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How Mathematicians Assign Homework Problems in Abstract Algebra Courses

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How Mathematicians Assign Homework Problems in Abstract Algebra Courses

While many aspects of the teaching and learning of advanced mathematics have been explored, the role, construction, and values of homework have been virtually ignored. This report draws on task-based interviews with six mathematicians to explore the relationship between an instructor's learning goals and factors considered when selecting homework problems. All participants viewed homework as critical to student learning, and the majority of the participants' claims focused on either the mathematics or how the problem would help students learn; no instructor gave primacy to evaluative reasons for homework. We highlight six themes used by participants to evaluate and select items for inclusion in homework. They are (1) knowing and recalling axioms and definitions, (2) developing an arsenal of examples, (3) developing new problem approaches, (4) remediating misconceptions, (5) making connections to prior and future material, and (6) valuing reading notes or text.

Keywords: abstract algebra, homework, instructional practice

1. Introduction

Most of the work on teaching advanced mathematics has focused on what occurs inside the mathematics classroom. However, we wonder whether this emphasis is appropriate. Consider, for example, Wu's description of learning undergraduate mathematics:

> Learning mathematics is a long and arduous process, and no matter how one defines "learning", it is not possible to learn all the required material of any mathematics course in 45 hours of discussion. … The professor gives an outline of what and how much students should learn, and students do the work on their own outside of the 45 hours of class meetings (1999, p. 5).

Wu's point is that the mathematics classroom is not, and cannot be, the central venue for mathematical learning. Most learning of advanced mathematics occurs outside the classroom. Indeed, in the United States (US), students are usually recommended to spend two or more hours outside of the classroom studying the material for every hour they spend inside the classroom. Consequently, even in inquiry-based classrooms where students are actively learning and doing mathematics for most of the class meetings, the majority of their mathematical work will take place outside the classroom. Clearly, then, a central aspect of instruction is providing students with the structure and support to learn mathematics outside the classroom. There are many ways that such instruction might proceed, such as providing students with insights into how mathematics should be read or studied and directing students to different resources to understand the material. All of these would be worthy topics of investigation for a mathematics educator. However, the most common way in which students engage with mathematics outside of class is by working on homework problems. Perhaps surprisingly, the assignment of homework problems has largely been overlooked by researchers who are interested in how advanced mathematics courses are taught.

The goal of this paper is to explore how mathematicians assign homework problems and what mathematicians' rationales are for their choices. We believe that exploring these topics can make two contributions to the literature. First, our work is an initial step in understanding instructors' beliefs about the role that homework plays in students' mathematical development in advanced mathematics courses. Consequently, this work can improve our understanding of mathematicians' pedagogical goals in advanced mathematics courses as well as our understanding of how they try to achieve these goals. Second, our work will be useful for generating hypotheses on what types of pedagogical actions induce students' learning. If mathematicians assign certain types of homework problems to help students acquire particular knowledge or engage in certain types of reasoning, follow-up studies can then investigate: When students work on these types of homework problems, what do they learn and what types of reasoning processes are elicited? Does this learning and reasoning align with the rationales that mathematicians provide for assigning these problems?

2. Related Literature

While the research on the assignment of homework problems in undergraduate mathematics classes is largely absent, there has been a recent increase in studies that look at the in-class instructional practices of mathematicians. In addition to providing an overview of relevant work in this area, we will also highlight the few relevant studies that have been conducted around mathematics homework.

2.1 Classroom-focused Instruction Studies

How do mathematicians typically teach in advanced mathematics courses? What is their rationale for teaching the way that they do? How might instructors teach advanced mathematics more effectively? And what effects do instructors have on student engagement? These questions have received significant attention amongst mathematics educators, especially in the last decade. However, most research in this area has focused on what has transpired *in the classroom*, with a focus on how mathematicians lecture or how students respond to tasks in inquiry-oriented curricula.

Regarding the first question, there have been several detailed case studies showing that mathematicians' instruction is nuanced and often based on coherent belief systems and a good deal of thought (e.g., Lew et al., 2016; Pinto & Karsenty, 2018; Weber, 2004). Other investigations have explored particular aspects of teaching, such as mathematicians' use of examples (e.g., Fukawa-Connelly & Newton, 2014; Mills, 2014), questioning (Paoletti et al., 2018), and their presentation of heuristics (e.g., Fukawa-Connelly et al., 2017).

Regarding the second question, there have been many interview studies on how mathematicians view teaching advanced mathematics (e.g., Nardi, 2008), their goals for instruction (e.g., Alcock, 2010; Hemmi, 2010), and why they present material in the way that they do (e.g., Weber, 2012). Surveys and interviews have also revealed that mathematicians claim their instruction is not monolithic; mathematicians claim to employ some student-centered instruction such as group problem-solving at least occasionally in their lectures (Johnson et al., 2018). Further, mathematicians rely on lecture because they believe it is the best way to teach (e.g., Johnson et al., 2018; Woods & Weber, 2020).

Regarding the third question, a number of researchers have advocated using forms of more student-centered, inquiry-based instruction (e.g., Laursen & Rasmussen, 2019). In abstract algebra, for instance, researchers have described instructional sequences based on local instructional theories in groups and isomorphisms (Larsen, 2013), quotient groups (Larsen & Lockwood, 2013), and rings (e.g., Cook, 2014); entire special issues of the *Journal of Mathematical Behavior* have been devoted to Dubinsky's (1997) ISETL and Larsen's TAAFU

(Larsen, Johnson, & Weber, 2013) curricula. In a different vein, some researchers have sought to identify instructional practices that might make traditional lectures more effective (e.g., Alcock, 2018; Gabel & Dreyfus, 2017; Iannone & Miller, 2019).

Finally, with regard to the last question, researchers studying lecture have illustrated how students may misinterpret, or otherwise miss the point, of what their lecturers are attempting to convey (e.g., Fukawa-Connelly et al., 2017; Krupnik et al., 2018; Lew et al., 2016). Students' failure to grasp what their instructors attempt to impart may offer a partial explanation for why so many undergraduates struggle to understand advanced mathematical content (e.g., Rasmussen & Wawro, 2017) and to write and evaluate proofs (e.g., Stylianides, Stylianides, & Weber, 2017), despite attending lectures in which these topics and concepts are discussed in a manner that is clear to mathematically knowledgeable observers.

2.2 Homework-focused Studies

In this paper we take the perspective that mathematics instruction is not limited to the time spent in class. As stated previously, in the US, it is general guidance that for every hour an undergraduate student spends in the classroom they should be spending two to three hours studying and doing homework outside of class. Thus, for a typical course in abstract algebra in the US, students spend three hours per week in class and should be spending six to nine hours per week reading the textbook, looking over their notes, and completing their homework assignments. As homework and out of class assignments are an extension of the learning environment, it is reasonable to expect that the instructor's goals and rationales that influence their in-class decision-making would therefore extend to their decision-making around homework. Here we provide an overview of the research literature that has looked at homework, with particular attention paid, when possible, to research that has conceived of assigning homework as an instructional practice.

