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The Quaternion Algebra and Its Connections to Medical Imaging

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1 Abstract

Pure mathematics topics have widely been regarded as having few practical applications; however, over time, many applications have arisen. One such application is using the quaternions, an abstract algebraic structure and extension of the complex number system, to enhance image quality. Quaternion numbers take the form $z = a + bi + cj + dk$, where a, b, c, d are real numbers, and i, j, k are distinct square roots of -1. By having three distinct square roots of -1, rather than just one (as in the standard complex number system), unique mathematical properties and practical uses arise. Quaternions have often been used in various aspects of imaging, including improving quality [5]. In this project, we focus on improving image quality, using Fourier transforms [3, 5].

2 Algebraic Properties of $\mathbb{H}$

The set of quaternions, denoted by $\mathbb{H}$, is a division ring over $\mathbb{R}$; notably, it is not commutative, and its basis elements $\{1, i, j, k\}$ satisfy the following equations:

$$\begin{aligned} ij &= -ji = k \\ jk &= -kj = i \\ ki &= -ik = j. \end{aligned}$$

Each element of $\mathbb{H}$ also has a unique multiplicative inverse. We can calculate the inverse of some quaternion of the form $a + bi + cj + dk$, for $a, b, c, d \in \mathbb{R}$, as follows:

$$\begin{aligned} \frac{1}{a+bi+cj+dk} &= \frac{a-bi-cj-dk}{(a+bi+cj+dk)(a-bi-cj-dk)} \\ &= \frac{a-bi-cj-dk}{a(a-bi-cj-dk)+bi(a-bi-cj-dk)+cj(a-bi-cj-dk)+dk(a-bi-cj-dk)} \\ &= \frac{a-bi-cj-dk}{(a^2-abi-acj-adk)+(abi+b^2-bcij-bdik)+(acj-bcji+c^2-cdjk)+(adk-bdki-cdkj+d^2)} \\ &= \frac{a-bi-cj-dk}{(a^2-abi-acj-adk)+(abi+b^2-bcij-bdik)+(acj+bcij+c^2-cdjk)+(adk+bdik+cdjk+d^2)} \\ &= \frac{a-bi-cj-dk}{a^2+b^2+c^2+d^2}. \end{aligned}$$

This value is clearly unique, as well, since every step taken here was reversible. Furthermore, we could easily verify that the multiplicative inverse of $\dfrac{a-bi-cj-dk}{a^2+b^2+c^2+d^2}$ is $a+bi+cj+dk$, by seeing that their product is 1.

3 Matrix Representations of $\mathbb{H}$

We can also represent elements of $\mathbb{H}$ as 2×2 matrices in $\mathbb{C}$. One such way, as shown in [4] is as follows: map $a + bi + cj + dk$ to

$$a \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b \begin{pmatrix} \boldsymbol{i} & 0 \\ 0 & -\boldsymbol{i} \end{pmatrix} + c \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + d \begin{pmatrix} 0 & \boldsymbol{i} \\ \boldsymbol{i} & 0 \end{pmatrix} = \begin{pmatrix} a + b\boldsymbol{i} & c + d\boldsymbol{i} \\ -c + d\boldsymbol{i} & a - b\boldsymbol{i} \end{pmatrix},$$

for $a, b, c, d \in \mathbb{R}$, and where $\boldsymbol{i}$ is the imaginary number with magnitude 1 in $\mathbb{C}$, equivalent to $\sqrt{-1}$. This form can also be used to find the multiplicative inverse of a quaternion. Starting with what we found for the inverse of the scalar form of quaternions, we can write $\frac{a-bi-cj-dk}{a^2+b^2+c^2+d^2}$ in this matrix form as

$$\frac{1}{a^2 + b^2 + c^2 + d^2} \left(a \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - b \begin{pmatrix} \boldsymbol{i} & 0 \\ 0 & -\boldsymbol{i} \end{pmatrix} - c \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - d \begin{pmatrix} 0 & \boldsymbol{i} \\ \boldsymbol{i} & 0 \end{pmatrix} \right),$$

or

$$\frac{1}{a^2 + b^2 + c^2 + d^2} \begin{pmatrix} a - b\boldsymbol{i} & -c - d\boldsymbol{i} \\ c - d\boldsymbol{i} & a + b\boldsymbol{i} \end{pmatrix}.$$

