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Mathematicians' Beliefs, Instruction and Students' Beliefs: How Related Are They?

It is generally accepted that teachers' beliefs impact their instructional choices, but characterizations of that relationship are limited in college settings. Furthermore, examinations of instructor beliefs, instruction, and student beliefs together in one setting are rarely described. Based on interviews with two Abstract Algebra instructors, classroom video from three units of instruction, and survey and interview data from students in the classes, this paper examines instructors' stated beliefs, ways these beliefs manifested in their teaching, and students' beliefs across the course. Both instructors made curricular choices clearly aligned with their stated views of the nature of mathematics, learning, and teaching. Day-to-day instructional choices reflected these stated beliefs as well, but the difficulty of material and tensions with other beliefs like the importance of interactivity manifested. Characterizations of the interactivity of classes and placement of the mathematical authority in class are provided through descriptive and quantitative measures. These characterizations of instruction provide nuanced portrayals of classroom norms and changes in those norms throughout the semester. Furthermore, subtle shifts in student beliefs about teaching and learning are noticeable, suggesting students' beliefs about teaching and learning math can be influenced even by modest changes in instructional practice.

Keywords: instructional practice, teacher beliefs, student beliefs, belief systems

Instructional practice is at the centre of what teachers do and is a crucial connection between teachers' beliefs and students' outcomes. While research has examined teachers' practices extensively at the K-12 level, limited research has examined college instructors' and mathematicians' teaching practice. Previous research has characterized what an Abstract Algebra lecture could look like but has examined general practices (e.g., Fukawa-Connelly, 2012) or relied on instructors' self-reported practices (Johnson, Keller, & Fukawa-Connelly, 2018). Blended instruction based on lecture and activity sessions have not been examined in detail.

Additionally, much of the research on college instruction has provided short snapshots of teaching during a single week or unit (e.g., AUTHOR; Lew et al., 2016). However, instructional practice can vary based on the form of a lesson or can be influenced by observation (Weston, Hayward, & Laursen, 2020). Mathematicians' research areas can also influence their approaches to teaching specific material, which is an effect not applicable in the K-12 setting (Fortune & Keene, 2021). Thus, examination of multiple units of college-level instruction is important for increased accuracy in characterizing instructional practices and to consider reasons why instruction may change from unit to unit.

Furthermore, abstract algebra instruction specifically is important because it is a class required for graduation for 89% of math departments training secondary pre-service teachers, and either required or generally taken in 95% of programs in the United States (Blair, Kirkman, & Maxwell 2013, p. 54). For math majors in general, Abstract Algebra is required by more than 85% of math departments across types of institutions (Blair, et al. 2013, p. 53). Not only is Abstract Algebra important, but it might also be considered a “best case scenario” for implementing and studying a variety of types of instruction. Many research-based reform materials, including the Inquiry-Oriented Abstract Algebra (IOAA) curriculum (Larsen, Johnson, & Bartlo 2013), have been developed for Abstract Algebra, providing more materials for active instruction in this course than for many other upper-level math courses (Johnson et al., 2018). Additionally, Abstract Algebra is often a terminal course, which potentially allows more flexibility with the course material.

The goals of this paper are, first, to highlight the affordances of two different ways of measuring instruction, the Inquiry Oriented Instructional Measure (IOIM) and the Toolkit for Assessing Mathematical Instruction—Observation Protocol (TAMI-OP).

Second, this paper is intended as a proof of concept for the usability of the IOIM in a non-Inquiry Oriented classroom. Third, giving more information about what non-lecture classes can look like is intended to aid those researching and seeking to implement such instruction. Finally, examining teachers' and students' beliefs in light of these descriptions of instruction permits examination of connections between students' beliefs and the instruction they experienced, which has not been studied extensively at the college level.

