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Characterization of table algebras by their multiplicities

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ABSTRACT

CHARACTERIZATION OF TABLE ALGEBRAS BY THEIR MULTIPLICITIES

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The structure of a standard integral table algebra can sometimes be determined by the multiplicities of the irreducible characters for that table algebra. In particular, it has been shown that a standard table algebra whose multiplicities are all equal (except for the trivial one) must be commutative. This dissertation seeks to extend the technique of using multiplicities to determine the structure of standard table algebras. For a commutative standard table algebra, we show that there exists exactly one character with nontrivial multiplicity if and only if the table basis is the wreath product of a two-dimensional subalgebra and an abelian group. Additionally, we show that a noncommutative standard integral table algebra with exactly one character (of degree two) that has nontrivial multiplicity must have one of two structures, both corresponding to a partial wreath product $(\mathbf{B}, \mathbf{D}, \mathbf{C})$, where $|\mathbf{D}| \in \{6, 8\}$, where $|\mathbf{C}| \in \{2, 3\}$, and where the structure constants are explicitly determined by certain parameters that are bounded by a function of the nontrivial multiplicity.

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CHAPTER 1

INTRODUCTION

This thesis studies a general class of algebras, called *table algebras*, which have a distinguished basis that satisfies certain axioms. These table algebras generalize group algebras of finite groups, the centers of group algebras, and the adjacency algebras of association schemes. Table algebras are reality-based algebras whose structure constants are nonnegative real numbers.

The finite set of irreducible characters of a table algebra generates a vector space of functions, called feasible trace maps. One feasible trace map, in particular, is called the standard feasible trace map. The coefficient with which a particular irreducible character appears in the standard feasible trace map is called the multiplicity of that irreducible character. Within the set of irreducible characters, there is a unique linear irreducible character which sends each basis element to a positive real value. This character is called the degree map, and it is known to have multiplicity equal to 1 [8, Corollary 6]. The study of the multiplicities of a table algebra can provide information about the structure of the table algebra itself. In 2009, Blau proved that a reality-based algebra with integer structure constants whose multiplicities are all equal (except possibly the multiplicity associated with the degree map) must be commutative [8, Theorem 1].

This thesis seeks to extend the concept of using multiplicities to determine the structure of standard table algebras. Standard table algebras are table algebras where certain structure constants are equal to values of the degree map. This additional property does not restrict the study of table algebras, as any table algebra can be rescaled so that it is standard. Additionally, many of the structures which are generalized by table algebras, such as association schemes and group algebras over finite groups, are standard using the traditionally chosen basis elements. One result of this dissertation is the classification of commutative standard table algebras with at most one nontrivial multiplicity. A similar theorem for finite groups had previously been published by Seitz [18], and provided a characterization of finite groups with exactly one irreducible representation of degree greater than one. The main result shows that there exists exactly one nontrivial multiplicity if and only if the table basis is the wreath product of a two-dimensional subalgebra and an abelian group. The theorem applies to adjacency algebras of commutative association schemes with exactly one primitive idempotent matrix of rank greater than one. The theorem of Seitz for finite groups is also a corollary.

Beyond the commutative case, the next natural question is: can a characterization of noncommutative standard table algebras with at most one nontrivial multiplicity be determined? This thesis describes the answer to this question when certain restrictions are met. In particular, we restrict the study to standard integral table algebras (SITA), standard table algebras whose structure constants are integers. Again, many of the interesting examples of table algebras satisfy this constraint as well, including association schemes and finite group algebras. We also restrict the study to the case where the character with nontrivial multiplicity has degree 2. In this case, the algebra can have one of two possible structures. Each of the possible structures is either determined by the multiplicity or it is determined by a set of parameters whose values are bounded by a function of the multiplicity.

CHAPTER 2

PRELIMINARIES

2.1 Definitions and Examples

Definition 2.1.1. [10, Definition 1.16] Let $\mathbf{B} = \{b_0, b_1, \ldots, b_d\}$ be a basis of a finite dimensional associative algebra A over $\mathbb{C}$, with identity element $1_A = b_0$. Then $(A, \mathbf{B})$ is called a *reality-based algebra* (and $\mathbf{B}$ is a *distinguished basis*) if:

(i) for all i, j, m, $b_i b_j = \sum_{m=0}^{d} \beta_{ijm} b_m$ with $\beta_{ijm} \in \mathbb{R}$;

(ii) there is an algebra anti-automorphism (denoted by *) of A whose order divides 2, such that $b_i \in \mathbf{B}$ implies $b_i{}^* \in \mathbf{B}$ (then i^* is defined by $b_{i^*} = b_i{}^*$);

(iii) for all i, j, $\beta_{ij0} \neq 0$ if and only if $j = i^*$; and $\beta_{i^*i0} = \beta_{ii^*0} > 0$.

Furthermore, if $\beta_{ijm} \geq 0$ for all i, j, m, then $(A, \mathbf{B})$ is called a *table algebra* (or, a *positive reality-based algebra*). The β_{ijm} are called the *structure constants* of $\mathbf{B}$.

Each reality-based algebra is semi-simple [10, Proposition 2.3]. Let $\{X_s : A \to M_{n_s}(\mathbb{C})\}$ denote a maximal (finite) set of inequivalent irreducible representations of A. Let $\mathrm{Irr}(A)$ denote the set of irreducible characters $\chi_s := tr(X_s)$. Then $\chi_s(b_{i^*}) = \overline{\chi_s(b_i)}$, where $^-$ denotes complex conjugation. The *degree* of the character χ_s is $\chi_s(b_0) = n_s$. Any character of degree 1 is called *linear*. Note that any linear character is also a homomorphism from A to $\mathbb{C}$. Any reality-based algebra has at most one linear character, called a *degree map* χ_0, that has a positive real value on each basis element. These values are called the *degrees* of $\mathbf{B}$. If $(A, \mathbf{B})$ is a table algebra, it always has a degree map. If $\chi_0(b_i) = \beta_{ii^*0}$ for all $b_i \in \mathbf{B}$, then $(A, \mathbf{B})$ is called a *standard table algebra* (STA). Let $\mathbb{Z}_{\geq 0}$ denote the nonnegative integers. Then $(A, \mathbf{B})$ is called a *standard integral table algebra* (SITA) if $(A, \mathbf{B})$ is a STA with structure constants in $\mathbb{Z}_{\geq 0}$. Note that the degree of a character is different from the degree of an element. For any nonempty subset $\mathbf{C} \subseteq \mathbf{B}$, $\mathbf{C}^+$ is defined by $\mathbf{C}^+ := \sum_{b_i \in \mathbf{C}} b_i$, and the *complex linear span of* $\mathbf{C}$, denoted $\mathbb{C}\mathbf{C}$ or $\langle \mathbf{C} \rangle$, consists of all sums of complex multiples of elements in $\mathbf{C}$. In addition, when $(A, \mathbf{B})$ is a STA, the *order* of $\mathbf{C}$ is the positive real number $o(\mathbf{C}) := \sum_{b_i \in \mathbf{C}} \chi_0(b_i)$.

Definition 2.1.2. [7, Definition 2.2] Let $(A, \mathbf{B})$ and $(U, \mathbf{V})$ be table algebras. A *table algebra homomorphism* $\psi : (A, \mathbf{B}) \to (U, \mathbf{V})$ is an algebra homomorphism $\psi : A \to U$ which sends 1_A to 1_U such that for every $b \in \mathbf{B}$, $\psi(b) = \alpha_b v$ for some $v \in \mathbf{V}$ and $\alpha_b \in \mathbb{R}_{>0}$. ψ is called an epimorphism, monomorphism, or isomorphism if it is such as an algebra map. $(A, \mathbf{B})$ and $(U, \mathbf{V})$ are called *exactly isomorphic* (denoted $\mathbf{B} \cong_x \mathbf{V}$) if there exists a bijection (an *exact isomorphism*) between $\mathbf{B}$ and $\mathbf{V}$ that preserves structure constants.

Recall that a *relation* on a set X is a subset of $X \times X$. If R is a relation on X, then ${}^tR = \{(y, x) \mid (x, y) \in R\}$ and is called the *transpose of* R.

Definition 2.1.3. Let X be a finite set. Let S be a collection of non-empty relations which partition $X \times X$. Then $\mathcal{S} := (X, S)$ is called an *association scheme* if it satisfies the following:

(i) the diagonal relation, $R_0 := \{(x, x) \mid x \in X\}$, is contained in S,

(ii) for any $R_i \in S$, ${}^tR_i \in S$; we denote ${}^tR_i$ by R_{i^*},

(iii) for any $R_i, R_j, R_k \in S$ and any pair $(x, y) \in R_k$, the nonnegative integer given by $|\{z \in X \mid (x, z) \in R_i \text{ and } (z, y) \in R_j\}|$ is independent of the choice of x and

y. This number is denoted ρ_{ijk}, and these numbers are called the *intersection numbers* of $\mathcal{S}$. The intersection number given by ρ_{ii^*0} is positive and is called the *valency* of R_i. It is denoted k_i.

Example 2.1.1. Let G be a finite group and A a subgroup of $Aut(G)$. Let $T_0 = \{1\}, T_1, T_2, ..., T_d$ be the distinct orbits of A on G. For $j \in \{0, 1, ..., d\}$, define $R_j \subseteq G \times G$ by $(x, y) \in R_j$ iff $xy^{-1} \in T_j$. Let $S = \{R_j \mid 0 \leq j \leq d\}$. Then $\mathcal{G} := (G, S)$ is an association scheme.

If $A = \{id\}$, the trivial subgroup of $Aut(G)$ consisting only of the identity map, then the orbits of A on G each consist of a single group element, so we may index the orbits (and relations) by the group elements themselves. Thus $(x, y) \in R_g$ iff $xy^{-1} = g$. Let $\widetilde{G} = \{R_g \mid g \in G\}$. The scheme $(G, \widetilde{G})$ is called a *group scheme*.

Definition 2.1.4. See [4, Definition 2.3]. Let (X, H) be an association scheme. Then the *algebraic automorphism group of* (X, H) is defined as

$$Aut(H) := \{\sigma \in Sym(H) \mid \rho_{abc} = \rho_{a^\sigma b^\sigma c^\sigma} \text{ for all } R_a, R_b, R_c \in H\}.$$

Definition 2.1.5. Let $\mathcal{S} = (X, \{R_j\}_{0 \leq j \leq d})$ be an association scheme. Define the *j'th adjacency matrix* A_j of $\mathcal{S}$ to be the matrix of degree $|X|$, whose rows and columns are indexed (in a fixed order) by elements of X and whose xy entry is 0 if $(x, y) \notin R_j$ and 1 if $(x, y) \in R_j$.

Let $\mathcal{A}(\mathcal{S})$ be the set of adjacency matrices A_j. Then the vector space span $\mathbb{C}\mathcal{A}(\mathcal{S})$ is an algebra under matrix multiplication, called the *adjacency algebra*, or the *Bose-Mesner algebra*, of $\mathcal{S}$.

Example 2.1.2. The adjacency algebra of an association scheme forms a SITA, $(\mathbb{C}\mathcal{A}(\mathcal{S}), \mathcal{A}(\mathcal{S}))$, where the anti-automorphism is matrix transpose. The structure constants for the adjacency algebra are precisely the intersection numbers of the scheme, and the degree of an adjacency matrix is its valency, or row sum, which is the valency of the corresponding relation of the scheme. Therefore, $o(\mathcal{A}(\mathcal{S})) = |X|$, where X is the underlying set of the scheme.

Definition 2.1.6. Let $(A, \mathbf{B})$ be a table algebra and $b_i, b_j \in \mathbf{B}$. Define the *support* of b_ib_j by $\mathrm{Supp}(b_ib_j) := \{b_k \in \mathbf{B} \mid \beta_{ijk} \neq 0\}$. Let $\mathbf{S}, \mathbf{T}$ be subsets of $\mathbf{B}$. Define $\mathbf{S}^* := \{s^* \mid s \in \mathbf{S}\}$. The *set product* $\mathbf{ST}$ is defined as the union, over all $s \in \mathbf{S}$ and $t \in \mathbf{T}$, of $\mathrm{Supp}(st)$. If $\mathbf{S} = \{b\}$, a singleton set, then $\mathbf{ST}$ is denoted as $b\mathbf{T}$. Set product is associative.

Definition 2.1.7. Let $(A, \mathbf{B})$ be a table algebra. A nonempty subset $\mathbf{C}$ of $\mathbf{B}$ is called a *closed subset* if $\mathbf{C}^*\mathbf{C} \subseteq \mathbf{C}$. A closed subset $\mathbf{C}$ of $\mathbf{B}$ is called *normal* if $b_i\mathbf{C} = \mathbf{C}b_i$ for all $0 \leq i \leq d$ and is called *strongly normal* if $b_i\mathbf{C}b_{i^*} \subseteq \mathbf{C}$ for all $0 \leq i \leq d$.

Definition 2.1.8. Let $(A, \mathbf{B})$ be a table algebra, and suppose $\mathbf{S}$ is a nonempty subset of $\mathbf{B}$. Define the *closure of* $\mathbf{S}$ to be the smallest closed subset containing $\mathbf{S}$.

Note that the closure of a set $\mathbf{S}$ is a well-defined object, as it is simply the intersection of all closed subsets of $\mathbf{B}$ which contain $\mathbf{S}$.

Remark 2.1.1. Suppose $\mathbf{C}$ is a nonempty subset of a STA basis $\mathbf{B}$. $\mathbf{C}$ is closed iff $\mathbf{CC} \subseteq \mathbf{C}$ [11, Corollary 2.6]. If $\mathbf{C}$ is closed, then $\mathbf{C}^* = \mathbf{C}$ ([8, Proposition 2.19], [1, Proposition 2.7]), and hence $(<\mathbf{C}>, \mathbf{C})$ is a table algebra. Also, the cosets and double cosets of $\mathbf{C}$ partition $\mathbf{B}$; that is, for all $b_i, b_j \in \mathbf{B}$, $b_i\mathbf{C} = b_j\mathbf{C}$ iff $b_j \in b_i\mathbf{C}$, and $\mathbf{C}b_i\mathbf{C} = \mathbf{C}b_j\mathbf{C}$ iff $b_j \in \mathbf{C}b_i\mathbf{C}$ ([8, Proposition 4.2]; [11, Proposition 2.8]). The degree map for $\mathbf{C}$ is $\chi_0\!\downarrow_{\mathbf{C}}$, where χ_0 is the degree map for $\mathbf{B}$; hence a closed subset of a STA is standard. Finally, if $\mathbf{C}$ is a strongly normal subset of a table algebra, then $\mathbf{C}$ is normal [10, Corollary 3.10].

Definition 2.1.9. Let $(A, \mathbf{B})$ be a STA and $\mathbf{C}$ a closed subset of $\mathbf{B}$. For all $b_i \in \mathbf{B}$, define $b_i//\mathbf{C} := o(\mathbf{C})^{-1}(\mathbf{C}b_i\mathbf{C})^+$, $\mathbf{B}//\mathbf{C} := \{b_i//\mathbf{C} \mid b_i \in \mathbf{B}\}$, and $A//\mathbf{C} :=< \mathbf{B}//\mathbf{C} >$ (the linear span). (Note that if $\mathbf{C}$ is a normal closed subset of $\mathbf{B}$, then $b_i//\mathbf{C} := o(\mathbf{C})^{-1}(b_i\mathbf{C})^+$.)

Remark 2.1.2. $(A//\mathbf{C}, \mathbf{B}//\mathbf{C})$ is a STA, called the *quotient algebra*. The degree map of $A//\mathbf{C}$ is the restriction of χ_0, where χ_0 is the degree map for A; $(b_i//\mathbf{C})^* = b_{i^*}//\mathbf{C}$; and $o(\mathbf{B}//\mathbf{C}) = \frac{o(\mathbf{B})}{o(\mathbf{C})}$ [3, Theorem 4.9].

Remark 2.1.3. Let $(A, \mathbf{B})$ be a STA and $\mathbf{C}$ a closed subset of $\mathbf{B}$. Then $\mathbf{C}$ is strongly normal iff $b_i//\mathbf{C} \cdot b_{i^*}//\mathbf{C} = b_0//\mathbf{C}$ for all $b_i \in \mathbf{B}$ iff $\mathbf{B}//\mathbf{C}$ is a group.

Example 2.1.3. Let G be a finite group. Then the center of the group algebra, $Z(\mathbb{C}G)$, with basis $\mathrm{Cla}(G)$, the sums over the conjugacy classes, forms a commutative SITA. Let $\{{K_i}^+\}_{i=0}^k$ denote the basis. Then $(Z(\mathbb{C}G), \mathrm{Cla}(G))$ has degree map $\chi_0({K_i}^+) = |K_i|$. So, $o(\{{K_i}^+\}_{i=0}^k) = \sum_{i=0}^k |K_i| = |G|$. The irreducible characters of $Z(\mathbb{C}G)$ are the central characters ω_{χ_j}, in bijection with the irreducible characters of G, where $\omega_{\chi_j}({K_i}^+) = \frac{|K_i|\chi_j(g_i)}{\chi_j(1)}$, $g_i \in K_i$ [17, pg. 78]. Any closed subset $\mathbf{C}$ is $\{{K_i}^+ \mid K_i \subseteq N\}$ for some fixed $N \lhd G$, and $\mathbf{B}//\mathbf{C} \cong_{\mathrm{x}} \mathrm{Cla}(G/N)$.

Definition 2.1.10. Let $(A, \mathbf{B})$ be a table algebra. An element $b_i \in \mathbf{B}$ is called *linear* if $\mathrm{Supp}(b_i b_{i^*}) = \{b_0\}$. The set of all linear elements in $\mathbf{B}$ is denoted $\mathrm{L}(\mathbf{B})$.

Remark 2.1.4. [10, Proposition 3.18; Proposition 3.19] Suppose $\mathbf{B}$ is standard. Then $\chi_0(b) \geq 1$ for all $b \in \mathbf{B}$, with equality iff b is linear. Also, $\mathrm{L}(\mathbf{B})$ is a group. Additionally, if $a \in \mathrm{L}(\mathbf{B})$ and $b \in \mathbf{B}$, then $ab \in \mathbf{B}$ and $ba \in \mathbf{B}$.

Definition 2.1.11. Let $(A, \mathbf{B})$ be a table algebra. A *feasible trace* is a linear transformation $\phi : A \to \mathbb{C}$ with $\phi(xy) = \phi(yx)$ for all $x, y \in A$. The set of feasible traces is a vector space over $\mathbb{C}$ with basis $\mathrm{Irr}(A)$ [16, Proposition 5.1].

Definition 2.1.12. Let $(A, \mathbf{B})$ be a table algebra. Define $\tau : A \to \mathbb{C}$ to be the linear transformation given by $\tau(b_0) = o(\mathbf{B})$ and $\tau(b_i) = 0$ for $1 \leq i \leq d$. τ is a feasible trace called the *standard feasible trace map*, and so $\tau = \sum_{j=0}^{k} m_j \chi_j$ where $\mathrm{Irr}(A) = \{\chi_0, \chi_1, ..., \chi_k\}$ [3, pg. 37]. The numbers $m_0, m_1, ..., m_k$ are called the *multiplicities* of the algebra.

For any closed subset $\mathbf{C}$ of $\mathbf{B}$, let ${}_{\mathbf{C}}\tau$ denote the τ function of $\mathbf{C}$. It is clear that ${}_{\mathbf{C}}\tau = \frac{o(\mathbf{C})}{o(\mathbf{B})} \tau{\downarrow_{\mathbf{C}}}$.

Remark 2.1.5. [6, Proposition 1; Corollary 6] $m_j \in \mathbb{R}_{>0}$ for $1 \leq j \leq k$ and $m_0 = 1$.

Remark 2.1.6. For $(\mathbb{C}G, G)$, a group algebra over a finite group, τ is the regular character. Thus $m_j = \chi_j(b_0)$ for all j. Also $\tau{\downarrow_{Z(\mathbb{C}G)}}$ is clearly the τ function for $Z(\mathbb{C}G)$. Since $\tau{\downarrow_{Z(\mathbb{C}G)}} = \sum_{j=0}^{k} (\chi_j(b_0))^2 \omega_{\chi_j}$, the multiplicities in $Z(\mathbb{C}G)$ are the squares of the degrees of the irreducible characters of $\mathbb{C}G$.

Remark 2.1.7. [5, Proposition II.3.4, Theorem II.5.9] For commutative association schemes, the multiplicities are the ranks of the primitive idempotent matrices in the adjacency algebra.

Definition 2.1.13. Let $(A, \mathbf{B})$ be a table algebra. Let χ_0 be the degree map and χ_s be any irreducible character with degree n_s. Define

$$\begin{aligned} \mathrm{Z}(\chi_s) &:= \{b_i \in \mathbf{B} : |\chi_s(b_i)| = \chi_0(b_i) n_s\}, \\ \mathrm{Ker}(\chi_s) &:= \{b_i \in \mathbf{B} : \chi_s(b_i) = \chi_0(b_i) n_s\}. \end{aligned}$$

Note that while $\mathrm{Ker}(\chi)$ is a closed subset of the basis $\mathbf{B}$, it is not necessarily normal in $\mathbf{B}$ ([13, Theorem 3.2]; [13, Example 3.3]). However, it will be shown that $\mathrm{Z}(\chi)$ is closed and $\mathrm{Ker}(\chi)$ is strongly normal in $\mathrm{Z}(\chi)$.

Definition 2.1.14. See [21, Definition 2.2; Corollary 2.4]. Let $(A, \mathbf{B})$ be a STA and $\mathbf{C} \subseteq \mathbf{B}$ be closed. Then we say that $(A, \mathbf{B})$ is the *wreath product* of $A//\mathbf{C}$ and $\mathbb{C}\mathbf{C}$ (denoted $(A//\mathbf{C}, \mathbf{B}//\mathbf{C}) \wr (\mathbb{C}\mathbf{C}, \mathbf{C})$) if $cb = bc = \chi_0(c)b$ for all $b \notin \mathbf{C}$ and all $c \in \mathbf{C}$.

Remark 2.1.8. The wreath product is determined up to exact isomorphism by $(A//\mathbf{C}, \mathbf{B}//\mathbf{C})$ and $(\mathbb{C}\mathbf{C}, \mathbf{C})$.

Suppose $(A, \mathbf{B})$ is a reality-based algebra and G is a group that acts as automorphisms of A, permuting $\mathbf{B}$. Then for any $g \in G$, $b_0^g = b_0 \in \mathrm{Supp}(b^* b)^g = \mathrm{Supp}(b^{*g} b^g)$. This implies $b^{*g} = b^{g*}$.

Definition 2.1.15. Let $\mathbf{B}$ be the basis of a table algebra, A, and let G be a group acting as automorphisms on A, permuting $\mathbf{B}$. Define $\mathbf{B} \rtimes G = \{(b, g) \mid b \in \mathbf{B} \text{ and } g \in G\}$ as a formal basis for a vector space over $\mathbb{C}$, such that for all $b, c \in \mathbf{B}$ and $g, h \in G$,

$$(b, g) \cdot (c, h) = \left(b^h c, gh\right),$$

where $\left(b^h c, gh\right) = \sum_{d \in \mathbf{B}} \beta_d (d, gh)$ if $b^h c = \sum_{d \in \mathbf{B}} \beta_d d$. Also define $(b, g)^* = (b^{*g^{-1}}, g^{-1})$. Then $\mathbf{B} \rtimes G$ is called the *semidirect product* of the table basis $\mathbf{B}$ and the group G.

Proposition 2.1.1. The complex linear span, $\mathbb{C}(\mathbf{B} \rtimes G)$, with multiplication in $\mathbf{B} \rtimes G$ extended bilinearly is an associative algebra, * is an anti-automorphism, and $(\mathbb{C}(\mathbf{B} \rtimes G), \mathbf{B} \rtimes G)$ is a table algebra.

Proof. By definition of $\mathbb{C}(\mathbf{B} \rtimes G)$, we only need to show associativity and conditions (ii) and (iii) of Definition 2.1.1.

Pick any $b, c, d \in \mathbf{B}$ and $f, g, h \in G$. Then

$$\begin{aligned}((b,f)(c,g))\,(d,h) &= (b^g c, fg)\,(d,h) \\ &= \left((b^g c)^h d, (fg)h\right) \\ &= \left(b^{gh} c^h d, f(gh)\right) \\ &= (b,f)\left(c^h d, gh\right) \\ &= (b,f)\,((c,g)(d,h))\,.\end{aligned}$$

Thus $\mathbb{C}(\mathbf{B} \rtimes G)$ is associative. Also, for any $b, c \in \mathbf{B}$ and $g, h \in G$,

$$\begin{aligned}((b,g)\cdot(c,h))^* &= (b^h c, gh)^* \\ &= \left(\left(b^h c\right)^{*(gh)^{-1}}, (gh)^{-1}\right) \\ &= \left(\left(c^*(b^h)^*\right)^{h^{-1}g^{-1}}, h^{-1}g^{-1}\right) \\ &= \left(c^{*h^{-1}g^{-1}} b^{*hh^{-1}g^{-1}}, h^{-1}g^{-1}\right) \\ &= \left(c^{*h^{-1}g^{-1}} b^{*g^{-1}}, h^{-1}g^{-1}\right) \\ &= (c^{*h^{-1}}, h^{-1})(b^{*g^{-1}}, g^{-1}) = (c,h)^*(b,g)^*.\end{aligned}$$

Additionally,

$$((b,g)^*)^* = (b^{*g^{-1}}, g^{-1})^* = ((b^{*g^{-1}})^{*g}, g) = (b,g).$$

Therefore * is an anti-automorphism of order at most 2 for $\mathbb{C}(\mathbf{B} \rtimes G)$.

Also, the identity is $(b_0, 1)$, where 1 is the identity of the group G and b_0 is the identity of $\mathbb{C}\mathbf{B}$. Now $(b,g)(c,h) = (b^h c, gh)$ will have the identity in its decomposition if and only if $c = b^{h*}$ and $h = g^{-1}$ if and only if $c = b^{g^{-1}*} = b^{*g^{-1}}$ and $h = g^{-1}$

if and only if $(c,h) = (b^{*g^{-1}}, g^{-1}) = (b,g)^*$. Thus condition (iii) of Definition 2.1.1 is satisfied. □

Note that if $(A, \mathbf{B})$ is a STA, then $(\mathbb{C}(\mathbf{B} \rtimes G), \mathbf{B} \rtimes G)$ will also be a STA.

Remark 2.1.9. An alternative definition of semidirect product is as follows. Let $\mathbf{B}$ be the basis of a table algebra, and let G be a group acting as automorphisms on A, permuting $\mathbf{B}$. Define $S = \{(b,g) \mid b \in \mathbf{B} \text{ and } g \in G\}$ as a formal basis for a vector space over $\mathbb{C}$, such that for all $b, c \in \mathbf{B}$ and $g, h \in G$,

$$(b,g) \cdot (c,h) = \left(bc^{g^{-1}}, gh\right),$$

where $\left(bc^{g^{-1}}, gh\right) = \sum_{d \in \mathbf{B}} \beta_d (d, gh)$ if $bc^{g^{-1}} = \sum_{d \in \mathbf{B}} \beta_d d$. Also define $(b,g)^* = (b^{*g}, g^{-1})$. Then a similar argument to that in Proposition 2.1.1 shows that $(\mathbb{C}S, S)$ is a reality-based algebra.