Now, to check, we see that

$$\begin{aligned} \frac{1}{a^2 + b^2 + c^2 + d^2} \begin{pmatrix} a - b\boldsymbol{i} & -c - d\boldsymbol{i} \\ c - d\boldsymbol{i} & a + b\boldsymbol{i} \end{pmatrix} \begin{pmatrix} a + b\boldsymbol{i} & c + d\boldsymbol{i} \\ -c + d\boldsymbol{i} & a - b\boldsymbol{i} \end{pmatrix} &= \frac{1}{a^2 + b^2 + c^2 + d^2} \begin{pmatrix} a^2 + b^2 + c^2 + d^2 & 0 \\ 0 & a^2 + b^2 + c^2 + d^2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \end{aligned}$$

showing that $\frac{1}{a^2+b^2+c^2+d^2} \begin{pmatrix} a - b\boldsymbol{i} & -c - d\boldsymbol{i} \\ c - d\boldsymbol{i} & a + b\boldsymbol{i} \end{pmatrix}$ is indeed the inverse of $\begin{pmatrix} a + b\boldsymbol{i} & c + d\boldsymbol{i} \\ -c + d\boldsymbol{i} & a - b\boldsymbol{i} \end{pmatrix}$, the said matrix form of $a + bi + cj + dk$, showing that the general form of inverses of quaternions carries over to such form.

While a given matrix has a unique inverse, there are numerous matrix representations of elements of $\mathbb{H}$, and for each of these representations, the representation of each quaternion has a unique inverse. One clear way to do this is to conjugate this matrix representation by an invertible matrix. If we have a map $f : \mathbb{H} \to \mathbb{C}^{2\times 2}$, with $f(x) = \begin{pmatrix} a + b\boldsymbol{i} & c + d\boldsymbol{i} \\ -c + d\boldsymbol{i} & a - b\boldsymbol{i} \end{pmatrix}$, for $x = a + bi + cj + dk \in \mathbb{H}$, then consider $g : \mathbb{H} \to \mathbb{C}^{2\times 2}$, where $g(x) = Q \begin{pmatrix} a + b\boldsymbol{i} & c + d\boldsymbol{i} \\ -c + d\boldsymbol{i} & a - b\boldsymbol{i} \end{pmatrix} Q^{-1}$, for some invertible matrix $Q \in \mathbb{C}^{2\times 2}$. Using properites of matrices, it follows that

$$g^{-1}(x) = (Q^{-1})^{-1} \begin{pmatrix} a + b\boldsymbol{i} & c + d\boldsymbol{i} \\ -c + d\boldsymbol{i} & a - b\boldsymbol{i} \end{pmatrix}^{-1} Q^{-1} = Q \begin{pmatrix} a + b\boldsymbol{i} & c + d\boldsymbol{i} \\ -c + d\boldsymbol{i} & a - b\boldsymbol{i} \end{pmatrix}^{-1} Q^{-1}.$$

This inverse is uniquely determined by Q and the values of a, b, c, d, hereby showing that for any invertible matrix $Q \in \mathbb{C}^{2\times 2}$, we generate a unique representation of $\mathbb{H}$. This shows how there are not only many copies of $\mathbb{C}$ in $\mathbb{H}$ (as is clear from a basis for $\mathbb{H}$ containing many square roots of -1), but that there are also many copies of $\mathbb{H}$ in $\mathbb{C}^{2\times 2}$.

4 Quaternions and Imaging

In signal processing, each pixel in an image or signal typically is comprised of a combination of three colors: red, green, and blue. As described in [2], pure quaternions (quaternions with a real part of 0), pixels can have three components, with the coefficients of i, j, and k being the amounts of each color. Then, on a 2D image, for a point located at (n, m), we can write a quaternion representation of the color as follows:

$$f[n, m] = r[n, m]i + g[n, m]j + b[n, m]k,$$

where f represents the overall color and r, g, and b represent the amounts of red, green, and blue, respectively.

5 Standard (Complex) Fourier Transform

The quaternion Fourier transform is a variation of a traditional Fourier transform in the complex numbers. For a function $f : \mathbb{R} \to \mathbb{C}$, we denote the function that is its Fourier transform by F, writing it at some ω as follows:

$$\mathcal{F}\{f\}(\omega) = F(\omega) = \int_{-\infty}^{\infty} f(x)e^{-i\omega x}dt.$$

Similarly, we have an analogous traditional 2D Fourier transform:

$$\mathcal{F}\{f\}(\omega, \nu) = F(\omega, \nu) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)e^{-i(\omega x+\nu y)}\ dx\ dy.$$

6 The Quaternion Fourier Transform

Before being able to understand the quaternion Fourier transform, it is important to define exponentials of quaternions. For a pure quaternion ξ, write

$$e^{\xi} = \sum_{n=0}^{\infty} \frac{\xi^n}{n!} = \sum_{n=0}^{\infty} \frac{|\xi|^n \boldsymbol{\xi}^n}{n!},$$

where $\boldsymbol{\xi} = \frac{\xi}{|\xi|}$, with $|\xi|$ being the magnitude of ξ, defined for a quaternion $a+bi+cj+ck$ as $\sqrt{a^2+b^2+c^2+d^2}$. This will become important when considering discretizations of quaternion Fourier transforms, which are essential in imaging, since images consider discrete pixels, not continuous spaces (the pixels are so small that on an actual image, the picture appears continuous).