2.2.1 Students' approaches to homework

While the research literature on in-class instruction has begun to address factors like instructors' beliefs and goals, the extant research on homework largely focuses on students. In their literature review on homework in the K12 setting, Epstein and Van Voorhis (2001) noted that most research on homework explores what students do and how homework affects student achievement, but not how homework assignments are chosen to reflect instructors' goals. This trend is present in research on mathematics homework at the university level as well. For instance, in Dorko's (2020) review of the extant literature on the online homework systems used in undergraduate mathematics classes, all of the papers she cited were focused on the student experience in some way. Additionally, a search in the undergraduate mathematics research literature reveals an overwhelming focus on exploring the relationship between students' performance on homework and their exam or course grades, the effect of online vs. on-paper homework, and students' experiences while doing homework (Dorko, 2020). For example, Maciejewski and Merchant's work (2016) provided an analysis of the relationship between study approaches, task demand, and course grades across four years of undergraduate mathematics. They argued that first-year courses require, primarily, process-focused proficiencies, while in upper-division courses, the tasks have a greater emphasis on recall and understanding.

In a series of papers, Lithner (2003, 2004, 2008) explored the reasoning that students used to complete calculus homework. Through a careful analysis of students' behavior when completing calculus homework, he found that students generally completed their homework assignments by adapting worked examples presented in the textbook and avoiding conceptual reasoning. In checking all exercises in a calculus textbook, Lithner (2004) found that most could be completed by mimicking the solutions from worked examples in the text. Alternatively, Lithner (2003, 2008) described hypothesized types of reasoning that students might plausibly use to complete homework tasks that might provide students with more learning opportunities (see also Sloan & Stephens, 1981).

Moore (1994) and Lew and Zazkis (2019) investigated how students completed homework in proof-based mathematics classes. Moore (1994) found that students struggled to write proofs when completing homework assignments (as well as in other settings) because they had a poor understanding of the concepts, challenges with interpreting language and notation, and difficulties getting started on assignments. Lew and Zazkis (2019) explored how 17 students completed the following "prove or disprove" task: "True or False and why: if *a* and *b* are both irrational, then a^b is irrational". Lew and Zazkis found that students engaged in a variety of types of activity, including how one student made cyclic attempts to both prove and disprove the statement by drawing on different types of examples. They also noted that students did not always stop work once a relevant proof or counterexample was found—they sometimes sought either a cleaner proof or presentation or a more general counterexample. Finally, Lew and Zazkis noted that time spent on a single task was in some cases quite significant, with some participants generating pages of work.

2.2.2 Teachers' views of mathematics homework

Moore (2016) explored how professors grade student proofs—finding that they attended to logical correctness, clarity, fluency, and what he called understanding of the proof, which implied that they were creating models of student thinking based on the work presented. Thoma and Nardi (2018) explored problems on an end-of-course exam in undergraduate mathematics and compared the expectations of undergraduate students to the expectations of students in school mathematics. They found, for example, that in the undergraduate context, students are expected to specify the set of numbers in which they are operating while in school mathematics, this is uncommon. They then attribute student difficulties with the exam items to commognitive conflict.

Research into how mathematicians choose homework assignments is comparatively sparse. The work that we found was primarily limited work in the K12 setting, a paper on how undergraduate mathematics instructors think about exam questions, and a paper on how undergraduate mathematics instructors think about calculus homework. Epstein (2001) claimed that in the process of designing homework in K12 settings, teachers draw on their understandings and beliefs about their students' abilities and needs. Across multiple studies at the K12 level, Epstein (1988, 2001) found 10 purposes cited for giving homework, only two of which—participation and practice—appear to be relevant to the university setting. Participation is a means of increasing students' involvement in learning, while homework set for practice is meant to increase speed, mastery, or maintenance of skills. Moreover, those were presented rather unproblematically, assuming that practicing relevant skills on homework is important, but with little discussion of how the selection of items or structuring of the assignment promotes those goals. Berqvist (2012) explored the modes of reasoning found on calculus exams in Sweden, finding that primarily imitative reasoning was needed, and that the instructors intended that to be the case because they believed students would otherwise fail. However, the goals of exams and homework are typically distinct: where exams are often summative evaluations, homework usually has an instructional function. Dorko (2019) observed that calculus instructors intended to engage students in current material, connect to prior and future material, and enhance students' familiarity with notation. However, it is not known if instructors of upper-level proof courses share those goals or have additional goals for their students.

3. Theoretical Perspective

In the last decade, mathematics educators have investigated common teaching practices in advanced mathematics classrooms, focusing on what pedagogical actions are commonly taken during lectures and mathematicians' rationales for engaging in these actions (see Gabel, 2019, for a comprehensive review of this literature). A general finding from this research is that mathematicians justify their pedagogical actions by claiming these actions will help them achieve their pedagogical goals (e.g., Weber, 2004) and, indeed, variance in mathematicians' pedagogical goals can explain variance in their pedagogical actions (Pinto, 2019). To illustrate, mathematicians frequently ask their classes questions in their lectures (Paoletti et al., 2018) because they believe questions will increase student engagement (Woods & Weber, 2020). In this paper, we will organize teachers' selection of homework problems based on the pedagogical goals they think these problems will help them obtain. Consequently, we organize our analysis around teachers' pedagogical goals, using Alcock's (2010) framework for proof-related pedagogical goals as a guiding theoretical framework.

Alcock's framework consisted of four pedagogical goals, which she labelled structural thinking, instantiation, creative thinking, and critical thinking goals. The mathematicians' *structural thinking* goal for students concerned having students develop competence and fluency in the "syntactic" components (Weber & Alcock, 2004, 2009), or the logical and mechanical aspects of proof-writing that can be made without appealing to the semantic meaning of the concepts involved. These syntactic skills included recalling and stating relevant definitions, making deductions by applying rules of logic, and applying an appropriate proof framework (Selden & Selden, 1995) in which the hypotheses and conclusions in the proof allow the proof to establish the theorem. Included in Alcock's structural thinking goal was the use of precision when expressing mathematical statements and common templates or schemas for writing certain types of proofs (e.g., proof by induction).

A second goal in Alcock's framework was *instantiation*, "to meaningfully understand a mathematical statement by thinking about its referent objects" (p. 79). This goal involves students having the disposition to instantiate concepts when encountering a new definition or working on a proving task with the aim of reducing the abstraction of the material, as well as having an arsenal of examples to consider.