Proposition 2.1.2. Let $\mathbf{B}$ be the basis of a table algebra, A, and let G be a group acting as automorphisms on A, permuting $\mathbf{B}$. Let $S = \{(b,g) \mid b \in \mathbf{B} \text{ and } g \in G\}$, a formal basis for a vector space over $\mathbb{C}$. Suppose multiplication among elements in S and * are defined as in Remark 2.1.9. (Note that S and $\mathbf{B} \rtimes G$ are the same set.) Then $S \cong_{\mathrm{x}} \mathbf{B} \rtimes G$.

Proof. Define $\phi : S \to \mathbf{B} \rtimes G$ by $\phi((b,g)) = (b^g, g)$ and extend it linearly to $\mathbb{C}S$. Then ϕ is a vector space isomorphism. Also,

$$\begin{aligned} \phi((b,g)(c,h)) = \phi((bc^{g^{-1}}, gh)) &= (b^{gh} c^{g^{-1}gh}, gh) \\ &= (b^{gh} c^h, gh) = (b^g, g)(c^h, h) = \phi((b,g))\phi((c,h)). \end{aligned}$$

Therefore ϕ is an exact isomorphism between S and $\mathbf{B} \rtimes G$. □

Remark 2.1.10. As a special case of the above semidirect product, suppose $(A, \mathbf{B})$ is a table algebra and $\mathbf{C}$ a commutative closed subset of $\mathbf{B}$ such that $\mathbf{B} = \mathbf{C} \cup b\mathbf{C}$ where $b^2 = b_0$ and $c_i b = b c_{i^*}$ for all $c_i \in \mathbf{C}$ (hence, $b\mathbf{C} = \mathbf{C}b$). Suppose also that $G = \langle z \rangle = \mathbb{Z}_2$ where $c^z = c^*$ for all $c \in \mathbf{C}$. Then $\mathbf{B} \cong_x \mathbf{C} \rtimes \mathbb{Z}_2$, where $c_i \leftrightarrow (c_i, 1)$ and $c_i b \leftrightarrow (c_i, z)$.

The following definition of a semidirect product of association schemes is a specialization of the definition provided by Bang, Hirasaka, and Song to the case where one of the schemes is given by a group scheme.

Definition 2.1.16. ([4, Definition 3.1]; see also [22, Section 2.7]). Let G be a group and $(G, \widetilde{G})$ be the group scheme defined by $(x, y) \in R_g$ iff $xy^{-1} = g$. Let (Y, K) be another scheme. Let $\pi : G \to Aut(K)$ be a group homomorphism. Let $g_0 \in G$ be a fixed element and $Z := \{(g, y) \mid g \in G, y \in Y\}$. Let $\widetilde{G} \ltimes_{\pi, g_0} K := \{g \cdot k \mid R_g \in \widetilde{G}, k \in K\}$ where relations $g \cdot k \subset Z \times Z$ are defined by:

$$((g_1, y_1), (g_2, y_2)) \in g \cdot k \text{ iff } g_1 g_2^{-1} = g \text{ and } (y_1, y_2) \in k^{\pi(g_2 g_0^{-1})}$$

for $(g_1, y_1), (g_2, y_2) \in Z$. Then $(Z, \widetilde{G} \ltimes_{\pi, g_0} K)$ is a scheme. This scheme is called the *(external) semidirect product of* (Y, K) *by* $(G, \widetilde{G})$.

Remark 2.1.11. For any $h \neq g_0 \in G$, $(Z, \widetilde{G} \ltimes_{\pi, g_0} K)$ is isomorphic to $(Z, \widetilde{G} \ltimes_{\pi, h} K)$ [4, Lemma 3.2]. Thus the selection of the group element in G used to generate the semidirect product is arbitrary. However, a different choice of homomorphism from G to $Aut(K)$ could produce a non-isomorphic semidirect product [4, Example 3.2].

Proposition 2.1.3. See [4, Proposition 3.3]. Let $(Z, \widetilde{G} \ltimes_\pi K)$ be the semidirect product of (Y, K) by $(G, \widetilde{G})$, where $(G, \widetilde{G})$ is the group scheme defined by $(x, y) \in R_g$ iff $xy^{-1} = g$. Let $\{A_k \mid k \in K\}$ be the set of adjacency matrices of (Y, K). Then the set of adjacency matrices for $(Z, \widetilde{G} \ltimes_\pi K)$ is $\{A_{g\cdot k} \mid g \cdot k \in \widetilde{G} \ltimes_\pi K\}$, where $A_{g\cdot k}$ is a $|G| \times |G|$ block matrix in which the (x_i, x_j)-block is given by a $|Y| \times |Y|$ zero-one matrix:

$$(A_{g\cdot k})_{(x_i,x_j)} = \begin{cases} \mathcal{O}_{|Y|} & \text{if } x_i {x_j}^{-1} \neq g \\ A_{k^{\pi(x_j)}} & \text{if } x_i {x_j}^{-1} = g \end{cases}$$

for any $g \cdot k \in \widetilde{G} \ltimes_\pi K$, where $\mathcal{O}_{|Y|}$ is the $|Y| \times |Y|$ zero matrix.

Proposition 2.1.4. Let G be a group and $\mathcal{G} := (G, \widetilde{G})$ be the group scheme defined by $(x, y) \in R_g$ iff $xy^{-1} = g$. Let $\mathcal{K} := (Y, K)$ be another scheme. Let $\pi : G \to Aut(K)$ be a group homomorphism and $g_0 \in G$ be a fixed element. Suppose $\mathcal{Z} := (Z, \widetilde{G} \ltimes_{\pi,g_0} K)$, the semidirect product of (Y, K) by $(G, \widetilde{G})$. Then $\mathcal{A}(\mathcal{Z}) \cong_{\mathrm{x}} \mathcal{A}(\mathcal{K}) \rtimes G$, where $\mathcal{A}(\mathcal{K}) \rtimes G$ generates a semidirect product of table algebras.

Proof. By Remark 2.1.11, we may assume $g_0 = 1_G$, the identity for the group G. We will denote $\pi(g)$ by ^{g}. Define $\phi : \mathcal{A}(\mathcal{K}) \rtimes G \to \mathcal{A}(\mathcal{Z})$ by $\phi((A_k, g)) = A_{g\cdot k}$, and extend it linearly. Then ϕ is a vector space isomorphism. Pick $A_k, A_l \in \mathcal{A}(\mathcal{K})$ and $g, h \in G$. Then for any $x_r, x_s \in G$, there exists an element $x_t \in G$ with $x_r {x_t}^{-1} = g$ and $x_t {x_s}^{-1} = h$ iff $x_r {x_s}^{-1} = gh$, and in this case $x_t = hx_s$. Thus

$$(\phi((A_k, g))\phi((A_l, h)))_{(x_r,x_s)} = (A_{g\cdot k} A_{h\cdot l})_{(x_r,x_s)}$$

$$= \begin{cases} 0 & \text{if } x_r {x_s}^{-1} \neq gh \\ A_{k^{hx_s}} A_{l^{x_s}} & \text{if } x_r {x_s}^{-1} = gh \end{cases}$$

$$= \begin{cases} 0 & \text{if } x_r {x_s}^{-1} \neq gh \\ \sum_{m^{x_s} \in K} \rho_{k^{hx_s} l^{x_s} m^{x_s}} A_{m^{x_s}} & \text{if } x_r {x_s}^{-1} = gh. \end{cases}$$

Also $\phi((A_k,g)(A_l,h))_{(x_r,x_s)} = \phi((A_{k^h}A_l,gh))_{(x_r,x_s)} = \sum\limits_{m\in K}\rho_{k^h lm}\phi((A_m,gh))_{(x_r,x_s)}$

$$= \begin{cases} 0 & \text{if } x_r{x_s}^{-1} \neq gh \\ \sum\limits_{m\in K}\rho_{k^h lm}A_{m^{x_s}} & \text{if } x_r{x_s}^{-1} = gh. \end{cases}$$

By Definition 2.1.4 $\rho_{k^h lm} = \rho_{k^{hx_s}l^{x_s}m^{x_s}}$ since $x_s \in Aut(K)$. Therefore, we have shown $\mathcal{A}(\mathcal{Z}) \cong_{\mathrm{x}} \mathcal{A}(\mathcal{K}) \rtimes G$. □

2.2 Preliminary Results

Theorem 2.2.1. [19, Theorem 3.6] Let $(A,\mathbf{B})$ be a STA and $\mathbf{C}$ be a normal closed subset of $\mathbf{B}$. Then $\mathrm{Irr}(A//\mathbf{C})$ is in bijection with $\{\chi \in \mathrm{Irr}(A) \text{ with } \mathbf{C} \subseteq \mathrm{Ker}(\chi)\}$, and $\mathrm{Irr}(A//\mathbf{C}) = \{\chi{\downarrow_{A//\mathbf{C}}} : \chi \in \mathrm{Irr}(A) \text{ and } \mathbf{C} \subseteq \mathrm{Ker}(\chi)\}$. Moreover, $\chi(\mathbf{B}//\mathbf{C}) = 0$ if $\chi \in \mathrm{Irr}(A)$ and $\mathbf{C} \nsubseteq \mathrm{Ker}(\chi)$.

Proposition 2.2.2. [19, Proposition 4.1] Let $(A,\mathbf{B})$ be a table algebra with degree map χ_0. Let $\chi_j \in \mathrm{Irr}(A)$. Then for any $b_i \in \mathbf{B}$, $|\chi_j(b_i)| \leq \chi_0(b_i)\chi_j(b_0)$.

Remark 2.2.1. To prove Proposition 2.2.2, Xu considers a representation, X_j, which affords χ_j and observes that any eigenvalue, λ, of $X_j(b_i)$ satisfies $|\lambda| \leq \chi_0(b_i)$. This was also noted in the commutative case in Lemma 2.6 of [2].

Lemma 2.2.3. If $(A,\mathbf{B})$ is a table algebra, χ_s is an irreducible character of A, $0 \neq a = \sum_{c\in\mathbf{B}}\beta_c c$ with all $\beta_c \geq 0$, and $\chi_s(a) = \alpha n_s\chi_0(a)$ for some $\alpha \in \mathbb{C}$ with $|\alpha| = 1$, then $\chi_s(c) = \alpha n_s\chi_0(c)$ for all c with $\beta_c \neq 0$.

Proof. Proposition 2.2.2 implies $n_s\chi_0(a) = |\chi_s(a)| \leq \sum\beta_c|\chi_s(c)| \leq \sum\beta_c n_s\chi_0(c) = n_s\chi_0(a)$. Thus the second inequality implies $|\chi_s(c)| = n_s\chi_0(c)$ for all $c \in \mathrm{Supp}(a)$, and the first inequality implies $\{\chi_s(c) : \beta_c > 0\}$ have the same argument in the complex plane; thus there exists $\hat{\alpha}$ with $|\hat{\alpha}| = 1$ and $\chi_s(c) = \hat{\alpha} n_s \chi_0(c)$ for all $c \in \mathrm{Supp}(a)$. Then $\alpha n_s \chi_0(a) = \chi_s(a) = \sum \beta_c \hat{\alpha} n_s \chi_0(c) = \hat{\alpha} n_s \chi_0(a)$. Thus $\hat{\alpha} = \alpha$. $\square$

Definition 2.2.1. [10, Definition 3.16] Let $(A, \mathbf{B})$ be a table algebra, and pick $b \in \mathbf{B}$. Define the closed subset generated by b, $\mathbf{B}_b := \cup_{m>0} \mathrm{Supp}(b^m)$.

Proposition 2.2.4. Suppose $(A, \mathbf{B})$ is a table algebra with degree map χ_0. Let $\chi_j \in \mathrm{Irr}(A)$ and $b \in \mathrm{Z}(\chi_j)$. Then $\chi_j \downarrow_{\mathbf{B}_b} = n_j \eta$, where η is a linear character of $(<\mathbf{B}_b>, \mathbf{B}_b)$.

Proof. Let X_j be a representation affording χ_j. Recall that $|\lambda| \leq \chi_0(b)$ for any eigenvalue, λ, of $X_j(b)$ (by Remark 2.2.1). Note also that $\chi_j(b)$ equals the sum of the n_j eigenvalues of $X_j(b)$. Thus $|\chi_j(b)| = \chi_0(b) n_j$ implies that $|\lambda| = \chi_0(b)$ for all eigenvalues λ of $X_j(b)$ and that all of the eigenvalues have the same argument in the complex plane. Thus there exists a representation X_j, affording χ_j, such that $X_j(b)$ is upper triangular with constant $\alpha_b \chi_0(b)$ along the diagonal, where $|\alpha_b| = 1$. Then for any $m > 0$ in $\mathbb{Z}$, $X_j(b^m) = (X_j(b))^m$ is upper triangular with constant ${\alpha_b}^m \chi_0(b^m)$ along the diagonal.

So suppose $b^m = \sum \beta_c c$, where $\beta_c \geq 0$. Since $\chi_j(b^m) = {\alpha_b}^m \chi_0(b^m) n_j$, Lemma 2.2.3 implies

$$\chi_j(c) = {\alpha_b}^m \chi_0(c) n_j \text{ for all } c \in \mathrm{Supp}(b^m). \tag{2.2.1}$$

Now define $\eta :< \mathbf{B}_b > \to \mathbb{C}$ by $\eta(c) = {\alpha_b}^m \chi_0(c)$, where $c \in \mathrm{Supp}(b^m)$, and extend it linearly. Note that by (2.2.1) and the definition of η, for each $c \in \mathbf{B}_b$, $\eta(c)$ is the same for all positive integers m with $c \in \mathrm{Supp}(b^m)$. Take any $c_1, c_2 \in \mathbf{B}_b$. Then there exists $v, w \in \mathbb{Z}_{\geq 0}$ with $c_1 \in \mathrm{Supp}(b^v)$ and $c_2 \in \mathrm{Supp}(b^w)$. For any $c \in \mathrm{Supp}(c_1 c_2)$,

$c \in \mathrm{Supp}(b^{v+w})$. Supposing $c_1c_2 = \sum \beta_c c$, $\eta(c_1c_2) = \sum \beta_c \eta(c) = \sum \beta_c {\alpha_b}^{v+w}\chi_0(c) = {\alpha_b}^{v+w}\chi_0(c_1c_2) = {\alpha_b}^{v}\chi_0(c_1){\alpha_b}^{w}\chi_0(c_2) = \eta(c_1)\eta(c_2)$. Thus η is a homomorphism.

So for any $c \in \mathbf{B}_b$, (2.2.1) implies that $\chi_j(c) = n_j\eta(c)$. Therefore, $\chi_j{\downarrow_{\mathbf{B}_b}} = n_j\eta$. □

Remark 2.2.2. As a result of the above proposition and the semi-simplicity of table algebras, for any $b \in \mathrm{Z}(\chi_j)$, there exists a matrix representation, X_j, which affords χ_j such that $X_j(b)$ is a scalar matrix. In particular, $X_j(b) = \eta(b)I_{n_j}$.

Lemma 2.2.5. Suppose $(A, \mathbf{B})$ is a table algebra with degree map χ_0. Let χ_s be any irreducible character of A. Then $\mathrm{Z}(\chi_s)$ is a closed subset of the basis, and $b\,\mathrm{Ker}(\chi_s)\,b^* \subseteq \mathrm{Ker}(\chi_s)$ for all $b \in \mathrm{Z}(\chi_s)$, and hence $\mathrm{Ker}(\chi_s)$ is strongly normal in $\mathrm{Z}(\chi_s)$.

Proof. Pick any $b_i, b_j \in \mathrm{Z}(\chi_s)$. By Remark 2.2.2, we can choose a representation X_s, which affords χ_s, such that $X_s(b_i) = \alpha_i\chi_0(b_i)I_{n_s}$, where $|\alpha_i| = 1$. Now $b_j \in \mathrm{Z}(\chi_s)$ also implies $\chi_s(b_j) = \alpha_j\chi_0(b_j)n_s$ where $|\alpha_j| = 1$. Thus

$$\chi_s(b_ib_j) = tr[X_s(b_i)X_s(b_j)] = tr\,[\alpha_i\chi_0(b_i)X_s(b_j)] = \alpha_i\chi_0(b_i)\chi_s(b_j) = \alpha_i\alpha_j\chi_0(b_ib_j)n_s.$$

Thus $|\chi_s(b_ib_j)| = \chi_0(b_ib_j)n_s$, and so Lemma 2.2.3 implies $|\chi_s(b_k)| = n_s\chi_0(b_k)$ for all b_k with $\beta_{ijk} \neq 0$. Therefore $\mathrm{Z}(\chi_s)$ is closed.

Now choose any $b \in \mathrm{Z}(\chi_s)$ and $c \in \mathrm{Ker}(\chi_s)$. From Remark 2.2.2, there exists a representation that affords χ_s, denote it X, with $X(b) = \alpha\chi_0(b)I_{n_s}$, where $|\alpha| = 1$. Then

$$\chi_s(bcb^*) = tr[X(b)X(c)X(b^*)] = \alpha\overline{\alpha}(\chi_0(b))^2 n_s\chi_0(c) = n_s\chi_0(bcb^*).$$

By Lemma 2.2.3, $\mathrm{Supp}(bcb^*) \subseteq \mathrm{Ker}(\chi_s)$. So, $\mathrm{Ker}(\chi_s)$ is strongly normal in $\mathrm{Z}(\chi_s)$. □

Lemma 2.2.6. Let $(A, \mathbf{B})$ be a STA. Let $T \subseteq \mathrm{Irr}(A)$, $\mathbf{C} := \bigcap_{\chi_s \in T} \mathrm{Ker}(\chi_s)$, and $\mathbf{D} := \bigcap_{\chi_s \in T} \mathrm{Z}(\chi_s)$. Then $\mathbf{D}//\mathbf{C}$ is a group.

Proof. By Lemma 2.2.5, $b\,\mathbf{C}\,b^* \subseteq \mathbf{C}$ for all $b \in \mathbf{D}$ (hence $\mathbf{C}$ is strongly normal in $\mathbf{D}$). This implies $(b//\mathbf{C})\,(b^*//\mathbf{C}) = b_0//\mathbf{C}$ and $\mathbf{D}//\mathbf{C}$ is a group. □

Proposition 2.2.7. [3, Theorem 3.6(iv)] Suppose $(A, \mathbf{B})$ is a STA, χ_0 is the degree map, and χ_s is an irreducible character of A. Suppose m_s is the multiplicity and e_s is the central primitive idempotent corresponding to χ_s. Then $e_s = \sum_{i=0}^{d} \frac{\frac{m_s}{o(\mathbf{B})}\chi_s(b_i^*)b_i}{\beta_{ii^*0}}$.

Lemma 2.2.8. Suppose $(A, \mathbf{B})$ is a STA with degree map χ_0. Also suppose $\tau = \sum_{j=0}^{k} m_j \cdot \chi_j$. For any irreducible character χ_s of A, $m_s n_s \geq 1$; and $\mathrm{Z}(\chi_s) = \mathbf{B}$ if and only if $m_s n_s = 1$.

Proof. Recall $(A, \mathbf{B})$ standard implies $\chi_0(b_i) = \beta_{ii^*0}$ for any i and $o(\mathbf{B}) = \sum \beta_{ii^*0}$.

By Proposition 2.2.7 and Proposition 2.2.2,

$$\begin{aligned} n_s &= \chi_s(e_s) = \chi_s\left(\sum_{i=0}^{d} \frac{\frac{m_s}{o(\mathbf{B})}\chi_s(b_i^*)b_i}{\beta_{ii^*0}}\right) = \frac{m_s}{o(\mathbf{B})}\sum_{i=0}^{d}\frac{|\chi_s(b_i)|^2}{\beta_{ii^*0}} \\ &\leq \frac{m_s}{o(\mathbf{B})}\sum_{i=0}^{d}\frac{(n_s\chi_0(b_i))^2}{\beta_{ii^*0}} = \frac{m_s {n_s}^2}{o(\mathbf{B})}\sum_{i=0}^{d}\beta_{ii^*0} = m_s {n_s}^2. \end{aligned}$$

The first claim is proved. The above inequality is an equality if and only if $|\chi_s(b_i)| = n_s\chi_0(b_i)$ for all i if and only if $m_s n_s = 1$. The lemma follows. □

Corollary 2.2.9. Suppose $(A, \mathbf{B})$ is a STA with degree map χ_0. Also suppose $\tau = \sum_{j=0}^{k} m_j \cdot \chi_j$. For any irreducible linear character χ_s of A, $m_s \geq 1$.

Proof. This follows from Lemma 2.2.8 with $n_s = 1$. □

Proposition 2.2.10. ([3, Proposition 4.8],[10, Proposition 3.3]) Suppose $(A, \mathbf{B})$ is a reality based algebra with degree map χ_0. Assume that $\mathbf{B}$ is standard. Then for all $b_i \in \mathbf{B}$, $b_i\mathbf{B}^+ = \chi_0(b_i)\mathbf{B}^+ = \mathbf{B}^+ b_i$.

Suppose $(A, \mathbf{B})$ is a STA with a closed subset $\mathbf{C}$ such that $\mathbf{B}//\mathbf{C}$ is an abelian group. From Remark 2.1.1, the cosets of $\mathbf{C}$ partition $\mathbf{B}$, so that $\mathbf{B}^+$ can be written $(d_1\mathbf{C})^+ + (d_2\mathbf{C})^+ + ... + (d_k\mathbf{C})^+$, where $\{d_1, d_2, ..., d_k\}$ make up the coset representatives. Additionally, since $\mathbf{B}//\mathbf{C}$ is a group, for any $b_i \in \mathbf{B}$ and any coset $b_k\mathbf{C}$ of $\mathbf{C}$ in $\mathbf{B}$, there is a unique coset $b_j\mathbf{C}$ such that $b_i(b_j\mathbf{C}) \subseteq b_k\mathbf{C}$. Thus the following is an immediate consequence of Proposition 2.2.10.

Corollary 2.2.11. Suppose $(A, \mathbf{B})$ is a STA. Suppose $\mathbf{C}$ is a closed subset of $\mathbf{B}$ such that $\mathbf{B}//\mathbf{C}$ is an abelian group. Pick $b_i//\mathbf{C}, b_j//\mathbf{C} \in \mathbf{B}//\mathbf{C}$. Suppose $b_i//\mathbf{C} \cdot b_j//\mathbf{C} = b_k//\mathbf{C}$. Then $b_i(b_j\mathbf{C})^+ = \chi_0(b_i)(b_k\mathbf{C})^+ = (b_j\mathbf{C})^+b_i$.

Remark 2.2.3. Suppose $(A, \mathbf{B})$ is a STA. Suppose $\mathbf{C}$ is a closed subset of $\mathbf{B}$ such that $\mathbf{B}//\mathbf{C}$ is an abelian group. The above corollary shows that sums over the cosets of $\mathbf{C}$ in $\mathbf{B}$ are contained in the center of the algebra $(A, \mathbf{B})$.

Remark 2.2.4. As a result of the above remark, if $(A, \mathbf{B})$ is a STA with closed subset $\mathbf{C}$ such that $\mathbf{B}//\mathbf{C}$ is an abelian group, then all elements of $\mathbf{B}$ whose cosets over $\mathbf{C}$ consist of just one element are contained in the center of the algebra.

The following is a corollary of Proposition 3.4 in [10].

Proposition 2.2.12. Suppose $(A, \mathbf{B})$ is a STA. Suppose $\mathbf{C}$ is a closed subset of $\mathbf{B}$ such that $b_{i^*}b_i \subseteq \mathbf{C}$ for every $b_i \in \mathbf{B}$. Then $o(b_i\mathbf{C}) = o(\mathbf{C})$ for all cosets $b_i\mathbf{C}$ in $\mathbf{B}$.

Proposition 2.2.13. [20, Lemma 2.2] Let $(A, \mathbf{B})$ be a STA and $\chi_s \in \mathrm{Irr}(A)$. Let n_s and m_s be the degree and multiplicity of χ_s, respectively. Then

$$n_s \leq m_s.$$

Lemma 2.2.14. Suppose $(A, \mathbf{B})$ is a standard integral table algebra. Let χ_0 be the degree map and $\{\chi_i \mid 0 \leq i \leq k\}$ be the irreducible characters of A. Suppose $\tau = \sum_{0 \leq i \leq k-1} \chi_i + m\chi_k$, and let $\mathbf{C} := \cap_{i=0}^{k-1} \mathrm{Ker}(\chi_i)$. Suppose $|\mathbf{C}| > 1$. Then

$$\chi_k(b) = 0 \text{ for } b \notin \mathbf{C}.$$

Additionally, if $\chi_k(b_0) > 1$, then

$$\chi_k(b_i) \in \mathbb{Z} \text{ for all } b_i \in \mathbf{B}.$$

Proof. First note that for $0 \leq i \leq k-1$, $m_i = 1$ implies $\chi_i(b_0) = 1$ (Proposition 2.2.13). Pick any $b \notin \mathbf{C}$. Then $b_0 \notin b\mathbf{C}$. Pick any $c \in \mathbf{C}$. Since $c \in \cap_{i=0}^{k-1} \mathrm{Ker}(\chi_i)$, $\tau(bc) = 0$, and χ_i is linear for $0 \leq i \leq k-1$, we have $-m\chi_k(bc) = \chi_0(c) \sum_{i=0}^{k-1} \chi_i(b)$. Therefore $-m\chi_k(b\mathbf{C}^+) = o(\mathbf{C}) \sum_{i=0}^{k-1} \chi_i(b)$.

Since $m_i = n_i = 1$ for χ_i with $0 \leq i \leq k-1$, Lemma 2.2.8 implies $Z(\chi_i) = \mathbf{B}$ for $0 \leq i \leq k-1$. Then, by Lemma 2.2.6, $\mathbf{B}//\mathbf{C}$ is a group. Since $\cap_{i=0}^{k} \mathrm{Ker}(\chi_i) = \{b_0\}$, $\mathbf{C} \nsubseteq \mathrm{Ker}(\chi_k)$. By Theorem 2.2.1 and Corollary 2.2.11, $0 = \chi_k(b\mathbf{C}^+) = \frac{o(\mathbf{C})}{-m} \sum_{i=0}^{k-1} \chi_i(b)$. Thus $\sum_{i=0}^{k-1} \chi_i(b) = 0$. Then $\tau(b) = 0$ implies $\chi_k(b) = 0$.

Suppose now that $\chi_k(b_0) > 1$. Pick any element $\sigma \in Aut(\mathbb{C})$. It is known that σ permutes the elements in $\mathrm{Irr}(A)$. Since $\chi_k^{\sigma}(b_0) = \sigma(\chi_k(b_0)) = \chi_k(b_0) > 1$, χ_k is fixed by any automorphism of the complex numbers. $\chi_k(b_i)$ is an algebraic integer [10, Proposition 3], so $\chi_k(b_i) \in \mathbb{Z}$ for any $b_i \in \mathbf{B}$.