Literature Review

Beliefs impact the ways people perceive, interpret, and behave in situations (Pajares, 1992). Thus, numerous studies have been done on teachers' beliefs, including three summary chapters on teachers' beliefs in math education (Thompson, 1992; Richardson 1996; Philipp, 2007). However, less is known about mathematicians' beliefs or their impact on instruction. Furthermore, connections between instruction and students' beliefs have not been examined in many college contexts.

Teachers' Beliefs and Instruction

Some frameworks have been developed to model teachers' beliefs and beliefs systems, as well as their relation to instruction. Ernest (1991) conceptualized teachers' espoused beliefs of learning and teaching mathematics as being influenced by their overall epistemological and ethical perspective and their view of the nature of mathematics. Anderson, White, and Sullivan (2005) noted the influence of the social context of teaching on beliefs, including constraints and opportunities based on the students' development, knowledge, and understanding. Beliefs directly impact teachers' plans, which indirectly impact classroom actions and student performance (e.g., Romberg 1984). Ernest (1991) noted teachers' espoused models of teaching and

learning impacted their enacted beliefs of teaching and learning as mediated by social context. Wilkins (2008) noted that teachers' beliefs had the strongest direct effect on instructional practice while also mediating knowledge and attitudes.

One of the major distinctions between studies incorporating teachers' beliefs is how researchers address perceived inconsistencies in teachers' statements and actions. In many early studies, the purpose was to show differences between what a teacher claimed about their practice and what they did (e.g., Cohen, 1990; Cooney, 1985). Later studies have emphasized the importance of examining both teachers' beliefs and practices before drawing conclusions about instruction. Teachers should be observed for a long period of time before coming to conclusions on the relationship between practice and beliefs (e.g., Skott, 2001). Others have noted the importance of studying both beliefs and practices before making claims about how they are connected to each other (Schoenfeld, 2003; Speer, 2005; Speer, 2008).

While extensive research has been conducted on pre-service math teachers' beliefs (e.g., Jao, 2017; Phillipou and Christou, 1998) and in-service K-12 math teachers' beliefs (e.g., Beswick, 2012; Cross, 2009; Wilkins, 2008), fewer studies have examined teachers' beliefs and instructional choices at the university level. Many mathematicians highlight the importance of students actively doing mathematics when learning (e.g., Nardi, 2008) and instructors make different instructional choices to help students make meaning for themselves across different instructional settings (e.g., Jaworski, Mali, and Petropoulou, 2017). Weber (2004) noted that even though a real analysis professor taught in a Definition-Theorem-Proof format, the style of teaching varied based on the material and was taught in different ways for specific reasons.

Other research has focused specifically on the challenges and beliefs involved in implementing inquiry-oriented (IO) instruction. Wagner, Speer, and Rossa (2007)

examined an instructor's challenges with adopting an IO curriculum, including determining appropriate pacing. Johnson, Caughman, Fredericks, and Gibson (2013) examined teachers' priorities for instruction while using IO materials, specifically noting concerns of content coverage, goals for student learning, and student opportunities to discover mathematics. Fortune and Keene (2021) noted differences in students' freedom to explore the mathematics in an IO class between units closely related to the professor's research area and those less related, with more student freedom in the sections less aligned with the professor's research.

Some work at the university level has specifically focused on Abstract Algebra instruction. In their survey of Abstract Algebra instructors in the United States, Johnson et al. (2018) examined lecturers' perceptions of constraints on adopting non-lecture practices. Many constraints given, like the ability to utilize professional development and department coverage pressure, seemed to be internalized constraints. Highly influential experiences (ordered by decreasing frequency) were instructors' experience as a teacher, experience as a student, and talking to colleagues. Furthermore, they determined instructors who think lecture is best for instruction would lecture, but a factor was missing for the decision to not lecture. In a follow-up study, instead of dividing instructors into lecturers and non-lecturers, Johnson et al. (2019) divided professors into three categories according to instructional practices: alternative, mixed, and lecture. They discovered there were differences in how professors believed students learn and in students' capabilities. Interest in research also differed, with "lecturers" indicating the most interest in Abstract Algebra research and "alternative" teachers indicating the most interest in education research. Despite some trends, the missing variables in the models show there is still much to learn about the influence of teachers'

beliefs on their instruction. This study looks more carefully at two professors who self-identified as “mixed” and “alternative” under Johnson et al.’s (2019) framing.

