{x(a)}{(t-a)^{\alpha-2}} + \int_a^t \frac{x'(s)}{(t-s)^{\alpha-2}}ds\right] = \Big(I^{3-\alpha}(x')\Big)(t).$$

Differentiating both sides twice and using (1.1.12) with $\gamma = \alpha - 1$ we have

$$\Big(D^{\alpha}_{a+}x\Big)(t) = \frac{d^2}{dt^2}\Big(D^{\alpha-2}_{a+}x\Big)(t) = \frac{d^2}{dt^2}\Big(I^{3-\alpha}(x')\Big)(t) = (D^{\alpha-1}_{a+}x')(t).$$

Hence (8.1.1) and (8.1.2) lead to

$$-\Big(D_{a^+}^{\alpha-1}x'\Big)(t) = q(t)x, \quad x'(a) = x'(b) = 0.$$

Using Lemma 8.2.1 with $u = x'$ and $\beta = \alpha - 1$ we have

$$x'(t) = \int_a^b G_{\alpha-1}(t,s)q(s)x(s)ds,$$

Since $x(a) = 0$, it follows that

$$x(t) = \int_a^t \int_a^b G_{\alpha-1}(\tau,s)q(s)x(s)ds\, d\tau.$$

Denote $m = \max\{x(t) : \ t \in [a,b]\}$. Note that $G_{\alpha-1}(t,s) \geq 0$, $0 \leq x(t) \leq m$ and $x(t) \not\equiv m$, and $q(t) \leq q_+(t)$, we have

$$m < m \int_a^b \int_a^b G_{\alpha-1}(\tau,s)q_+(s)dsd\tau = m \int_a^b \Big(\int_a^b G_{\alpha-1}(\tau,s)d\tau\Big)q_+(s)ds.$$

canceling m from both sides and applying (8.2.4) with $\beta = \alpha - 1$, we obtain (8.1.7).

(ii) We assume $x(t)$ satisfies (8.1.3). By Rolle's theorem, there exists a $c \in (a,b)$ such that $x'(c) = 0$. Hence $x(t)$ satisfies BC (8.1.2) with b replaced by c. Now applying Part (i) we obtain

$$\int_a^b q_+(t)dt \geq \int_a^c q_+(t)dt > \frac{(\alpha-1)^{\alpha-1}\Gamma(\alpha)}{(\alpha-2)^{\alpha-2}(c-a)^{\alpha-1}} > \frac{(\alpha-1)^{\alpha-1}\Gamma(\alpha)}{(\alpha-2)^{\alpha-2}(b-a)^{\alpha-1}}.$$

□

Proof of Theorem 8.1.2. (i) We assume $x(t)$ satisfies (8.1.3) and without loss of generality, let that $x(t) \geq 0$ for $t \in [a,b]$. We note that $\left(D_{a^+}^{\alpha}x\right)(t) = \left(D_{a^+}^{\alpha-2}x\right)''(t)$. Hence (8.1.1) and (8.1.4) lead to

$$-\left(D_{a^+}^{\alpha-2}x\right)'' = q(t)x \quad \text{with} \quad \left(D_{a^+}^{\alpha-2}x\right)(a) = \left(D_{a^+}^{\alpha-2}x\right)(b) = 0.$$

Using Lemma 8.2.1 with $u = D_{a^+}^{\alpha-2}x$ and $\beta = 2$ we have

$$\left(D_{a^+}^{\alpha-2}x\right)(t) = \int_a^b G_2(t,s)q(s)x(s)ds, \tag{8.2.5}$$

where

$$G_2(t,s) = \frac{1}{b-a}\begin{cases}(s-a)(b-t), & a \leq s \leq t \leq b \\ (t-a)(b-s), & a \leq t \leq s \leq b\end{cases}$$

satisfying

$$0 \leq G_2(t,s) \leq G_2(s,s) = (s-a)(b-s)/(b-a). \tag{8.2.6}$$

It follows that

$$x(t) = \left(I_{a^+}^{\alpha-2}\left(D_{a^+}^{\alpha-2}x\right)\right)(t) + C(t-a)^{\alpha-3}$$

with $C \in \mathbb{R}$. Now $x(a) = 0$ implies $C = 0$ and thus

$$x(t) = \left(I_{a^+}^{\alpha-2}\left(D_{a^+}^{\alpha-2}x\right)\right)(t) = \frac{1}{\Gamma(\alpha-2)}\int_a^t (t-s)^{\alpha-3}\left(D_{a^+}^{\alpha-2}x\right)(s)ds.$$