□

Definition 2.2.2. Suppose $(A, \mathbf{B})$ is a reality-based algebra with distinguished basis $\mathbf{B} = \{b_0, b_1, ..., b_d\}$. Define the function $\langle \, , \, \rangle : Hom_{\mathbb{C}}(A, \mathbb{C}) \rightarrow Hom_{\mathbb{C}}(A, \mathbb{C})$ by

$$\langle \theta, \phi \rangle := \sum_{i=0}^{d} \frac{\theta(b_i)\phi(b_{i^*})}{\beta_{ii^*0}}.$$

Note that $\langle \, , \, \rangle$ is a symmetric bilinear form on $Hom_{\mathbb{C}}(A, \mathbb{C})$.

Theorem 2.2.15. (*Orthogonality Relations*, [3, Theorem 3.6]) Suppose $(A, \mathbf{B})$ is a reality-based algebra with distinguished basis $\mathbf{B} = \{b_0, b_1, ..., b_d\}$, degree map χ_0, and $\mathrm{Irr}(A) = \{\chi_0, \chi_1, ...\chi_k\}$. Then for all $0 \leq s, t \leq k$,

$$\langle \chi_s, \chi_t \rangle = \delta_{st} \frac{n_s o(\mathbf{B})}{m_s},$$

where δ_{st} is the Kronecker delta function.

For the remainder of this paper, the phrase "$b_i\mathbf{C}$ commutes with $b_j\mathbf{C}$" means $\hat{b}\bar{b} = \bar{b}\hat{b}$ for any $\hat{b} \in b_i\mathbf{C}$ and any $\bar{b} \in b_j\mathbf{C}$.

Lemma 2.2.16. Suppose $(A, \mathbf{B})$ is a STA with closed subset $\mathbf{C}$ such that $\mathbf{B}//\mathbf{C}$ is an abelian group. Suppose there exists $b_i, b_j \in \mathbf{B}$ with $b_i//\mathbf{C} \cdot b_j//\mathbf{C} = b_k//\mathbf{C}$. Then the following hold:

(i) If $|b_k\mathbf{C}| = 1$, then $b_i\mathbf{C}$ commutes with $b_j\mathbf{C}$.

(ii) If $b_{j^*}\mathbf{C}$ commutes with $b_k\mathbf{C}$, then $b_i\mathbf{C}$ commutes with $b_j\mathbf{C}$.

(iii) If $|b_i\mathbf{C}| = 1$, then $b_{j^*}\mathbf{C}$ commutes with $b_k\mathbf{C}$.

(iv) Suppose $|b_k\mathbf{C}| = 2$ and pick $b \in b_k\mathbf{C}$. If the coefficient of b in b_ib_j is the same as the coefficient of b in b_jb_i, then $b_ib_j = b_jb_i$.

(v) If $b_i\mathbf{C} \cup b_j\mathbf{C} \cup b_k\mathbf{C}$ consists of real elements, then $b_i\mathbf{C}$ commutes with $b_j\mathbf{C} \cup b_k\mathbf{C}$ and $b_j\mathbf{C}$ commutes with $b_k\mathbf{C}$.

Proof. (i) Suppose $b_k\mathbf{C} = \{b_k\}$. Pick any $\bar{b} \in b_i\mathbf{C}$ and any $\hat{b} \in b_j\mathbf{C}$. Then $\mathrm{Supp}(\bar{b}\hat{b}) = \{b_k\}$. Since $\mathbf{B}//\mathbf{C}$ is an abelian group, $\mathrm{Supp}(\hat{b}\bar{b}) = \{b_k\}$. Applying χ_0 to $\bar{b}\hat{b} = \alpha b_k$ and $\hat{b}\bar{b} = \beta b_k$ yields $\alpha = \beta$. Thus $\bar{b}\hat{b} = \hat{b}\bar{b}$ and $b_i\mathbf{C}$ commutes with $b_j\mathbf{C}$.

(ii) Suppose $b_{j^*}\mathbf{C}$ commutes with $b_k\mathbf{C}$. Pick any $\hat{b} \in b_i\mathbf{C}$ and any $\bar{b} \in b_j\mathbf{C}$. Then $\mathrm{Supp}(\hat{b}\bar{b}) \subseteq b_k\mathbf{C}$. Pick any $d \in b_k\mathbf{C}$. Then $d\bar{b}^* = \bar{b}^* d$. Therefore,

$$(\hat{b}\bar{b}, d) = (\hat{b}, d\bar{b}^*) = (\hat{b}, \bar{b}^* d) = (\bar{b}\hat{b}, d).$$

Thus $\hat{b}\bar{b} = \bar{b}\hat{b}$. Therefore $b_i\mathbf{C}$ commutes with $b_j\mathbf{C}$.

(iii) Suppose $b_i\mathbf{C} = \{b_i\}$. Since $\mathbf{B}//\mathbf{C}$ is a group, $b_k//\mathbf{C} \cdot b_{j^*}//\mathbf{C} = b_i//\mathbf{C}$. From (i), $b_k\mathbf{C}$ commutes with $b_{j^*}\mathbf{C}$.

(iv) Suppose $b_k\mathbf{C} = \{b, d\}$, and suppose $\alpha = (b_i b_j, b) = (b_j b_i, b)$. Then $\exists\, \beta, \gamma \in \mathbb{Z}_{\geq 0}$ with $b_i b_j = \alpha b + \beta d$ and $b_j b_i = \alpha b + \gamma d$. Applying χ_0 shows $\beta = \gamma$. Thus $b_i b_j = b_j b_i$.

(v) Suppose $b_i\mathbf{C} \cup b_j\mathbf{C} \cup b_k\mathbf{C}$ consists of real elements. Then $b^* = b$ for any $b \in b_i\mathbf{C} \cup b_j\mathbf{C} \cup b_k\mathbf{C}$. Pick any $\bar{b} \in b_i\mathbf{C}$ and any $\hat{b} \in b_j\mathbf{C}$. Applying * to $\bar{b}\hat{b} = \sum_{d \in b_k\mathbf{C}} \beta_d d$ produces $\hat{b}\bar{b} = \sum_{d \in b_k\mathbf{C}} \beta_d d = \bar{b}\hat{b}$. So $b_i\mathbf{C}$ commutes with $b_j\mathbf{C}$. By (ii), $b_k\mathbf{C}$ commutes with $b_{j^*}\mathbf{C} = b_j\mathbf{C}$ and $b_k\mathbf{C}$ commutes with $b_{i^*}\mathbf{C} = b_i\mathbf{C}$.

□

Remark 2.2.5. Suppose $(A, \mathbf{B})$ is a STA with closed subset $\mathbf{C}$ such that $\mathbf{B}//\mathbf{C}$ is an abelian group. Then for any $b_i//\mathbf{C} \in \mathbf{B}//\mathbf{C}$, $b_i//\mathbf{C} \cdot b_{i^*}//\mathbf{C} = b_0//\mathbf{C}$ and

$b_0//\mathbf{C} \cdot b_i//\mathbf{C} = b_i//\mathbf{C}$. So, by Lemma 2.2.16, $\mathbf{C}$ commutes with $b_i\mathbf{C}$ iff $b_i\mathbf{C}$ commutes with $b_{i^*}\mathbf{C}$.

Definition 2.2.3. Let $\psi : (A, \mathbf{B}) \to (U, \mathbf{V})$ be a table algebra homomorphism. Define

$$\mathrm{Ker}(\psi) := \{b_i \in \mathbf{B} \mid \psi(b_i) = \beta_i 1_U \text{ for some } \beta_i \in \mathbb{R}_{>0}\},$$

$$\psi_V(\mathbf{B}) := \{v_j \in \mathbf{V} \mid \psi(b_{i_j}) = \beta_{i_j} v_j \text{ for some } b_{i_j} \in \mathbf{B} \text{ and } \beta_{i_j} \in \mathbb{R}_{>0}\}.$$

Note that $< \psi_V(\mathbf{B}) >= \psi(A)$, $\psi_V(\mathbf{B})$ is a closed subset of $\mathbf{V}$, and $\mathrm{Ker}(\psi)$ is a normal closed subset of $\mathbf{B}$.

Definition 2.2.4. [9] Let $(A, \mathbf{B})$ be a table algebra and $\emptyset \neq \mathbf{Q} \subseteq \mathbf{C} \subseteq \mathbf{B}$, with $\mathbf{C}$ a closed subset of $\mathbf{B}$ and $\mathbf{Q}$ a normal closed subset of $\mathbf{C}$. Suppose χ_0 is the degree map for $(A, \mathbf{B})$. We say $(\mathbf{B}, \mathbf{C}, \mathbf{Q})$ is a *partial wreath product* if $bq = qb = \chi_0(q)b$ for any $q \in \mathbf{Q}$ and all $b \in \mathbf{B}\backslash\mathbf{C}$.

The wreath product $(A//\mathbf{C}, \mathbf{B}//\mathbf{C}) \wr (\mathbb{C}\mathbf{C}, \mathbf{C})$ for a table algebra $(A, \mathbf{B})$ with closed subset $\mathbf{C}$ can now be viewed as the partial wreath product $(\mathbf{B}, \mathbf{C}, \mathbf{C})$.

Theorem 2.2.17. [7, Theorem 2.9] Let $\psi : (A, \mathbf{B}) \to (U, \mathbf{V})$ be a table algebra homomorphism. Assume $(A, \mathbf{B})$ and $(\psi(A), \psi_V(\mathbf{B}))$ are standard. Let $\mathbf{C} = \mathrm{Ker}(\psi)$, as in Definition 2.2.3. Define $\mathbf{B} \circ_\psi \mathbf{V} := \mathbf{B} \cup (\mathbf{V}\backslash\psi_V(\mathbf{B}))$, a formal disjoint union. Then $A \circ_\psi U$ is defined, and $(A \circ_\psi U, \mathbf{B} \circ_\psi \mathbf{V})$ is a table algebra such that each of the following holds:

(i) $1_{A\circ_\psi U} = 1_A$.

(ii) Each $v_k \in \psi_V(\mathbf{B})$ is replaced by $b//\mathbf{C} = o(\mathbf{C})^{-1}(b\mathbf{C})^+$ in $A \circ_\psi U$, where $b \in \mathbf{B}$ with $\psi(b) = \beta_k v_k$ for some $\beta_k > 0$. In particular, if $v_i, v_j \in \mathbf{V}\backslash\psi_V(\mathbf{B})$ and v_k has coefficient ν_{ijk} in the decomposition of $v_i v_j$ in terms of $\mathbf{V}$ (in U), then

$\nu_{ijk} o(\mathbf{C})^{-1}(b\mathbf{C})^+$ appears in the decomposition of $v_i v_j$ in terms of $\mathbf{B} \circ_\psi \mathbf{V}$ (in $A \circ_\psi U$).

(iii) For each $v_i, v_j \in \mathbf{V} \backslash \psi_V(\mathbf{B})$, the decomposition of $v_i v_j$ in terms of $\mathbf{B} \circ_\psi \mathbf{V}$ is exactly as in terms of $\mathbf{V}$ (in U), except for the replacements as in (ii) above. For each $b_i, b_j \in \mathbf{B}$, the decomposition of $b_i b_j$ in terms of $\mathbf{B} \subseteq \mathbf{B} \circ_\psi \mathbf{V}$ is the same in $A \circ_\psi U$ as in A. So except for the noted replacements, the structure constants for these products are unchanged from $\mathbf{B}$ or $\mathbf{V}$ to $\mathbf{B} \circ_\psi \mathbf{V}$.

(iv) For all $b_i \in \mathbf{B}$ and $v_j \in \mathbf{V} \backslash \psi_V(\mathbf{B})$, $b_i v_j = \psi(b_i) v_j$ and $v_j b_i = v_j \psi(b_i)$ (in $A \circ_\psi U$). Hence, $b_i v_j = v_j b_i = \chi_0(b_i) v_j$ for all $b_i \in \mathbf{C}$.

Remark 2.2.6. It follows from (iv) that $(\mathbf{B} \circ_\psi \mathbf{V}, \mathbf{B}, \mathbf{C})$ is a partial wreath product.

Corollary 2.2.18. See [7, Proposition 2.11]. Suppose $(A, \mathbf{B})$ is a STA with a closed subset $\mathbf{C}$ of $\mathbf{B}$ such that $\mathbf{B}//\mathbf{C}$ is an abelian group. Let $\mathbf{D}$ be the closure of the set $\{b_i \mid |b_i \mathbf{C}| > 1\}$. Define $\psi : (< \mathbf{D} >, \mathbf{D}) \to (< \mathbf{B}//\mathbf{C} >, \mathbf{B}//\mathbf{C})$ by $\psi(d) = \chi_0(d) \cdot d//\mathbf{C}$ for any $d \in \mathbf{D}$. Then ψ is a table algebra homomorphism, $\mathbf{B} \cong_{\mathrm{x}} \mathbf{D} \circ_\psi \mathbf{B}//\mathbf{C}$, and $(\mathbf{B}, \mathbf{D}, \mathbf{C})$ forms a partial wreath product.

Proof. Define $\psi : (< \mathbf{D} >, \mathbf{D}) \to (< \mathbf{B}//\mathbf{C} >, \mathbf{B}//\mathbf{C})$ by $\psi(d) = \chi_0(d) \cdot d//\mathbf{C}$. Pick $b_i, b_j \in \mathbf{D}$. Suppose $b_i//\mathbf{C} \cdot b_j//\mathbf{C} = b_m//\mathbf{C}$. Then $b_i b_j = \sum_{k=0}^{d} \beta_{ijk} b_k$, where $\beta_{ijk} = 0$ unless $b_k \in b_m \mathbf{C}$. Applying χ_0, $\chi_0(b_i b_j) = \chi_0(b_i)\chi_0(b_j) = \sum_{k=0}^{d} \beta_{ijk} \chi_0(b_k)$. This equation implies that ψ is a homomorphism.

Then Theorem 2.2.17 implies $\mathbf{D} \circ_\psi \mathbf{B}//\mathbf{C}$ is a table algebra with the following properties:

(i) each $d//\mathbf{C}$ is replaced by $\frac{1}{o(\mathbf{C})}(d\mathbf{C})^+$;

(ii) for $b_i//\mathbf{C}, b_j//\mathbf{C} \in \mathbf{B}//\mathbf{C} \setminus \mathbf{D}//\mathbf{C}$, if $b_i//\mathbf{C} \cdot b_j//\mathbf{C} \notin \mathbf{D}//\mathbf{C}$, then the multiplication in $\mathbf{D} \circ_\psi \mathbf{B}//\mathbf{C}$ is the same as in $\mathbf{B}//\mathbf{C}$; otherwise $b_i//\mathbf{C} \cdot b_j//\mathbf{C} = b_k//\mathbf{C} \in \mathbf{D}//\mathbf{C}$, where $b_k//\mathbf{C} = \frac{1}{o(\mathbf{C})}(d\mathbf{C})^+$ for some $d \in \mathbf{D}$;

(iii) for any $d_i, d_j \in \mathbf{D}$, multiplication in $\mathbf{D} \circ_\psi \mathbf{B}//\mathbf{C}$ is the same as in $\mathbf{D}$;

(iv) for any $d_i \in \mathbf{D}$ and $b_j//\mathbf{C} \in \mathbf{B}//\mathbf{C} \setminus \mathbf{D}//\mathbf{C}$, $d_i \cdot b_j//\mathbf{C} = \psi(d_i) \cdot b_j//\mathbf{C} = \chi_0(d_i)d_i//\mathbf{C} \cdot b_j//\mathbf{C}$ and $b_j//\mathbf{C} \cdot d_i = \chi_0(d_i)b_j//\mathbf{C} \cdot d_i//\mathbf{C}$.

Note that by Corollary 2.2.11, $d_i(b_j\mathbf{C})^+ = \chi_0(d_i)(b_k\mathbf{C})^+$ for any $d_i \in \mathbf{D}$ with $d_i//\mathbf{C} \cdot b_j//\mathbf{C} = b_k//\mathbf{C}$. Now define $\theta : \mathbf{B} \to \mathbf{D} \cup (\mathbf{B}//\mathbf{C} \setminus \mathbf{D}//\mathbf{C})$ by

$$\theta(b_i) = \begin{cases} b_i & \text{if } b_i \in \mathbf{D} \\ \chi_0(b_i)b_i//\mathbf{C} & \text{if } b_i \notin \mathbf{D}. \end{cases}$$

Then $\theta(b_ib_j) = \theta(b_i)\theta(b_j)$ for any $b_i, b_j \in \mathbf{D}$.

If $b_i, b_j \notin \mathbf{D}$, then $|b_i\mathbf{C}| = 1$ and $|b_j\mathbf{C}| = 1$. Thus $\chi_0(b_i) = \chi_0(b_j) = o(\mathbf{C})$, so $\chi_0(b_i)b_i//\mathbf{C} = b_i$. Also $b_ib_j = o(\mathbf{C})(b_k\mathbf{C})^+$ for some coset $b_k\mathbf{C}$, where $|b_k\mathbf{C}|$ may or may not be equal to 1. Then

$$\theta(b_ib_j) = \theta(o(\mathbf{C})(b_k\mathbf{C})^+) = \begin{cases} o(\mathbf{C})^2 \cdot b_k//\mathbf{C} & \text{if } b_k\mathbf{C} \not\subseteq \mathbf{D} \\ o(\mathbf{C})(b_k\mathbf{C})^+ & \text{if } b_k\mathbf{C} \subseteq \mathbf{D}. \end{cases}$$

Also, $\theta(b_i)\theta(b_j) = o(\mathbf{C})^2 b_i//\mathbf{C} \cdot b_j//\mathbf{C}$

$$= \begin{cases} o(\mathbf{C})^2 \cdot b_k//\mathbf{C} & \text{if } b_k\mathbf{C} \not\subseteq \mathbf{D} \\ o(\mathbf{C})^2 \frac{1}{o(\mathbf{C})}(b_k\mathbf{C})^+ = o(\mathbf{C})(b_k\mathbf{C})^+ & \text{if } b_k\mathbf{C} \subseteq \mathbf{D}. \end{cases}$$

Finally suppose $b_i \in \mathbf{C}$ and $b_j \notin \mathbf{D}$ (so that $\chi_0(b_j) = o(\mathbf{C})$ and $b_i//\mathbf{C} \cdot b_j//\mathbf{C} = b_j//\mathbf{C}$), and suppose $b_i b_j = \sum_{k=0}^{d} \beta_{ijk} b_k$. Then $\chi_0(b_i)\chi_0(b_j) = \sum_{k=0}^{d} \beta_{ijk}\chi_0(b_k)$, where $\beta_{ijk} = 0$ unless $b_k \in b_j\mathbf{C}$. Therefore

$$\begin{aligned}\theta(b_i b_j) &= \theta(\sum_{k=0}^{d} \beta_{ijk} b_k) = \sum_{k=0}^{d} (\beta_{ijk}\, \chi_0(b_k) \cdot b_j//\mathbf{C}) \\ &= b_j//\mathbf{C} \cdot \sum_{k=0}^{d} \beta_{ijk}\chi_0(b_k) = \chi_0(b_i) \cdot \chi_0(b_j) \cdot b_j//\mathbf{C}.\end{aligned}$$

Also,

$$\begin{aligned}\theta(b_i)\theta(b_j) &= o(\mathbf{C}) b_i \cdot b_j//\mathbf{C} = o(\mathbf{C})\psi(b_i) b_j//\mathbf{C} = o(\mathbf{C})\chi_0(b_i) b_i//\mathbf{C} \cdot b_j//\mathbf{C} \\ &= \chi_0(b_j)\, \chi_0(b_i)\, b_i//\mathbf{C}\, b_j//\mathbf{C} = \chi_0(b_i)\, \chi_0(b_j)\, b_j//\mathbf{C}.\end{aligned}$$

Thus θ is a homomorphism. It is obviously one-to-one and onto. Thus it is an isomorphism between $\mathbf{B}$ and $\mathbf{D} \circ_\psi \mathbf{B}//\mathbf{C}$. Then $(\mathbf{B}, \mathbf{D}, \mathbf{C})$ is a partial wreath product by Remark 2.2.6. □

CHAPTER 3

THE COMMUTATIVE CASE

3.1 Main Result

Theorem 3.1.1. Let $(A, \mathbf{B})$ be a commutative standard table algebra. All multiplicities are trivial if and only if $(A, \mathbf{B})$ is an abelian group algebra. There is exactly one nontrivial multiplicity if and only if there exists a closed subset $\mathbf{C}$ of $\mathbf{B}$ such that $(A, \mathbf{B}) \cong_{\mathrm{x}} (A//\mathbf{C}, \mathbf{B}//\mathbf{C}) \wr (\mathbb{C}\mathbf{C}, \mathbf{C})$, where $\mathbf{B}//\mathbf{C}$ is an abelian group and $\mathbb{C}\mathbf{C}$ is a table algebra of dimension 2.

Proof. $(A, \mathbf{B})$ commutative implies $|\mathbf{B}| = |\mathrm{Irr}(A)|$ and all irreducible characters are linear. Thus, we may index the irreducible characters as $\{\chi_i : 0 \leq i \leq d\}$.

First suppose $\mathbf{B}$ is an abelian group. For any irreducible character χ_i, Remark 2.1.6 implies $m_i = \chi_i(b_0)$ for all i, which is 1 for an abelian group algebra.

Now assume $\tau = \sum_{i=0}^{d} \chi_i$. By Lemma 2.2.8, $\mathrm{Z}(\chi_s) = \mathbf{B}$ for any irreducible character χ_s. Thus $\bigcap_{\chi_s \in \mathrm{Irr}(A)} \mathrm{Z}(\chi_s) = \mathbf{B}$ and $\bigcap_{\chi_s \in \mathrm{Irr}(A)} \mathrm{Ker}(\chi_s) = \{b_0\}$. Therefore $\mathbf{B}//\{b_0\} = \mathbf{B}$ is a group, by Lemma 2.2.6.

Now suppose $\exists$ a closed subset $\mathbf{C} \subseteq \mathbf{B}$ with $(A, \mathbf{B}) \cong_{\mathrm{x}} (A//\mathbf{C}, \mathbf{B}//\mathbf{C}) \wr (\mathbb{C}\mathbf{C}, \mathbf{C})$, where $\mathbf{B}//\mathbf{C}$ is an abelian group and $\mathbb{C}\mathbf{C}$ is a table algebra of dimension 2. By Definition 2.1.14, $b_i\mathbf{C} = \{b_i\}$ for $b_i \notin \mathbf{C}$, so $|\mathbf{B}//\mathbf{C}| = |\mathbf{B}| - |\mathbf{C}| + 1 = |\mathbf{B}| - 1$. Also since $\mathbf{B}//\mathbf{C}$ is commutative, $|\mathrm{Irr}(\mathbf{B}//\mathbf{C})| = |\mathbf{B}//\mathbf{C}| = |\mathbf{B}| - 1$. Thus, according to Theorem 2.2.1, exactly one $\chi \in \mathrm{Irr}(A)$ has $\mathbf{C} \not\subseteq \mathrm{Ker}(\chi)$. Reorder the characters so that $\chi = \chi_d$, where $d = |\mathrm{Irr}(A)| - 1$. Also, let χ_0 be the degree map.

Since $\mathbf{B}//\mathbf{C}$ is a group, $\mathrm{Supp}(b_i b_{i^*}) \subseteq \mathbf{C}$ for all $b_i \in \mathbf{B}$. So, for any $j \neq d$, since $\mathbf{C} \subseteq \mathrm{Ker}(\chi_j)$, $\chi_j(b_i b_{i^*}) = \chi_0(b_i b_{i^*}) = (\chi_0(b_i))^2$.

This, along with Proposition 2.2.7, implies that for $0 \leq s \leq d-1$,

$$1 = \chi_s(e_s) = \frac{m_s}{o(\mathbf{B})} \sum_{i=0}^{d} \frac{\chi_s(b_{i^*})\chi_s(b_i)}{\chi_0(b_i)} = \frac{m_s}{o(\mathbf{B})} \sum_{i=0}^{d} \frac{\chi_s(b_{i^*}b_i)}{\chi_0(b_i)} = \frac{m_s}{o(\mathbf{B})} \sum_{i=0}^{d} \frac{\chi_0(b_i)^2}{\chi_0(b_i)} = m_s.$$

Thus $\tau = \sum_{j=0}^{d-1} \chi_j + m_d \chi_d$. Note that $m_d \neq 1$ since $(A, \mathbf{B})$ is not an abelian group algebra.

Now suppose that $\tau = \sum_{i=0}^{d-1} \chi_i + m_d \chi_d$ with $m_d \neq 1$. Let $\mathbf{C} = \bigcap_{i=0}^{d-1} \mathrm{Ker}(\chi_i)$. Lemma 2.2.8 implies $\bigcap_{i=0}^{d-1} \mathrm{Z}(\chi_i) = \mathbf{B}$. Thus $\mathbf{B}//\mathbf{C}$ is a group by Lemma 2.2.6. So, $|\mathbf{C}| > 1$. Also, Theorem 2.2.1 yields $|\mathbf{B}//\mathbf{C}| = |\{\chi \in \mathrm{Irr}(A) : \mathbf{C} \subseteq \mathrm{Ker}(\chi)\}| = d = |\mathbf{B}| - 1$. Thus $|\mathbf{B}| - 1 = |\mathbf{B}//\mathbf{C}| \leq |\mathbf{B}| - |\mathbf{C}| + 1$, where equality holds exactly when each nontrivial coset consists of just one element. This inequality implies $|\mathbf{C}| \leq 2$. So $|\mathbf{C}| = 2$, and $|\mathbf{B}//\mathbf{C}| = |\mathbf{B}| - |\mathbf{C}| + 1$. Therefore $(A, \mathbf{B}) \cong_{\mathrm{x}} (A//\mathbf{C}, \mathbf{B}//\mathbf{C}) \wr (\mathbb{C}\mathbf{C}, \mathbf{C})$, with $\mathbf{B}//\mathbf{C}$ a group and $|\mathbf{C}| = 2$.

□

The following corollary is an immediate consequence of Theorem 3.1.1 and Remark 2.1.7.

Corollary 3.1.2. Let $(A, \mathbf{B})$ be the adjacency algebra of a commutative association scheme. All of the primitive idempotent matrices of A are rank 1 iff the scheme is thin, and there is exactly one primitive idempotent matrix with rank greater than 1 iff the association scheme is the wreath product of an abelian group and an association scheme of class 2.

Corollary 3.1.3. (*Seitz's Theorem*, [18]) A group G has exactly one irreducible representation of degree greater than 1 if and only if one of the following occurs:

(i) $|G| = 2^l$, $G' = Z(G)$, and $|G'| = 2$ or

(ii) G is a Frobenius group with kernel an elementary abelian group of order $p^n \neq 2$ and complement a cyclic group of order $p^n - 1$.