Students’ Beliefs

A number of studies have examined students’ epistemological beliefs, their beliefs about the nature, basis, and breadth of knowledge and how we justify those beliefs (Honderich, 1995). In this section, we focus on prior research on students’ beliefs about mathematics and their development of beliefs.

Epistemological beliefs about mathematics

Epistemological beliefs about mathematics include what Muis (2004) called “nonavailing” beliefs about mathematics: beliefs that are not correlated with more learning gains (as opposed to “availing” beliefs that are correlated with higher learning gains). Some nonavailing beliefs students have about mathematics include believing mathematical knowledge as a field does not change, mathematical knowledge can be passively received from authorities, and that people have or do not have the ability to do mathematics. This idea of who can and cannot do mathematics is related to self-theories of intelligence, which include students’ views of the malleability of their intelligence: as a fixed entity or as a malleable quality that can incrementally grow over time (e.g., Boaler, 2016; Yeager & Dweck, 2012). Multiple studies have examined students’ views of mathematical intelligence and have noted that students with a fixed mindset often have more performance-oriented goals that can feed into stereotype threat for African Americans (Aronson, Fried, & Good, 2002) and women (Dweck, 2007; Dweck & Master, 2009), leading to lower performance.

Epistemological beliefs about mathematics also include students’ views on the nature of mathematics. A number of studies have asked students and faculty members to

describe their views of mathematics directly or through metaphors. Some studies have focused on animal-based metaphors for math (Markovits & Forgasz; 2017) while others allowed students to make their own metaphors (Schinck et al., 2008). Both studies emphasized that directly asking about students' attitudes and beliefs toward math might have been difficult, but this method allowed students to express their thoughts in a creative manner. Other studies have focused on what mathematics is in general education math courses (Szydlik, 2013) or math appreciation courses (Ward et al., 2010). These studies involving college students' statements and descriptions of math gave students an opportunity to critically consider their beliefs about math. In addition, by asking about students' math beliefs, researchers had an opportunity to consider whether those beliefs are availing or not, and whether interventions should be designed to change students' beliefs.

Students' beliefs about the nature of mathematics can affect their beliefs about what math classes should look like, including the forms of classroom norms and sociomathematical norms. Boaler (1999) noted students in "traditional" classes scored more poorly on standardized exams than students in a class focused on contextualizing math, and theorized this discrepancy arose because students in the "traditional" class saw math in real-life and classroom math as being disconnected. Gresalfi (2009) noted the difference in students' approaches and explanations to classmates in need of help based on the established norms for group work in each classroom, especially noting stronger explanations from the class where the burden of understanding was placed on the explainer. Similarly, Cobb, Gresalfi, and Hodge (2009) noted students' different types of agency that were activated in two different classes. In an Algebra I class, students were given only disciplinary agency, in which they could explain solutions with given solution methods, whereas in a Statistics class, students were also given

conceptual agency, in which they could choose methods and develop meanings between concepts. Students who were in both classes experienced very different sociomathematical norms in the two classes. Although the researchers had hoped that students would develop a need to fully understand their work in both classes, this did not happen. Thus, the classroom norms established through instruction can influence students' beliefs. How students' beliefs are affected, however, is not well understood.

Development of beliefs

Unfortunately, a number of students adhere to nonavailing beliefs of mathematics, and these nonavailing beliefs are often at least partially a result of instruction. Boaler (2000) observed that students felt the math classroom was monotonous and lacking meaning. Schoenfeld (1988) noted dangers could arise even from "well-taught" classes in which students participated and performed well as measured by standardized tests, but also generated a number of nonavailing beliefs. These nonavailing beliefs included proof processes not being connected to discovery, students thinking it should be possible to solve any problem in five minutes, a belief that only geniuses could truly understand math and others should just accept procedures "from above", and that success is measured by following the teacher's directions instead of by learning.