From (8.2.5) it follows that

$$
\begin{aligned}
x(t) &= \frac{1}{\Gamma(\alpha-2)}\int_a^t (t-s)^{\alpha-3}\int_a^b G_2(s,\tau)q(\tau)x(\tau)d\tau ds \\
&= \frac{1}{\Gamma(\alpha-2)}\int_a^t\int_a^b (t-s)^{\alpha-3}G_2(s,\tau)q(\tau)x(\tau)d\tau ds \\
&= \frac{1}{\Gamma(\alpha-2)}\int_a^b \left(\int_a^t (t-s)^{\alpha-3}G_2(s,\tau)ds\right) q(\tau)x(\tau)d\tau.
\end{aligned}
$$

Denote $m = \max\{x(t):\ t\in[a,b]\}$. Note that $0 \le G_2(s,\tau) \le G_2(\tau,\tau)$, $0 \le x(t) \le m$ and $x(t) \not\equiv m$, and $q(t) \le q_+(t)$, from (8.2.6) we have that for $t \in [a,b]$

$$
m < \frac{m}{(b-a)\Gamma(\alpha-2)}\int_a^b \left(\int_a^t (t-s)^{\alpha-3}ds\right)(\tau-a)(b-\tau)q_+(\tau)d\tau. \tag{8.2.7}
$$

Note that

$$
\int_a^t (t-s)^{\alpha-3}ds = \frac{(t-a)^{\alpha-2}}{\alpha-2} \le \frac{(b-a)^{\alpha-2}}{\alpha-2},\ t\in[a,b]
$$

and

$$
(\tau-a)(b-\tau) \le \frac{(b-a)^2}{4},\ \tau\in[a,b].
$$

Then (8.1.8) follows from (8.2.7).

(ii) We assume $x(t)$ satisfies (8.1.5). Note that $0 < \alpha-2 \le 1$, by Lemma 8.2.3 with $\gamma = \alpha-2$, there exists a $c \in (a,b)$ such that $\left(D_{a^+}^{\alpha-2}x\right)(c) = 0$. Hence $x(t)$ satisfies BC (8.1.4) with b replaced by c. Now applying Part (i) we obtain

$$
\int_a^b q_+(t)dt \ge \int_a^c q_+(t)dt > \frac{4\Gamma(\alpha-1)}{(c-a)^{\alpha-1}} > \frac{4\Gamma(\alpha-1)}{(b-a)^{\alpha-1}}.
$$

$\square$

Proof of Theorem 8.1.3. Without loss of generality we assume that $x(t) \geq 0$ for $t \in [a, b]$. We rewrite Eq. (8.1.1) as

$$\left(D_{a^+}^{\alpha-1}x\right)'(t) + q(t)x = 0.$$

Integrating both sides from ξ to t and using the fact that $\left(D_{a^+}^{\alpha-1}x\right)(\xi) = 0$ we have

$$-\left(D_{a^+}^{\alpha-1}x\right) = \int_\xi^t q(\tau)x(\tau)d\tau.$$

By Lemma 8.2.1 with $u = x$ and $\beta = \alpha - 1$ we see that

$$x(t) = \int_a^b G_{\alpha-1}(t,s)\int_\xi^s q(\tau)x(\tau)d\tau ds = \int_a^b\int_\xi^s G_{\alpha-1}(t,s)q(\tau)x(\tau)d\tau ds.$$