Proof. Recall that the multiplicities in $Z(\mathbb{C}G)$ are the squares of the degrees of the irreducible characters of $\mathbb{C}G$. Therefore there will be one nonlinear irreducible character for $\mathbb{C}G$ exactly when there is one nontrivial multiplicity for $Z(\mathbb{C}G)$. Let $\mathbf{B}$ denote $\{{K_i}^+\}_{i=0}^k$. Theorem 3.1.1 implies that $Z(\mathbb{C}G)$ has just one nontrivial multiplicity exactly when there exists a closed subset $\mathbf{C}$ of $\mathbf{B}$ such that $|\mathbf{C}| = 2$ and $(Z(\mathbb{C}G), \mathbf{B}) \cong_{\mathrm{x}} (Z(\mathbb{C}G)//\mathbf{C}, \mathbf{B}//\mathbf{C}) \wr (\mathbb{C}\mathbf{C}, \mathbf{C})$, where $\mathbf{B}//\mathbf{C}$ is an abelian group.

Suppose that either (i) G is a 2-group with $G' = Z(G)$ and $|G'| = 2$; or (ii) $G = VH$, a Frobenius group with kernel V, an elementary abelian group of order p^n, and complement H, a cyclic group of order $p^n - 1$ (hence, $V = G'$). In either case, $G' = \{1\} \cup K_1$, where K_1 is a single conjugacy class, and for any $g_i \in G \backslash G'$, ${g_i}^G = G' g_i$. Thus, for $K_i = {g_i}^G$, $K_1 K_i = K_i$. So, if $\mathbf{C} = \{1, {K_1}^+\}$, then $\mathbf{B}//\mathbf{C} \cong \mathrm{Cla}(G/G')$ is an abelian group and $\mathbf{B} \cong_{\mathrm{x}} (\mathbf{B}//\mathbf{C}) \wr \mathbf{C}$.

Now suppose that there exists a closed subset $\mathbf{C}$ of $\mathbf{B}$ such that

$$(Z(\mathbb{C}G), \mathbf{B}) \cong_{\mathrm{x}} (Z(\mathbb{C}G)//\mathbf{C}, \mathbf{B}//\mathbf{C}) \wr (\mathbb{C}\mathbf{C}, \mathbf{C}),$$

where $\mathbf{B}//\mathbf{C}$ is an abelian group and $|\mathbf{C}| = 2$. Let $\mathbf{C} = \{1, {K_1}^+\}$. Since $\mathbf{C}$ is closed, $V := \{1\} \cup K_1$ is a normal p-subgroup of G for some prime p. For any $K_i \not\subseteq V$, $K_1 K_i = K_i$. Therefore V is the unique minimal normal subgroup of G, and thus is elementary abelian; $|K_i| > 1$; and for any $h \in G$, $Vh \subseteq h^G$. Since $\mathbf{B}//\mathbf{C}$ is a group,

${K_i}^+{K_{i^*}}^+ \in \mathbb{C}\mathbf{C}$ for all K_i. So for any $x, h \in G$, $x(hx^{-1}h^{-1}) \in V$. Thus $G' = V$ and for any $h \in G$, $G'h = h^G$.

Suppose $|G'| = 2$. Then $G' = Z(G)$ and G is nilpotent. Since G' is the unique minimal normal subgroup of G, $|G| = 2^l$ for some $l > 0$.

Now suppose $|G'| = p^n > 2$. Pick any $h \in G$. Suppose there exists $g \in G' \backslash \{1\}$ with $g^h = g$. Pick any other element in $G' \backslash \{1\}$, say $y^{-1}gy$ for $y \in G$. Since $G/C_G(G')$ is abelian, $(g^y)^h = g^{yh} = g^{hy} = (g^h)^y = g^y$. So, $h \in C_G(G')$. Therefore $G/C_G(G')$ acts transitively on $G' \backslash \{1\}$, each $t \in G \backslash C_G(G')$ acts fixed-point-freely, and $|G : C_G(G')| = p^n - 1$. Then $t^{G'} = G't = t^G$. So $G = C_G(t)G'$. Let $C = C_G(G')$. Thus $C = C_C(t)G'$. Since G' is central in C, $C' = C_C(t)' \subseteq G' \cap C_C(t)$. As t acts fixed-point-freely on $G' \backslash \{1\}$, we have $C' = \{1\}$. Thus C is abelian.

Now if there exists $g \in C_G(G') \backslash G'$, then $C_G(G') \subseteq C_G(g)$ implies $p^n = |G'g| = |G : C_G(g)| \leq |G : C_G(G')| = p^n - 1$. This contradiction implies $G' = C_G(G')$ with $|G'| = p^n$ and $|G : G'| = p^n - 1$. Now the Schur-Zassenhaus Theorem implies there exists a complement T of G' with $|T| = p^n - 1$ and where T acts fixed-point-freely on G'. Then T is cyclic by Theorem 3.3.3 of [12]. $\square$

CHAPTER 4

A REPRESENTATION OF A DEGREE-TWO CHARACTER

For this chapter, we assume the following hypothesis.

Hypothesis 4.0.1. Let $(A, \mathbf{B})$ be a noncommutative standard integral table algebra, and let χ_0 be the degree map and $\{\chi_i : 0 \leq i \leq k\}$ be the irreducible characters of $(A, \mathbf{B})$. Assume that $\tau = \sum_{i=0}^{k-1} \chi_i + m\chi_k$ with $\chi_k(b_0) = 2$. Let $\mathbf{C} = \bigcap_{i=0}^{k-1} \operatorname{Ker}(\chi_i)$. Suppose $|\mathbf{C}| = 3$, and let $\mathbf{C} = \{b_0, b_1, b_2\}$.

Note that Lemma 2.2.14 implies $\chi_k(b) = 0$ for $b \notin \mathbf{C}$ and $\chi_k(b_i) \in \mathbb{Z}$ for all $b_i \in \mathbf{B}$. Theorem 2.2.1 implies $\sum_{c \in \mathbf{C}} \chi_k(c) = 0$, and so $-\sum_{b_0 \neq c \in \mathbf{C}} \chi_k(c) = 2$. Also, for any $c \in \mathbf{C} \backslash \{b_0\}$, $\tau(c) = 0$ implies $\chi_k(c) = \frac{-k}{m}\chi_0(c)$. Then since m and $\chi_0(c)$ are positive, $\chi_k(c) < 0$. So, $\chi_k(c) = -1$ for $b_0 \neq c \in \mathbf{C}$ and $\chi_0(b_1) = \chi_0(b_2) = \frac{m}{k}$.

Also note that Lemma 2.2.8 and Lemma 2.2.6 imply $\mathbf{B}//\mathbf{C}$ is a group and $\mathbf{C}$ is normal. In addition, $|\mathbf{C}| = 3$ implies $\mathbf{C}$ is commutative.

Proposition 4.0.4. Suppose Hypothesis 4.0.1 holds. Then there exists a representation $X : A \to M_2(\mathbb{C})$ affording χ_k with

$$X(b_0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$X(b_1) = \begin{pmatrix} \theta_1 & 0 \\ 0 & -1-\theta_1 \end{pmatrix},$$

$$X(b_2) = \begin{pmatrix} -1-\theta_1 & 0 \\ 0 & \theta_1 \end{pmatrix}, \text{ and}$$

$$\text{for any } b_i \notin \mathbf{C},\ X(b_i) = \begin{pmatrix} 0 & r_i \\ s_i & 0 \end{pmatrix},$$

for $\theta_1, r_i, s_i \in \mathbb{C}$ with $\theta_1 \neq -\frac{1}{2}$.

Proof. Since $\mathbf{C}$ is commutative of dimension 3, $\mathrm{Irr}(\mathbf{C}) = \{\chi_0{\downarrow_{\mathbf{C}}}, \theta, \zeta\}$, where θ and ζ are linear characters. From ${}_{\mathbf{C}}\tau = \frac{o(\mathbf{C})}{o(\mathbf{B})}\,\tau{\downarrow_{\mathbf{C}}}$ and $\chi_i{\downarrow_{\mathbf{C}}} = \chi_0{\downarrow_{\mathbf{C}}}$ for all $i < k$, it follows that

$$\chi_k{\downarrow_{\mathbf{C}}} = \theta + \zeta.$$

Therefore, $\chi_k(b_1) = -1$ implies

$$\zeta(b_1) = -1 - \theta(b_1).$$

Since $(\langle \mathbf{C} \rangle, \mathbf{C})$ is a SITA, the Orthogonality Relations (Theorem 2.2.15) imply $0 = \langle \zeta, \chi_0 \rangle = 1 + \zeta(b_1) + \zeta(b_2)$. Thus

$$\zeta(b_2) = -1 - \zeta(b_1) = \theta(b_1).$$

Similarly,

$$\theta(b_2) = -1 - \theta(b_1).$$

Let $X : A \to M_2(\mathbb{C})$ afford χ_k. Since $\chi_k(b_0) = 2$, $X(A) \cong M_2(\mathbb{C})$. Now we may assume

$$X(b_0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$X(b_1) = \begin{pmatrix} \theta(b_1) & 0 \\ 0 & \zeta(b_1) \end{pmatrix} = \begin{pmatrix} \theta(b_1) & 0 \\ 0 & -1-\theta(b_1) \end{pmatrix}, \text{ and}$$

$$X(b_2) = \begin{pmatrix} \theta(b_2) & 0 \\ 0 & \zeta(b_2) \end{pmatrix} = \begin{pmatrix} -1-\theta(b_1) & 0 \\ 0 & \theta(b_1) \end{pmatrix}.$$

Note that since b_1 and b_2 are linearly independent, and since the representations for all other characters in $\mathrm{Irr}(A)$ evaluate the same on b_1 and b_2, $X(b_1) \neq X(b_2)$. Thus $\theta(b_1) \neq -1 - \theta(b_1)$ (i.e., $\theta(b_1) \neq -\frac{1}{2}$), and $X(b_1)$ and $X(b_2)$ are nonscalar diagonal matrices. We can now denote $\theta(b_1)$ by θ_1.

Now pick any $b_i \notin \mathbf{C}$. Since $\chi_k(b_i) = 0$ (Lemma 2.2.14), we must have

$$X(b_i) = \begin{pmatrix} q_i & r_i \\ s_i & -q_i \end{pmatrix},$$

for some $q_i, r_i, s_i \in \mathbb{C}$. Since $\mathrm{Supp}(b_1 b_i) \nsubseteq \mathbf{C}$, Lemma 2.2.14 implies $0 = \chi_k(b_1 b_i) = tr(X(b_1)X(b_i))$. Thus

$$0 = tr\left(\begin{pmatrix} \theta_1 & 0 \\ 0 & -1-\theta_1 \end{pmatrix}\begin{pmatrix} q_i & r_i \\ s_i & -q_i \end{pmatrix}\right) = tr\left(\begin{pmatrix} \theta_1 q_i & \theta_1 r_i \\ (-1-\theta_1)s_i & (-1-\theta_1)(-q_i) \end{pmatrix}\right).$$

Thus $0 = q_i\theta_1 + (1+\theta_1)q_i = q_i(2\theta_1 + 1)$. Since $\theta_1 \neq -\frac{1}{2}$, $q_i = 0$. Therefore

$$X(b_i) = \begin{pmatrix} 0 & r_i \\ s_i & 0 \end{pmatrix}.$$

□

Corollary 4.0.5. Suppose Hypothesis 4.0.1 holds. Pick $b_i \notin \mathbf{C}$. Suppose $b_i = b_{i^*}$. Then

$$b_i b_i = \chi_0(b_i)b_0 + \frac{\chi_0(b_i)^2 - \chi_0(b_i)}{2\chi_0(b_1)}(b_1 + b_2) \text{ and}$$

$$\beta_{i1i} = \beta_{i2i} = \beta_{1ii} = \beta_{2ii} = \frac{\chi_0(b_i) - 1}{2}.$$

Proof. Pick $b_i \notin \mathbf{C}$, and suppose $b_i = b_{i^*}$. Let X be a representation affording χ_k with evaluation as in Proposition 4.0.4. Then there exists $\theta_1, r_i, s_i \in \mathbb{C}$ such that $\theta_1 \neq -\frac{1}{2}$ and $X(b_1) = \begin{pmatrix} \theta_1 & 0 \\ 0 & -1-\theta_1 \end{pmatrix}$, $X(b_2) = \begin{pmatrix} -1-\theta_1 & 0 \\ 0 & \theta_1 \end{pmatrix}$, and $X(b_i) = \begin{pmatrix} 0 & r_i \\ s_i & 0 \end{pmatrix}$.

Then

$$X(b_i^2) = X(b_i)X(b_i) = \begin{pmatrix} r_i s_i & 0 \\ 0 & r_i s_i \end{pmatrix}.$$

Since $b_i = b_{i^*}$, $X(b_i^2)$ also equals

$$\chi_0(b_i)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \beta_{ii1}\begin{pmatrix} \theta_1 & 0 \\ 0 & -1-\theta_1 \end{pmatrix} + \beta_{ii2}\begin{pmatrix} -1-\theta_1 & 0 \\ 0 & \theta_1 \end{pmatrix}.$$

Therefore $\beta_{ii1}\theta_1 + \beta_{ii2}(-1-\theta_1) = \beta_{ii1}(-1-\theta_1) + \beta_{ii2}\theta_1$, and so $\beta_{ii1} = \beta_{ii2}$. Applying χ_0 to $b_i b_i = \chi_0(b_i)b_0 + \beta_{ii1}b_1 + \beta_{ii1}b_2$, we get

$$b_i b_i = \chi_0(b_i)b_0 + \frac{\chi_0(b_i)^2 - \chi_0(b_i)}{2\chi_0(b_1)}(b_1 + b_2).$$

Then $(b_i, b_i b_1) = (b_i b_i, b_1) = (b_i, b_1 b_i)$ implies

$$\beta_{i1i} = \beta_{1ii} = \frac{\chi_0(b_1)}{\chi_0(b_i)}\beta_{ii1} = \frac{\chi_0(b_i) - 1}{2}.$$

Similarly,

$$\beta_{i2i} = \beta_{2ii} = \frac{\chi_0(b_i) - 1}{2}.$$

□

Proposition 4.0.6. Suppose Hypothesis 4.0.1 holds. If $b_1 = b_{1^*}$ and $b_2 = b_{2^*}$, then

$$\begin{aligned} b_1 b_2 = b_2 b_1 &= \frac{\chi_0(b_1)}{2} b_1 + \frac{\chi_0(b_1)}{2} b_2, \\ b_1^2 &= \chi_0(b_1) b_0 + \left(\frac{\chi_0(b_1)}{2} - 1 \right) b_1 + \frac{\chi_0(b_1)}{2} b_2, \text{ and} \\ b_2^2 &= \chi_0(b_1) b_0 + \frac{\chi_0(b_1)}{2} b_1 + \left(\frac{\chi_0(b_1)}{2} - 1 \right) b_2. \end{aligned}$$

If $b_1 = b_{2^*}$, then

$$\begin{aligned} b_1^2 &= \frac{\chi_0(b_1) - 1}{2} b_1 + \frac{\chi_0(b_1) + 1}{2} b_2, \\ b_2^2 &= \frac{\chi_0(b_1) + 1}{2} b_1 + \frac{\chi_0(b_1) - 1}{2} b_2, \text{ and} \\ b_1 b_2 &= \chi_0(b_1) b_0 + \frac{\chi_0(b_1) - 1}{2} (b_1 + b_2) = b_2 b_1. \end{aligned}$$

Proof. Let $X : A \to M_2(\mathbb{C})$ be a representation affording χ_k with evaluation as in Proposition 4.0.4. Thus

$$X(b_1) = \begin{pmatrix} \theta_1 & 0 \\ 0 & -1 - \theta_1 \end{pmatrix} \text{ and}$$

$$X(b_2) = \begin{pmatrix} -1 - \theta_1 & 0 \\ 0 & \theta_1 \end{pmatrix},$$

for some constant $\theta_1 \in \mathbb{C}$ with $\theta_1 \neq -1 - \theta_1$. Let $\zeta_1 = -1 - \theta_1$. Note, then, that $\zeta_1 \neq \theta_1$ and $\zeta_1 + \theta_1 = -1$.

Then

$$X(b_1b_1) = X(b_1)X(b_1) = \begin{pmatrix} \theta_1 & 0 \\ 0 & \zeta_1 \end{pmatrix} \begin{pmatrix} \theta_1 & 0 \\ 0 & \zeta_1 \end{pmatrix} = \begin{pmatrix} \theta_1^2 & 0 \\ 0 & \zeta_1^2 \end{pmatrix}.$$

Also, if $b_1 = b_{1^*}$,

$$X(b_1b_1) = \chi_0(b_1) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \beta_{111} \begin{pmatrix} \theta_1 & 0 \\ 0 & \zeta_1 \end{pmatrix} + \beta_{112} \begin{pmatrix} \zeta_1 & 0 \\ 0 & \theta_1 \end{pmatrix}.$$

If $b_1 \neq b_{1^*}$,

$$X(b_1b_1) = \beta_{111} \begin{pmatrix} \theta_1 & 0 \\ 0 & \zeta_1 \end{pmatrix} + \beta_{112} \begin{pmatrix} \zeta_1 & 0 \\ 0 & \theta_1 \end{pmatrix}.$$

Thus

$$\begin{aligned} \theta_1^2 &= \chi_0(b_1) + \beta_{111}\theta_1 + \beta_{112}\zeta_1 \quad &&(\text{or } \theta_1^2 = \beta_{111}\theta_1 + \beta_{112}\zeta_1) \text{ and} \\ \zeta_1^2 &= \chi_0(b_1) + \beta_{111}\zeta_1 + \beta_{112}\theta_1 \quad &&(\text{or } \zeta_1^2 = \beta_{111}\zeta_1 + \beta_{112}\theta_1, \text{ respectively}). \end{aligned}$$

In either case, subtracting the equations yields $\theta_1^2 - \zeta_1^2 = (\beta_{111} - \beta_{112})(\theta_1 - \zeta_1)$. Since $\theta_1 \neq \zeta_1$, $-1 = \theta_1 + \zeta_1 = \beta_{111} - \beta_{112}$. Therefore

$$\beta_{112} = \beta_{111} + 1.$$

Similarly,

$$\beta_{221} = \beta_{222} + 1.$$

Now suppose $b_1 = b_{1^*}$ and $b_2 = b_{2^*}$. Then applying χ_0 to $b_1b_1 = \chi_0(b_1)b_0 + \beta_{111}b_1 + (\beta_{111}+1)b_2$ and to $b_2b_2 = \chi_0(b_2)b_0 + (\beta_{222}+1)b_1 + \beta_{222}b_2$ yields

$$\beta_{111} = \frac{\chi_0(b_1) - 2}{2} = \beta_{222}.$$

Also,

$$\begin{aligned}\chi_0(b_1)\beta_{121} &= (b_1b_2, b_1) = (b_2, b_1b_1) = \chi_0(b_1)\frac{\chi_0(b_1)}{2} \text{ and}\\ \chi_0(b_1)\beta_{122} &= (b_1b_2, b_2) = (b_1, b_2b_2) = \chi_0(b_1)\frac{\chi_0(b_1)}{2}.\end{aligned}$$

Therefore

$$\begin{aligned}b_1b_2 = b_2b_1 &= \frac{\chi_0(b_1)}{2}b_1 + \frac{\chi_0(b_1)}{2}b_2,\\ b_1^2 &= \chi_0(b_1)b_0 + \left(\frac{\chi_0(b_1)}{2} - 1\right)b_1 + \frac{\chi_0(b_1)}{2}b_2, \text{ and}\\ b_2^2 &= \chi_0(b_1)b_0 + \frac{\chi_0(b_1)}{2}b_1 + \left(\frac{\chi_0(b_1)}{2} - 1\right)b_2.\end{aligned}$$

Now suppose $b_1 = b_{2^*}$. Then applying χ_0 to $b_1b_1 = \beta_{111}b_1 + (\beta_{111}+1)b_2$ and to $b_2b_2 = (\beta_{222}+1)b_1 + \beta_{222}b_2$ yields

$$\beta_{111} = \frac{\chi_0(b_1) - 1}{2} = \beta_{222}.$$

Also,

$$\begin{aligned}\chi_0(b_1)\beta_{121} &= (b_1b_2, b_1) = (b_1, b_1b_1) = \chi_0(b_1)\frac{\chi_0(b_1) - 1}{2} \text{ and}\\ \chi_0(b_1)\beta_{122} &= (b_1b_2, b_2) = (b_2, b_2b_2) = \chi_0(b_1)\frac{\chi_0(b_1) - 1}{2}.\end{aligned}$$

Therefore

$$
\begin{aligned}
b_1^2 &= \frac{\chi_0(b_1) - 1}{2} b_1 + \frac{\chi_0(b_1) + 1}{2} b_2, \\
b_2^2 &= \frac{\chi_0(b_1) + 1}{2} b_1 + \frac{\chi_0(b_1) - 1}{2} b_2, \text{ and} \\
b_1 b_2 &= \chi_0(b_1) b_0 + \frac{\chi_0(b_1) - 1}{2} (b_1 + b_2) = b_2 b_1.
\end{aligned}
$$

□

CHAPTER 5

THE NONCOMMUTATIVE CASE

5.1 Hypothesis and Examples

In this section we attempt to expand the results of the commutative section to the noncommutative case. Examples will be constructed that satisfy the following hypothesis, which will be assumed in our main theorem in the next section.

Hypothesis 5.1.1. Let $(A, \mathbf{B})$ be a noncommutative standard integral table algebra, and let χ_0 be the degree map and $\{\chi_i : 0 \leq i \leq k\}$ be the irreducible characters of $(A, \mathbf{B})$. Assume that $\tau = \sum_{i=0}^{k-1} \chi_i + m\chi_k$ with $\chi_k(b_0) = 2$.

Example 5.1.1. Let $\mathbf{D} = \{b_0, b_1, b_{1^*}, b_3, b_4, b_5\}$, a formal basis for a six-dimensional vector space A over $\mathbb{C}$. Define $\chi_0 : A \to \mathbb{C}$ to be a linear transformation such that $\chi_0(b_0) = 1$, and χ_0 has arbitrary positive real values on the other elements of $\mathbf{D}$, but subject to two conditions:

(i) $\chi_0(b_1) = \chi_0(b_{1^*})$

(ii) $2\chi_0(b_1) + 1 = \chi_0(b_3) + \chi_0(b_4) + \chi_0(b_5)$.

Define a bilinear multiplication in A such that $b_0 = 1_A$, and the products of all other pairs of basis elements are as follows:

$$b_1 b_1 = \frac{\chi_0(b_1) - 1}{2} b_1 + \frac{\chi_0(b_1) + 1}{2} b_{1^*},$$
$$b_1 b_{1^*} = b_{1^*} b_1 = \chi_0(b_1) b_0 + \frac{\chi_0(b_1) - 1}{2} (b_1 + b_{1^*}),$$

$$
\begin{aligned}
b_{1^*}b_{1^*} &= \frac{\chi_0(b_1)+1}{2}b_1 + \frac{\chi_0(b_1)-1}{2}b_{1^*},\\
b_1b_3 = b_3b_{1^*} &= \frac{\chi_0(b_3)-1}{2}b_3 + \left(\frac{\chi_0(b_3)}{2} + \frac{1}{2}\sqrt{\frac{\chi_0(b_3)\chi_0(b_5)}{\chi_0(b_4)}}\right)b_4\\
&\quad + \left(\frac{\chi_0(b_3)}{2} - \frac{1}{2}\sqrt{\frac{\chi_0(b_3)\chi_0(b_4)}{\chi_0(b_5)}}\right)b_5,\\
b_1b_4 = b_4b_{1^*} &= \left(\frac{\chi_0(b_4)}{2} - \frac{1}{2}\sqrt{\frac{\chi_0(b_4)\chi_0(b_5)}{\chi_0(b_3)}}\right)b_3 + \frac{\chi_0(b_4)-1}{2}b_4\\
&\quad + \left(\frac{\chi_0(b_4)}{2} + \frac{1}{2}\sqrt{\frac{\chi_0(b_3)\chi_0(b_4)}{\chi_0(b_5)}}\right)b_5,\\
b_1b_5 = b_5b_{1^*} &= \left(\frac{\chi_0(b_5)}{2} + \frac{1}{2}\sqrt{\frac{\chi_0(b_4)\chi_0(b_5)}{\chi_0(b_3)}}\right)b_3\\
&\quad + \left(\frac{\chi_0(b_5)}{2} - \frac{1}{2}\sqrt{\frac{\chi_0(b_3)\chi_0(b_5)}{\chi_0(b_4)}}\right)b_4 + \frac{\chi_0(b_5)-1}{2}b_5,\\
b_{1^*}b_3 = b_3b_1 &= \frac{\chi_0(b_3)-1}{2}b_3 + \left(\frac{\chi_0(b_3)}{2} - \frac{1}{2}\sqrt{\frac{\chi_0(b_3)\chi_0(b_5)}{\chi_0(b_4)}}\right)b_4\\
&\quad + \left(\frac{\chi_0(b_3)}{2} + \frac{1}{2}\sqrt{\frac{\chi_0(b_3)\chi_0(b_4)}{\chi_0(b_5)}}\right)b_5,\\
b_{1^*}b_4 = b_4b_1 &= \left(\frac{\chi_0(b_4)}{2} + \frac{1}{2}\sqrt{\frac{\chi_0(b_4)\chi_0(b_5)}{\chi_0(b_3)}}\right)b_3\\
&\quad + \frac{\chi_0(b_4)-1}{2}b_4 + \left(\frac{\chi_0(b_4)}{2} - \frac{1}{2}\sqrt{\frac{\chi_0(b_3)\chi_0(b_4)}{\chi_0(b_5)}}\right)b_5,\\
b_{1^*}b_5 = b_5b_1 &= \left(\frac{\chi_0(b_5)}{2} - \frac{1}{2}\sqrt{\frac{\chi_0(b_4)\chi_0(b_5)}{\chi_0(b_3)}}\right)b_3\\
&\quad + \left(\frac{\chi_0(b_5)}{2} + \frac{1}{2}\sqrt{\frac{\chi_0(b_3)\chi_0(b_5)}{\chi_0(b_4)}}\right)b_4 + \frac{\chi_0(b_5)-1}{2}b_5,\\
b_3b_3 &= \chi_0(b_3)b_0 + \frac{\chi_0(b_3)(\chi_0(b_3)-1)}{2\chi_0(b_1)}(b_1+b_{1^*}),\\
b_4b_4 &= \chi_0(b_4)b_0 + \frac{\chi_0(b_4)(\chi_0(b_4)-1)}{2\chi_0(b_1)}(b_1+b_{1^*}),
\end{aligned}
$$

$$
\begin{aligned}
b_5b_5 &= \chi_0(b_5)b_0 + \frac{\chi_0(b_5)(\chi_0(b_5)-1)}{2\chi_0(b_1)}(b_1 + b_{1^*}),\\
b_3b_4 &= \left(\frac{\chi_0(b_3)\chi_0(b_4)}{2\chi_0(b_1)} - \frac{1}{2\chi_0(b_1)}\sqrt{\chi_0(b_3)\chi_0(b_4)\chi_0(b_5)}\right) b_1\\
&\quad + \left(\frac{\chi_0(b_3)\chi_0(b_4)}{2\chi_0(b_1)} + \frac{1}{2\chi_0(b_1)}\sqrt{\chi_0(b_3)\chi_0(b_4)\chi_0(b_5)}\right) b_{1^*},\\
b_3b_5 &= \left(\frac{\chi_0(b_3)\chi_0(b_5)}{2\chi_0(b_1)} + \frac{1}{2\chi_0(b_1)}\sqrt{\chi_0(b_3)\chi_0(b_4)\chi_0(b_5)}\right) b_1\\
&\quad + \left(\frac{\chi_0(b_3)\chi_0(b_5)}{2\chi_0(b_1)} - \frac{1}{2\chi_0(b_1)}\sqrt{\chi_0(b_3)\chi_0(b_4)\chi_0(b_5)}\right) b_{1^*},\\
b_4b_3 &= \left(\frac{\chi_0(b_3)\chi_0(b_4)}{2\chi_0(b_1)} + \frac{1}{2\chi_0(b_1)}\sqrt{\chi_0(b_3)\chi_0(b_4)\chi_0(b_5)}\right) b_1\\
&\quad + \left(\frac{\chi_0(b_3)\chi_0(b_4)}{2\chi_0(b_1)} - \frac{1}{2\chi_0(b_1)}\sqrt{\chi_0(b_3)\chi_0(b_4)\chi_0(b_5)}\right) b_{1^*},\\
b_5b_3 &= \left(\frac{\chi_0(b_3)\chi_0(b_5)}{2\chi_0(b_1)} - \frac{1}{2\chi_0(b_1)}\sqrt{\chi_0(b_3)\chi_0(b_4)\chi_0(b_5)}\right) b_1\\
&\quad + \left(\frac{\chi_0(b_3)\chi_0(b_5)}{2\chi_0(b_1)} + \frac{1}{2\chi_0(b_1)}\sqrt{\chi_0(b_3)\chi_0(b_4)\chi_0(b_5)}\right) b_{1^*},\\
b_4b_5 &= \left(\frac{\chi_0(b_4)\chi_0(b_5)}{2\chi_0(b_1)} - \frac{1}{2\chi_0(b_1)}\sqrt{\chi_0(b_3)\chi_0(b_4)\chi_0(b_5)}\right) b_1\\
&\quad + \left(\frac{\chi_0(b_4)\chi_0(b_5)}{2\chi_0(b_1)} + \frac{1}{2\chi_0(b_1)}\sqrt{\chi_0(b_3)\chi_0(b_4)\chi_0(b_5)}\right) b_{1^*},\\
b_5b_4 &= \left(\frac{\chi_0(b_4)\chi_0(b_5)}{2\chi_0(b_1)} + \frac{1}{2\chi_0(b_1)}\sqrt{\chi_0(b_3)\chi_0(b_4)\chi_0(b_5)}\right) b_1\\
&\quad + \left(\frac{\chi_0(b_4)\chi_0(b_5)}{2\chi_0(b_1)} - \frac{1}{2\chi_0(b_1)}\sqrt{\chi_0(b_3)\chi_0(b_4)\chi_0(b_5)}\right) b_{1^*}.
\end{aligned}
$$

A Mathematica computation shows that this multiplication on A is associative. Define * to be the linear transformation on A such that $b_j^* = b_j$ for $j \in \{0,3,4,5\}$; $b_1^* = b_{1^*}$ and $b_{1^*}^* = b_1$. Inspection of the given products reveals that * is an anti-automorphism of A of order 2, and that $(A, \mathbf{D})$ is then a reality-based algebra with degree map χ_0.

