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Frobenius groups

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ABSTRACT

FROBENIUS GROUPS

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We discuss the nature of Frobenius groups. We consider group representations, Schur's Lemma, and group characters. We define irreducible representations and prove Maschke's Theorem. We examine induced representations and Frobenius reciprocity. Using these results and other basic notions from character theory, we show that a Frobenius group has a Frobenius kernel. We then obtain several equivalent characterizations of Frobenius groups.

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DEDICATION

I am grateful for the love, encouragement, and tolerance of my husband, the man who has made all the difference in my life. Without his patience and sacrifice, I could not have completed this thesis. A special word of thanks also goes to my family for their continuous support and encouragement.

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CHAPTER 1

INTRODUCTION AND BASIC DEFINITIONS FROM GROUP THEORY

Frobenius groups are named after the German mathematician Ferdinand Georg Frobenius. Group theory was one of Frobenius' principal interests, and he made several advances in the study of simple groups. Frobenius helped develop the theory of group representations and characters, which have become fundamental tools for studying the structure of groups.

Frobenius was interested in finding conditions which could determine whether or not a group was simple; equivalently, whether it had a nontrivial normal subgroup. In 1901 [2] he proved the following theorem.

Theorem (Frobenius). *If a group G contains a nontrivial subgroup $H \leq G$ such that $gHg^{-1} \cap H = \{1\}$ for all $g \in G \setminus H$, then there exists a normal subgroup $K \trianglelefteq G$ such that $H \cap K = \{1\}$. Further, G is the semi-direct product of subgroups K and H.*

If we have such a group, where G, H, and K are as in the theorem, then we call K the **Frobenius kernel** of G and H the **Frobenius complement** of K in G. In his proof, Frobenius used his Reciprocity Theorem. The key to this proof is a method of constructing characters of G from those of H in such a way that the resulting characters of G then restrict to irreducible characters of H. Then K is recognized as the kernel of a suitable character of G. (See [5], for example). Later study by Thompson [7] would show that the Frobenius kernel is nilpotent.

We assume a familiarity with the basics of group theory (see [1], for example), but briefly a **group** is a set G with an associative multiplication $\cdot$ which has an identity and in which every element has an inverse. We say that the group G is **abelian** if $g \cdot h = h \cdot g$ for all $g, h \in G$. We usually denote the product $g \cdot h$ in G by gh.

A **subgroup** of a group G is a subset H of G which is also a group under the operation inherited from G. We denote this relationship by $H \leq G$. We denote the trivial subgroup (consisting of the identity alone) of a group G by $\langle 1 \rangle$. The **cosets** of H in G are the subsets gH (left cosets) or Hg (right cosets). We note that the cosets form a partition of the group.

Recall that a subgroup $H \leq G$ is said to be **normal** in G if $g^{-1}Hg = H$ for all $g \in G$. Equivalently, H is a normal subgroup of G, denoted $H \trianglelefteq G$, if $gH = Hg$ for all $g \in G$, or for any $h_1 \in H$ and $g \in G$ there is $h_2 \in H$ such that $gh_1 = h_2 g$.

The collection of all cosets of a normal subgroup H of G is denoted G/H; G/H is a group under the multiplication $g_1Hg_2H = g_1g_2H$ and called the **quotient group**. Common examples include the **center** of the group G, $Z(G) = \{g \in G | gh = hg \quad \text{for all } h \in G\}$, which is easily seen to be a normal subgroup of G.

Let A be any set. The collection of all bijections (one-to-one and onto maps) from A to itself forms a group under composition called the **symmetric group** over A. In general the symmetric group is denoted $\mathrm{Sym}(A)$, and if $A = \{1, 2, \ldots, n\}$ we write S_n. A **permutation group** is a subgroup of $\mathrm{Sym}(A)$ or S_n. If H is a subgroup of a group G and K is a normal subgroup of G, then we can define the **semi-direct product** of H and K as follows. First, let $\mathrm{Auto}(K)$ denote the automorphism group of K. If we have a group homomorphism $\Theta : H \to \mathrm{Auto}(K)$, then we define a multiplication on $H \times K$ by $(h, k) \cdot (h', k') = (hh', \Theta(h')(k)k')$. Commonly H acts on K by conjugation (an inner automorphism): $\Theta(h)(k) = hkh^{-1}$.

An important part of this discussion depends on **group actions** on a set.