It follows that

$$\begin{aligned}
x(t) &= \int_a^\xi\int_\xi^s G_{\alpha-1}(t,s)q(\tau)x(\tau)d\tau ds + \int_\xi^b\int_\xi^s G_{\alpha-1}(t,s)q(\tau)x(\tau)d\tau ds \\
&= \int_a^\xi\int_s^\xi G_{\alpha-1}(t,s)(-q(\tau))x(\tau)d\tau ds + \int_\xi^b\int_\xi^s G_{\alpha-1}(t,s)q(\tau)x(\tau)d\tau ds \\
&= \int_a^\xi\left(\int_a^\tau G_{\alpha-1}(t,s)ds\right)(-q(\tau))x(\tau)d\tau + \int_\xi^b\left(\int_\tau^b G_{\alpha-1}(t,s)ds\right)q(\tau)x(\tau)d\tau.
\end{aligned}$$

Denote $m = \max\{x(t) :\ t \in [a,b]\}$. Note that $G_{\alpha-1}(t,s) \geq 0$, $0 \leq x(t) \leq m$ and $x(t) \not\equiv m$ on $[a,b]$, $-q(t) \leq q_-(t)$, and $q(t) \leq q_+(t)$. Hence, we have that for $t \in [a,b]$

$$m < m\int_a^b G_{\alpha-1}(t,s)ds\left(\int_a^\xi q_-(\tau)d\tau + \int_\xi^b q_+(\tau)d\tau\right). \tag{8.2.8}$$

Lemma 8.2.2 with $\beta = \alpha - 1$ lead to

$$\int_a^b G_{\alpha-1}(t,s)ds = \frac{1}{\Gamma(\alpha)}\max_{t\in[a,b]}(t-a)^{\alpha-2}(b-t) = \frac{(\alpha-2)^{\alpha-2}(b-a)^{\alpha-1}}{(\alpha-1)^{\alpha-1}\Gamma(\alpha)}.$$

Then (8.1.9) follows from (8.2.8). (8.1.10) follows from (8.1.9) immediately. □

8.3 Multivariate Lyapunov-type Inequalities

In the last section, we show how the Lyapunov-type inequalities in Section 8.1 can be extended to fractional multivariate equations. To avoid redundancy, we only give the extension of Theorem 8.1.3. To present our results, we introduce some notations.

For $N \geq 2$, we denote

$$S^{N-1} := \{u \in \mathbb{R}^N : |u| = 1\}$$

as the unit sphere in $\mathbb{R}^N$. It is well known that the surface area of S^{N-1} is

$$\int_{S^{N-1}} d\omega = \frac{2\pi^{N/2}}{\Gamma(N/2)},$$

where Γ stands for the Gamma function as given in Section 2. Note that every $u \in \mathbb{R}^N \setminus \{0\}$ has a unique representation of the form $u = r\omega$ with $|u| = r$ for some $r > 0$ and $\omega \in S^{N-1}$.

Assume that $a, b, \xi \in C(S^{N-1}, \mathbb{R})$ and $0 < a(\omega) \leq \xi(\omega) \leq b(\omega)$ for all $\omega \in S^{N-1}$. We define a doubly connected region A in $\mathbb{R}^N$ as

$$A := \{u = r\omega : r \in (a(\omega), b(\omega)), \omega \in S^{N-1}\},$$

together with its subregions

$$A_1 := \{u = r\omega : r \in (a(\omega), \xi(\omega)), \omega \in S^{N-1}\}$$

and

$$A_2 := \{u = r\omega : r \in (\xi(\omega), b(\omega)), \omega \in S^{N-1}\}.$$

Clearly, $A = A_1 \cup A_2$. Let the corresponding boundaries be denoted by

$$B_a = \{u = r\omega : r = a(\omega), \omega \in S^{N-1}\},$$

$$B_b = \{u = r\omega : r = b(\omega), \omega \in S^{N-1}\},$$

and

$$B_\xi = \{u = r\omega : r = \xi(\omega), \omega \in S^{N-1}\}.$$

The following gives a graphical interpretation of the region A.