Although many nonavailing beliefs can be developed through instruction, availing beliefs can also be developed through instruction. De Corte, Op't Eynde, and Verschaffel (2002) cited studies in which students' beliefs about mathematical problem solving were improved by teaching through non-traditional, challenging problems and having students discuss the problems. Gresalfi (2009) noted that two successful students in two classes had similar beliefs about participation at the beginning of the year, but that the ways in which the students interacted with their classmates changed by the end

of the year. In the class where the burden of understanding was placed on the explainer, the student improved explanations by the end of the year, whereas the other student's explanations underwent limited change. By structuring group work, the teacher was able to help students develop mathematical communication and reasoning skills rather than only focusing on answers.

Some authors have also noted more availing beliefs about math arise as students become more educated (Muis, 2004). Muis et al. (2016) examined the beliefs of psychology students from secondary school (15-16 years old) to graduate school (23-30 years old) on math, psychology, and general knowledge. Students regularly had absolutist beliefs about math, meaning they expressed views of math knowledge as certain, relied on experts for the source of math knowledge, believed in an objective mathematical reality, and were certain of their beliefs. However, they were less absolutist about general knowledge and even less absolutist about psychology. The authors noted that students at increasing educational levels held less absolutist beliefs, especially the graduate students. The authors posited that this was because the graduate students were linking knowledge of the uncertainty of their own domain of psychology to what high-level mathematics might be like. Similarly, Szydlik (2013) examined the views of junior/senior math majors and math faculty as well as students in a problem-based class and observed similar increases in availing beliefs. Additionally, Schommer (1990) observed that age was correlated with believing the ability to learn is acquired and taking more college-level math classes was correlated with believing knowledge is tentative. Like Szydlik (2013), this study examines beliefs of upper-level math majors and faculty members. However, this is combined with an analysis of instruction to see what, if any, influence on these beliefs about the nature of math arise from class instruction.

Theoretical Perspectives

Beliefs are made evident through what is said about math, teaching, learning and what is done in the classroom. The main lens used for analysing beliefs in this study is Leatham's (2006) construct of sensible systems, though Ernest's (1989) views of the nature of mathematics are also incorporated. Because the IOIM is used to measure instruction, the central tenets of IO instruction, which underpin the IOIM, are also explained.

Researchers have emphasized that teachers' belief systems make sense to teachers and stating that a teacher's beliefs are inconsistent likely means analysis has been oversimplified. Leatham (2006) outlined the theoretical framework of "sensible systems," based on Green's (1971) belief system framework. Because teachers' beliefs could be clustered in ways that do not cause beliefs that seem inconsistent to an outsider to come into contact with each other, a teacher could consider their behaviour perfectly consistent. Furthermore, certain beliefs could be held as ideal while others are considered more appropriate for specific situations. Generally, he suggested that if a researcher concluded that a teacher's beliefs were inconsistent, the researcher did not have all of the information.

Ernest (1991) conceptualized teachers' espoused beliefs of learning and teaching mathematics as being influenced by their overall epistemological and ethical perspective and their view of the nature of mathematics. In his 1989 paper, he noted three main views of the nature of mathematics: the problem-solving view, the Platonist view, and the instrumentalist view. The problem-solving view pictures math as a continually changing and growing field that can be created and revised. The Platonist view pictures math as an unchanging, unified entity to be discovered. The instrumentalist view pictures math as a useful but disconnected collection of facts and

tools. These views of mathematics can influence teachers' views of learning or students' mathematics. For example, a teacher with an instrumentalist view could focus on one solution path as the only valid solution path for a problem based on a belief that a specific tool or algorithm should be used to address the problem. This lens is used to compare teachers' and students' views of the nature of mathematics.