Definition. *A group G acts on a set A if for each $g \in G$, there is a map $a \mapsto g \cdot a \in A$ such that for all $g_1, g_2 \in G$, and for every $a \in A$,*

$$g_1 \cdot (g_2 \cdot a) = (g_1 g_2) \cdot a,$$

and for all $a \in A$

$$1 \cdot a = a.$$

If G acts on the set A and $a \in A$, then the **orbit** of a is $\{g \cdot a : a \in A\}$ and the **stabilizer** of $a \in A$ is $G_a = \{g \in G : g \cdot a = a\}$. Note that G_a is a subgroup of G. Also, given a group action on A we naturally have an action on n-tuples of elements of A: $g(a_1, \ldots, a_n) = (g \cdot a_1, \ldots, g \cdot a_n)$. A useful concept related to orbits is *transitivity*.

Definition. *A group G acting on a set A is n-transitive (or acts n-transitively) on the set A if for any $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ in A there is a $g \in G$ such that $g \cdot (a_1, \ldots, a_n) = (b_1, \ldots, b_n)$.*

It is easy to see that $G_{a,b}$, the stabilizer of the pair (a, b), is $G_a \cap G_b$.

Following Passman [5], we also define notions of 1/2 and 3/2-transitivity.

Definition. *A group G acting on a finite set A is called* ***1/2-transitive*** *on A if all the orbits of the group action on A have the same size.*

Definition. *We say that G is* ***3/2-transitive*** *if it is transitive and, for each $a \in A$, $G_a \neq \langle 1 \rangle$ is 1/2-transitive on $A \setminus \{a\}$.*

We note that the action by left multiplication of a group G on itself is transitive, 1/2-transitive, and 3/2-transitive.

Definition. *A finite group G is called a Frobenius group if there is a non-trivial subgroup H of G, called the Frobenius complement, such that for all $g \in G \setminus H$ we have $gHg^{-1} \cap H = \langle 1 \rangle$.*

In this case Frobenius kernel is the subgroup of G consisting of the identity element together with all other elements that are not conjugate to any element of H.

We note that a Frobenius group is a semi-direct product of its complement and kernel. In [5] it is shown that our previous definition of a Frobenius group is equivalent to the following characterization in terms of group actions on a set.

Proposition ([5, Page 29])**.** *A group G is a Frobenius group if and only if $G_a \neq \langle 1 \rangle$, but $G_{a,b} = \langle 1 \rangle$ for all distinct a and b. Also, a Frobenius group is 3/2-transitive on some finite set A.*

CHAPTER 2

GROUP REPRESENTATIONS

2.1 Irreducible Representations

For us here, a **linear representation** of a group G is group homomorphism

$$\rho : G \longrightarrow \mathrm{GL}(V),$$

where $\mathrm{GL}(V)$ is the general linear group of a finite-dimensional complex vector space V. Note that this is equivalent to representing G by a homomorphism into a group of invertible matrices. Note also that if we choose an ordered basis for V, then the representation (ρ, V) induces a homomorphism (using coordinates) from G to $\mathrm{GL}_n(\mathbb{C})$, where the dimension of V is n.

In our proof of Frobenius' Theorem we will construct a representation of any Frobenius group G so that the kernel of the representation (homomorphism) is precisely the subgroup K, thus establishing that K is a normal subgroup of G. We first discuss some necessary basics of representation theory of finite groups over complex vector spaces. All vector spaces occurring below are over $\mathbb{C}$ (in many places this can be weakened, but this suffices for our purposes).

A subspace W of V is called a G**-invariant** subspace if $\rho(g)(w) \in W$ for all $g \in G$ and for all $w \in W$. If such a W exists, we call $(\rho|_W, W)$ a sub-representation of V. The representation (ρ, V) is said to be **irreducible** if V has no proper G-invariant subspaces. We next examine several important results concerning irreducible representations.

Suppose that $W \subseteq V$ is a proper G-invariant subspace of V of dimension m and set $\dim(V) = n$. Choose a basis $\{v_, \ldots, v_m\}$ for W and extend this to a basis $\{v_1, \ldots, v_n\}$ for V. Then for any $g \in G$, $\rho(g)$ corresponds to a matrix M_g where the i^{th} column is the coordinates of $\rho(g)(v_i)$. Now notice that the j^{th} coordinate of a $\rho(g)(v_i)$, $i \leq m$, will be zero for all $j > m$ since W is G-invariant. We thus obtain a block matrix of the following form:

$$M_g = \begin{bmatrix} M_g' & \star \\ 0 & M_g'' \end{bmatrix}.$$

We see that M_g' gives the matrix of the sub-representation $\rho|_W$. If we let U denote the span of $v_{m+1}, \ldots, v_n$, then $V = W \oplus U$ but U may not be G-invariant, i.e., there may be non-zero entries in the starred block of the matrix M_g above. We will construct a complement U to W in V (that is, choose the vectors $v_{m+1}, \ldots, v_n$ above) which is also G-invariant. The matrix M_g will then have a simpler form, with blocks on the diagonal corresponding to our two sub-representations and zeros elsewhere. This then reduces our representation to an isomorphic representation via the sum of two representations over vector spaces of smaller dimension. We can pursue this and show that any representation is the sum of irreducible representations, and we can then apply results proven about irreducible representations to the general case. The necessary construction is found in Maschke's Theorem.