A third goal in Alcock's framework is *creative thinking*. Alcock described creative thinking as a process of examining instantiations of objects "in order to identify a property or set of manipulations that can form the crux of a proof" (p. 82). Creative thinking might be 'direct' in which an example is explored in the hopes of finding a set of generalizable claims or logical steps that would yield a proof. Alternatively, creative thinking might be indirect; exploring an instantiation in order to generate a contradiction or find a counterexample.

The final goal of Alcock's framework was *critical thinking.* Alcock (2010) described critical thinking as students having the inclination and capacity to check the correctness of a proof. To Alcock, checking the correctness of a proof did not only involve verifying the logical correctness of each step in the proof. Validation also required the student to explore the semantic meanings of statements in the proof. Students should have the ability to identify errors in a proof by considering example objects which may show a statement in a proof is false or an inference within a proof does not generally hold. Further, students should develop the disposition to engage in this process. Alcock summarized, stating that students should be "searching for possible counterexamples, checking for implied properties that are false and/or checking for properties that should be preserved" (p. 83).

We use Alcock's theoretical frame as the backdrop of a semi-open coding scheme to analyze our data. Alcock's framework is a valuable tool to make sense of our data by providing us with a theoretical orientation to recognize and organize mathematicians' rationales for choosing the homework problems that they did. At the same time, we use our data to offer a refinement of Alcock's framework. Alcock (2010) developed her framework based on interviews with five mathematicians in the context of a transition-to-proof course. Transition-to-proof courses typically have limited conceptual content so that students can focus on mastering the mechanics of proving. Our research occurs in the context of an abstract algebra course, where developing a conceptual understanding of algebraic concepts is presumably a prominent goal of teaching the course. Consequently, our work in this context can be used to expand Alcock's framework from transition-to-proof courses to content-based proof-oriented courses. Further, Alcock's mathematicians cited relatively few pedagogical strategies that mathematicians used to promote instantiation, creative thinking, and critical thinking, even though the mathematicians identified each of these learning goals as important. In the majority of the cases, the mathematicians in Alcock's study said instantiation, creative thinking, and critical thinking were encouraged through the homework exercises that students were assigned. Our work augments Alcock's by identifying how specific exercises might promote the goals that Alcock's mathematicians identified as important.

4. Methods

The primary aim of this study is to identify how mathematicians develop homework assignments to support student learning of abstract algebra via exploring the reasons they gave for choosing particular problems. We explore the learning goals that their problems and assignments support, and the reasons they believe those problems will support students in meeting their learning goals. To explore this issue, we asked the mathematicians to construct homework assignments for three sections of a common abstract algebra textbook (Gallian, 2013) for a first-semester abstract algebra class. This had the value of engaging the participants in a regular pedagogical practice while allowing for reflection and responding to interviewer questions.

4.1 Participants and Context

We solicited participants from two research universities on the east coast of the United States by asking any faculty member who had taught introductory abstract algebra within the last seven years to participate in the study. We secured the participation of six mathematicians who met this criterion, three at each university. Two participants were female and four were male. Two were members of underrepresented minorities, and four were white. Four were tenure stream faculty at the time of their interview while two were long-term instructional faculty. All but one had taught the course at least three times. We assigned all participants pseudonyms with gender-neutral pronouns in order to protect their anonymity.

In the United States, abstract algebra is an upper-division course. The population of the course consists predominantly of mathematics majors (i.e., university students pursuing a mathematics degree) in their third or fourth year of study. Abstract algebra is typically a required course for mathematics majors to complete their degree. Although schools usually offer a second course of abstract algebra, many students who complete a first abstract algebra course will not take the sequel. Courses are usually taught in a lecture format with small class sizes—typically under 40 students. Homework is generally required to be submitted for a grade. All of these typicalities held for the six participants in our study. When the participants in this study taught abstract algebra, they would meet with students for three hours a week over the course of a 15-week semester. Participants were given a large amount of autonomy when teaching their abstract algebra course, including the prerogative to create their own homework schedule and choose their own homework problems.

4.2 Data Collection Procedures

Prior to the interview, we emailed each participant a photocopy of the relevant sections of Gallian's undergraduate abstract algebra textbook (2013). If they did not arrive with the homework assignments already prepared, we presented each chapter separately and sequentially and asked them to develop an assignment for the relevant chapter. Due to time constraints, some of the participants only constructed homework sets for two of the chapters and so we concentrate our analysis on the chapters for which all of the participants constructed assignments. The two common chapters were about Groups and Group Isomorphisms. Each participant met individually with an interviewer for a task-based interview lasting between one and two hours. The interviews were audio-recorded and any written notes the participants made were scanned.

The goals of the interview prompts were to explore which items participants chose to assign for homework and participants' rationales for choosing the problems that they did, including what learning goals the items might achieve and how they anticipated students would engage with the problems. Specifically, for each chapter, questions focused on how participants grouped problems, why each problem was chosen, why other problems were not chosen, and which problems were most interesting to them. We hoped that by having participants engage with the same sections of the same text, we would observe common problem choices. We could then use these selections to gain insight into the constellation of what our participants wanted their students to learn about groups and isomorphisms (and related concepts), how students can learn these things by completing homework, and how the specific problems selected by our participants would be conducive to achieving their pedagogical goals. Overarching questions about the role and importance of homework were asked at the end of the interview. The goal of the overarching questions was to solicit larger scale beliefs about homework and homework practices more generally; this included exploring, beyond "habit," why the participants would assign homework, and how homework complements the in-class activity.

4.3 Data Analysis Procedures

After the interviews were transcribed, our first step was to parse the transcript into discrete chunks using the following rules:

- All contiguous text about one particular numbered textbook problem should stay together
- When the content of the oral text switches to another numbered textbook problem, we create a new chunk of text
- Text that concurrently addresses multiple textbook problems should stay together.

Similarly, we chunked any "general" claims the participants made (e.g., describing overall goals or approaches, but not discussing an individual item) while discussing individual homework assignments and separated them from discussions of specific items. We also chunked complete responses to each of the final set of ‘general’ questions.

In our initial pass through the data, we noticed that mathematicians frequently justified their choice of homework problems based on the particular learning goals that they had for their students. To better understand the relationship between the selection of homework problems and learning goals, we coded participants’ reasons for choosing homework problems using a semi-open coding scheme, with Alcock’s (2010) four proof-related pedagogical goals as our background theory. Because Alcock’s pedagogical goals were derived from interviews with instructors of a transition-to-proof course, which in the US context is generally ‘content free,’ we reserved the right to create new categories to ensure our analysis was grounded in our data. We identified six learning goals that participants had for assigning homework, four of which aligned well with the four goals from Alcock’s framework.