Because quaternions are not commutative, we must consider both left-sided and right-sided quaternion Fourier transforms, defined as follows in the 1D case, respectively, where μ is a pure unit quaternion (unit meaning $\mu^2 = -1$):

$$\mathcal{F}^L_{\mp\mu}\{f\}(\omega) = F^L(\omega) = \kappa_- \int_{-\infty}^{\infty} e^{-\mp\mu\omega x} f(x),$$

$$\mathcal{F}^R_{\mp\mu}\{f\}(\omega) = F^R(\omega) = \kappa_- \int_{-\infty}^{\infty} f(x)e^{-\mp\mu\omega x}.$$

In these expressions, κ_- is a scale factor, and in the inverse transforms, the scale factor κ_+ satisfies $\kappa_-\kappa_+ = \dfrac{1}{2\pi}$.

There is a 2D quaternion Fourier transform, as well. Following is the formula for the right-sided 2D quaternion Fourier transform; the left-sided equivalent is defined analogously:

$$\mathcal{F}^R_{\mp\mu}\{f\}(\omega, \nu) = F^R(\omega, \nu) = \int_{\mathbb{R}^2} f(x, y)e^{\mp\mu(\omega x+\nu y)}\ dx\ dy.$$

7 Discretization of 1D Quaternion Fourier Transform

We must discretize QFTs when applying them to imaging, in order to consider individual pixels. For instance, for a 1D left-sided QFT (with μ being a pure unit quaternion, hence having magnitude 1), a discretization is:

$$\begin{aligned}
F(\omega) &= \sum_{n=0}^{N-1} \exp\left(-\mu 2\pi \frac{n\omega}{N}\right) f(n) \\
&= \sum_{n=0}^{N-1} \exp\left(\left(-2\pi \frac{n\omega}{N}\right)\mu\right) f(n) \\
&= \sum_{n=0}^{N-1} \left[\left(\sum_{k=0}^{\infty}(-1)^k \frac{|\mu|^{2k}\left(-2\pi\frac{n\omega}{N}\right)^{2k}}{(2k)!} + \mu \sum_{k=0}^{\infty}(-1)^k \frac{|\mu|^{2k+1}\left(-2\pi\frac{n\omega}{N}\right)^{2k+1}}{(2k+1)!}\right) f(n)\right] \\
&= \sum_{n=0}^{N-1} \left[\left(\sum_{k=0}^{\infty}(-1)^k \frac{\left(-2\pi\frac{n\omega}{N}\right)^{2k}}{(2k)!} + \mu \sum_{k=0}^{\infty}(-1)^k \frac{\left(-2\pi\frac{n\omega}{N}\right)^{2k+1}}{(2k+1)!}\right) f(n)\right].
\end{aligned}$$

2D quaternion Fourier transforms can be similarly discretized, though this will be omitted here. In the next section, we will see an application of discretizations of quaternion Fourier transforms.

8 Use of Convolution in Imaging

Quaternion convolution is used in image construction using quaternion Fourier transforms. A convolution mask is a filter that is used on an image, with the goal of sharpening its quality. Convolution masks are applied upon the taking of a Fourier transform, in order to alter the image ever so slightly, prior to taking the inverse to get an enhanced version of the original image. Convolution masks can be applied such that the enhancement is to certain pixels.

For a left convolution mask h_L, and $f[n,m]$ being the input image, the left convolution of f is given by

$$(h_L \circ f)[n,m] = \sum_{i=0}^{N-1}\sum_{j=0}^{M-1} h_L[i,j] f[n-i, m-j],$$

and the right convolution is defined analogously, with the mask applied on the right. The bi-correlation of f is defined similarly, except both left and right convolution masks are applied:

$$(h_L \circ f \circ h_R)[n,m] = \sum_{i=0}^{N-1}\sum_{j=0}^{M-1} h_L[i,j] f[n-i, m-j] h_R[i,j].$$

When applying convolution to a quaternion Fourier transform of an input image, we are able to produce enhanced images.

9 Basic Experiments with Transform Axis

One unique aspect of quaternion Fourier transforms is that the value of μ can change, and is chosen when the Fourier transform is taken. Some preliminary computations were done, varying μ, using the Quaternion Toolbox for MATLAB [6]. We interpret μ as the transform axis; that is, the axis which determines the spectrum of images that can be created using quaternion Fourier transforms.