Theorem (**Maschke**, [1, Chapter 18, Theorem 1])**.** *Let G be a finite group and let (ρ, V) be a representation of G. If $W \subset V$ is G-invariant, then there exists a subspace $U \subset V$, which is also G-invariant, such that $V = W \oplus U$.*

Proof. Suppose W is a G-invariant subspace and let π be any projection from V to W. For $v \in V$, define

$$\pi_W(v) = \frac{1}{|G|} \sum_{g \in G} \rho(g)\pi\big(\rho(g)^{-1}(v)\big).$$

Then $\pi_W(v) \in W$ because π_W is a projection. If $h \in G$, we have (setting $g' = h^{-1}g$)

$$\begin{aligned}
\pi_W\big(\rho(h)(v)\big) &= \frac{1}{|G|}\sum_{g\in G}\rho(g)\pi\big(\rho(g)^{-1}\rho(h)(v)\big) \\
&= \frac{1}{|G|}\sum_{g\in G}\rho(g)\pi\big(\rho(g^{-1}h)(v)\big) \\
&= \frac{1}{|G|}\sum_{g'\in G}\rho(hg')\pi\big(\rho(g')^{-1}(v)\big) \\
&= \frac{1}{|G|}\rho(h)\sum_{g'\in G}\rho(g')\pi\big(\rho(g')^{-1}(v)\big) \\
&= \rho(h)\pi_W(v).
\end{aligned}$$

Now set $U = \mathrm{Ker}(\pi_W)$. Then $V = W \oplus U$ and if $v \in U$ and $h \in G$, then $\pi_W\big(\rho(h)v\big) = \rho(h)\big(\pi_W(v)\big) = \rho(h)(0) = 0$. Thus, U is G-invariant. $\square$

As remarked earlier, this means that in a suitable basis the matrix representation M_g will be of the form

$$\begin{bmatrix} M'_g & 0 \\ 0 & M''_g \end{bmatrix}$$

where M''_g is now the matrix subrepresentation of $\rho|_{\mathrm{Ker}(\pi_W)}$.

Consider an arbitrary representation of a finite group. If it is not irreducible, then we can decompose the representation into a direct sum of representations as above, and continue on until we have a decomposition into a direct sum of irreducible representations. Thus, we may focus on irreducible representations from here on.

2.2 The Induced Representation

Let G be a group and H a subgroup of G, and let (ρ, W) be a representation of H. In other words, we have a homomorphism $\rho : H \to \mathrm{GL}(W)$. We want a natural construction which extends ρ to a representation of the entire group G.

Note that for any action of a group G on a set X we get a homomorphism $G \to \mathrm{Sym}(X)$. If we take $X = G/H$, the set of left cosets of H in G, then we get a homomorphism $G \to \mathrm{Sym}(G/H)$. Also, if $|X| = n$ then $\mathrm{Sym}(X) \cong S_n$.

We have a natural representation of S_n, and whence of any permutation group, via permutation matrices as follows. Let $\{e_1, \ldots, e_n\}$ denote the canonical basis for $\mathbb{C}^n$ and define $\rho(\sigma)$ by setting $\rho(\sigma)(e_i) = e_{\sigma(i)}$ for all $i = 1, \ldots, n$. For example, the element $(1, 2) \in S_3$ has the matrix representation

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Now suppose H is a subgroup of G of index n. Since G acts on the left cosets via left multiplication, we may compose the resulting homomorphism taking G into S_n described above with the natural representation. We therefore obtain a representation of G acting on $\mathbb{C}^n$. This generalizes, giving us a powerful idea.

To get an induced representation we start with a subgroup H and a representation $\rho : H \to \mathrm{GL}(W)$. To get a representation of G from this information, we construct a vector space and describe how G acts on that vector space. Our construction of an extension uses module techniques. The reader is referred to Passman [3] for a general discussion of modules and tensor products in this situation. We briefly show how this extension can be realized by extending a basis for W to one containing a distinct copy of the basis for each left coset of H.