5. Results

5.1 Overview

Across our participants, the most important reason for assigning homework was to help students learn the course content. We highlight three points to justify this claim: homework was described as highly important for learning, homework played a communicative role for both students and faculty, and summative assessment was only mentioned once as a value of homework.

Five of the six participants indicated that homework is an essential activity for learning mathematics. Eli, Lennon, and Parker described homework as “very important” and August and Dakota referred to homework as “the most important thing”. As August noted, “I think that’s where the learning happens”. Some participants also described specific learning goals. For instance, Dakota described the importance of assigning homework as the chance to apply the definition of abstract concepts to concrete examples:

> They [the students] need to work out concrete examples and they need to also test their understanding of what can and cannot happen in the abstract situations they’re involved with. You’re given a definition of a group. There’s not too much to it. There’s not much there. But to start to think about what can and cannot happen in groups is important and you learn that through the examples. You learn that through the homework.

Our interpretation of this excerpt is that the definition of a group can be stated succinctly, but that to experience the richness of the group concept, learners need to explore various groups to see what properties a group can or cannot have. Homework is an opportunity to engage students in this exploration. Together, these quotes support our contention that instructors view homework as a critical part of students’ learning experience and an important part of mathematics instruction.

Two participants—Hayden and Eli—both described two other purposes for homework, as providing feedback for the student and the instructor. In terms of providing feedback to the student, Hayden remarked that, “A student needs to know, ‘I misunderstood this concept. I can correct it, and then I can give new results based on this idea that I understand now’” and Eli remarked that instructor feedback could provide students with “a more accurate sense of their understanding”. In terms of providing feedback to the instructor, Hayden claimed that students’ performance on homework might provide “feedback that they’re giving to me as to problems that I thought that they would be successful at and they weren’t successful at, tells me that I need to do something a little differently”, that is, if students collectively perform poorly on an assignment, different teaching practices may be necessary.

We note one potential purpose of assigning homework was hardly mentioned: that of summative assessment. Indeed, only one participant mentioned summative assessment at all in responding to the question. Hayden felt that as students have different needs, the importance of homework varies by student:

> I have an obligation to let ... to communicate to them that the number of problems on this sheet is an arbitrary number that I’ve picked, that some of you are gonna need more than these, and some of you won’t need this many. And assigning a common set of homework for all students is not doing an equal job of facilitating learning in all students. It’s just the way that we do things is I have to grade everybody by a common set of criteria, and so I have to pick a common set of problems. But if you need more than these people or less than these people, that’s not a weakness on your part, it’s just a different way of approaching the material. And I’m here to support you if you need to do that.….So, I think it is important but I think its importance is relative, and not equal for all students.

Here Hayden cites the institutional obligation of assigning grades as a justification for giving a common set of problems for all students. This was the only mention of grading or summative assessment across our corpus of data. In general, the participants viewed the primary purpose of homework as providing students with learning opportunities.

Because the participants claimed that supporting student learning was their primary reason for assigning homework, we primarily organize our results by the types of learning that their problems were intended to provoke in their students. We have created meta-categories for these learning goals, that we call themes. These themes illustrate how the participants attended to learning goals, student thinking, broad mathematical habits, and more. In the following sections we present and illustrate each of the themes and follow with some additional findings and observations.

5.2 Identified Themes among the Provided Rationales

In our analysis, we identified six themes, which are listed in Table 1. Our analysis focuses on connecting these themes to Alcock’s (2010) framework and highlighting new themes that Alcock did not identify from her interviews of instructors in introduction to proofs courses. We provide examples of these themes below.

Themes	Number of Participants	Which participants
Knowing and recalling axioms and definitions	6	All
Developing an arsenal of examples	6	All
Developing new problem approaches	6	All
Remediating misconceptions	4	Eli, Hayden, Lennon, Parker
Making connections to prior	5	August, Dakota, Eli, Hayden,

and future material		Lennon
Valuing reading the notes or text	4	August, Eli, Hayden, Lennon

Table 1. Description of which participants used each theme.

5.2.1 Structural thinking: Knowing and recalling axioms and definitions

The learning goals aligned with Alcock's (2010) description of structural thinking include that students develop fluency in the logical and mechanical aspects of proof writing. This includes recalling and stating relevant definitions. All six of the participants provided this rationale for at least one item's selection. This expectation consisted primarily of recalling the statements of definitions and axioms and applying them to direct and routine tasks, which we interpreted to mean logical manipulations to arrive at desired results.

August explained that, in order to prepare a homework set, it is important to identify the needed definitions and then seek problems that directly address them.

> I would go back to the beginning and look at, all right, what are the definitions that they're supposed to absorb from this section, because definitions are your bread and butter. I even tell them, when you come to office hours, I want you to know your definitions. I can't teach you words. I can teach you how to put words together into sentences and maybe put sentences into paragraphs, but I can't teach you the words. I want to assign them exercises that help drill those words in their head, and also have an arsenal of examples.

We interpret this to mean that August sees knowing the exact formulation of the definition as an essential first step toward writing proofs.

Beyond Alcock's (2010) framework, substantial work has shown that knowing definitions is critical to mathematical thinking, conjecturing, and proving (Alcock & Inglis, 2009; Alcock & Weber, 2010; Edwards & Ward, 2004, 2008; Lay, 2009), though generally students do not understand the purpose of definitions well (Edwards & Ward, 2004, 2008). August attempts to address this by choosing problems, such as one asking students to prove that $\phi(x) = \sqrt{x}$ is an automorphism of the positive real numbers, "because this is going to force students to go through the definition." In other words, because the problem "forces" students to state and use the definition, students will have the opportunity to learn what the definition is and how it can be applied.

For Hayden, the goal of knowing and using definitions and axioms has a unique importance. On the first day of class, Hayden assigns a short set of problems focused almost entirely on this foundational knowledge to be due on the second day of class. The reason for focusing on this foundational knowledge is to facilitate the group work that students will participate in on the second day of class:

> For that second class, I don't want to have students in the groups that are not carrying their load. I would like to have some way of assessing that they have done a minimal level of preparation. If I assign some problems that require a minimal level of preparation, can you read the book, can you take this definition and work it, apply that definition to this specific example, at least they know what they're talking about. I feel like I'd just get a better, better second class out of the section, as a result of assigning some things from the first day.

Hayden cited the example of exercise 6 from the isomorphism chapter as one problem that was chosen expressly for this purpose. The prompt reads, "Prove that the notion of group isomorphism is transitive. That is, if G, H, and K are groups and $G \cong H$ and $H \cong K$, then $G \cong K$" (Gallian, 2013, p. 130).

> One of the first-round problems I chose for the isomorphism was actually just to prove that isomorphism is transitive, which again is just definition pushing. It's not a challenging proof, but it is a proof, at this stage, that by the time we get to section six in the course, I'm not expecting that asking them to... prove something that is entirely based on the definition, should challenge them at that point.