Proposition 5.1.1. Let $(A, \mathbf{D}, \chi_0)$ be a reality-based algebra as in Example 5.1.1. Then $(A, \mathbf{D}, \chi_0)$ is a SITA iff the degrees of $\mathbf{D}$ satisfy:

1. $\chi_0(b_3)$, $\chi_0(b_4)$, and $\chi_0(b_5)$ are positive odd integers,
2. each of $\frac{\chi_0(b_3)\chi_0(b_5)}{\chi_0(b_4)}$, $\frac{\chi_0(b_3)\chi_0(b_4)}{\chi_0(b_5)}$, and $\frac{\chi_0(b_4)\chi_0(b_5)}{\chi_0(b_3)}$ is a square integer, and
3. $2\chi_0(b_1) \mid \chi_0(b_j)(\chi_0(b_j)-1)$ for $j \in \{3,4,5\}$.

In this case, $\mathbf{C} := \{b_0, b_1, b_{1^*}\}$ is a normal closed subset with $\mathbf{D}//\mathbf{C} \cong_{\mathrm{x}} \mathbb{Z}_2$, and $\psi : d \to \chi_0(d)d//\mathbf{C}$ for all $d \in \mathbf{D}$ is a table algebra homomorphism.

Proof. Suppose $(A, \mathbf{D}, \chi_0)$ is a SITA. Then $\frac{\chi_0(b_3)}{2} - \frac{1}{2}\sqrt{\frac{\chi_0(b_3)\chi_0(b_4)}{\chi_0(b_5)}} \in \mathbb{Z}$ and $\chi_0(b_3) \in \mathbb{Z}$ imply $\sqrt{\frac{\chi_0(b_3)\chi_0(b_4)}{\chi_0(b_5)}} \in \mathbb{Z}$. Thus $\frac{\chi_0(b_3)\chi_0(b_4)}{\chi_0(b_5)}$ is a square integer. Similarly, $\frac{\chi_0(b_3)\chi_0(b_5)}{\chi_0(b_4)}$ and $\frac{\chi_0(b_4)\chi_0(b_5)}{\chi_0(b_3)}$ are square integers. Pick any $j \in \{3,4,5\}$, then $\frac{\chi_0(b_j)(\chi_0(b_j)-1)}{2\chi_0(b_1)} \in \mathbb{Z}$ and $\frac{\chi_0(b_j)-1}{2} \in \mathbb{Z}$ imply $2\chi_0(b_1) \mid \chi_0(b_j)(\chi_0(b_j)-1)$ and $\chi_0(b_j)$ is odd for $j \in \{3,4,5\}$.

Now suppose conditions 1 through 3 hold. It is obvious that many of the structure constants will be non-negative integers. We only need to show that $\chi_0(b_1)$ is an odd positive integer and that β_{ij1} and β_{ij1^*} are non-negative integers for $i, j \in \{3,4,5\}$ with $i \neq j$. From $2\chi_0(b_1) + 1 = \chi_0(b_3) + \chi_0(b_4) + \chi_0(b_5)$ and the fact that $\chi_0(b_i)$ is odd for $3 \leq i \leq 5$, $2\chi_0(b_1) \in \mathbb{Z}$ is an even positive integer. Thus $\chi_0(b_1)$ is a positive integer. If $\chi_0(b_1)$ were even, then condition 3 implies that $\chi_0(b_j)$ is congruent to $1 \pmod 4$ for $3 \leq j \leq 5$. Then $\chi_0(b_3)+\chi_0(b_4)+\chi_0(b_5)$ is congruent to $3 \pmod 4$, which contradicts $2\chi_0(b_1)+1$ congruent to $1 \pmod 4$. So $\chi_0(b_1)$ is odd.

Let i, j, k be distinct values in $\{3,4,5\}$. Now $\sqrt{\frac{\chi_0(b_i)\chi_0(b_j)}{\chi_0(b_k)}} \in \mathbb{Z} \geq 1$ implies $\sqrt{\chi_0(b_i)\chi_0(b_j)} \geq \sqrt{\chi_0(b_k)}$. Therefore $\frac{\chi_0(b_i)\chi_0(b_j)}{2\chi_0(b_1)} - \frac{1}{2\chi_0(b_1)}\sqrt{\chi_0(b_i)\chi_0(b_j)\chi_0(b_k)} \geq 0$. Also, since $2\chi_0(b_1) = \chi_0(b_i)+\chi_0(b_j)+\chi_0(b_k)-1$, $2\chi_0(b_i)\chi_0(b_1) - (\chi_0(b_i)^2 - \chi_0(b_i)) = \chi_0(b_i)\chi_0(b_j)+\chi_0(b_i)\chi_0(b_k)$. From condition 3, $2\chi_0(b_1) \mid (\chi_0(b_i)\chi_0(b_j)+\chi_0(b_i)\chi_0(b_k))$. Similarly, $2\chi_0(b_1)$ divides $\chi_0(b_j)\chi_0(b_i)+\chi_0(b_j)\chi_0(b_k)$ and $\chi_0(b_k)\chi_0(b_i)+\chi_0(b_k)\chi_0(b_j)$.

Therefore $2\chi_0(b_1) \mid ((\chi_0(b_i)\chi_0(b_j) + \chi_0(b_i)\chi_0(b_k)) + (\chi_0(b_j)\chi_0(b_i) + \chi_0(b_j)\chi_0(b_k)) - (\chi_0(b_k)\chi_0(b_i) + \chi_0(b_k)\chi_0(b_j)))$. Thus, $2\chi_0(b_1) \mid 2\chi_0(b_i)\chi_0(b_j)$. So,

$$\chi_0(b_1) \mid \chi_0(b_i)\chi_0(b_j).$$

From condition 3, $2\chi_0(b_1) \mid (\chi_0(b_i) - 1)\chi_0(b_i)\sqrt{\frac{\chi_0(b_j)\chi_0(b_k)}{\chi_0(b_i)}}$. Thus, $2\chi_0(b_1) \mid (\chi_0(b_i) - 1)\sqrt{\chi_0(b_i)\chi_0(b_j)\chi_0(b_k)}$. A similar argument can be made for j and k. Therefore, $2\chi_0(b_1) \mid (\chi_0(b_i) + \chi_0(b_j) + \chi_0(b_k) - 3)\sqrt{\chi_0(b_i)\chi_0(b_j)\chi_0(b_k)}$, which implies $2\chi_0(b_1) \mid (2\chi_0(b_1) - 2)\sqrt{\chi_0(b_i)\chi_0(b_j)\chi_0(b_k)}$. So

$$\chi_0(b_1) \mid \sqrt{\chi_0(b_i)\chi_0(b_j)\chi_0(b_k)}.$$

Therefore, $\frac{\chi_0(b_i)\chi_0(b_j)}{2\chi_0(b_1)} - \frac{1}{2\chi_0(b_1)}\sqrt{\chi_0(b_i)\chi_0(b_j)\chi_0(b_k)} \in \mathbb{Z}$. Thus all structure constants in $\mathbf{D}$ are nonnegative integers. It is obvious that $\mathbf{C}$ is a normal closed subset such that $\mathbf{D}//\mathbf{C} \cong \mathbb{Z}_2$. Additionally, if we define $\psi : \mathbf{D} \to \mathbf{D}//\mathbf{C}$ by $\psi(d) = \chi_0(d) \cdot d//\mathbf{C}$ for all $d \in \mathbf{D}$, then ψ is a table algebra homomorphism by Corollary 2.2.18. □

Definition 5.1.1. We say that the STAs $(A, \mathbf{D}, \chi_0)$ as in Proposition 5.1.1 are of type $NR(6)$. Note that a STA of type $NR(6)$ is six-dimensional.

Remark 5.1.1. S_3, the symmetric group on three letters, is an example of a basis for a SITA of type $NR(6)$.

Example 5.1.2. Let $\mathbf{C} = \{b_0, b_1, b_2\}$ be a formal basis for a three-dimensional vector space A over $\mathbb{C}$. Let $\chi_0(b_1)$ be any positive real number. Define a bilinear multiplication in A such that $b_0 = 1_A$, and the products of all other pairs of basis elements are as follows:

$$b_1b_2 = b_2b_1 = \frac{\chi_0(b_1)}{2}b_1 + \frac{\chi_0(b_1)}{2}b_2,$$

$$b_1^2 = \chi_0(b_1)b_0 + \left(\frac{\chi_0(b_1)}{2} - 1\right) b_1 + \frac{\chi_0(b_1)}{2} b_2, \text{ and}$$
$$b_2^2 = \chi_0(b_1)b_0 + \frac{\chi_0(b_1)}{2} b_1 + \left(\frac{\chi_0(b_1)}{2} - 1\right) b_2.$$

Then multiplication on A is associative, and $(A, \mathbf{C})$ is a reality-based algebra with * equal to the identity map on A. Note that $(A, \mathbf{C})$ is a SITA iff $\chi_0(b_1)$ is an even positive integer. Symmetry in the structure constants shows that A has an automorphism of order 2 that permutes b_1 and b_2. So by Definition 2.1.15, $\mathbf{D} := \mathbf{C} \rtimes \mathbb{Z}_2$ exists. Also, $\mathbf{C}$ is a normal closed subset of $\mathbf{D}$, $\mathbf{D}//\mathbf{C} \cong_{\mathrm{x}} \mathbb{Z}_2$, and $\psi : \mathbf{D} \to \mathbf{D}//\mathbf{C}$ given by $\psi(d) = \chi_0(d) \cdot d//\mathbf{C}$ for all $d \in \mathbf{D}$ is a table algebra homomorphism.

Definition 5.1.2. We call such $(\langle \mathbf{D} \rangle, \mathbf{D})$ from Example 5.1.2 a SITA of type $R(6)$.

Remark 5.1.2. The notation NR from Definition 5.1.1 (respectively, R from Definition 5.1.2) denotes that the non-identity elements of the closed subset $\mathbf{C}$ are non-real (respectively, real).

Proposition 5.1.2. Suppose that G is an abelian group with subgroup H. Suppose that one of the following hold:

(i) $H \cong \mathbb{Z}_2$ and $(A, \mathbf{D})$ is a SITA of type $NR(6)$ or $R(6)$ with $\mathbf{C} \subseteq \mathbf{D}$ as in Proposition 5.1.1 or Example 5.1.2;

(ii) $H \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, $(A, \mathbf{D})$ is a group algebra over $\mathbf{D}$, the quaternion group or the dihedral group of order 8, and $\mathbf{C} \subseteq \mathbf{D}$ is a normal subgroup of order 2.

Let $\psi : (A, \mathbf{D}) \to (A//\mathbf{C}, \mathbf{D}//\mathbf{C} \cong_{\mathrm{x}} H)$ be the table algebra homomorphism where $\psi(d) = \chi_0(d) \cdot d//\mathbf{C}$ for all $d \in \mathbf{D}$. Then $(A \circ_\psi G, \mathbf{D} \circ_\psi G)$ satisfies Hypothesis 5.1.1.

Proof. Suppose either (i) or (ii) holds. Let $\mathbf{B} = \mathbf{D} \circ_\psi G$. Note that, in either case, $\mathbf{C} = \mathrm{Ker}(\psi)$, $\mathbf{B}//\mathbf{C} \cong_{\mathrm{x}} G$, and $|b\mathbf{C}| = 1$ for $b \notin \mathbf{D}$. Suppose $|\mathbf{B}//\mathbf{C}| = t$. Since

$\mathbf{B}//\mathbf{C}$ is an abelian group, Theorem 2.2.1 implies $\mathbf{C} \subseteq \mathrm{Ker}(\chi)$ for t irreducible linear characters $\chi \in \mathrm{Irr}(A \circ_\psi G)$. If (i) holds, then there are $t-2$ cosets with one element and the remaining elements are in $\mathbf{D}$, which has dimension 6. If (ii) holds, then there are $t-4$ cosets with one element and the remaining elements are in $\mathbf{D}$, which has dimension 8. Thus, in either case, $|\mathbf{B}| = t+4$. Also $|\mathbf{B}| = \sum_{\chi \in \mathrm{Irr}(A \circ_\psi G)} \chi(b_0)^2$. Since there are at least t linear characters and since $\mathbf{B}$ is noncommutative, $(A \circ_\psi G, \mathbf{B})$ must have exactly one nonlinear character, which has degree 2. So by $|\mathbf{B}| = t+4$, there are exactly $t+1$ irreducible characters, t of which are linear. Furthermore, all linear characters in $\mathrm{Irr}(A \circ_\psi G)$ must have $\mathbf{C}$ in their kernels. Relabel the characters so that $\chi_t(b_0) = 2$. Since χ_i is linear for $0 \leq i \leq t-1$, and $\mathbf{B}//\mathbf{C}$ is an abelian group with χ_i in $\mathrm{Irr}(\mathbf{B}//\mathbf{C})$, then $Z(\chi_i) = \mathbf{B}$. Hence, $m_i = 1$ by Lemma 2.2.8. Now

$$t + 4 = |\mathbf{B}| \leq o(\mathbf{B}) = \tau(b_0) = t + 2m_t$$

implies that $m_t \geq 2$. (The last inequality also follows from Proposition 2.2.13.) $\square$

5.2 Main Result

Theorem 5.2.1. Let $(A, \mathbf{B})$ be a noncommutative standard integral table algebra, χ_0 be the degree map, and $\{\chi_i : 0 \leq i \leq k\}$ be the irreducible characters of $(A, \mathbf{B})$. Then $\tau = \sum_{i=0}^{k-1} \chi_i + m\chi_k$ with $\chi_k(b_0) = 2$ if and only if one of the following holds:

(i) $\mathbf{B} \cong_{\mathrm{x}} \mathbf{D} \circ_\psi G$ where G is an abelian group of order k with subgroup H of order 2, $(\mathbb{C}\mathbf{D}, \mathbf{D})$ is a SITA of type $NR(6)$ or $R(6)$, $\psi : \mathbf{D} \to H$ is a table algebra epimorphism, and $o(\mathbf{D}) = 4\frac{m}{k} + 2$.

(ii) $\mathbf{B} \cong_{\mathrm{x}} \mathbf{D} \circ_{\psi} G$ where G is an abelian group of order k with a subgroup $H \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbf{D}$ is either the dihedral or quaternion group of order 8, and $\psi : \mathbf{D} \to H$ is a group epimorphism.

Proof. First note that by Proposition 2.2.13, $\chi_i(b_0) = 1$ for $0 \le i \le k-1$. Let $\mathbf{C} = \bigcap_{i=0}^{k-1} \mathrm{Ker}(\chi_i)$. By Lemma 2.2.8, $\mathrm{Z}(\chi_i) = \mathbf{B}$ for $\chi_i \in \{\chi_0, \cdots, \chi_{k-1}\}$. So, $\mathbf{B} = \bigcap_{i=0}^{k-1} \mathrm{Z}(\chi_s)$. By Lemma 2.2.6, $G := \mathbf{B}//\mathbf{C}$ is a group and $\mathbf{C}$ is normal. If $|\mathbf{C}| = 1$, then $\mathbf{B}$ is a group which is isomorphic to G. Let $T := \{\chi \in \mathrm{Irr}(G) | \chi(b_0) = 1\}$; for groups, $G' = \bigcap_{\chi \in T} \mathrm{Ker}(\chi)$, and so $|\mathbf{C}| = |G'| = 1$. Thus $\mathbf{B}$ is an abelian group, which contradicts our assumptions. So, $|\mathbf{C}| > 1$ and $\mathbf{C} \not\subseteq \mathrm{Ker}(\chi_k)$.

From Theorem 2.2.1, $\chi_j(b_0//\mathbf{C})$ is zero unless $\mathbf{C} \subset \mathrm{Ker}(\chi_j)$. Thus

$$\chi_j(b_0//\mathbf{C}) = \begin{cases} \chi_j(b_0) = 1 & \text{if } j \in \{0, \cdots, k-1\} \\ 0 & \text{if } j = k. \end{cases}$$

Therefore $\sum_{c \in \mathbf{C}} \chi_k(c) = 0$, which implies $-\sum_{b_0 \neq c \in \mathbf{C}} \chi_k(c) = 2$. Also, $|\mathrm{Irr}(\mathbf{B}//\mathbf{C})| = k$ implies $\mathbf{B}//\mathbf{C}$ is an abelian group of order k.

Using τ on $b_0 \neq c \in \mathbf{C}$, $\chi_k(c) = \frac{-\chi_0(c)k}{m}$. Since $\chi_0(c)$ and m are both positive, $\chi_k(c) < 0$. Also, Lemma 2.2.14 implies $\chi_k(c) \in \mathbb{Z}$. Then since $-\sum_{b_0 \neq c \in \mathbf{C}} \chi_k(c) = 2$, there are at most 2 non-identity elements in $\mathbf{C}$, i.e. $|\mathbf{C}| - 1 \le 2$. Therefore, either $|\mathbf{C}| = 2$ or $|\mathbf{C}| = 3$. Thus $\mathbf{C}$ is commutative.

Suppose $|\mathbf{C}| = 3$, and label $\mathbf{C} = \{b_0, b_1, b_2\}$. Now $\chi_k(b_1) = \chi_k(b_2) = -1$. Applying τ to b_1 and b_2 yields $\chi_0(b_1) = \chi_0(b_2) = \frac{m}{k}$.

Since $|\mathbf{B}| = k + 4$ and $|\mathbf{B}//\mathbf{C}| = k$, there are at most two cosets (not equal to $\mathbf{C}$) with more than one element. We refer to such cosets as the *nontrivial cosets*.

Suppose there is one nontrivial coset with more than one element, and label it $b_3\mathbf{C}$. Then $\mathbf{C} \cup \{b_i\mathbf{C} : |b_i\mathbf{C}| = 1\}$ makes up $3 + (k-2) = k+1$ elements. This implies that the nontrivial coset consists of 3 elements. Label those elements $\{b_3, b_4, b_5\}$. Since $(b_i//\mathbf{C})^{-1} = b_{i^*}//\mathbf{C}$ and $|b_{i^*}\mathbf{C}| = |b_i\mathbf{C}|$, $b_3//\mathbf{C}$ must be an element with group order 2 in $\mathbf{B}//\mathbf{C}$. Thus $\mathbf{D} := b_3\mathbf{C} \cup \mathbf{C}$ forms a closed subset of $\mathbf{B}$ of dimension 6. Note that Remark 2.2.4 implies that $\mathbf{B}\backslash\mathbf{D}$ is contained in the center of the algebra. So, since $(A, \mathbf{B})$ is noncommutative, $(<\mathbf{D}>, \mathbf{D})$ must be noncommutative.

Now either $b_1 = b_{2^*}$ or b_1 and b_2 are both real. Suppose $b_1 = b_{2^*}$. By Proposition 4.0.6,

$$\begin{aligned}
b_1^2 &= \frac{\chi_0(b_1) - 1}{2} b_1 + \frac{\chi_0(b_1) + 1}{2} b_2, \\
b_2^2 &= \frac{\chi_0(b_1) + 1}{2} b_1 + \frac{\chi_0(b_1) - 1}{2} b_2, \text{ and} \\
b_1 b_2 &= \chi_0(b_1) b_0 + \frac{\chi_0(b_1) - 1}{2}(b_1 + b_2) = b_2 b_1.
\end{aligned}$$

Also, either $b_3\mathbf{C}$ contains only one real element or $b_3\mathbf{C}$ contains all real elements. Suppose $b_3\mathbf{C}$ contains only one real element. Without loss of generality, assume $b_3 = b_{5^*}$. Therefore, $\mathbf{D} = \{b_0, b_1, b_{1^*}, b_3, b_{3^*}, b_4\}$. Applying * and χ_0 to $b_3 b_{3^*}$ and $b_{3^*} b_3$ yields $\beta_{33^*1} = \beta_{33^*2} = \beta_{3^*31} = \beta_{3^*32}$. So, $b_3 b_{3^*} = b_{3^*} b_3$. From Corollary 2.2.11, $b_3(b_3\mathbf{C})^+ = (b_3\mathbf{C})^+ b_3$. Written another way, $b_3(b_3 + b_{3^*} + b_4) = (b_3 + b_{3^*} + b_4)b_3$. Since b_3 commutes with b_{3^*}, $b_3 b_4 = b_4 b_3$. Thus $b_3\mathbf{C}$ commutes with itself. Now since $b_3//\mathbf{C} = b_{3^*}//\mathbf{C}$ and $b_0//\mathbf{C} \cdot b_3//\mathbf{C} = b_3//\mathbf{C}$, Lemma 2.2.16 implies $\mathbf{C}$ commutes with $b_3\mathbf{C}$. Therefore $\mathbf{D}$ commutes, a contradiction. Thus $b_3\mathbf{C}$ must consist of all real elements.