Towards that end, we observe that the representation $\rho : H \to \mathrm{GL}(W)$ gives the vector space W the structure of a $\mathbb{C}H$-module. That means we have $\mathbb{C}H$ acting on W (on the left). We also may view $\mathbb{C}G$ and a $\mathbb{C}H$-module (on the right). Thus we have the tensor product $\mathbb{C}G \otimes_{\mathbb{C}H} W$. The cosets of H form a partition of G, and we can write G as the disjoint union of the cosets of H,

$$G = g_1H \cup g_2H \cup \cdots \cup g_nH,$$

where $n = [G : H]$. In fact, as $\mathbb{C}H$-modules we have

$$\mathbb{C}G = \mathbb{C}g_1H \oplus \cdots \oplus \mathbb{C}g_nH.$$

Now since tensor products commute with direct sums, we have

$$\begin{aligned}\mathbb{C}G \otimes_{\mathbb{C}H} W &= \big(\mathbb{C}g_1H \oplus \cdots \oplus \mathbb{C}g_nH\big) \otimes_{\mathbb{C}H} W \\ &= \big(\mathbb{C}g_1H \otimes_{\mathbb{C}H} W\big) \oplus \cdots \oplus \big(\mathbb{C}g_nH \otimes_{\mathbb{C}H} W\big).\end{aligned}$$

Our **induced representation**, $\mathrm{Ind}_H^G(\rho)$, will be a homomorphism into the general linear group of the vector space

$$V = g_1W \oplus \cdots \oplus g_nW.$$

Choosing a basis $w_1, \ldots, w_k$ of W gives us a basis $g_1w_1, \ldots, g_1w_k, \ldots, g_lw_1, \ldots, g_lw_k$ for V. To define $\mathrm{Ind}_H^G(W)$ it suffices to show how elements of G act on our extended basis.

Consider the action of G on G/H by left multiplication. Let $g \in G$. For any left coset g_iH we have $gg_iH = g_jH$ for some j. This means there is a unique $h_{g,i} \in H$ such that $gg_i = g_jh_{g,i}$. We define the induced representation by setting

$$\mathrm{Ind}_H^G(\rho)(g)\big(g_iw_l\big) = g_j\rho(h_{g,i})\big(w_l\big)$$

for each $i = 1, \ldots, n$ and $l = 1, \ldots, k$. Note that $\dim(\mathrm{Ind}_H^G(\rho)) = [G : H]\dim(W)$.

2.3 Schur's Lemma

Lemma (Schur). *Let G be a group and let $\rho : G \to \mathrm{GL}(V)$ and $\rho' \to \mathrm{GL}(W)$ be finite-dimensional irreducible representations of G. Let $T : V \to W$ be a $\mathbb{C}G$-linear homomorphism.*

1. *Either T is an isomorphism or $T = 0$.*
2. *If $V = W$, then $T = \lambda \cdot \mathrm{id}_V$ for some $\lambda \in \mathbb{C}$.*

Proof. (1) Suppose that $T \neq 0$. Since $\mathrm{Ker}(T)$ is a G-invariant subspace of V, and V is irreducible, we deduce that $\mathrm{Ker}(T)$ is either $\{0\}$ or V. Since T is assumed to be nonzero, it follows that $\mathrm{Ker}(T) = \{0\}$, and so T is injective. Similarly, since $\mathrm{Im}(T)$ is a G-invariant subspace of W, and W is irreducible, we deduce that $\mathrm{Im}(T)$ is either $\{0\}$ or W. Since T is assumed to be nonzero, it follows that $\mathrm{Im}(T) = W$, and so T is surjective. Hence T is an isomorphism.

(2) Suppose that $V = W$. Since $\mathbb{C}$ is algebraically closed, T must have an eigenvalue, say λ. It follows that the $\mathbb{C}G$-linear homomorphism $T - \lambda \cdot \mathrm{id}_V$ has a nonzero kernel. By part (a), it follows that $T - \lambda \cdot \mathrm{id}_V = 0$, equivalently, $T = \lambda \cdot \mathrm{id}_V$. □

Suppose we have representations of G as in Schur's Lemma, though not necessarily irreducible. We denote by $\mathrm{Hom}_G(V,W)$ the space of all $\mathbb{C}G$-linear homomorphisms from V to W. It follows from Schur's Lemma that, if V and W are irreducible, then

$$\mathrm{Hom}_G(V,W) \cong \begin{cases} 0 & \text{if } V \ncong W \\ \mathbb{C} & \text{if } V \cong W. \end{cases}$$