Again, we view this as Hayden identifying students knowing definitions and axioms and then "definition pushing" as an important starting point in students' progress toward learning about isomorphism and writing proofs about isomorphisms, which suggests an attempt to promote structural knowledge within the context of abstract algebra.

As a final example, Lennon's rationale for selecting exercise 26, in Chapter 2, which asks students to "Prove that if $(ab)^2=a^2b^2$ in a group G, then $ab=ba$" was:

> First of all, battle with the axioms, as much as, given a relation, what can you deduce from that? Things like that. There are a bunch of problems like that, for example, number 26 is one of those. Just straight from the axioms, write down the equation, manipulate it…

Here, Lennon's formulation is perfectly aligned with Alcock's structural thinking, starting with the definitions, or, in this case, the equation, and performing logical manipulations to arrive at the desired result.

5.2.2 Instantiation: Developing an arsenal of examples

Alcock described mathematicians as wanting students to engage in *instantiation*, "the goal of which is to meaningfully understand a mathematical statement by thinking about its referent objects" (2010, p. 79), which requires students to have an arsenal of examples of particular constructs. All six of the participants provided a rationale for selecting at least one item that aligned with this learning goal. Five explicitly declared that learning a collection of examples was an overarching learning goal of their homework assignments. The instructors' comments indicate that, through these problems, they are developing the students' capacities to instantiate.

August claimed the following as goals for student outcomes from the assignment about groups. (Part of this quote also illustrates and is used in section 5.2.1.)

> They're not going to understand what a group is deeply right off the bat. I'm not expecting that. I wouldn't expect that until they know how to teach it or do research on it. That's not the goal. I want them to know the definition, that's their bread and butter, know the definition. Those are your words and have an arsenal of examples, that's it. That's already hard enough.

In this quote, August claims that students should have many examples at the conclusion of the assignment, without any additional specific claims about what the students should be able to do with them or why having the "arsenal" might be valuable. August explicitly used the exposure to examples theme, for example, in assigning item 22 from the groups-based homework. The item reads, "Give an example of a group with 105 elements. Give two examples of groups with 44 elements" (Gallian, 2013, p. 55). August's rationale was:

> I like this number 22. It's very clear cut because you're providing examples. You know what? They could probably even go through their arsenal of examples that the author has provided and even reread them more to figure out which ones fit this criterion.

Notice August included item 22 in order to support student exposure to examples, so that, after completing the assignment, a student would have "an arsenal of examples."

Hayden gave another variation on this theme for selecting homework items, including item 2, which reads, "Referring to example 13, verify that subtraction is not associative" (Gallian, 2013, p. 54). (Example 13 reads, "The set of integers under subtraction is not a group, since the operation is not associative" (p. 46).)

> I can assign a problem that says, "Verify something from example 13." I at least know everybody in the class has read example 13 and that I have a common example that we can use for building into that. Sometimes I would be attracted, for that first group, to problems that specifically tied into a specific thing from the book. Here's one where I did this.

That is, Hayden hoped that students would learn an example and that this would also ensure that students would have a shared example that, in subsequent class discussions, could be used in a variety of ways to support student understanding of group-theoretic content and subsequent instantiation opportunities.

5.2.3 Creative thinking: Developing new problem approaches

Alcock (2010) described creative thinking as the process of examining instantiations of objects "in order to identify a property or set of manipulations that can form the crux of a proof" (p. 82). Creative thinking might be 'direct', in which an example is explored in the hopes of finding a set of generalizable claims or logical steps that would yield a proof. Alternatively, creative thinking might be indirect: exploring an instantiation in order to generate a contradiction or find a counterexample. All of the participants provided a rationale for selecting at least one item that aligned with this learning goal. In general, participants attempted to support students in learning insights and important ways of thinking about group theory by assigning items that would require novel approaches. Their rationale appeared to be that if students are able to successfully complete these problems, they will have necessarily engaged in this productive way of thinking, giving them the opportunity to be exposed to, and witness the utility of, that way of thinking.

This type of item choice commonly occurred in the context of isomorphism. Lennon, Parker, and Dakota justified choosing a problem about non-isomorphic groups because they felt it important for students to think about algebraic structures in terms of invariant properties. Demonstrating that two groups are not isomorphic requires students to identify invariant properties of groups that are present in one group and absent in the other. Parker assigned the following problem:

> Suppose that G is a finite abelian group and G has no element of order 2. Show that the mapping $g \rightarrow g^2$ is an automorphism of G. Show, by example, that if G is infinite, the mapping need not be an automorphism.

Parker justified this choice of problem by saying, "the idea is to use, 'a map of a finite set to itself is one-to-one, if and only if it's onto,' which is a trick which is used many times in group theory, so they should be well-aware of this trick". Observe that Parker considered this method of showing that a mapping of a finite set to itself is a bijection to be a common trick, and therefore useful to include.

As a second illustrative example, consider Eli's claims about item 40 in the isomorphism section. Eli provided the following rationale for assigning the item:

> Show that every automorphism five from the rationals to itself has that four and five X equals X times five Y....This question is beautiful because it involves several concepts. … You have to be really careful on how you tackle this problem. ... You see one over Q as one rational element, but it happens to be the invertible element of Q with respect to multiplication, but it's not the invertible element in respect to addition. That's the tricky part in this problem. How do we manage the case when we're dealing with one over Q? ...What does that mean when you multiply by Q? This is crucial here. When you multiply by Q and Q happens to be positive, multiplying by Q means you are adding that amount Q times. That's something that students don't see immediately here.

Here, we interpreted Eli's claim as indicating that this item required a new problem approach. When combined with the explanation of the mathematical content of the item, what the invertible element is, and the challenge with multiplication, Eli's rationale for assigning this item suggests that students will engage in a process of examining the structures and objects carefully in order to identify the crux of the proof. That is, we read Eli's explanation as an example, grounded in the content of algebra, of Alcock's (2010) construct of creative thinking.

Three participants highlighted the difficulty some students might face with certain problems but seemed to view problems that not everyone could do as an acceptable form of differentiation. Parker assigned basic group theory problems and computations about the groups U(n) to help students realize how group theoretic operations are more general than addition and multiplication on Z, Q, and R. In particular, squares of many elements can be 1. Parker noted:

> Being able to do problems on groups, *if they can manage to do the problems*, and they'll presumably have some idea what a group is, so that's what I do. So I think *most students*, by doing these problems, will get to know what a group is. [Italics were our emphasis.]