In these experiments, four "images" were considered, which were quaternions representing color distributions. In one of them, the colors were given equal intensity; in the others, two colors were given almost no intensity, whereas one was very intense relative to the others. The results were as follows:

Image	Transform Axis	Discrete Quaternion Fourier Transform Output
$7i + 7j + 7k$	$(i + j + k)/\sqrt{3}$	$28i + 28j + 28k$
$19i + j + k$	$(i + j + k)/\sqrt{3}$	$16.39i - 32.78j + 16.39k$
$i + 19j + k$	$(i + j + k)/\sqrt{3}$	$-12i + 24j - 12k$
$i + j + 19k$	$(i + j + k)/\sqrt{3}$	$-4.392i + 8.785j - 4.392k$
$7i + 7j + 7k$	$(5i + j + k)/\sqrt{27}$	$28i + 28j + 28k$
$19i + j + k$	$(5i + j + k)/\sqrt{27}$	$13.86 + 9.494i - 32.78j + 9.464k$
$i + 19j + k$	$(5i + j + k)/\sqrt{27}$	$-12i + 24j - 12k$
$i + j + 19k$	$(5i + j + k)/\sqrt{27}$	$-13.86 + 2.536i + 8.785j + 2.536k$
$7i + 7j + 7k$	$(i + 5j + k)/\sqrt{27}$	$28i + 28j + 28k$
$19i + j + k$	$(i + 5j + k)/\sqrt{27}$	$23.32i - 18.93j + 23.32k$
$i + 19j + k$	$(i + 5j + k)/\sqrt{27}$	$-12i + 24j - 12k$
$i + j + 19k$	$(i + 5j + k)/\sqrt{27}$	$-11.32i - 5.072j - 11.32k$
$7i + 7j + 7k$	$(i + j + 5k)/\sqrt{27}$	$28i + 28j + 28k$
$19i + j + k$	$(i + j + 5k)/\sqrt{27}$	$-13.86 + 9.494i - 32.78j + 9.464k$
$i + 19j + k$	$(i + j + 5k)/\sqrt{27}$	$-12i + 24j - 12k$
$i + j + 19k$	$(i + j + 5k)/\sqrt{27}$	$13.86 + 2.536i + 8.785j + 2.536k$

Interestingly, from these outputs, we see that when the image had equal strengths of the three colors (equal coefficients of i, j, and k), the discrete quaternion Fourier transform output was the same, regardless of the transform output. Likewise, that output had equal strengths of the colors. Additionally, there are striking similarities between the output with transform axis of $(5i + j + k)/\sqrt{27}$ and $(i + j + 5k)/\sqrt{27}$; the only differences are the signs of the real parts in the outputs with nonzero real parts. Interestingly, those outputs are where the image mainly consists of one color, and that color corresponds to either i or k.

While it is not yet clear why these phenomena occur, these preliminary results provide insight into what must be considered in more advanced experiments. In many cases, the coefficients of some of the imaginary terms differ from other such coefficients by a factor of 2; this will also be worthwhile to explore.

10 Future Work

Most importantly, it will be critical to apply quaternion Fourier transforms to more practical situations, especially with changes to the transform axis. When applied to actual image data, it will be clearer what considerations need to be given before determining what methods should be used to enhance the images, as well as how to alter the transform axis. Applying quaternion Fourier transforms to specific medical situations is also a crucial step. There is limited prior work on this, as the field is fairly new; however, one such work is [5]. One possible application is to imaging for proton therapy cancer treatments, for which the idea of applying quaternions was discussed at the 6th Annual Loma Linda Workshop [1].

11 Appendix: MATLAB Code

Following is the MATLAB code used in the experiments in Section 9:

```
image = [quaternion(7,7,7) quaternion(19,1,1) quaternion(1,19,1) quaternion(1,1,19)];
mu1 = unit(quaternion(1,1,1));
mu2 = unit(quaternion(5,1,1));
mu3 = unit(quaternion(1,5,1));
mu4 = unit(quaternion(1,1,5));


out1 = qfft2(image,mu1,'L');
display(out1(1))
display(out1(2))
display(out1(3))
display(out1(4))


out2 = qfft2(image,mu2,'L');
display(out2(1))
display(out2(2))
display(out2(3))
display(out2(4))


out3 = qfft2(image,mu3,'L');
display(out3(1))
display(out3(2))
display(out3(3))
display(out3(4))


out4 = qfft2(image,mu4,'L');
display(out4(1))
display(out4(2))
display(out4(3))
display(out4(4))
```

.

References

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[2] Todd A Ell, Nicolas Le Bihan, and Stephen J Sangwine. *Quaternion Fourier transforms for signal and image processing.* John Wiley & Sons, 2014.

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