More generally, if

$$V = \bigoplus_{i=1}^{d} S_i^{a_i} \qquad \text{and} \qquad W = \bigoplus_{j=1}^{d} S_j^{b_j},$$

where $S_1, \ldots, S_d$ are irreducible representations of G, then

$$\begin{aligned} \mathrm{Hom}_G(V,W) = \mathrm{Hom}_G\left(\bigoplus_{i=1}^{d} S_i^{a_i}, \bigoplus_{j=1}^{d} S_j^{b_j}\right) &\cong \bigoplus_{i=1}^{d}\bigoplus_{j=1}^{d} \mathrm{Hom}_G\left(S_i^{a_i}, S_j^{b_j}\right) \\ &\cong \bigoplus_{i=1}^{d} \mathrm{Hom}_G\left(S_i^{a_i}, S_j^{b_i}\right) \\ &\cong \bigoplus_{i=1}^{d} \mathbb{C}^{a_i \cdot b_i}, \end{aligned}$$

and so

$$\dim \mathrm{Hom}_G(V,W) = \sum_{i=1}^{d} a_i \cdot b_i.$$

2.4 Orthogonality Relations

Definition. *Let G be a group and let $\rho : G \to \mathrm{GL}(V)$ be a representation of G. The* ***character*** *of ρ is the function $\chi_\rho : G \to \mathbb{C}$ defined by $\chi_\rho(g) = \mathrm{Trace}(\rho(g))$. The character χ_ρ is a said to be* ***irreducible*** *if ρ is.*

Our goal in this section is to obtain the so-called orthogonality relations among the irreducible characters of a finite group. Let G be a finite group and let $\rho : G \to \mathrm{GL}(V)$ be a representation of G. The **fixed points** of V is the space

$$V^G = \{v \in V \mid \rho(g)(v) = v \text{ for all } g \in G\}.$$

For $g \in G$, the endomorphism $\rho(g) : V \to V$ is, in general, not a $\mathbb{C}G$-linear homomorphism. However, the map

$$\phi = \frac{1}{|G|}\sum_{g\in G}\rho(g)$$

is a $\mathbb{C}G$-linear homomorphism. For notational convenience, we will occasionally denote $(\rho(g))(v)$ by $g \cdot v$.

Proposition. *Let G, ρ, V, and ϕ be as above.*

1. *We have $\phi \circ \phi = \phi$ and $\operatorname{Im}\phi = V^G$, and so $V = \operatorname{Ker}\phi \oplus V^G$.*

2. $\dim V^G = \frac{1}{|G|}\sum_{g\in G}\chi_\rho(g)$.

Proof. (1) We have

$$\phi \circ \phi = \frac{1}{|G|}\sum_{g\in G}\left(\frac{1}{|G|}\sum_{h\in G}\rho(gh)\right) = \frac{1}{|G|}\sum_{g\in G}\left(\frac{1}{|G|}\sum_{h\in G}\rho(h)\right) = \frac{1}{|G|}\sum_{g\in G}\phi = \phi.$$

Then $V = \operatorname{Ker}\phi \oplus \operatorname{Im}\phi$. It is clear that $\operatorname{Im}\phi \subseteq V^G$. To see that $V^G \subseteq \operatorname{Im}\phi$, let $v \in V^G$. Then $g \cdot v = v$ for all $g \in G$, and so

$$v = \frac{1}{|G|}\sum_{g\in G} v = \frac{1}{|G|}\sum_{g\in G} g\cdot v = \frac{1}{|G|}\sum_{g\in G}(\rho(g))(v) = \phi(v),$$

showing that $v \in \operatorname{Im}\phi$. Hence $V^G \subseteq \operatorname{Im}\phi$. It follows that $\operatorname{Im}\phi = V^G$.

(2) Applying part (1), we get

$$\dim V^G = \dim \operatorname{Im}\phi = \operatorname{Trace}(\phi) = \frac{1}{|G|}\sum_{g\in G}\chi_\rho(g). \qquad \square$$

Now let $\rho' : G \to \mathrm{GL}(W)$ be another representation of G. Then there is a natural action of G on the space $\operatorname{Hom}(V, W)$, and

$$\operatorname{Hom}(V, W)^G = \operatorname{Hom}_G(V, W).$$

Suppose that both ρ and ρ' are irreducible. Then it follows from Schur's Lemma and the equality above that

$$\dim \operatorname{Hom}(V, W)^G = \begin{cases} 1 & \text{if } V \cong W \\ 0 & \text{if } V \not\cong W. \end{cases}$$

We apply the proposition above to the character $\overline{\chi_\rho} \cdot \chi_{\rho'}$ on $\operatorname{Hom}(V, W)^G$. We get

$$\frac{1}{|G|}\sum_{g\in G}\overline{\chi_\rho(g)} \cdot \chi_{\rho'}(g) = \begin{cases} 1 & \text{if } V \cong W \\ 0 & \text{if } V \not\cong W. \end{cases}$$

For any pair χ and χ' of characters of G (irreducible or not), we set

$$[\chi, \chi'] = \frac{1}{|G|} \sum_{g \in G} \overline{\chi(g)} \cdot \chi'(g).$$

This gives an inner product on the set of all characters of G. Our assertion above can now be rephrased as

$$[\chi, \chi'] = \begin{cases} 1 & \text{if } V \cong W, \\ 0 & \text{otherwise,} \end{cases}$$

for all irreducible characters χ, χ' of G. These relations are called the **orthogonality relations**.