We interpret Parker as saying that students who cannot make progress on the problems assigned will not be able to learn what was desired from the problems. Hayden chose to use some of these problems as objects for discussion in the following class so more students could be exposed to productive ways of thinking. Lennon similarly acknowledged this challenge when assigning the problem of proving that Z under addition is not isomorphic to Q under addition because it requires thinking about the group theoretic structure in each group. Lennon acknowledged that, "I'm not sure everyone can do it but I would assign it. At least they would think about infinite groups. So just counting elements won't do that problem". Lennon did not elaborate on this point, but seems to suggest that even if students fail to complete the problem in question, they will at least be thinking about infinite groups and potentially develop productive ways of thinking about proof with infinite groups. We might interpret Lennon's claims as saying the students will be engaging in 'productive struggle' and they will benefit from doing so (c.f., Warshauer, 2015) even if only some students successfully complete the problem. In short, participants chose problems that were intended to foster opportunities to use a novel approach that could also be a useful technique in subsequent proof contexts.

5.2.4 Critical thinking: Remediating misconceptions

Alcock (2010) described critical thinking as students having the inclination and capacity to check the correctness of a proof by checking potential counterexamples, exploring the meaning of statements, checking whether any properties used were incorrectly assumed, or looking for properties of objects that should or should not be preserved. None of the participants directly claimed that they wanted students to learn to check the correctness of proofs via the homework items, but four of the participants claimed learning goals that aligned with aspects of critical thinking. In particular, we interpreted the participants' claim that they were assigning items to force students to confront possible misconceptions, especially assumptions of commutativity, as a type of checking whether any properties were incorrectly assumed.

The misconception that was most cited was that students commonly believe that all groups are abelian, based on much of students' K12 experience involving commutative operations. The participants wanted students to understand that non-abelian groups can exist and that some techniques they had used in the past do not "work" in the case of non-commutative groups (e.g., that properties are not preserved in a more generic space). Three participants specifically picked homework problems that they believed would require students to confront these issues (Chapter 2, numbers 6, 8, 15, 17, and 26). Hayden also wanted to remediate the misconception that all well-defined operations are associative. Eli selected an item to confront misconceptions about the zero exponent in new structures. Finally, Lennon selected an item that illustrated a "surprising result," which we took to be conceptually similar in focus and intent. In total, four faculty used a 'misconceptions' theme for selecting homework items on at least one of the assignments.

For Parker, Hayden and Eli, who selected items to help students confront the misconception that all groups are abelian, they explained the importance of eliminating unwarranted assumptions of commutativity by focusing on subsequent problems the misconception would cause in understanding concepts or proving theorems. Parker noted that students will have difficulties with homomorphisms if they assume that all groups are abelian, claiming, "These power maps can only be homomorphisms in the abelian case. Sometimes students think they can be a homomorphism, even if the group's non-abelian." Hayden similarly noted the difficulties that students might face in dealing with normality if they assume that all groups are commutative—in that then all subgroups are normal. Eli noted that "Students tend to get lost when they try to change the conditions," and explained that one of the goals for the assignment was to remediate student misconceptions:

> As I said, they usually assume everything is associative, everything is commutative, so it's better to get rid of these type of mistakes at the very beginning. That's why I picked those examples. Someone might say, "Well, checking the subtraction is not associative [is] straightforward." Yes, but you do it by your own, you will never forget it.

We interpreted Eli as claiming that through the process of doing work, students will learn the desired concepts in a way that they would not from lecture and remediate the misconceptions that all operations are associative and commutative.

Two instructors used the same problem in the first chapter to help students learn to not assume structures are commutative. Eli instantiated this by choosing to include item 6, from Chapter 2, which stated, "Give an example of group elements *a* and *b* with the property that $a^{-1}ba \neq b$. Eli's rationale was:

> I like number six because students usually get confused with respect to an operation in the sense that they just feel it's commutativity all the time. It's very common to see that a student wants to cancel out terms immediately just by interchanging the order of the elements. That's very common. Even if you emphasize that in class with an example, some of them are so attached to the commutativity of the set of the real numbers that they resist to come up with their own examples of something non-commutative. That's why, to emphasize that particular property, I pick problem number six. They need to come up

with an example. They need to create their own example. That also pushes the student to think about new possible sets with new possible operations, not something that they already know.

Eli believed that students frequently assume that all operations are commutative and that by requiring students to generate an example, they would have to work with a group they likely have limited experience with: a group where $a^{-1}ba \neq b$. This provides students with the learning opportunity to see that the group the student has generated is not abelian and to see that group operations can expand beyond the common operations that students already know.

Parker had similar goals for the homework assignment and chose to include item 6 on the homework for similar reasons—to remediate students' common assumption that all groups are commutative. Parker claimed:

> Yes. So, one reason for this is that I often like to sort of emphasize that groups are non-abelian and things don't commute, and this is often, beginners get muddled by this. I see so often in [course number], they have an expression like the one given there, and they just say that's b.

Here, Parker included item 6, because it invites students to perceive that "things don't commute" and help students develop mathematically normative conceptions of groups. That is, these four participants assigned problems that forced students to confront what they believed to be a common unwarranted assumption of commutativity, which is a form of engaging in "searching for possible counterexamples, checking for implied properties that are false and/or checking for properties that should be preserved" (Alcock, 2010, p. 83). The participants suggested that this would also encourage students to consider what other hidden assumptions they make about how groups work that they have (incorrectly) assumed from their experiences with real number operations.

5.2.5 Making connections to prior and future material

Three participants used homework problems to ground current material in earlier work from the course and prior courses. August wanted to make sure students had really internalized skills from their introduction to proofs course and would test their abilities accordingly. Lennon would highlight problems that used earlier skills, such as using a problem about dihedral groups after an earlier homework had focused on dihedral groups. However, of our instructors, Hayden was the most focused on connecting to earlier material, especially to linear algebra and proof techniques.

> So, in some cases because they tied into something that's coming up or something... that they've seen before, that if I can build a linkage between this topic and something they've seen before, I feel like there's some value to that. So the example of the matrix problem that ties into something they've already seen in linear algebra, or the problems that are setting up ideas that we're going to see later on, like the conjugacy problem or the baby isomorphism problem in this section. I feel like those are pretty valuable. Another thing that I try to do more in this very first section [groups], that I didn't do as much of in [the isomorphism] and [homomorphism sections], is try to come up with sort of a quick rundown of things that they may have seen in intro to proofs of an example of a problem that would be easiest to solve with a direct proof, a problem that would be easiest to solve with an indirect proof. A problem with a quantifier, some different kinds of proof techniques that we could ... this reading early in the course, that that would give me a chance to sort of bring in a discussion of different kinds of proof techniques for that.

Hayden wanted to connect to earlier material from linear algebra and build on students' knowledge of proof in early parts of the course while laying a foundation for later algebra material, like isomorphism and quotient groups, to help tie the course together. We interpreted this as designed to create a more coherent learning experience for students.