Now let $\mathbf{D} = \{b_0, b_1, b_{1^*}, b_3, b_4, b_5\}$, where b_3, b_4, and b_5 are real. Pick any $b_i \in \mathbf{D}\backslash\mathbf{C}$. From Corollary 4.0.5,

$$b_i b_i = \chi_0(b_i) b_0 + \frac{\chi_0(b_i)^2 - \chi_0(b_i)}{2\chi_0(b_1)}(b_1 + b_2) \text{ and}$$

$$\beta_{i1i} = \beta_{i2i} = \beta_{1ii} = \beta_{2ii} = \frac{\chi_0(b_i) - 1}{2}.$$

Therefore, we have $b_1 b_3 = \frac{\chi_0(b_3)-1}{2} b_3 + \beta_{134} b_4 + \beta_{135} b_5$. Applying χ_0 yields $\chi_0(b_1)\chi_0(b_3) = \frac{\chi_0(b_3)-1}{2}\chi_0(b_3) + \beta_{134}\chi_0(b_4) + \beta_{135}\chi_0(b_5)$. Thus

$$\beta_{134}\chi_0(b_4) + \beta_{135}\chi_0(b_5) = \frac{\chi_0(b_3)}{2}(2\chi_0(b_1) - \chi_0(b_3) + 1).$$

Let $F = \frac{\chi_0(b_3)}{2}(2\chi_0(b_1) - \chi_0(b_3) + 1)$. From Proposition 2.2.12, $2\chi_0(b_1) + 1 = o(\mathbf{C}) = o(\mathbf{D}) - o(\mathbf{C}) = \chi_0(b_3) + \chi_0(b_4) + \chi_0(b_5)$. Then $F = \frac{\chi_0(b_3)}{2}(\chi_0(b_4) + \chi_0(b_5))$ and

$$(*) \quad \beta_{134}\chi_0(b_4) + \beta_{135}\chi_0(b_5) = F.$$

Additionally,

$$(b_1 b_3, b_1 b_3)$$

$$= \left(\frac{\chi_0(b_3) - 1}{2} b_3 + \beta_{134}\chi_0(b_4) + \beta_{135}\chi_0(b_5), \frac{\chi_0(b_3) - 1}{2} b_3 + \beta_{134}\chi_0(b_4) + \beta_{135}\chi_0(b_5)\right)$$

$$= \left(\frac{\chi_0(b_3) - 1}{2}\right)^2 \chi_0(b_3) + \beta_{134}^2\chi_0(b_4) + \beta_{135}^2\chi_0(b_5).$$

Also,

$$\begin{aligned}
&(b_1 b_3, b_1 b_3) = (b_{1^*} b_1, b_3^2) \\
&= \left(\chi_0(b_1) b_0 + \frac{\chi_0(b_1) - 1}{2}(b_1 + b_{1^*}), \chi_0(b_3) b_0 + \frac{\chi_0(b_3)(\chi_0(b_3) - 1)}{2\chi_0(b_1)}(b_1 + b_{1^*})\right) \\
&= \chi_0(b_1)\chi_0(b_3) + \frac{(\chi_0(b_1) - 1)}{2}\frac{(\chi_0(b_3) - 1)\chi_0(b_3)}{2\chi_0(b_1)} 2\chi_0(b_1) \\
&= \chi_0(b_1)\chi_0(b_3) + \frac{\chi_0(b_3)}{2}(\chi_0(b_1) - 1)(\chi_0(b_3) - 1).
\end{aligned}$$

Thus

$$\beta_{134}^2\chi_0(b_4)+\beta_{135}^2\chi_0(b_5)$$

$$\begin{aligned}
&=\chi_0(b_1)\chi_0(b_3)+\frac{\chi_0(b_3)}{2}(\chi_0(b_1)-1)(\chi_0(b_3)-1)-\frac{\chi_0(b_3)}{2}\frac{(\chi_0(b_3)-1)^2}{2}\\
&=\frac{\chi_0(b_3)}{4}\left[4\chi_0(b_1)+2(\chi_0(b_1)-1)(\chi_0(b_3)-1)-(\chi_0(b_3)-1)^2\right]\\
&=\frac{\chi_0(b_3)}{4}\left[4\chi_0(b_1)+(\chi_0(b_3)-1)(2\chi_0(b_1)-\chi_0(b_3)-1)\right]\\
&=\frac{\chi_0(b_3)}{4}\left[4\chi_0(b_1)+(\chi_0(b_3)-1)\left(\frac{2}{\chi_0(b_3)}F-2\right)\right]\\
&=\frac{\chi_0(b_3)}{4}\left[4\chi_0(b_1)-2\chi_0(b_3)+2+(\chi_0(b_3)-1)\frac{2}{\chi_0(b_3)}F\right]\\
&=\frac{\chi_0(b_3)}{4}\left[\frac{4}{\chi_0(b_3)}F+(\chi_0(b_3)-1)\frac{2}{\chi_0(b_3)}F\right]\\
&=F+\frac{\chi_0(b_3)-1}{2}F=\frac{F(\chi_0(b_3)+1)}{2}.
\end{aligned}$$

Thus we have the system of equations:

$$\begin{cases}\beta_{134}^2\chi_0(b_4)+\beta_{135}^2\chi_0(b_5)=\frac{F(\chi_0(b_3)+1)}{2}\\[2ex] \beta_{134}\chi_0(b_4)+\beta_{135}\chi_0(b_5)=F.\end{cases}$$

Therefore, $\beta_{135}=\frac{F-\beta_{134}\chi_0(b_4)}{\chi_0(b_5)}$. Substituting this into the first equation,

$$\beta_{134}^2\chi_0(b_4)+\frac{(F-\beta_{134}\chi_0(b_4))^2}{\chi_0(b_5)}=\frac{F(\chi_0(b_3)+1)}{2}$$

$$\implies (\beta_{134})^2\chi_0(b_4)(\chi_0(b_4)+\chi_0(b_5))-2F\chi_0(b_4)\beta_{134}+F^2-\frac{F(\chi_0(b_3)+1)\chi_0(b_5)}{2}=0.$$

Since $F=\frac{\chi_0(b_3)}{2}(\chi_0(b_4)+\chi_0(b_5))$, we have

$$(\beta_{134})^2\frac{2\chi_0(b_4)F}{\chi_0(b_3)}-2F\chi_0(b_4)\beta_{134}+F^2-\frac{F(\chi_0(b_3)+1)\chi_0(b_5)}{2}=0.$$

Dividing by $\frac{F}{\chi_0(b_3)}$ yields

$$(\beta_{134})^2 2\chi_0(b_4) - 2\chi_0(b_3)\chi_0(b_4)\beta_{134} + \frac{\chi_0(b_3)^2}{2}\chi_0(b_4) - \frac{\chi_0(b_3)}{2}\chi_0(b_5) = 0.$$

So,

$$\beta_{134} = \frac{2\chi_0(b_4)\chi_0(b_3) \pm \sqrt{4\chi_0(b_4)^2\chi_0(b_3)^2 - 8\chi_0(b_4)\left(\frac{\chi_0(b_3)^2}{2}\chi_0(b_4) - \frac{\chi_0(b_3)}{2}\chi_0(b_5)\right)}}{4\chi_0(b_4)}$$

$$= \frac{\chi_0(b_3)}{2} \pm \frac{1}{2}\sqrt{\frac{\chi_0(b_3)\chi_0(b_5)}{\chi_0(b_4)}}.$$

Then

$$\begin{aligned}\beta_{135} &= \frac{F - \beta_{134}\chi_0(b_4)}{\chi_0(b_5)} \\ &= \frac{\frac{\chi_0(b_3)}{2}(\chi_0(b_4) + \chi_0(b_5)) - \frac{\chi_0(b_3)}{2}\left(\chi_0(b_4) \pm \sqrt{\frac{\chi_0(b_4)\chi_0(b_5)}{\chi_0(b_3)}}\right)}{\chi_0(b_5)} \\ &= \left(\frac{\chi_0(b_3)\chi_0(b_5)}{2} \mp \frac{\chi_0(b_3)}{2}\sqrt{\frac{\chi_0(b_4)\chi_0(b_5)}{\chi_0(b_3)}}\right)/\chi_0(b_5) \\ &= \frac{\chi_0(b_3)}{2} \mp \frac{1}{2}\sqrt{\frac{\chi_0(b_3)\chi_0(b_4)}{\chi_0(b_5)}}.\end{aligned}$$

By choice of indices, we may assume

$$\beta_{134} = \frac{\chi_0(b_3)}{2} + \frac{1}{2}\sqrt{\frac{\chi_0(b_3)\chi_0(b_5)}{\chi_0(b_4)}} \text{ and}$$

$$\beta_{135} = \frac{\chi_0(b_3)}{2} - \frac{1}{2}\sqrt{\frac{\chi_0(b_3)\chi_0(b_4)}{\chi_0(b_5)}}.$$

Now from $\mathbf{C}^+ b_3 = \chi_0(b_3)(\mathbf{C}b_3)^+$, $(b_0 + b_1 + b_{1^*})b_3 = \chi_0(b_3)(b_3 + b_4 + b_5)$. Thus $\beta_{134} + \beta_{1^*34} = \chi_0(b_3)$ and $\beta_{135} + \beta_{1^*35} = \chi_0(b_3)$. Therefore

$$\beta_{1^*34} = \chi_0(b_3) - \beta_{134} = \frac{\chi_0(b_3)}{2} - \frac{1}{2}\sqrt{\frac{\chi_0(b_3)\chi_0(b_5)}{\chi_0(b_4)}} \text{ and}$$

$$\beta_{1^*35} = \chi_0(b_3) - \beta_{135} = \frac{\chi_0(b_3)}{2} + \frac{1}{2}\sqrt{\frac{\chi_0(b_3)\chi_0(b_4)}{\chi_0(b_5)}}.$$

Now $\beta_{143}\chi_0(b_3) = (b_1 b_4, b_3) = (b_4, b_{1^*} b_3) = \beta_{1^*34}\chi_0(b_4)$, and so

$$\beta_{143} = \frac{\chi_0(b_4)}{\chi_0(b_3)}\beta_{1^*34} = \frac{\chi_0(b_4)}{2} - \frac{1}{2}\sqrt{\frac{\chi_0(b_4)\chi_0(b_5)}{\chi_0(b_3)}}.$$

Then

$$b_1 b_4 = \frac{\chi_0(b_4) - 1}{2} b_4 + \beta_{143} b_3 + \beta_{145} b_5.$$

Thus

$$\chi_0(b_1)\chi_0(b_4) = \frac{\chi_0(b_4) - 1}{2}\chi_0(b_4) + \beta_{143}\chi_0(b_3) + \beta_{145}\chi_0(b_5) \implies$$

$$\beta_{145}\chi_0(b_5) = \chi_0(b_1)\chi_0(b_4) - \frac{(\chi_0(b_4) - 1)\chi_0(b_4)}{2} - \beta_{143}\chi_0(b_3)$$

$$= \frac{(\chi_0(b_3) + \chi_0(b_4) + \chi_0(b_5) - 1)}{2}\chi_0(b_4) - \frac{\chi_0(b_4)^2}{2} + \frac{\chi_0(b_4)}{2} - \beta_{143}\chi_0(b_3)$$

$$= \frac{(\chi_0(b_3) + \chi_0(b_5))}{2}\chi_0(b_4) - \beta_{143}\chi_0(b_3).$$

Therefore,

$$\beta_{145} = \frac{\chi_0(b_3)\chi_0(b_4)}{2\chi_0(b_5)} + \frac{\chi_0(b_4)}{2} - \left(\frac{\chi_0(b_4)\chi_0(b_3)}{2\chi_0(b_5)} - \frac{\chi_0(b_3)}{2\chi_0(b_5)}\sqrt{\frac{\chi_0(b_4)\chi_0(b_5)}{\chi_0(b_3)}}\right)$$
$$= \frac{\chi_0(b_4)}{2} + \frac{1}{2}\sqrt{\frac{\chi_0(b_3)\chi_0(b_4)}{\chi_0(b_5)}}.$$

Similarly,

$$\beta_{153} = \frac{\chi_0(b_5)}{2} + \frac{1}{2}\sqrt{\frac{\chi_0(b_4)\chi_0(b_5)}{\chi_0(b_3)}} \text{ and}$$
$$\beta_{154} = \frac{\chi_0(b_5)}{2} - \frac{1}{2}\sqrt{\frac{\chi_0(b_3)\chi_0(b_5)}{\chi_0(b_4)}}.$$

Following a similar process, one can conclude

$$\beta_{1^*43} = \frac{\chi_0(b_4)}{2} + \frac{1}{2}\sqrt{\frac{\chi_0(b_4)\chi_0(b_5)}{\chi_0(b_3)}},$$
$$\beta_{1^*45} = \frac{\chi_0(b_4)}{2} - \frac{1}{2}\sqrt{\frac{\chi_0(b_3)\chi_0(b_4)}{\chi_0(b_5)}},$$
$$\beta_{1^*53} = \frac{\chi_0(b_5)}{2} - \frac{1}{2}\sqrt{\frac{\chi_0(b_4)\chi_0(b_5)}{\chi_0(b_3)}}, \text{ and}$$
$$\beta_{1^*54} = \frac{\chi_0(b_5)}{2} + \frac{1}{2}\sqrt{\frac{\chi_0(b_3)\chi_0(b_5)}{\chi_0(b_4)}}.$$

Now pick any $b_i \neq b_j \in \mathbf{D} \setminus \mathbf{C}$. Then $\beta_{ij1}\chi_0(b_1) = (b_ib_j, b_1) = (b_i, b_1b_j) = \beta_{1ji}\chi_0(b_i)$. Thus

$$\beta_{ij1} = \frac{\beta_{1ji}\chi_0(b_i)}{\chi_0(b_1)}.$$

Therefore,

$$\beta_{ij1} = \frac{\chi_0(b_i)\chi_0(b_j)}{2\chi_0(b_1)} + \frac{1}{2\chi_0(b_1)}\sqrt{\chi_0(b_3)\chi_0(b_4)\chi_0(b_5)} \text{ or}$$

$$\beta_{ij1} = \frac{\chi_0(b_i)\chi_0(b_j)}{2\chi_0(b_1)} - \frac{1}{2\chi_0(b_1)}\sqrt{\chi_0(b_3)\chi_0(b_4)\chi_0(b_5)},$$

depending on β_{1ji}. Also, since the $\mathrm{Supp}(b_{1^*}b_i) \subseteq \mathbf{D}\backslash\mathbf{C}$ and $\mathrm{Supp}(b_1b_j) \subseteq \mathbf{D}\backslash\mathbf{C}$, which consists of real elements, applying * shows $b_{1^*}b_i = b_ib_1$ and $b_1b_j = b_jb_{1^*}$. Thus $\beta_{i1j} = \beta_{1^*ij}$ and $\beta_{j1^*i} = \beta_{1ji}$. Therefore the structure constants in $\mathbf{D}$ are equal to those in Example 5.1.1. Thus $(\mathbb{C}\mathbf{D}, \mathbf{D})$ is a SITA of type $NR(6)$. Let $G = \mathbf{B}//\mathbf{C}$ and $H = \mathbf{D}//\mathbf{C}$. Then G is an abelian group with $|G| = k$, and H is a subgroup of G with $|H| = 2$. If $\psi : \mathbf{D} \to G$ is defined by $\psi(d) = \chi_0(d)d//\mathbf{C}$, then ψ maps $\mathbf{D}$ onto H and Corollary 2.2.18 implies $\mathbf{B} \cong_{\mathrm{x}} \mathbf{D} \circ_\psi G$.

Now suppose $b_1 = b_{1^*}$ and $b_2 = b_{2^*}$. If $\mathbf{D}\backslash\mathbf{C}$ consists of real elements, then $\mathbf{D}$ is commutative. But since $\mathbf{B}\backslash\mathbf{D}$ is in the center of the algebra, by Remark 2.2.4, this is a contradiction. Thus $\mathbf{D}\backslash\mathbf{C}$ must have the form $\{b_3, b_4, b_{4^*}\}$.

Since $b_3 = b_{3^*}$, Corollary 4.0.5 implies

$$b_3b_3 = \chi_0(b_3)b_0 + \frac{\chi_0(b_3)^2 - \chi_0(b_3)}{2\chi_0(b_1)}(b_1 + b_2) \text{ and}$$

$$\beta_{313} = \beta_{323} = \beta_{133} = \beta_{233} = \frac{\chi_0(b_3) - 1}{2}.$$

Also, applying χ_0 to b_1b_3 yields $\chi_0(b_1)\chi_0(b_3) = \beta_{133}\chi_0(b_3) + (\beta_{134} + \beta_{134^*})\chi_0(b_4)$. Since Proposition 2.2.12 implies $\chi_0(b_3) + 2\chi_0(b_4) = 2\chi_0(b_1) + 1$, we have

$$\begin{aligned}\beta_{134} + \beta_{134^*} &= (\chi_0(b_1) - \beta_{133})\frac{\chi_0(b_3)}{\chi_0(b_4)} \\ &= \frac{\chi_0(b_3)}{\chi_0(b_4)}\left(\chi_0(b_1) - \frac{\chi_0(b_3) - 1}{2}\right) = \frac{\chi_0(b_3)}{\chi_0(b_4)}\frac{(2\chi_0(b_1) + 1 - \chi_0(b_3))}{2}\end{aligned}$$

$$= \frac{\chi_0(b_3)}{\chi_0(b_4)} \frac{(2\chi_0(b_4) + \chi_0(b_3) - \chi_0(b_3))}{2} = \chi_0(b_3).$$

Thus

$$\beta_{134} + \beta_{134^*} = \chi_0(b_3).$$

Now

$$\begin{aligned}
&(b_1 b_3, b_1 b_3)\\
&= (\beta_{133} b_3 + \beta_{134} b_4 + \beta_{134^*} b_{4^*}, \beta_{133} b_3 + \beta_{134} b_4 + \beta_{134^*} b_{4^*})\\
&= \beta_{133}^2 \chi_0(b_3) + (\beta_{134}^2 + \beta_{134^*}^2)\chi_0(b_4).
\end{aligned}$$

Also, $(b_1 b_1, b_3 b_3)$

$$\begin{aligned}
&= (\chi_0(b_1) b_0 + \beta_{111} b_1 + (\chi_0(b_1) - 1 - \beta_{111}) b_2, \chi_0(b_3) b_0 + \beta_{331} b_1 + \beta_{332} b_2)\\
&= \chi_0(b_1)\chi_0(b_3) + \beta_{111}\beta_{331}\chi_0(b_1) + (\chi_0(b_1) - 1 - \beta_{111})\beta_{332}\chi_0(b_1)\\
&= \chi_0(b_1)\chi_0(b_3) + (\chi_0(b_1) - 1)\beta_{332}\chi_0(b_1)\\
&= \chi_0(b_1)\chi_0(b_3) + (\chi_0(b_1) - 1)\frac{(\chi_0(b_3)^2 - \chi_0(b_3))}{2}.
\end{aligned}$$

Thus

$$\begin{aligned}
\beta_{134}^2 + \beta_{134^*}^2 &= \frac{\chi_0(b_3)}{\chi_0(b_4)} \left(-\beta_{133}^2 + \chi_0(b_1) + (\chi_0(b_1) - 1)\frac{(\chi_0(b_3) - 1)}{2} \right)\\
&= \frac{\chi_0(b_3)}{\chi_0(b_4)} \left(-\left(\frac{\chi_0(b_3) - 1}{2} \right)^2 + \chi_0(b_1) + (\chi_0(b_1) - 1)\frac{(\chi_0(b_3) - 1)}{2} \right)\\
&= \frac{\chi_0(b_3)}{\chi_0(b_4)} \left(\left(\frac{\chi_0(b_3) - 1}{2} \right) \left(\frac{2\chi_0(b_1) - 2 - (\chi_0(b_3) - 1)}{2} \right) + \chi_0(b_1) \right)\\
&= \frac{\chi_0(b_3)}{\chi_0(b_4)} \left(\left(\frac{\chi_0(b_3) - 1}{2} \right) \left(\frac{\chi_0(b_3) + 2\chi_0(b_4) - \chi_0(b_3) - 2}{2} \right) + \chi_0(b_1) \right)
\end{aligned}$$

$$= \frac{\chi_0(b_3)}{\chi_0(b_4)}\left(\left(\frac{\chi_0(b_3)-1}{2}\right)(\chi_0(b_4)-1)+\frac{\chi_0(b_3)+2\chi_0(b_4)-1}{2}\right)$$
$$= \frac{\chi_0(b_3)}{\chi_0(b_4)}\frac{(\chi_0(b_3)+1)\chi_0(b_4)}{2} = \frac{\chi_0(b_3)(\chi_0(b_3)+1)}{2}.$$

Then we have the system of equations:

$$\begin{cases} \beta_{134}^2+\beta_{134^*}^2 = \frac{\chi_0(b_3)(\chi_0(b_3)+1)}{2} \\ \beta_{134}+\beta_{134^*} = \chi_0(b_3). \end{cases}$$

Substituting $\beta_{134^*} = \chi_0(b_3) - \beta_{134}$ into the first equation,

$$\beta_{134}^2 + (\chi_0(b_3)-\beta_{134})^2 = \frac{\chi_0(b_3)(\chi_0(b_3)+1)}{2}, \text{ so}$$

$$2\beta_{134}^2 - 2\chi_0(b_3)\beta_{134} + \frac{\chi_0(b_3)^2}{2} - \frac{\chi_0(b_3)}{2} = 0.$$

Thus β_{134} equals

$$\frac{2\chi_0(b_3) \pm \sqrt{4\chi_0(b_3)^2 - 4(\chi_0(b_3)^2-\chi_0(b_3))}}{4}$$

$$= \frac{\chi_0(b_3) \pm \sqrt{\chi_0(b_3)}}{2}.$$

Thus $\beta_{134} = \frac{\chi_0(b_3)+\sqrt{\chi_0(b_3)}}{2}$ or $\beta_{134} = \frac{\chi_0(b_3)-\sqrt{\chi_0(b_3)}}{2}$. Without loss of generality,

$$\beta_{134} = \frac{\chi_0(b_3)+\sqrt{\chi_0(b_3)}}{2}.$$

Thus $\beta_{134^*} = \frac{\chi_0(b_3)-\sqrt{\chi_0(b_3)}}{2}$.

Now $\beta_{134}\chi_0(b_4) = (b_1b_3, b_4) = (b_1, b_4b_3) = \chi_0(b_1)\beta_{431}$. Thus

$$\beta_{431} = \frac{\chi_0(b_4)}{\chi_0(b_1)}\beta_{134} = \frac{\chi_0(b_4)}{\chi_0(b_1)}\left(\frac{\chi_0(b_3)+\sqrt{\chi_0(b_3)}}{2}\right).$$

Also $\mathbf{C}$ real implies $b_3b_{4^*} = b_4b_3$, so

$$\beta_{143}\chi_0(b_3) = (b_1b_4, b_3) = (b_1, b_3b_{4^*}) = \chi_0(b_1)\beta_{34^*1} = \chi_0(b_1)\beta_{431}.$$

Thus

$$\beta_{143} = \frac{\chi_0(b_1)}{\chi_0(b_3)}\beta_{431} = \frac{\chi_0(b_4)}{\chi_0(b_3)}\left(\frac{\chi_0(b_3)+\sqrt{\chi_0(b_3)}}{2}\right) = \frac{\chi_0(b_4)}{2}\left(1+\frac{1}{\sqrt{\chi_0(b_3)}}\right).$$

So,

$$\beta_{143} = \frac{\chi_0(b_4)}{2}\left(1+\frac{1}{\sqrt{\chi_0(b_3)}}\right) \in \mathbb{Z}_{\geq 0}.$$

This implies $\chi_0(b_4) + \frac{\chi_0(b_4)}{\sqrt{\chi_0(b_3)}} \in \mathbb{Z}_{\geq 0}$. Thus $\frac{\chi_0(b_4)}{\sqrt{\chi_0(b_3)}} \in \mathbb{Z}_{\geq 0}$. Therefore, for any prime p such that $p \mid \chi_0(b_3)$, we have $p \mid \chi_0(b_4)$. Since $2\chi_0(b_4) + \chi_0(b_3) = 2\chi_0(b_1) + 1$, $p \nmid \chi_0(b_1)$. Thus the greatest common divisor of $\chi_0(b_1)$ and $\chi_0(b_3)$ is 1.

Now since $\beta_{133} = \frac{\chi_0(b_3)-1}{2} \in \mathbb{Z}$, $\chi_0(b_3)$ is odd. Also, $\beta_{331} = \frac{\chi_0(b_3)(\chi_0(b_3)-1)}{2\chi_0(b_1)} \in \mathbb{Z}$, but $(\chi_0(b_3), 2\chi_0(b_1)) = 1$. Therefore $2\chi_0(b_1) \mid (\chi_0(b_3)-1)$. But $2\chi_0(b_1) = 2\chi_0(b_4) - 1 + \chi_0(b_3) > \chi_0(b_3)$. Thus $\chi_0(b_3) = 1$. Then from $2\chi_0(b_4)+1 = 2\chi_0(b_1)+1$, $\chi_0(b_4) = \chi_0(b_1)$. Now b_3 is linear and $\beta_{134} = \frac{\chi_0(b_3)+\sqrt{\chi_0(b_3)}}{2} = 1$, so $b_1b_3 = b_4$. Applying * yields $b_3b_1 = b_{4^*}$. Multiplying by b_3 shows $b_1 = b_4b_3 = b_3b_{4^*}$. Since

$\mathbf{C}b_3 = \{b_3, b_4, b_{4^*}\}$ and b_0b_3, b_1b_3, b_2b_3 are distinct, $b_2b_3 = b_{4^*}$. Applying * yields $b_3b_2 = b_4$. Also, multiplying by b_3 shows $b_2 = b_3b_4 = b_{4^*}b_3$. Thus we have

$$b_3b_3 = b_0,$$

$$b_1b_3 = b_4 = b_3b_2,$$

$$b_2b_3 = b_{4^*} = b_3b_1.$$

By Proposition 4.0.6,

$$b_1b_2 = b_2b_1 = \frac{\chi_0(b_1)}{2}b_1 + \frac{\chi_0(b_1)}{2}b_2,$$
$$b_1^2 = \chi_0(b_1)b_0 + \left(\frac{\chi_0(b_1)}{2} - 1\right) b_1 + \frac{\chi_0(b_1)}{2}b_2, \text{ and}$$
$$b_2^2 = \chi_0(b_1)b_0 + \frac{\chi_0(b_1)}{2}b_1 + \left(\frac{\chi_0(b_1)}{2} - 1\right) b_2.$$

So, there is an automorphism of $\mathbf{C}$ which permutes b_1 and b_2. Consider $z \in Aut(\mathbf{C})$ such that $b_1^z = b_2$, $b_2^z = b_1$, and $b_0^z = b_0$. Then z has order 2. Now define $\theta : \mathbf{D} \to \mathbf{C} \rtimes \mathbb{Z}_2$ by $\theta(b_ib_3) = (b_i, z)$ and $\theta(b_i) = (b_i, 1)$ for $b_i \in \mathbf{C}$. Then θ is an isomorphism. Thus $(\mathbb{C}\mathbf{D}, \mathbf{D})$ is a SITA of type $R(6)$. Let $G = \mathbf{B}//\mathbf{C}$ and $H = \mathbf{D}//\mathbf{C}$. Then G is an abelian group with $|G| = k$ and H is a subgroup of G with $|H| = 2$. Given $\psi : \mathbf{D} \to G$ by $\psi(d) = \chi_0(d)d//\mathbf{C}$, Corollary 2.2.18 implies $\mathbf{B} \cong_{\text{x}} \mathbf{D} \circ_\psi G$.

Recall that $\mathbf{C} = \{b_0, b_1, b_2\}$ and that there can be at most two nontrivial cosets. Now suppose there are exactly two nontrivial cosets of $\mathbf{C}$. Then the $k - 3$ cosets which contain 1 element and $\mathbf{C}$ make up k elements of $\mathbf{B}$. Thus the two nontrivial cosets must contain the remaining 4 elements. Therefore each of the cosets has 2 elements. Label the two nontrivial cosets $b\mathbf{C}$ and $d\mathbf{C}$. One of the following must occur: (a) $b\mathbf{C} = b^*\mathbf{C}$ and $d\mathbf{C} = d^*\mathbf{C}$ OR (b) $b\mathbf{C} = d^*\mathbf{C}$. Recall, from Remark 2.2.4, that cosets which contain just one element are in the center of the algebra. Thus noncommutativity must result from multiplication within $b\mathbf{C} \cup d\mathbf{C} \cup \mathbf{C}$.

Suppose (a). Now since $b\mathbf{C} = b^*\mathbf{C}$, $b//\mathbf{C}$ has group order 2 in $\mathbf{B}//\mathbf{C}$. Thus $b\mathbf{C}\cup\mathbf{C}$ forms a closed subset of dimension 5, and therefore must be commutative. Similarly $d\mathbf{C} \cup \mathbf{C}$ is commutative. Thus noncommutativity must come from multiplication between elements in $b\mathbf{C}$ with elements in $d\mathbf{C}$. However, $b//\mathbf{C}$ and $d//\mathbf{C}$ are their own inverses in $\mathbf{B}//\mathbf{C}$, and thus $b//\mathbf{C} \cdot d//\mathbf{C} = b_i//\mathbf{C}$, where $|b_i\mathbf{C}| = 1$. By Lemma 2.2.16, $b\mathbf{C}$ commutes with $d\mathbf{C}$. Therefore $(A, \mathbf{B})$ is commutative, which contradicts our assumptions.