2.5 Frobenius Reciprocity

Definition. *Let G be a group, let H be a subgroup, and let $\rho : G \to \mathrm{GL}(V)$ be a representation of G. The representation $\rho|_H : H \to \mathrm{GL}(V)$ of H is called the* ***restriction*** *of V to H, and denoted* $\mathrm{Res}_G^H(V)$.

Let G, H, and V be as above. If χ is the character of V, then the character of $\mathrm{Res}_G^H(V)$ is denoted χ_H. If W is a representation of H, then the character of the induced representation $\mathrm{Ind}_H^G(W)$ is denoted χ^G, and is given by the formula

$$\chi^G(g) = \frac{1}{|H|} \sum_{\substack{x \in G \\ x^{-1}gx \in H}} \chi(x^{-1}gx).$$

The following theorem gives a useful relation between induction and restriction of characters.

Theorem (Frobenius Reciprocity). *Let G be a finite group, let H be a subgroup of G, let χ be a character of G, and let ψ be a character of H. Then*

$$[\psi^G, \chi]_G = [\psi, \chi_G]_H,$$

where $[\ ,\ ]_G$ and $[\ ,\ ]_H$ denote the inner products on the set of characters of G and H, respectively.

Proof. We have

$$\begin{aligned}
[\psi^G, \chi]_G &= \frac{1}{|G|} \sum_{g \in G} \overline{\psi^G(g)} \cdot \chi(g) \\
&= \frac{1}{|G|} \frac{1}{|H|} \sum_{g \in G} \sum_{\substack{x \in G \\ x^{-1}gx \in H}} \overline{\psi(x^{-1}gx)} \cdot \chi(g) \\
&= \frac{1}{|G|} \frac{1}{|H|} \sum_{\substack{(g,x) \in G \times G \\ g \in xHx^{-1}}} \overline{\psi(x^{-1}gx)} \cdot \chi(x^{-1}gx),
\end{aligned}$$

where we used the fact that $\chi(g) = \chi(x^{-1}gx)$. Using the equality

$$\{(g, x) \in G \times G : g \in xHx^{-1}\} = \{(xhx^{-1}, x) : x \in G, h \in H\},$$

we obtain

$$[\psi^G, \chi]_G = \frac{1}{|G|} \frac{1}{|H|} \sum_{x \in G} \sum_{h \in H} \overline{\psi(h)} \cdot \chi(h) = \frac{1}{|H|} \sum_{h \in H} \overline{\psi(h)} \cdot \chi(h) = [\psi, \chi_G]_H. \qquad \square$$

CHAPTER 3

FROBENIUS' THEOREM

We have defined a Frobenius group as a group G with a non-trivial subgroup H such that $gHg^{-1} \cap H = \langle 1 \rangle$ for $g \in G \setminus H$. The subgroup H is called the Frobenius complement of G. In this section we show that a Frobenius group has a Frobenius kernel, that is, a non-trivial normal subgroup K, such that $K \cap H = \langle 1 \rangle$ and $KH = G$. We also characterize the Frobenius kernel of G. In particular, we show that K is a non-trivial normal subgroup of G, and we show that for the given H

$$K = \langle 1 \rangle \cup (G \setminus \bigcup_{g \in G} gHg^{-1}).$$

The proof relies on techniques from representation theory as developed in Chapter 2. As in Chapter 2, all vector spaces occurring below are finite dimensional and over the complex numbers.

The characters of a group G can be used to form a ring. The sum and product of two characters are also characters, and we call a difference of two characters a **generalized character**. The set of all generalized characters of a group G forms a ring $I(G)$. The subset of characters which vanish at the identity is denoted $I_0(G) = \{\alpha \in I(G) : \alpha(1) = 0\}$. Note that characters are class functions on G (i.e., constant on conjugacy classes), so the elements of $I_0(G)$ and $I(G)$ are also class functions.

We will need the following.

Lemma ([5, Theorem 16.8])**.** *If G is any finite group, then it has finitely many irreducible characters, and*

$$\bigcap_{\chi \text{ irreducible}} \mathrm{Ker}(\chi) = \langle 1 \rangle.$$

Theorem (**Frobenius**)**.** *If G is a Frobenius group, then G has a Frobenius kernel.*

Proof. Let G be a Frobenius group. In other words, G has a nontrivial subgroup H such that $gHg^{-1} \cap H = \langle 1 \rangle$ when $g \notin H$. This subgroup H is the Frobenius complement of G, and G can be written as the following disjoint union

$$G = K \cup \bigcup_{g \in G} (gHg^{-1} \setminus \langle 1 \rangle)$$

where K is as above. The set K is the Frobenius kernel, and we will show that it is a non-trivial normal subgroup of G.