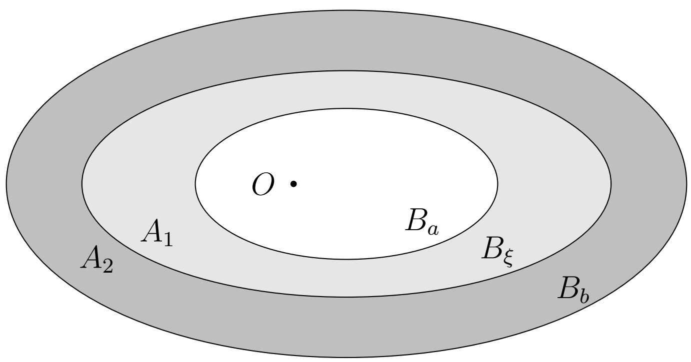

Figure 8.1: Region A

Let $\omega \in S^{N-1}$ be fixed. For any $\gamma > 0$ and $u = r\omega$ with $r > a(\omega)$, we denote by $(\mathcal{D}_r^\gamma y)(u)$ the γ-th order Riemann-Liouville directional derivative of $y(u)$ in the radial direction at $a(\omega)$, i.e.,

$$\left(\mathcal{D}_r^\gamma y\right)(u) := \frac{1}{\Gamma(n-\gamma)} \frac{\partial^n}{\partial r^n} \int_{a(\omega)}^{r} (r-s)^{n-\gamma-1} y(s\omega) ds, \tag{8.3.1}$$

where n and Γ are given in Section 8.1. Now, on the region A, we consider the equation

$$\left(\mathcal{D}_r^\alpha y\right)(u) + q(u)y = 0, \quad 2 < \alpha \leq 3, \tag{8.3.2}$$

where $q \in C(\overline{A})$, together with the BC

$$y(B_a) = y(B_b) = 0 \quad \text{and} \quad \left(\mathcal{D}_r^{\alpha-1} y\right)(B_\xi) = 0. \tag{8.3.3}$$

The following provides a Lyapunov-type inequality for BVP (8.3.2), (8.3.3).

Theorem 8.3.1. *Assume Eq. (8.3.2) has a nontrivial solution $y(u)$ satisfying (8.3.3) and $y(u)$ does not change sign on A . Then*

$$\int_{A_1} |u|^{1-N} q_-(u) du + \int_{A_2} |u|^{1-N} q_+(u) du > \frac{2\pi^{N/2}\Gamma(\alpha)(\alpha-1)^{\alpha-1}}{\Gamma(N/2)(\alpha-2)^{\alpha-2}(b-a)^{\alpha-1}}, \tag{8.3.4}$$

where $a = \min\{a(\omega) : \omega \in S^{N-1}\}$ and $b = \max\{b(\omega) : \omega \in S^{N-1}\}$. Consequently,

$$\int_A |u|^{1-N} |q(u)| du > \frac{2\pi^{N/2}\Gamma(\alpha)(\alpha-1)^{\alpha-1}}{\Gamma(N/2)(\alpha-2)^{\alpha-2}(b-a)^{\alpha-1}}. \tag{8.3.5}$$

Proof. For a fixed $\omega \in S^{N-1}$, we denote $z(r) := y(r\omega)$. By comparing (8.3.1) with (1.1.12) we see that $\left(\mathcal{D}_r^\gamma y\right)(u) = \left(D_{a(\omega)+}^\gamma z\right)(r)$ for $\gamma > 0$. Since $y(u)$ is a nontrivial solution of Eq. (8.3.2) satisfying BC (8.3.3) and $y(u)$ does not change sign on A, we have that for $\omega \in S^{N-1}$, $z(r)$ is a nontrivial solution of the equation

$$\left(D_{a(\omega)+}^\alpha z\right)(r) + q(r\omega) z = 0, \quad 2 < \alpha \leq 3 \tag{8.3.6}$$

satisfying the BC

$$z(a(\omega)) = z(b(\omega)) = 0 \quad \text{and} \quad \left(D_{a(\omega)+}^{\alpha-1} z\right)(\xi(\omega)) = 0, \tag{8.3.7}$$

and $z(r)$ does not change sign for $r \in [a(\omega), b(\omega)]$. Thus by Theorem 8.1.3,

$$\int_{a(\omega)}^{\xi(\omega)} q_-(r\omega)dr + \int_{\xi(\omega)}^{b(\omega)} q_+(r\omega)dr > \frac{(\alpha-1)^{\alpha-1}\Gamma(\alpha)}{(\alpha-2)^{\alpha-2}(b(\omega)-a(\omega))^{\alpha-1}}. \tag{8.3.8}$$