Additionally, five participants made some reference to using homework to preview future material in their course or in application to other courses. For example, Lennon focused on applications to physics and other aspects of algebra by taking a historical perspective.

> The connections to linear algebra and physics, symmetry groups, the dihedral groups, and number [theory] with the prime number. These are some important connections and some important roots of group theory, so this is what I would hope for them to see….but also richer problems which give you background, which lead you to battle with a group. Like the dihedral groups, Heisenberg group, integers mod a prime, and that eventually leads to the notion of a field.

Lennon's goal was to use the history of group theory to show natural extensions to developments in algebra as well as other courses.

Eli highlighted that some homework problems could extend to connections outside algebra, specifically quantum mechanics. Such problems could give opportunities for extensions of learning, but would be impractical to talk about in class because too much background knowledge was needed:

> Yeah, this [exercise 34 in Chapter 2] is the Heisenberg group. This group has a lot of applications in physics and quantum mechanics, actually. I will not go in depth through those properties, but I would mention that if someone is interested, come to my office to discuss about those. Yeah. The problem of discussing this in class would be expanding a lot of knowledge with respect to physics. That's why I prefer to assign this as a homework problem. If someone becomes interested, stop by and chat about it.

Instead of expecting everyone to learn the required physics background, Eli thought this type of problem might encourage students to come to office hours to learn more about applications. Other instructors highlighted connections to courses like cryptography (Hayden) and computer science (Dakota) or assigned problems because they would show students a trick that could be used frequently in later algebra studies (Parker). This claim aligns with Dorko's (2019) finding in the domain of calculus where instructors wanted purposeful connections to prior and future material.

Alcock did not identify this kind of connection-building as an explicit goal in her framework, and so in recognizing it we extend her framework. The professors in Alcock's study were asked about broader course goals and teaching strategies, and it may be that in pursuing finer-grain questions related to individual homework tasks, we have uncovered secondary goals that were simply not at the forefront of Alcock's professors' minds. It also seems likely that, with a content-specific focus, our instructors were more apt to reflect on the connections they naturally make between subjects, which would be less likely to come up in the discussion of a content-neutral proofs course.

5.2.6 Valuing reading the notes or text

An additional theme for selecting homework items was to select items that referenced the text or notes so that students would be forced to re-read them, meaning the participants wanted to promote reading the text and notes via the homework. Four of the participants, Eli, August, Hayden, and Lennon, used this theme to select items. Additionally, Parker and Lennon wanted homework items to make explicit connections to what happens in class, although Parker did not use an explicit theme of forcing students to read the text and notes. We illustrate what the participants said about why they used this theme with an example from August, who provided an explicit description of this theme for problem selection, along with some explanation for why problems that force reading of notes or text are valuable:

> A couple of easy to get started, most of them medium, really think about, check your understanding, if you have reread the text, you have to reread your notes, perhaps, to do the problems. … [One of my goals is] to force them to read through the notes again. I actually, I tend to lecture pretty fast. I write a lot on the board, though, and I ask them to take pictures. They never do it, though, because I want them to rewrite their notes if they're that type of learner who learns that way. … Whenever I ask, "How do you learn?" they always say, "Oh, I learn by writing." They always do that. Okay, those of you who write, rewrite your notes. For the homework I would want them to basically, where they would have to value their notes in a very careful way. Also reading a book is very helpful, so I like problems where you have to go back in the chapter and look at example number 16, read it carefully, and see where you can add on to that material.

August makes explicit reference to an overall goal of having students reread their notes and read the text. Furthermore, August viewed writing of class notes as a primary way for students to learn, so forcing them back into their notes, possibly even prompting them to re-write their notes, is a purposeful attempt to support and improve student learning. That is, for August, and the other faculty, this problem-selection theme of forcing students to re-read notes and to read the text, was explicitly to support and improve student learning.

Both Eli and August chose to include Item #10 from Chapter 2 on the homework: "In the notation of Example 16, verify that $T_{a,b}T_{c,d} = T_{a+c,\, b+d}$." They offered similar rationales for the inclusion; for example, August claimed, "This is one of those things where they have to go back and read the book and finish stuff. Yeah, I like that, so number 10, because anything where they have to not just passively read the examples, actually have to go through and add some stuff to them. Yeah, I like that, so number 10 is good." Notice August repeatedly referenced reading and adding to the text as a reason for selecting the item.

We note that both Parker and Lennon made a related claim that we chose to highlight—that their choice of items depends on what they have done in class. For example, Parker claimed, "It would depend on sort of how I was teaching the course. I briefly looked at the chapters it was on, so presumably I'll be teaching at least portions of that chapter. I try and look for problems which are close to what I've been teaching." We interpreted Parker's claim as meaning that what Parker does in class is in a dialectic with the choice of items for the homework assignment—that the structure of the chapter would affect the classroom instruction, and that the homework items would be chosen in a way that reflects what the instructor did in class, thus increasing students' opportunities to learn through repeated exposure to the material.

Alcock did not identify valuing reading the notes or text as an explicit goal in her framework either. Much like the previous theme, it is possible that the focus on abstract algebra encouraged participants to consider specific content that they would want students to remember and review after class. Also, since Alcock's focus was primarily on what occurred in the classroom, it is perhaps not surprising that this goal, which must be met outside of the classroom, did not come up in the discussions she held.

5.3 Further Observations

In this subsection we report on two observations that we believe are of interest but were not the primary focus of our investigation or analysis. The first is a general claim about the creation of homework assignments that is not directly related to student learning: each instructor made some claims about balancing the relative difficulties of tasks across the assignment. For example, Lennon claimed:

> Is there some little bit of challenge in it? There are problems at all levels of difficulty, so I try to, as I said, reorder those problems. I would go larger by difficulty. There should be problems that every student can do. There should be problems, ideally, that maybe only the better students can do. There should be a range in difficulty.

Here, Lennon was explicit about constructing homework sets with problems with a range of difficulties so as to provide graduated challenges, which we might call the relative cognitive demand of the items. Dakota explained that, “I don't like to put too many of those long involved in any one assignment, so I may have only picked those two.” In short, the instructors were attending to both the amount of time and the amount of effort required to complete problem sets, mixing ‘harder’ and ‘easier’ problems to create a balanced assignment where all students could do some of the items. The participants’ claims about categorizing items by difficulty can be interpreted as an evaluation of the difficulty for each item. The notion of the ‘length’ of the assignment is about the difficulty that might be generated from a variety of tasks eliciting a variety of types of thinking.