We first find the size of K. By the disjoint union above

$$|G| = |K| + [G : H](|H| - 1) = |K| + \frac{|G|}{|H|}(|H| - 1) = |K| + |G| - \frac{|G|}{|H|},$$

so that

$$|K| = [G : H] = \frac{|G|}{|H|}.$$

Now let $\phi_1, \dots, \phi_n$ denote the irreducible characters of H. We then extended these to elements $\chi_i \in I(G)$ in a particular fashion (explained below) so that $\chi_i|_H = \phi_i$. We set

$$N = \bigcap_{1 \le i \le n} \mathrm{Ker}(\chi_i) \trianglelefteq G.$$

Then by the Lemma above

$$N \cap H = \bigcap_{1 \le i \le n} \operatorname{Ker}(\psi_i) = \langle 1 \rangle.$$

This shows that

$$|N| = |NH|/|H| \le |G|/|H|.$$

Our method thus proceeds by first showing how to extend the characters ϕ_i in the desired manner. We then show that $K \subseteq N$ which, in conjunction with the inequality above on $|N|$ and our earlier computation of $|K|$, will complete the proof of Frobenius' Theorem.

Suppose $\psi \in I_0(H)$. Since $x^{-1}Hx \cap H = \langle 1 \rangle$ for all $x \in G \setminus H$ by the hypothesis on H, we see that for all $h \in H$

$$\begin{aligned}
\psi^G(h) &= \frac{1}{|H|} \sum_{\substack{x \in G \\ x^{-1}hx \in H}} \psi(x^{-1}hx) \\
&= \frac{1}{|H|} \sum_{\substack{x \in G \setminus H \\ x^{-1}hx = 1}} \psi(x^{-1}hx) + \frac{1}{|H|} \sum_{x \in H} \psi(x^{-1}hx) \\
&= \frac{1}{|H|} \sum_{x \in H} \psi(h) \\
&= \psi(h),
\end{aligned}$$

since $\psi(1) = 0$ and $\psi(x^{-1}hx) = \psi(h)$ for all $x \in H$.

Now let $\psi_1, \ldots, \psi_n$ be the irreducible characters of H above. For each $i = 1, \ldots, n$ set $f_i = \psi_i(1)$ and let $1_H \in I(H)$ be the trivial character. Set $\alpha_i = f_i 1_H - \psi_i \in I(H)$. We then have $\alpha_i \in I_0(H)$ by construction, so that $\alpha_i^G|_H = \alpha_i$ as shown above. Finally, for each $i = 1, \ldots, n$ we set

$$\chi_i = f_i 1_G - \alpha_i^G \in I(G).$$

We immediately see that

$$\chi_i|_H = f_i 1_G - \alpha_i^G|_H = f_i 1_H - \alpha_i = \psi_i$$

for all $i = 1, \ldots, n$. It remains to show that $K \subseteq N$. Towards that end, let $k \in K,\ k \neq 1$. Then by construction $k \notin gHg^{-1}$ for all $g \in G$, or what is the same, $x^{-1}kx \notin H$ for all $x \in G$. But this then implies that for all $i = 1, \ldots, n$

$$\chi_i(k) = (f_i 1_G - \alpha_i^G)(k) = (f_i 1_H - \alpha_i)^G(k) = \phi_i^G(k) = \frac{1}{|H|} \sum_{xkx^{-1} \in H} \phi_i(xkx^{-1}) = 0$$

since this is an empty sum. Hence, if $k \in K$ then $k \in \cap_i \operatorname{Ker}(\chi_i) = N$. This shows that $K \subseteq N$ and completes the proof. $\square$

CHAPTER 4

CHARACTERIZING FROBENIUS GROUPS

In this chapter we explore the relationship between Frobenius groups and Hall subgroups.

Definition. *For a finite group G, a subgroup $H \leq G$ is called a **Hall subgroup** if $[G : H]$ and $|H|$ are relatively prime.*

Example. *Every Sylow p-subgroup is a Hall subgroup.*

The significance of Hall subgroups in the study of groups, in particular the study of groups as extensions of simple groups, is indicated by the following result.

Theorem (**Schur-Zassenhaus**, [3, Chapter 6, Theorem 2.1])**.** *If G is a finite group and K is a normal Hall subgroup of G, then there exists a subgroup $H \leq G$, with $H \cap K = \langle 1 \rangle$ and $KH = G$.*

The result below characterizes Frobenius groups.