Recall that $r = |u|$ and for any $\Omega \subset \mathbb{R}^N$ and $f \in C(\Omega)$,

$$\int_\Omega f(u)du = \int_\Omega r^{N-1} f(r\omega)dr\, d\omega.$$

Hence

$$\int_{A_1} |u|^{1-N} q_-(u)du = \int_{S^{N-1}} \left(\int_{a(\omega)}^{\xi(\omega)} q_-(r\omega)dr \right) d\omega$$

and

$$\int_{A_2} |u|^{1-N} q_+(u)du = \int_{S^{N-1}} \left(\int_{\xi(\omega)}^{b(\omega)} q_+(r\omega)dr \right) d\omega.$$

Integrating both sides of (8.3.8) with respect to ω on S^{N-1} and noting that $0 < a \le a(\omega) < b(\omega) \le b$, we obtain

$$\begin{aligned}
&\int_{A_1} |u|^{1-N} q_-(u)du + \int_{A_2} |u|^{1-N} q_+(u)du \\
= \;& \int_{S^{N-1}} \left(\int_{a(\omega)}^{\xi(\omega)} q_-(r\omega)dr + \int_{\xi(\omega)}^{b(\omega)} q_+(r\omega)dr \right) d\omega \\
> \;& \frac{\Gamma(\alpha)(\alpha-1)^{\alpha-1}}{(\alpha-2)^{\alpha-2}} \int_{S^{N-1}} \frac{d\omega}{(b(\omega)-a(\omega))^{\alpha-1}} \ge \frac{\Gamma(\alpha)(\alpha-1)^{\alpha-1}}{(\alpha-2)^{\alpha-2}(b-a)^{\alpha-1}} \int_{S^{N-1}} d\omega \\
= \;& \frac{2\pi^{N/2}\Gamma(\alpha)(\alpha-1)^{\alpha-1}}{\Gamma(N/2)(\alpha-2)^{\alpha-2}(b-a)^{\alpha-1}},
\end{aligned}$$

i.e., (8.3.4) holds. (8.3.5) follows from (8.3.4) immediately. □

Remark 8.3.1. (i) A comment similar to that in Remark 8.1.1 can be made to inequalities (8.3.4) and (8.3.5). We omit the detail.

(ii) Let $\alpha = 3$ and the region A be given as in Theorem 1.1.16 with $A_1 = B(0,\xi) \setminus \overline{B(0,a)}$ and $A_2 = B(0,b) \setminus \overline{B(0,\xi)}$. Then BVP (8.3.2), (8.3.3) becomes BVP (1.1.19), (1.1.20), and inequality (8.3.4) becomes

$$\int_{A_1} |u|^{1-N} q_-(u)du + \int_{A_2} |u|^{1-N} q_+(u)du > \frac{16\pi^{N/2}}{\Gamma(N/2)(b-a)^2}. \tag{8.3.9}$$

Since $|u| = r \geq a > 0$, it follows that

$$\int_{A_1} q_-(u)du + \int_{A_2} q_+(u)du > \frac{16\pi^{N/2}a^{N-1}}{\Gamma(N/2)(b-a)^2}. \tag{8.3.10}$$

We observe that even the weakened inequality (8.3.10) is sharper than (1.1.21) in Theorem 1.1.16. In fact, not only the $|q|$ on the left-hand side of (1.1.21) is replaced by q_- or q_+, but also the constant 8 on the right-hand side is strengthened to 16.

(iii) In addition to the assumptions in Part (ii), we let $a(\omega) = 0$ for $\omega \in S^{N-1}$, i.e., the region A is simply connected. In this case, (8.3.9) becomes

$$\int_{A_1} |u|^{1-N} q_-(u)du + \int_{A_2} |u|^{1-N} q_+(u)du > \frac{16\pi^{N/2}}{\Gamma(N/2)b^2},$$

but (1.1.21) provides no useful information.

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