Now suppose (b). Let $b\mathbf{C} = \{b_3, b_4\}$. Then $d\mathbf{C} = \{b_{3^*}, b_{4^*}\}$, so $b\mathbf{C} = b_3\mathbf{C}$ and $d\mathbf{C} = b_{3^*}\mathbf{C}$. Suppose, also, that $b_1 = b_{2^*}$ so that $\mathbf{C} = \{b_0, b_1, b_{1^*}\}$. Applying * to $b_3b_{3^*}$ yields $\beta_{33^*1} = \beta_{33^*1^*}$. Then applying χ_0 results in $b_3b_{3^*} = \chi_0(b_3)b_0 + \frac{\chi_0(b_3)^2-\chi_0(b_3)}{2\chi_0(b_1)}(b_1 + b_{1^*})$. Similarly, $b_{3^*}b_3 = \chi_0(b_3)b_0 + \frac{\chi_0(b_3)^2-\chi_0(b_3)}{2\chi_0(b_1)}(b_1 + b_{1^*})$. Thus $b_3b_{3^*} = b_{3^*}b_3$. By symmetry, b_4 also commutes with b_{4^*}. From Corollary 2.2.11, $b_3(b_{3^*}\mathbf{C})^+ = (b_{3^*}\mathbf{C})^+b_3$ and also $b_3(b_3\mathbf{C})^+ = (b_3\mathbf{C})^+b_3$. So, $b_3(b_{3^*}+b_{4^*}) = (b_{3^*}+b_{4^*})b_3$ and $b_3(b_3 + b_4) = (b_3 + b_4)b_3$. The commutativity of b_3 with b_{3^*} and also of b_3 with itself implies b_3 also commutes with b_{4^*} and b_4. Therefore $b_3\mathbf{C}$ commutes with $b_{3^*}\mathbf{C}$ and also with itself. By Corollary 2.2.5, $\mathbf{C}$ commutes with $b_3\mathbf{C}$ and also with $b_{3^*}\mathbf{C}$. Therefore $(A, \mathbf{B})$ is commutative, a contradiction.

Thus we may assume that $b\mathbf{C} = b_3\mathbf{C} = \{b_3, b_4\}$ and $d\mathbf{C} = b_{3^*}\mathbf{C} = \{b_{3^*}, b_{4^*}\}$ and that $\mathbf{C} = \{b_0, b_1, b_2\}$ where $b_1 = b_{1^*}$ and $b_2 = b_{2^*}$.

By Proposition 4.0.6,

$$b_1b_2 = b_2b_1 = \frac{\chi_0(b_1)}{2}b_1 + \frac{\chi_0(b_1)}{2}b_2,$$
$$b_1^2 = \chi_0(b_1)b_0 + \left(\frac{\chi_0(b_1)}{2} - 1\right)b_1 + \frac{\chi_0(b_1)}{2}b_2, \text{ and}$$

$$b_2^2 = \chi_0(b_1)b_0 + \frac{\chi_0(b_1)}{2}b_1 + \left(\frac{\chi_0(b_1)}{2} - 1\right) b_2.$$

Thus the structure constants from multiplication in $\mathbf{C}$ depend on $\chi_0(b_1) = \frac{m}{k}$.

Applying χ_0 to $b_3b_{3^*} = \chi_0(b_3)b_0 + \beta_{33^*1}b_1 + \beta_{33^*2}b_2$ produces the following:

$$\beta_{33^*2} = \frac{\chi_0(b_3)^2 - \chi_0(b_3)}{\chi_0(b_1)} - \beta_{33^*1}.$$

Also,

$$(*) \quad \beta_{33^*1}\chi_0(b_1) = (b_3b_{3^*}, b_1) = (b_3, b_1b_3) = \beta_{133}\chi_0(b_3).$$

Therefore, we have

$$\begin{aligned}
&(b_1b_3, b_1b_3) = \beta_{133}^2\chi_0(b_3) + \beta_{134}^2\chi_0(b_4) = (b_1^2, b_3b_{3^*})\\
&= \left(\chi_0(b_1)b_0 + \left(\frac{\chi_0(b_1)}{2} - 1\right) b_1 + \frac{\chi_0(b_1)}{2}b_2\,,\ \chi_0(b_3)b_0 + \beta_{33^*1}b_1 + \beta_{33^*2}b_2\right)\\
&= \chi_0(b_1)\chi_0(b_3) + \left(\frac{\chi_0(b_1)}{2} - 1\right)\beta_{33^*1}\chi_0(b_1) + \frac{\chi_0(b_1)}{2}\beta_{33^*2}\chi_0(b_1)\\
&= \chi_0(b_1)\chi_0(b_3) + \left(\frac{\chi_0(b_1)}{2} - 1\right)\beta_{33^*1}\chi_0(b_1) + \frac{\chi_0(b_1)^2}{2}\left(\frac{\chi_0(b_3)^2 - \chi_0(b_3)}{\chi_0(b_1)} - \beta_{33^*1}\right)\\
&= \chi_0(b_1)\chi_0(b_3) + \left(\frac{\chi_0(b_1)}{2} - 1 - \frac{\chi_0(b_1)}{2}\right)\beta_{33^*1}\chi_0(b_1) + \frac{\chi_0(b_1)}{2}(\chi_0(b_3)^2 - \chi_0(b_3))\\
&= \frac{\chi_0(b_1)\chi_0(b_3)}{2} + \frac{\chi_0(b_1)\chi_0(b_3)^2}{2} - \beta_{133}\chi_0(b_3) \text{ (by } (*)).
\end{aligned}$$

So for $X = \beta_{133}$ and $Y = \beta_{134}$, we have

$$X^2\chi_0(b_3) + Y^2\chi_0(b_4) + X\chi_0(b_3) = \frac{\chi_0(b_1)\chi_0(b_3)}{2}(1 + \chi_0(b_3)).$$

We can also apply χ_0 to b_1b_3 to get

$$X\chi_0(b_3) + Y\chi_0(b_4) = \chi_0(b_1)\chi_0(b_3).$$

Then, using the above and $o(\mathbf{C}) = 2\chi_0(b_1) + 1 = \chi_0(b_3) + \chi_0(b_4)$,

$$\frac{(\chi_0(b_1)\chi_0(b_3) - Y\chi_0(b_4))^2}{\chi_0(b_3)} + Y^2\chi_0(b_4) + (\chi_0(b_1)\chi_0(b_3) - Y\chi_0(b_4))$$

$$= \frac{\chi_0(b_1)\chi_0(b_3)}{2}(1 + \chi_0(b_3))$$

$$\implies Y^2\left(\frac{\chi_0(b_4)}{\chi_0(b_3)}(\chi_0(b_4) + \chi_0(b_3))\right) - 2Y\chi_0(b_1)\chi_0(b_4) - Y\chi_0(b_4) + \chi_0(b_1)^2\chi_0(b_3)$$

$$+\chi_0(b_1)\chi_0(b_3) - \frac{\chi_0(b_1)\chi_0(b_3)}{2}(1 + \chi_0(b_3)) = 0$$

$$\implies Y^2\frac{\chi_0(b_4)}{\chi_0(b_3)}o(\mathbf{C}) - Y\chi_0(b_4)o(\mathbf{C}) + \chi_0(b_1)\chi_0(b_3)\left(\frac{2\chi_0(b_1) + 1 - \chi_0(b_3)}{2}\right) = 0$$

$$\implies Y^2\frac{\chi_0(b_4)}{\chi_0(b_3)}o(\mathbf{C}) - Y\chi_0(b_4)o(\mathbf{C}) + \chi_0(b_1)\chi_0(b_3)\frac{\chi_0(b_4)}{2} = 0$$

$$\implies Y^2\frac{o(\mathbf{C})}{\chi_0(b_3)} - Yo(\mathbf{C}) + \frac{\chi_0(b_1)\chi_0(b_3)}{2} = 0.$$

Thus the discriminant is

$$o(\mathbf{C})^2 - 4\frac{o(\mathbf{C})}{\chi_0(b_3)}\frac{\chi_0(b_1)\chi_0(b_3)}{2}$$

$$= o(\mathbf{C})^2 - 2o(\mathbf{C})\chi_0(b_1)$$

$$= o(\mathbf{C})^2 - 2o(\mathbf{C})\left(\frac{o(\mathbf{C}) - 1}{2}\right) = o(\mathbf{C}).$$

Therefore

$$\beta_{134} = Y = \frac{o(\mathbf{C}) \pm \sqrt{o(\mathbf{C})}}{\frac{2o(\mathbf{C})}{\chi_0(b_3)}}$$

$$= \frac{\chi_0(b_3)}{2}\left(1 \pm \frac{1}{\sqrt{o(\mathbf{C})}}\right).$$

Then

$$\beta_{133} = X = \frac{\chi_0(b_1)\chi_0(b_3) - \beta_{134}\chi_0(b_4)}{\chi_0(b_3)}$$

$$= \chi_0(b_1) - \frac{\chi_0(b_4)}{2}\left(1 \pm \frac{1}{\sqrt{o(\mathbf{C})}}\right).$$

Thus $\beta_{33^*1} = \beta_{133}\frac{\chi_0(b_3)}{\chi_0(b_1)} = \chi_0(b_3) - \frac{\chi_0(b_4)\chi_0(b_3)}{o(\mathbf{C})-1}\left(1 \pm \frac{1}{\sqrt{o(\mathbf{C})}}\right)$. Therefore, using $\chi_0(b_3) - 1 = 2\chi_0(b_1) - \chi_0(b_4)$ and $2\chi_0(b_1) + 1 = o(\mathbf{C})$, we have

$$\beta_{33^*2} = \frac{\chi_0(b_3)^2 - \chi_0(b_3)}{\chi_0(b_1)} - \beta_{33^*1}$$

$$= \frac{\chi_0(b_3)^2 - \chi_0(b_3)}{\chi_0(b_1)} - \left(\chi_0(b_3) - \frac{\chi_0(b_4)\chi_0(b_3)}{o(\mathbf{C})-1}\left(1 \pm \frac{1}{\sqrt{o(\mathbf{C})}}\right)\right)$$

$$= \frac{\chi_0(b_3)}{2\chi_0(b_1)}\left[2(2\chi_0(b_1) - \chi_0(b_4)) - \left(2\chi_0(b_1) - \chi_0(b_4) \mp \frac{\chi_0(b_4)}{\sqrt{o(\mathbf{C})}}\right)\right]$$

$$= \frac{\chi_0(b_3)}{2\chi_0(b_1)}\left[2\chi_0(b_1) - \chi_0(b_4) \pm \frac{\chi_0(b_4)}{\sqrt{o(\mathbf{C})}}\right]$$

$$= \chi_0(b_3) - \frac{\chi_0(b_3)\chi_0(b_4)}{o(\mathbf{C})-1}\left(1 \mp \frac{1}{\sqrt{o(\mathbf{C})}}\right).$$

Thus we may assume, without loss of generality, that

$$\beta_{33^*1} = \chi_0(b_3) - \frac{\chi_0(b_4)\chi_0(b_3)}{\sqrt{o(\mathbf{C})}-1}\frac{1}{\sqrt{o(\mathbf{C})}} \text{ and}$$

$$\beta_{33^*2} = \chi_0(b_3) - \frac{\chi_0(b_4)\chi_0(b_3)}{\sqrt{o(\mathbf{C})}+1}\frac{1}{\sqrt{o(\mathbf{C})}}$$

$$= \chi_0(b_3) - \frac{\chi_0(b_4)\chi_0(b_3)}{o(\mathbf{C}) + \sqrt{o(\mathbf{C})}}.$$

Thus $\sqrt{o(\mathbf{C})} \in \mathbb{Q}$, which implies $\sqrt{o(\mathbf{C})} \in \mathbb{Z}$. Also, $\sqrt{o(\mathbf{C})} - 1$ and $\sqrt{o(\mathbf{C})} + 1$ divide $\chi_0(b_3)\chi_0(b_4)$, so $\frac{o(\mathbf{C})-1}{2} \mid \chi_0(b_3)\chi_0(b_4)$.

Now $\beta_{133} = \chi_0(b_1) - \frac{\chi_0(b_4)}{2}\left(1 + \frac{1}{\sqrt{o(\mathbf{C})}}\right)$ implies $\sqrt{o(\mathbf{C})} \mid \chi_0(b_4)$. Then $\sqrt{o(\mathbf{C})}$ divides $o(\mathbf{C}) - \chi_0(b_4) = \chi_0(b_3)$, and so $o(\mathbf{C}) \mid \chi_0(b_3)\chi_0(b_4)$. Thus $o(\mathbf{C})\frac{o(\mathbf{C})-1}{2}$ divides $\chi_0(b_3)\chi_0(b_4)$. However, since $o(\mathbf{C}) = 2\chi_0(b_1) + 1 = \chi_0(b_3) + \chi_0(b_4)$, one of $\chi_0(b_3)$ or $\chi_0(b_4)$ must be less than or equal to $\chi_0(b_1) = \frac{o(\mathbf{C})-1}{2}$. Furthermore, both of $\chi_0(b_3)$ and $\chi_0(b_4)$ must be smaller than $o(\mathbf{C})$. Thus $\chi_0(b_3)\chi_0(b_4) < o(\mathbf{C})\frac{o(\mathbf{C})-1}{2}$, a contradiction. Therefore, if $|\mathbf{C}| = 3$, there must be exactly one nontrivial coset of $\mathbf{C}$ in $\mathbf{B}$.

Now suppose $|\mathbf{C}| = 2$ and let $\mathbf{C} = \{b_0, b_1\}$. Applying τ to b_1 and noting that $\chi_k(b_1) = -\chi_k(b_0) = -2$ yields $\chi_0(b_1) = \frac{2m}{k}$.

Pick any $b_i \in \mathbf{B}$. Applying χ_0 to $b_i b_{i^*}$ and $b_{i^*} b_i$ yields:

$$b_i b_{i^*} = \chi_0(b_i) b_0 + \frac{(\chi_0(b_i))^2 - \chi_0(b_i)}{\chi_0(b_1)} b_1 = b_{i^*} b_i. \tag{5.2.1}$$

Recall that Lemma 2.2.16 implies

$(*)$ $|b_i\mathbf{C}| = 1$ implies b_i is in the center of $(A, \mathbf{B})$.

Pick any $b \in b_i\mathbf{C}$ and $d \in b_{i^*}\mathbf{C}$. If $b = d^*$, then equation (5.2.1) implies $bd = db$. If $b \neq d^*$, then $\text{Supp}(bd) \subseteq \{b_1\}$. By Lemma 2.2.16, $bd = db$. So,

$(**)$ $b_i\mathbf{C}$ commutes with $b_{i^*}\mathbf{C}$ for any $b_i \in \mathbf{B}$.

Therefore, Remark 2.2.5 implies

$$(***)\ \mathbf{C} \text{ is in the center of } (A, \mathbf{B}).$$

Since $|\mathbf{C}| = 2$, the $k-1$ non-identity cosets contain a total of $k+2$ elements. Thus there are at most three nontrivial cosets which contain more than one element. Since $|b_i\mathbf{C}| = |b_{i^*}\mathbf{C}|$, $(*), (**)$, and $(***)$ imply that there are either two or three such nontrivial cosets; otherwise, A is commutative, which is a contradiction.

Suppose there are exactly two nontrivial cosets of $\mathbf{C}$, say $b_2\mathbf{C}$ and $b_3\mathbf{C}$. Then $|\mathbf{B}| = k+4$ implies these cosets must contain a total of five elements. So, suppose $|b_2\mathbf{C}| = 2$ and $|b_3\mathbf{C}| = 3$. From $|b_i\mathbf{C}| = |b_{i^*}\mathbf{C}|$, $b_2\mathbf{C} = b_{2^*}\mathbf{C}$ and $b_3\mathbf{C} = b_{3^*}\mathbf{C}$. Thus $(b_2//\mathbf{C})^2 = (b_3//\mathbf{C})^2 = b_0//\mathbf{C}$. Therefore $\mathbf{C} \cup b_2\mathbf{C}$ forms a commutative closed subset of $\mathbf{B}$ of dimension 4, and $\mathbf{C} \cup b_3\mathbf{C}$ forms a commutative closed subset of $\mathbf{B}$ of dimension 5. Then $(*), (**)$, and $(***)$ imply non-commutativity of $(A, \mathbf{B})$ must result from elements in $b_2\mathbf{C}$ being multiplied to elements in $b_3\mathbf{C}$. However, $b_2//\mathbf{C} \cdot b_3//\mathbf{C} = b_i//\mathbf{C}$ for some $b_i \in \mathbf{B}$ with $|b_i\mathbf{C}| = 1$. Thus $b_2\mathbf{C}$ and $b_3\mathbf{C}$ commute. This is a contradiction.

Therefore, there must be exactly three nontrivial cosets, say $b_2\mathbf{C}$, $b_3\mathbf{C}$, and $b_4\mathbf{C}$. Furthermore, each of these three cosets must have two elements in them. At least one of these cosets satisfies $(b_i//\mathbf{C})^2 = b_0//\mathbf{C}$. Suppose $b_4//\mathbf{C} = b_{4^*}//\mathbf{C}$.

Now suppose $b_2//\mathbf{C} \neq b_{2^*}//\mathbf{C}$, and so $b_3//\mathbf{C} = b_{2^*}//\mathbf{C}$. By $(**)$, $b_2\mathbf{C}$ and $b_3\mathbf{C}$ commute. Now for any coset $b_i\mathbf{C}$, $b_i(b_i\mathbf{C})^+ = (b_i\mathbf{C})^+ b_i$ where $|b_i\mathbf{C}| \in \{1, 2\}$. Thus $b_i\mathbf{C}$ commutes with itself. Therefore we can assume without loss of generality that $b_2b_4 \neq b_4b_2$. Then we must have $b_4//\mathbf{C} \cdot b_2//\mathbf{C} = b_3//\mathbf{C}$ since they cannot multiply to yield a quotient element corresponding to a coset of size 1. However, since $b_{2^*}\mathbf{C} = b_3\mathbf{C}$, $b_{2^*}\mathbf{C}$ commutes with $b_3\mathbf{C}$. Lemma 2.2.16 implies $b_4\mathbf{C}$ commutes with $b_2\mathbf{C}$. So, $\mathbf{B}$ is commutative, a contradiction.

Thus we must have $b_i//\mathbf{C} = b_{i^*}//\mathbf{C}$ for all $i \in \{2,3,4\}$. If $b_2\mathbf{C} \cup b_3\mathbf{C} \cup b_4\mathbf{C}$ consists of real elements, then Lemma 2.2.16 implies that $(A, \mathbf{B})$ is commutative. So, suppose $b_i\mathbf{C} = \{b_i, b_{i^*}\}$ for some $i \in \{2,3,4\}$. From equation (5.2.1), $\frac{\chi_0(b_i)(\chi_0(b_i)-1)}{\chi_0(b_1)} \in \mathbb{Z}_{\geq 0}$. By Proposition 2.2.12, $2\chi_0(b_i) = o(\mathbf{C}) = 1 + \chi_0(b_1)$. So

$$\frac{\frac{o(\mathbf{C})}{2}\left(\frac{o(\mathbf{C})-2}{2}\right)}{o(\mathbf{C})-1} = \frac{o(\mathbf{C})(o(\mathbf{C})-2)}{4(o(\mathbf{C})-1)} \in \mathbb{Z}_{\geq 0}. \tag{5.2.2}$$

We now have $o(\mathbf{C})$ is greater than or equal to 2 and $o(\mathbf{C})-1$ divides $o(\mathbf{C})(o(\mathbf{C})-2)$. However, $o(\mathbf{C})-1$ is relatively prime to both $o(\mathbf{C})$ and $o(\mathbf{C})-2$ unless $o(\mathbf{C}) = 2$. Thus $o(\mathbf{C}) = 2$.

Now $o(\mathbf{C}) = 2$ implies $1 = \chi_0(b_1) = \frac{2m}{k}$. Thus $m = \frac{k}{2}$. So, let $\mathbf{D} := \mathbf{C} \cup b_2\mathbf{C} \cup b_3\mathbf{C} \cup b_4\mathbf{C}$. Then $\chi_0(b_1) = 1$ and Proposition 2.2.12 implies $\mathbf{D}$ is a subset of $L(\mathbf{B})$. Also, for any $b_i \notin \mathbf{D}$, $\chi_0(b_i) = o(\mathbf{C}) = 2$, and so $b_i \notin L(\mathbf{B})$. Thus $\mathbf{D} = L(\mathbf{B})$ forms a nonabelian group of order 8. Note that if $\mathbf{B}$ is itself a group, then $\mathbf{B} = L(\mathbf{B}) = \mathbf{D}$. Otherwise, define $\psi : (<\mathbf{D}>, \mathbf{D}) \rightarrow (<\mathbf{B}//\mathbf{C}>, \mathbf{B}//\mathbf{C})$ by $\psi(d) = \chi_0(d) \cdot d//\mathbf{C}$. Then $\psi(\mathbf{D}) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and $\mathbf{B} \cong_{\mathrm{x}} \mathbf{D} \circ_\psi \mathbf{B}//\mathbf{C}$ by Corollary 2.2.18, where $\mathbf{D}$ is isomorphic to the dihedral group of order 8 or the quaternions.

Thus, one direction of the theorem has been proved. Suppose, now, that one of the following holds:

(i) $\mathbf{B} \cong_{\mathrm{x}} \mathbf{D} \circ_\psi G$ where G is an abelian group of order k with subgroup H of order 2, $(\mathbb{C}\mathbf{D}, \mathbf{D})$ is a SITA of type $NR(6)$ or $R(6)$, $\psi : \mathbf{D} \rightarrow H$ is a table algebra epimorphism, and $o(\mathbf{D}) = 4\frac{m}{k} + 2$.

(ii) $\mathbf{B} \cong_{\mathrm{x}} \mathbf{D} \circ_{\psi} G$ where G is an abelian group of order k with a subgroup $H \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbf{D}$ is either the dihedral or quaternion group of order 8, and $\psi : \mathbf{D} \to H$ is a group epimorphism.

In either case, let $\mathbf{C} = \mathrm{Ker}(\psi)$. Then Proposition 5.1.2 implies $\tau = \sum_{i=0}^{k-1} \chi_i + m\chi_k$ with $\chi_k(b_0) = 2$. □

Remark 5.2.1. As a special case of (i) above, $\mathbf{B}$ might be isomorphic to the symmetric group on three letters, S_3. In this case, $G = H$ (so $|G| = |H| = 2$ and $\psi(\mathbf{D}) = G$) and $\mathbf{B} \cong_{\mathrm{x}} \mathbf{D} \cong_{\mathrm{x}} S_3$, where S_3 is a SITA of type $NR(6)$. Additionally, as a special case of (ii) above, $\mathbf{B}$ might be isomorphic to the quaternions or the dihedral group of order 8. In this case, $G = H$ and $\psi(\mathbf{D}) = G$.

CHAPTER 6

PARAMETERS OF $NR(6)$

In this chapter we investigate the restrictions required to conclude that a SITA is of type $NR(6)$. We first rewrite the restrictions based on equivalent conditions and then use those conditions to determine infinite families of parameters which determine a SITA of type $NR(6)$. While we have found several infinite families, we provide examples demonstrating that there are parameters outside of these families which also satisfy the conditions for a SITA of type $NR(6)$. Additionally, we identify an infinite family of association schemes which are SITAs of type $NR(6)$.

6.1 Infinite Families of SITA of Type $NR(6)$

Proposition 6.1.1. Suppose $\chi_0(b_3)$, $\chi_0(b_4)$, and $\chi_0(b_5)$ are positive real numbers. Then each of $\frac{\chi_0(b_3)\chi_0(b_5)}{\chi_0(b_4)}$, $\frac{\chi_0(b_3)\chi_0(b_4)}{\chi_0(b_5)}$, and $\frac{\chi_0(b_4)\chi_0(b_5)}{\chi_0(b_3)}$ is a square integer if and only if there exist positive integers a, b, and c with $\chi_0(b_3) = ab$, $\chi_0(b_4) = bc$, and $\chi_0(b_5) = ac$.

Proof. Suppose $\chi_0(b_3) = ab$, $\chi_0(b_4) = bc$, and $\chi_0(b_5) = ac$ for positive integers $a, b,$ and c. Then

$$\frac{\chi_0(b_3)\chi_0(b_5)}{\chi_0(b_4)} = \frac{abca}{bc} = a^2,$$
$$\frac{\chi_0(b_3)\chi_0(b_4)}{\chi_0(b_5)} = \frac{abbc}{ac} = b^2, \text{ and}$$
$$\frac{\chi_0(b_4)\chi_0(b_5)}{\chi_0(b_3)} = \frac{bcac}{ab} = c^2.$$

Conversely, if $\frac{\chi_0(b_3)\chi_0(b_5)}{\chi_0(b_4)} = a^2$, $\frac{\chi_0(b_3)\chi_0(b_4)}{\chi_0(b_5)} = b^2$, and $\frac{\chi_0(b_4)\chi_0(b_5)}{\chi_0(b_3)} = c^2$, where $a, b,$ and c are integers, then

$$\chi_0(b_3) = a^2\frac{\chi_0(b_4)}{\chi_0(b_5)} = \frac{a^2b^2}{\chi_0(b_3)}.$$

Thus $\chi_0(b_3)^2 = a^2b^2$, and therefore $\chi_0(b_3) = |a||b|$. Similarly, $\chi_0(b_4) = |b||c|$ and $\chi_0(b_5) = |a||c|$. □

Remark 6.1.1. The three conditions of Proposition 5.1.1 are satisfied if and only if there exist positive odd integers a, b, c with $\chi_0(b_3) = ab$, $\chi_0(b_4) = bc$, $\chi_0(b_5) = ca$,

$$(ab + bc + ac - 1) \mid ab(ab - 1),$$
$$(ab + bc + ac - 1) \mid ac(ac - 1), \text{ and}$$
$$(ab + bc + ac - 1) \mid bc(bc - 1).$$

Proposition 6.1.2. Suppose $a, b,$ and c satisfy the conditions of Remark 6.1.1. Then there exists $r_a, r_b,$ and $r_c \in \mathbb{Z}_{>0}$ such that $r_a \mid a$, $r_b \mid b$, $r_c \mid c$, and $ab + bc + ac - 1 = 2r_ar_br_c$. Additionally, $(r_a, bc) = (r_b, ac) = (r_c, ab) = 1$. Conversely, suppose there exist $r_a, r_b,$ and $r_c \in \mathbb{Z}_{>0}$ which are pairwise coprime and that there are odd $a, b,$ and c which satisfy $r_a \mid a$, $r_b \mid b$, $r_c \mid c$, and $ab + bc + ac - 1 = 2r_ar_br_c$. Then $a, b,$ and c satisfy the conditions of Remark 6.1.1.

Proof. First suppose $a, b,$ and c satisfy the conditions of Remark 6.1.1. Suppose that there exists a prime p such that $p^{n+1} \mid ab + bc + ac - 1$ and that $p^n \mid a$ but $p^{n+1} \nmid a$. Then $p^{n+1} \mid ab(ab-1)$ implies $p \mid b$. From $p \mid ab+bc+ac-1$, this implies $p \mid 1$, a contradiction. Therefore any prime factor of a, b, or c which also divides $ab + bc + ac - 1$ cannot show up with greater multiplicity in $ab + bc + ac - 1$.