Secondly, Hayden, Lennon, and August noted they normally taught homomorphism before isomorphism. This is contrary to the isomorphism-first ordering adopted by the four most common introductory abstract algebra texts in the US (Melhuish, 2015). We did not prompt for this information, and as a result, we can only report on those participants who specifically chose to comment on the ordering of the two morphism sections. This had some effect on the way that they engaged with the task of the interview. For example, Hayden noted a lack of comfort with choosing problems without knowing how Gallian (2013) developed the material to go from isomorphism to homomorphism. Lennon, for example, explicitly ruled out problems that were ‘in the wrong section,’ while August ‘hated’ the way that some problems were phrased. However, Hayden also noted the book chosen by the department would dictate instruction and all of the participants were willing to engage in the exercise of choosing problems from such a text. This lack of familiarity with the text and unfamiliar ordering can be understood as a limitation to the study in that it may have forced choices that the participants would not have made in a more naturalistic setting. Further research would be needed to explore the ways that mathematicians create assignments in each ordering of the content.

6. Discussion and Conclusion

The goal of this exploratory study was to analyze a previously unexplored aspect of teaching advanced mathematics: how do mathematicians choose homework assignments and what are their rationales for doing so? This study occurred in a specific context—in particular, with instructors of abstract algebra courses in the United States at universities that grant doctoral degrees. Clearly many of the findings that we reported here will be specific to this context, or to the modest sample of mathematicians who agreed to participate in our exploratory study. Nonetheless, we highlight some general phenomena from our data set that would be useful for future investigation.

Each of our participants highlighted that their primary purpose of assigning homework was to teach, not to evaluate students. That is, our participants highlighted how working on these homework problems was an essential component to having students learn abstract algebra. We believe that this finding is theoretically important for two reasons. First, this can resolve an apparent discrepancy between mathematicians' beliefs and practices when it comes to teaching advanced mathematics. On the one hand, mathematicians profess that students need to be actively engaged to learn mathematics (Johnson et al., 2018; Woods & Weber, 2020), but a weakness of lecture is it leads students to be passive recipients of content. On the other hand, most abstract algebra courses are taught via lecture and mathematicians do so because many believe lecture is the best way to teach (Johnson et al., 2018). A clear way to bridge this gap is that mathematicians believe that lecturing is only one component of how students learn. Another component, one that is perhaps more important, is for students to learn individually outside of class. The homework problems are meant to guide this learning. Second, we observe that mathematics educators seem to gauge the success of lectures primarily by the extent to which students can accurately interpret what the mathematician is trying to convey. For instance, Leron and Dubinsky (1995) claimed that nearly all students fail to learn from lectures because "*telling* students about mathematical process, objects, and relations is not sufficient to induce meaningful learning (hence the sorry state of affairs even with the best of lecturers)" (p. 241, italics are the authors' emphasis). Other scholars have documented that a lecture was unsuccessful because students could not recall the main points that a professor was trying to convey (e.g., Lew et al., 2016) or appeared to misinterpret specific utterances by the professor (e.g., Krupnik et al., 2018). We would encourage a broader lens for interpreting lecture success or failure. Perhaps a more important function of lecture is to prepare students to work outside of class to learn the content. Finally, we observe that some efforts to improve lectures in advanced mathematics involve finding ways to have the professor present content so it is clearer and more structured to the students (e.g., Gabel & Dreyfus, 2017). While we applaud such efforts, a future research direction might involve preparing students to learn on their own.

In our study, we used Alcock's (2010) framework of pedagogical goals for proving as an initial starting point to make sense of our data. We found six specific types of student learning that the participants wanted to promote via their problem choices: knowing and using axioms and definitions, developing an arsenal of examples, developing new problem approaches, remediating misconceptions, making connections to prior and future material, and valuing reading the notes and text in advanced mathematics classes. The first four of these learning goals can be understood as versions of the ways of thinking that Alcock (2010) described transition-to-proof instructors as wanting to promote. Our work corroborated the efficacy of Alcock's framework for analyzing the decision-making rationales for professors in advanced mathematics courses and extends her work beyond the transition-to-proof course from which the framework was developed. We believe we have elaborated upon Alcock's framework in two important ways. First, we have found two learning goals beyond the original framework. One of the learning goals involved implicitly teaching students effective study habits, with the hope that students will recognize that their textbook and course notes are a valuable resource. (Note that this is an instance of preparing students to learn outside of lectures.) The other learning goal involved forming links between students' prior learning and the content being taught, as well as preparing students for future learning. Second, we have already observed, like Alcock (2010), that mathematicians aimed to achieve many of their pedagogical learning goals through the assignment of problems. We extend this insight by exploring *how* mathematicians choose problems to achieve their learning goals. For instance, to teach students to recall and apply definitions, mathematicians assign basic problems that involve drawing basic consequences from the definition or verifying that objects satisfy a definition. Teaching critical reasoning could involve having students study concepts that violate their expectations.

These findings suggest three different directions for future research. The first involves checking the extent to which mathematicians' homework choices actually achieve their desired goals. For instance, does assigning problems that have students explore non-abelian groups reduce students' common inaccurate belief that the group operation will always commute? Does giving basic problems that involve performing standard verifications about a new concept actually help students recall and apply definitions? We imagine that mathematicians will find research in this area to be practical and important. If mathematicians give homework assignments to achieve particular pedagogical goals (as our participants claimed to do), then we presume that they would be interested in knowing which problems are most efficacious for doing so.

A second research direction involves investigating the extent to which our results generalize. For instance, is it a common practice to give students homework problems involving non-abelian groups to reduce students' propensity to believe that all groups are abelian? A confirmatory survey like Mejía-Ramos and Weber's (2014) would be one way to address this question. On the other hand, to what extent were our findings specific to the context that we studied—abstract algebra courses in the United States at doctoral-granting institutions? Would content matter? Perhaps so. Alcock and Simpson (2002) conjectured that real analysis was hard *because* students in real analysis were so familiar with the content that they naturally applied inferential schemes that were not generally valid. In this case, the goal of homework might not involve forming links to connect prior learning and the course content (as it did in our study) but helping students recognize the disruptions between their prior ways of knowing and their new learning. We also know that advanced mathematics courses taught outside the United States may differ considerably from our context—for instance, they may have much larger class sizes (which, if nothing else, may place practical constraints on grading and the availability of written lecture notes). Would these factors influence mathematicians' pedagogical goals or problem choices? These questions would be productive avenues for future research.

Finally, instructors may think deeply about the purpose of homework, but that does not necessarily imply that students understand or share their instructor's goals. More research is needed on how students perceive homework in their advanced mathematics classes. Do students agree with the participants that homework is primarily an (essential) opportunity to learn the course material? Or is homework viewed mainly as a way to accumulate points to offset potentially lower grades on exams? Perhaps implicitly, this study suggests the need for instructors to communicate their rationales for assigning homework. Alcock's framework may provide a structure for this discussion, if couched in student-friendly language. Nevertheless, more research on connections between instructors' and students' views of homework may provide useful insights into how this central component of mathematics classes can support student learning in a way that is clear to students.

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