Proposition. *Let G be a finite group. The following are equivalent.*

1. *G is Frobenius group with complement H of order m.*
2. *G contains a nontrivial proper normal subgroup K such that $C_G\big(k\big) \leq K$ for all $k \in K \setminus \langle 1 \rangle$. Moreover, $|G| = mn$, where $n = |K|$.*
3. *$|G| = mn$ for relatively prime m and n, and for all $g \in G$ either $g^m = 1$ or $g^n = 1$. Moreover, $K = \{g \mid g^n = 1\}$ is a nontrivial proper normal subgroup of G.*

Proof. ($1 \Longrightarrow 2$) Let G be a Frobenius group with complement H of order m. By Forbenius' Theorem, G contains a nontrivial proper normal subgroup K, such that $K \cap H = \langle 1 \rangle$ and $G = KH$. Then $|G| = mn$, where $n = |K|$, proving the second assertion in (2). To see that $C_G(k) \leq K$ for all $k \in K \setminus \langle 1 \rangle$, suppose that $C_G(k) \nleq K$ for some $k \in K \setminus \langle 1 \rangle$. Since G is the union of K and all conjugates of H in G, we may assume that there exists an element $h \in H \setminus \langle 1 \rangle$ with $h \in C_G(k)$. Then $h \in H \cap H^k$, so that $H \cap H^k \neq \langle 1 \rangle$, a contradiction. Hence $C_G(k) \leq K$ for all $k \in K \setminus \langle 1 \rangle$.

($2 \Longrightarrow 3$) Suppose that G contains a nontrivial proper normal subgroup K such that $C_G\big(k\big) \leq K$ for all $k \in K - \langle 1 \rangle$, and that $|G| = mn$ with $|K| = n$. We will first show that K is a Hall subgroup. To this end, let p be a prime dividing $|K|$, let P be a Sylow p-subgroup of K, and let P^* be a Sylow p-subgroup of G containing P. Since $Z(P^*) \subseteq C_G(P - \langle 1 \rangle)$, we deduce that $Z(P^*) \subseteq K$. The containment $P^* \subseteq C_G(Z(P^*) - \langle 1 \rangle)$ forces $P^* \subseteq K$. Hence $P = P^*$, and it follows that K is a Hall subgroup.

Since $|K| = n$ and $|G| = mn$, we have m and n are relatively prime. Normality of K in G forces the equality $K = \{g \in G \mid g^n = 1\}$. Finally, let $g \in G$, and suppose that $g^m \neq 1$. Then $(g^m)^n = 1$, and so $g^m \in K \setminus \langle 1 \rangle$. Our supposition implies that $g \in C_G(g^m) \subseteq K$, and hence $g^n = 1$.

($3 \Longrightarrow 1$) Suppose that $|G| = mn$ with m and n relatively prime, that for all $g \in G$ either $g^m = 1$ or $g^n = 1$, and that $K = \{g \mid g^n = 1\}$ is a nontrivial proper normal subgroup of G. We first show that K is a Hall subgroup. To this end, let p be any prime dividing $|K|$. Then K contains an element u of order p, and since $u^n = 1$ by our supposition, it follows that p divides n. Since m and n are relatively prime, we deduce that m and $|K|$ are relatively prime, and so K is a Hall subgroup. Since by supposition K is normal in G, we apply the

Schur-Zassenhaus Theorem to secure a subgroup H of G of order m, with $H \cap K = \langle 1 \rangle$ and $G = KH$.

Finally, we show that $H \cap gHg^{-1} = \langle 1 \rangle$ for all $g \in G - H$, from which it will follow that G is a Frobenius group with complement H. Suppose that $H \cap gHg^{-1} \neq \langle 1 \rangle$ for some $g \in G$. Since $G = KH$, we may assume that $g \in K$. Now choose a nontrivial element $h \in H \cap gHg^{-1}$. Then $h \in H$ and $ghg^{-1} \in H$, and so $ghg^{-1}h^{-1} \in H$. By normality of K in G, we get $hgh^{-1} \in N$, and so $ghg^{-1}h^{-1} \in K \cap H$. The equality $K \cap H = \langle 1 \rangle$ forces $gh = hg$. The elements g and h must have relatively prime orders, and so $|gh| = |g||h|$. Since by supposition each element of G has order dividing either m or n, nontriviality of h forces $|g| = 1$, equivalently, $g = 1$. It follows that $H \cap gHg^{-1} = \langle 1 \rangle$ for all $g \in G \setminus H$, proving that G is a Frobenius group. $\square$

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