Now suppose there exists a prime p such that $p \mid ab + bc + ac - 1$ and $(a, p) = (b, p) = (c, p) = 1$. From Remark 6.1.1, $p \mid ab - 1$. Thus $p \mid c(a + b)$. Then $(p, c) = 1$ implies $p \mid a+b$. Similarly, $p \mid a+c$ and $p \mid b+c$. Thus p divides $(a+b)-(a+c) = b-c$, and so p divides $(b - c) + (b + c) = 2b$. Since $(b, p) = 1$, $p = 2$.

Now we will show that $4 \nmid ab + bc + ac - 1$. For any odd integer x, $x^2 \equiv 1 \pmod 4$. Suppose $a \equiv b \equiv c \pmod 4$. Then $ab + bc + ac - 1 \equiv 2 \pmod 4$, and so $4 \nmid ab + bc + ac - 1$. Suppose, without loss of generality, that $a \equiv b \pmod 4$ but $c \not\equiv b \pmod 4$. Then

$$\begin{aligned} ab + bc + ac - 1 &\equiv c(a + b) \pmod 4 \\ &\equiv 2bc \pmod 4 \\ &\equiv 2 \cdot 3 \pmod 4 \\ &\equiv 2 \pmod 4. \end{aligned}$$

Thus $4 \nmid ab + bc + ac - 1$. Therefore $ab + bc + ac - 1 = 2r_a r_b r_c$ for factors $r_a \mid a$, $r_b \mid b$, and $r_c \mid c$.

$(r_a, bc) = (r_b, ac) = (r_c, ab) = 1$ follows as an obvious consequence of $ab + bc + ac - 1 = 2r_a r_b r_c$.

Conversely, suppose there exists $r_a, r_b,$ and $r_c \in \mathbb{Z}_{>0}$ which are odd and pairwise coprime, and suppose that there are odd $a, b,$ and c which satisfy $r_a \mid a$, $r_b \mid b$, $r_c \mid c$, and $ab+bc+ac-1 = 2r_a r_b r_c$. Then since r_c divides $ab+bc+ac-1$ and c and since

$ab-1$ is even, $2r_c$ divides $ab-1$. Thus $ab+bc+ac-1 = 2r_ar_br_c$ divides $ab(ab-1)$. Similarly $ab+bc+ac-1$ divides $ac(ac-1)$ and $bc(bc-1)$. Therefore, if we define $\chi_0(b_3) = ab$, $\chi_0(b_4) = bc$, and $\chi_0(b_5) = ca$, then a, b, and c satisfy the conditions in Remark 6.1.1. □

Proposition 6.1.3. Suppose a, b, and c satisfy the conditions of Remark 6.1.1. Suppose $ab+bc+ac-1 = 2r_ar_br_c$ for $r_a \mid a$, $r_b \mid b$, and $r_c \mid c$. Then $r_a = 1$ iff $a = min\{a,b,c\}$, $r_b = b$, and $r_c = c$. In this case, $ab+bc+ac-1 = 2bc$.

Proof. Suppose $r_a = 1$ and $r_b < b$. Since b is odd, $b = s_br_b$ where $s_b > 2$. Thus $bc > 2r_bc \geq 2r_br_c$. Then $ab+bc+ac-1 > 2r_br_c$, a contradiction. Therefore, $r_b = b$. Similarly $r_c = c$. Now, without loss of generality, suppose $a > b$. Then $2bc = ab+bc+ac-1 > (ab-1)+2bc$, a contradiction. Thus $a = min\{a,b,c\}$. Conversely, suppose that $a = min\{a,b,c\}$, $r_b = b$, and $r_c = c$. Suppose $r_a > 1$. Then $r_a \geq 3$ since r_a is odd, and so $2bc \geq a(b+c) = 2r_abc - bc + 1 \geq 5bc+1$, a contradiction. Thus $r_a = 1$. □

Remark 6.1.2. From Proposition 6.1.2, if $a = b$, then $r_a = r_b = 1$. Thus, Proposition 6.1.3 implies $a = b = 1$. Therefore, if two of $\{a,b,c\}$ are equal, they must equal 1.

Example 6.1.1. Suppose $(A, \mathbf{B})$ is a table algebra and $\mathbf{C} := \{c_0, c_1, c_{1^*}\}$ is a closed subset of $\mathbf{B}$ with

$$\begin{aligned} c_1c_1 &= \frac{\chi_0(c_1)-1}{2}c_1 + \frac{\chi_0(c_1)+1}{2}c_{1^*}, \\ c_1c_{1^*} = c_{1^*}c_1 &= \chi_0(c_1)c_0 + \frac{\chi_0(c_1)-1}{2}(c_1+c_{1^*}), \text{ and} \\ c_{1^*}c_{1^*} &= \frac{\chi_0(c_1)+1}{2}c_1 + \frac{\chi_0(c_1)-1}{2}c_{1^*}. \end{aligned}$$

Suppose also that $\mathbf{B} = \mathbf{C} \cup d\mathbf{C}$ where $d^2 = c_0$ and $c_i d = dc_{i^*}$ for all $c_i \in \mathbf{C}$. Then Remark 2.1.10 shows that $\mathbf{B} \cong_{\mathrm{x}} \mathbf{C} \rtimes \mathbb{Z}_2$, where $c_i \leftrightarrow (c_i, 1)$ and $c_i b \leftrightarrow (c_i, z)$. Note that this is a SITA of type $NR(6)$ with parameters $a = 1$ and $b = 1$ (so that $b_3 = d$ and $\chi_0(b_3) = 1$).

The following remark is a special case of Proposition 6.1.2 and is the case described in Proposition 6.1.3.

Remark 6.1.3. Suppose a, b, and c are any odd positive integers such that $ab + bc + ac - 1 = 2bc$. Suppose $\chi_0(b_3) = ab$, $\chi_0(b_4) = \chi_0(b_1) = bc$, and $\chi_0(b_5) = ac$. Then a, b, and c satisfy the conditions of Remark 6.1.1.

Proposition 6.1.4. Pick any odd positive integers a and c with $c > a$. Let n be the positive even integer such that $c = a + n$. Then

$$b = \frac{ac - 1}{c - a} \text{ iff } ab + bc + ac - 1 = 2bc.$$

Also, $b = \frac{ac-1}{c-a}$ is an odd integer iff $a^2 \equiv 1 \pmod{2n}$.

Proof. $b = \frac{ac-1}{c-a}$ iff $ab + bc + ac - 1 = 2bc$ is obvious. Suppose $b = \frac{ac-1}{c-a} = \frac{a(a+n)-1}{n}$ is an odd integer. Then $a^2 \equiv 1 \pmod{n}$. Now either $a^2 \equiv 1 \pmod{2n}$ or $a^2 \equiv n + 1 \pmod{2n}$. Suppose $a^2 \equiv n + 1 \pmod{2n}$. Since $2n \mid n(a + 1)$, $an + a^2 - 1 \equiv n(a + 1) \equiv 0 \pmod{2n}$. This implies $2 \mid \frac{a(a+n)-1}{n}$, or $2 \mid b$, a contradiction. Thus $a^2 \equiv 1 \pmod{2n}$. Conversely, if $b = \frac{ac-1}{c-a} = \frac{a^2+an-1}{n}$ and $a^2 \equiv 1 \pmod{2n}$, then $b \in \mathbb{Z}$. Also, $b = \frac{a^2-1}{n} + a$, which is the sum of an even and an odd. Thus b is odd. □

Note that this characterizes all positive odd integers a, b, and c which satisfy $ab + bc + ac - 1 = 2bc$.

The following corollary is a result of Proposition 6.1.4 and Remark 6.1.3.

Corollary 6.1.5. Pick any even positive integer n. Pick any positive odd integer x satisfying $x^2 \equiv 1 \pmod{2n}$. Then, $a = x$, $b = \frac{x(x+n)-1}{n}$, and $c = x + n$ satisfy the conditions of Remark 6.1.1 and, thus, generate a SITA of type $NR(6)$.

Remark 6.1.4. Observe that if $x = 1$, then $x^2 \equiv 1 \pmod{2n}$. Therefore, the last corollary shows that $a = 1$, $b = 1$, and $c = n + 1$ satisfy the conditions of Remark 6.1.1 and, thus, produce a SITA of type $NR(6)$.

For any positive even integer n, we can always find a positive odd integer x which satisfies $x^2 \equiv 1 \pmod{2n}$, namely $x \equiv 1 \pmod{n}$ or $x \equiv -1 \pmod{n}$. Thus the previous corollary provides an infinite class of parameters which produce a SITA of type $NR(6)$.

The following examples demonstrate that parameters can exist which satisfy the conditions of Remark 6.1.1 but which produce $ab+bc+ac-1 \neq 2bc$, where a is the minimum of $\{a, b, c\}$. Thus the equality in Proposition 6.1.4 is not required.

Example 6.1.2. Let $a = 31$, $b = 189$, and $c = 5353$. Then $ab + bc + ac - 1 = 2 \cdot 101 \cdot 31 \cdot 189 = 1183418$, which divides $ab(ab - 1)$, $ac(ac - 1)$, and $bc(bc - 1)$. So, defining $\chi_0(b_3) = ab$, $\chi_0(b_4) = \chi_0(b_1) = bc$, and $\chi_0(b_5) = ac$, then a, b, and c satisfy the requirements of Remark 6.1.1; however, $ab + bc + ac - 1 < 2bc$, where $2bc = 2023434$. In fact, $ab+bc+ac-1 = 2r_a r_b r_c$ where $r_a = a$, $r_b = b$, and $r_c = 101$.

Recall from Remark 6.1.2, that if two parameters are equal, they must equal 1. Noting that any odd integer can be written in the form $n+1$ for some even integer n, Remark 6.1.4 implies the following.

Example 6.1.3. For any odd $c > 1$, $a = 1$, $b = 1$, and c satisfy the conditions in Remark 6.1.1 and, thus, generate a SITA of type $NR(6)$. In this case $\chi_0(b_3) = 1$, $\chi_0(b_4) = c$, and $\chi_0(b_5) = c$. Therefore $o(\mathbf{B}) = 4c + 2$.

According to the list of association schemes of small order published by Hanaki and Miyamoto [14], schemes as14.10, as22.8, as30.73, as38.18, and as38.19 are examples of SITAs of type $NR(6)$. Note that these correspond to parameters $a = 1$, $b = 1$, and c equal to $3, 5, 7$, or 9.

In fact, Proposition 6.1.6 will demonstrate that if $2c + 1$ is a prime which is congruent to $3 \pmod 4$, then there exists an association scheme with order $2(2c+1)$ which is a SITA of type $NR(6)$. Note, however, that as30.73 is a SITA of type $NR(6)$ which corresponds to parameters $a = 1$, $b = 1$, and $c = 7$, and so $2c + 1 = 15$. Thus the converse does not hold.

Proposition 6.1.6. Suppose p is a prime which is congruent to $3 \pmod 4$. Then there exists an association scheme of order $2p$ which is a SITA of type $NR(6)$.

Proof. Consider ${\mathbb{Z}_p}^{\times}$, which is a cyclic group of order $p-1$. There exists an element $a \in {\mathbb{Z}_p}^{\times}$ with $o(a) = \frac{p-1}{2}$. Thus $\phi_a : \mathbb{Z}_p \to \mathbb{Z}_p$ given by $\phi_a(x) = ax$ is an automorphism of $(\mathbb{Z}_p, +)$. Let $A = \langle \phi_a \rangle$. Then A is a cyclic subgroup of $Aut(\mathbb{Z}_p)$ with order $\frac{p-1}{2}$. Thus there are two nontrivial orbits of A in $\mathbb{Z}_p$, call them T_1 and T_2. Let T_0 be the trivial orbit consisting of $[0]_p$. Define R_i by $(x, y) \in R_i$ iff $x - y \in T_i$, and also let $K := \{R_0, R_1, R_2\}$. By Example 2.1.1, $(\mathbb{Z}_p, K)$ is an association scheme.

Since $p \equiv 3 \pmod 4$, $\frac{p-1}{2}$ is odd. Then, for any i with $0 < i < \frac{p-1}{2}$, $a^{2i} \not\equiv 1 \pmod p$ since $o(a) = \frac{p-1}{2}$ in ${\mathbb{Z}_p}^{\times}$. Thus for any $x \neq 0$, $a^i x \not\equiv -x \pmod p$ for any i. Therefore, $x \in T_1$ iff $-x \in T_2$. This implies $T_2 = -T_1$. Thus $R_2 =^t R_1$. Note that the valencies of R_1 and R_2 are both $\frac{p-1}{2}$.

Since $\mathcal{K} := (\mathbb{Z}_p, K)$ is an association scheme, it generates a table algebra with basis elements given by the adjacency matrices. Let A_0, A_1, and A_2 be the adjacency matrices such that $A_i \leftrightarrow R_i$. Then since the valencies of R_i with $i \neq 0$ are both $\frac{p-1}{2}$ and since $A_1 = {A_2}^{tr}$, $A_1 A_2 = \frac{p-1}{2} A_0 + \alpha A_1 + \beta A_2$ and $\alpha + \beta = \frac{p-3}{2}$. Since

$(A_1A_2, A_1) = (A_2, A_2A_1) = (A_2, A_1A_2)$, $\alpha = \beta = \frac{p-3}{4} = \frac{\frac{p-1}{2}-1}{2}$. From $(A_1A_1, A_1) = (A_1, A_1A_2)$, $A_1A_1 = \frac{\frac{p-1}{2}-1}{2}A_1 + \frac{\frac{p-1}{2}+1}{2}A_2$. Similarly, $A_2A_2 = \frac{\frac{p-1}{2}+1}{2}A_1 + \frac{\frac{p-1}{2}-1}{2}A_2$. Thus the scheme $\mathcal{K}$ generates a table algebra of dimension 3 which is exactly isomorphic to the table algebra generated by the normal closed subset $\mathbf{C}$ in Example 5.1.1. Here, $\chi_0(A_1) = \frac{p-1}{2}$.

Let $G = \mathbb{Z}_2$. Then $\mathcal{A}(\mathcal{K}) \rtimes G$ is a SITA of type $NR(6)$ as in Example 6.1.1, with $a = 1$, $b = 1$, and $c = p$. Let $(G, \widetilde{G})$ be the group scheme defined by $(x, y) \in R_g$ iff $xy^{-1} = g$. Define $\pi : G \to Aut(K)$ by $\pi([0]_2) = id_K$ and $\pi([1]_2)$ switches R_1 and R_2 (or, equivalently, takes the transpose of the input relation). Then π is a homomorphism. Let $g_0 = [0]_2$, the identity in $\mathbb{Z}_2$. Let $Z := \{(g, x) \mid g \in G, x \in \mathbb{Z}_p\}$, and suppose $\mathcal{Z} := (Z, \widetilde{G} \ltimes_{\pi, g_0} K)$, the semidirect product of $(\mathbb{Z}_p, K)$ by $(G, \widetilde{G})$. Then $o(\mathcal{Z}) = 2p$. Furthermore, $\mathcal{A}(\mathcal{Z}) \cong_{\mathrm{x}} \mathcal{A}(\mathcal{K}) \rtimes G$ by Proposition 2.1.4.

□

6.2 Other Examples of SITA of Type $NR(6)$

In the previous section, we provided restrictions on the possible values of the positive odd parameters a, b, and c which satisfy the conditions of Remark 6.1.1. In particular, $ab + bc + ac - 1 = 2r_ar_br_c$, where $r_a \mid a$, $r_b \mid b$, $r_c \mid c$, and $(r_a, bc) = (r_b, ac) = (r_c, ab) = 1$. In Proposition 6.1.4, we also created a set of requirements which characterize a, b, and c when $ab + bc + ac - 1 = 2bc$.

In this section, we will show other examples of parameters which generate SITA of type $NR(6)$. This will demonstrate the variety of types of parameters which might generate such a table algebra. For the remainder of this section, we assume $r_a \mid a$, $r_b \mid b$, and $r_c \mid c$.

First, in Example 6.1.2, we found parameters a, b, and c such that $ab+bc+ac-1 = 2 \cdot r_c \cdot ab$. One might suspect that $ab + bc + ac - 1$ will be a multiple of two of the three parameters. First note that by Proposition 6.1.2, these two parameters would have to be relatively prime. In other words, in this case, $ab + bc + ac - 1$ would have to be a multiple of ab, bc, or ac. From the following example, we will see that $ab + bc + ac - 1$ need not be a multiple of even one of the parameters.

Example 6.2.1. Suppose $a = 91 = 7 \cdot 13$, $b = 125 = 5 \cdot 25$, and $c = 17061 = 3 \cdot 5687$. Then $ab + ac + bc - 1 = 2 \cdot 5687 \cdot 25 \cdot 13 = 3696550$, which divides $ab(ab - 1) = 35 \cdot 3696550$, $ac(ac - 1) = 652071 \cdot 3696550$, and $bc(bc - 1) = 1230360 \cdot 3696550$. However, $ab + ac + bc - 1$ is not a multiple of either a, b, or c.

All of the examples of parameters meeting the conditions of Remark 6.1.1 which have been provided thus far have satisfied $(a, bc) = (b, c) = 1$. In the next example, we show that this coprime condition is not necessary.

Example 6.2.2. Suppose $a = 15 = 3 \cdot 5$, $b = 141 = 3 \cdot 47$, and $c = 3171 = 3 \cdot 1057$. Then $ab+ac+bc-1 = 2 \cdot 1057 \cdot 47 \cdot 5 = 496790$, which divides $ab(ab-1) = 9 \cdot 496790$, $ac(ac - 1) = 4554 \cdot 496790$, and $bc(bc - 1) = 402399 \cdot 496790$. However, 3 divides a, b, and c.

The following is another example of parameters a, b, and c which are all divisible by 3 and which satisfy Remark 6.1.1.

Example 6.2.3. Suppose $a = 21 = 3 \cdot 7$, $b = 39 = 3 \cdot 13$, and $c = 1227 = 3 \cdot 409$. Then $ab + ac + bc - 1 = 2 \cdot 409 \cdot 13 \cdot 7 = 74438$, which divides $ab(ab - 1)$, $ac(ac - 1)$, and $bc(bc - 1)$. Also, 3 divides a, b, and c.

Note that $\frac{a}{r_a} = \frac{b}{r_b} = \frac{c}{r_c} = 3$ for both Example 6.2.2 and Example 6.2.3.

In Example 6.1.2, we found a, b, and c which satisfied the requirements of Remark 6.1.1 but where $ab+bc+ac-1 < 2bc$ (and a is the minimum of $\{a, b, c\}$), showing that the equality in Remark 6.1.3 is not necessary. The next example demonstrates a set of parameters satisfying the conditions of Remark 6.1.1 but where $ab+bc+ac-1 > 2bc$.

Example 6.2.4. Let $a = 243 = 9 \cdot 27$, $b = 245 = 5 \cdot 49$, and $c = 1445 = 5 \cdot 289$. Then $ab+bc+ac-1 = 2 \cdot 27 \cdot 49 \cdot 289 = 764694$, which divides $ab(ab-1)$, $ac(ac-1)$, and $bc(bc-1)$. So, defining $\chi_0(b_3) = ab$, $\chi_0(b_4) = bc$, and $\chi_0(b_5) = ac$, then a, b, and c satisfy the requirements of Remark 6.1.1; however, $ab + bc + ac - 1 > 2bc$, where $2bc = 708050$. In this case, $r_a < a$, $r_b < b$, and $r_c < c$.

Remark 6.2.1. Suppose $a = r_a s_a$, $b = r_b s_b$, and $c = r_c s_c$. Suppose that $ab + bc + ac - 1 = 2r_a r_b r_c$. Then $s_a s_b r_a r_b + s_a s_c r_a r_c + s_b s_c r_b r_c - 1 = 2r_a r_b r_c$. Thus

$$s_a s_b < 2r_c,$$

$$s_a s_c < 2r_b,$$

$$s_b s_c < 2r_a.$$

Furthermore, we can assume without loss of generality that $s_a s_b < r_c$ and $s_a s_c < r_b$.

Note that Example 6.1.2 shows that it is not necessary for the third inequality, $s_b s_c < r_a$, to hold. However, Example 6.2.4 shows that the third inequality may hold.

The above examples were established using trial and error (with the aid of a computer program), the inequalities in Remark 6.2.1, and the equality in Proposition 6.1.2.

CHAPTER 7

OPEN PROBLEMS

While we have provided the structure of the SITA of type $NR(6)$ in terms of three parameters, an exact description of these parameters has yet to be completed. We have shown that they must be positive odd integers satisfying the requirement $ab + ac + bc - 1 = 2r_a r_b r_c$ where $r_a \mid a$, $r_b \mid b$, and $r_c \mid c$, and we have provided a description for a particular sub-case (when $ab + ac + bc - 1 = 2bc$). We have also presented several other examples of parameters which satisfy the requirements for a SITA of type $NR(6)$. However a complete description of the types of solutions for the parameters a, b, and c has yet to be established. Furthermore, it is yet to be determined exactly which of the SITAs in the conclusion of Theorem 5.2.1 arise from association schemes. Proposition 6.1.6 is only a partial result.

Another question for future research is whether the structure of a SITA can be determined when we relax the condition of the degree of the unique nonlinear character with nontrivial multiplicity. Xu [20] has shown that if there is exactly one irreducible character whose degree does not equal its multiplicity, then $(A, \mathbf{B})$ has a fused-center and the center of A is a wreath product of an abelian group and a table algebra of dimension 2; however, one would like to expand on this result to obtain more information about the structure of the SITA.

An additional question worth investigating is whether structure can be determined in the case when there are precisely two nontrivial multiplicities. Even when there are just 3 irreducible characters, the structure has not yet been determined. Such a result for table algebras would apply to association schemes, and, in particular, to association schemes of rank 6. Such association schemes have recently been studied by Hanaki and Zieschang [15], where some characterizations were determined for the imprimitive case, under certain conditions.

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APPENDIX

A NOTE ON XU'S THEOREM ON TABLE ALGEBRAS WITH ONE NONTRIVIAL MULTIPLICITY

The following theorem of Xu was received by the author after most of this dissertation was written.

Theorem A.0.1. [20, Theorem 1.4] Let $(A, \mathbf{B})$ be a STA. Then the following are equivalent.

(i) There is exactly one $\chi_s \in \mathrm{Irr}(A)$ such that $m_s \neq n_s$.

(ii) $(A, \mathbf{B})$ has a fused-center, and

$$(Z(A), \mathrm{Cla}(\mathbf{B})) \cong_{\mathrm{x}} (Z(A//O^{\mathcal{V}}(\mathbf{B})), \mathrm{Cla}(\mathbf{B}//O^{\mathcal{V}}(\mathbf{B})) \wr (\mathbb{C}\mathbf{D}, \mathbf{D}),$$

where $(\mathbb{C}\mathbf{D}, \mathbf{D})$ is a table algebra of dimension 2.

We present here a short proof of one direction of a specialization of the above theorem to the case where all characters with $n_i = m_i$ also satisfy $n_i = m_i = 1$. This proof is based directly on our results.

Definition A.0.1. Let $(A, \mathbf{B})$ be a table algebra. If there exists a partition of the basis, $\{\mathbf{B}_0, \mathbf{B}_1, ..., \mathbf{B}_k\}$, such that $\{\mathbf{B}_0{}^+, \mathbf{B}_1{}^+, ..., \mathbf{B}_k{}^+\}$ is a basis for $Z(A)$, the center of A, then we say that $(A, \mathbf{B})$ has a *fused-center.*

Definition A.0.2. The *thin residue of* $\mathbf{B}$, denoted $O^{\mathcal{V}}(\mathbf{B})$, is the intersection of all strongly normal closed subsets of $\mathbf{B}$.

Note that $O^{\mathcal{V}}(\mathbf{B})$ must, itself, be strongly normal, and so it is the smallest normal closed subset of $\mathbf{B}$ with the property that $\mathbf{B}//O^{\mathcal{V}}(\mathbf{B})$ is a group.

We will now show that if $(A, \mathbf{B})$ is a STA such that $\tau = \sum_{i=0}^{k-1} \chi_i + m_k \chi_k$, with $m_k \neq n_k$, then $(i) \implies (ii)$ in Theorem A.0.1.

Proof. Let $\mathbf{C} = \bigcap_{i=0}^{k-1} \mathrm{Ker}(\chi_i)$. By Lemma 2.2.8, $\mathrm{Z}(\chi_i) = \mathbf{B}$ for $\chi_i \in \{\chi_0, \cdots, \chi_{k-1}\}$. So, $\mathbf{B} = \bigcap_{i=0}^{k-1} \mathrm{Z}(\chi_i)$. By Lemma 2.2.6, $\mathbf{B}//\mathbf{C}$ is a (abelian) group and $\mathbf{C}$ is normal.

If $|\mathbf{C}| = 1$, then $\mathbf{B}$ is a group and $|G'| = 1$. Thus $\mathbf{B}$ is an abelian group, which contradicts our assumptions. So, $|\mathbf{C}| > 1$ and $\mathbf{C} \not\subseteq \mathrm{Ker}(\chi_k)$. Note, here, that $O^{\mathcal{V}}(\mathbf{B}) = \mathbf{C}$ since $k = |\mathbf{B}//\mathbf{C}| \leq |\mathbf{B}//O^{\mathcal{V}}(\mathbf{B})| < k+1$ by Theorem 2.2.1.

By Remark 2.2.3, the k coset sums are in $Z(A)$. Thus $\{b_0, (\mathbf{C}\backslash\{b_0\})^+\} \bigcup \{(b_i\mathbf{C})^+ \mid b_i \notin \mathbf{C}\}$ forms a linearly independent set of $k+1$ elements in $Z(A)$. Note that $\{b_0, (\mathbf{C}\backslash\{b_0\})^+\}$ form a closed subset in $Z(A)$. Since A is semisimple,

$$A \cong \begin{pmatrix} M_{n_1}(\mathbb{C}) & & 0 \\ & \ddots & \\ 0 & & M_{n_k}(\mathbb{C}) \end{pmatrix} \cong \begin{pmatrix} \mathbb{C} & & & 0 \\ & \ddots & & \\ & & \mathbb{C} & \\ 0 & & & M_{n_k}(\mathbb{C}) \end{pmatrix}.$$

Thus $|Z(A)| = k+1$. Therefore, the $k+1$ linearly independent sums above (denoted $\mathrm{Cla}(\mathbf{B})$) form a basis for $Z(A)$, and, thus, $Z(A)$ has a fused-center. Furthermore, for any $b_i \in \mathbf{B}\backslash\mathbf{C}$, $b_i//\mathbf{C} \cdot (\mathbf{C}\backslash\{b_0\})^+ = (o(\mathbf{C}) - 1)b_i//\mathbf{C}$. Thus, letting $\mathbf{D} = \{b_0, (\mathbf{C}\backslash\{b_0\})^+\}$, $(Z(A), \mathrm{Cla}(\mathbf{B})) \cong_{\mathrm{x}} (Z(A)//\mathbf{D}, \mathrm{Cla}(\mathbf{B})//\mathbf{D}) \wr (\mathbb{C}\mathbf{D}, \mathbf{D})$ with $|\mathbf{D}| = 2$.

$\square$