Although lecture is the dominant instructional practice used for teaching Abstract Algebra, the IOAA curriculum provides a well-researched alternative to lecture. IO instruction is based on four guiding principles: *generating* student ways of reasoning, *building* on student contributions, developing a *shared understanding*, and *connecting* to standard mathematical language and notation (Kuster, Johnson, Keene, and Andrews-Larson, 2017). Generating student ways of reasoning includes engaging students in mathematical tasks so their thinking is shared and explored with the class. Building on student contributions involves taking students' ideas and using them to direct class discussion. Developing a shared understanding describes helping individual students understand one another's thinking and notation so a common experience can be "taken-as-shared" in the classroom (Stephan and Rasmussen, 2002). Connecting to standard mathematical language and notation involves transitioning students from the idiosyncratic mathematical notation and terms used in class to standard descriptions and notation, such as "groups". The three units of the IOAA curriculum focus on groups (Larsen, 2013), isomorphism (Larsen, 2013), and quotient groups (Larsen and Lockwood, 2013).

Methods

This study utilized a case study approach using two sections of Abstract Algebra as the two cases. According to Yin (2009), "A case study is an empirical inquiry that investigates a contemporary phenomenon in depth and within its real-life context,

especially when the boundaries between phenomenon and context are not clearly evident” (p. 18). Because instruction is difficult to study outside its real-life context, a case study approach is appropriate. Approval for this case study was given by the Virginia Tech Institutional Review Board (#17-1200).

The instructor participants, Alex and Bailey (gender-neutral pseudonyms), were two faculty members at a land-grant university in the Mid-Atlantic United States teaching an introductory Abstract Algebra course. Abstract Algebra is typically taken by math majors and courses include 15-30 students at this institution. These instructors were chosen from the instructors teaching Abstract Algebra in the 15-week data collection semester. Instructor Alex was a tenure-track professor who used the IOAA materials in class. Instructor Bailey was a full-time instructor who used a mixture of lecture days and “lab” days based on worksheets they¹ wrote. Both instructors had PhDs in math but not with an algebra focus, and both had taught the course in the manner described at least once before.

The student participants were chosen from Alex and Bailey’s classes based on a survey conducted in Weeks 3-4. Four student interview participants were chosen in each class from those who had been willing to participate and had given clear explanations of their thinking in the survey. (One student initially dropped out of the interviews and was replaced later in the semester to maintain eight interview participants.) Aeren, Abi, Ace and Andi were in Alex’s class, and Baker, Blaine, Blake, and Bryce were in Bailey’s class.

¹ The gender-neutral pronouns of “they/their/them” are used in place of he/she etc. for each participant throughout the text.

Participants engaged in semi-structured interviews (Fylan, 2005) lasting roughly one hour each. The two relevant interviews with each instructor occurred in Week 3 and Week 11 or 12 of the semester, and the two relevant interviews with students occurred in Week 5, 6, 7, or 13 and Week 14 or 15. Interviews were audio and video recorded and any written work was collected. The interviews were transcribed and coded in alignment with thematic analysis (Braun and Clarke, 2006). Statements relating to the instructors' views of math, math learning, and math teaching were coded using structural coding (Saldaña, 2016). These beliefs were identified based on responses to questions like "What is mathematics?" to identify their beliefs and "In what ways does your teaching style reflect your beliefs about mathematics, teaching, and learning?" to examine how they perceived their beliefs were reflected in their teaching. This allowed instructors to self-identify what they considered to be a belief as well as how it could be observed in instruction. Parallel questions to those of the instructors, like "What is mathematics?", as well as questions about changes during the course, like "In what ways, if any, have your beliefs about math, teaching, and learning changed or been strengthened by this class?" were asked of students. These allowed students to reveal their beliefs as well as reflect on how the course might have affected those beliefs.

Classroom data were collected during three units in each class and were analyzed based on audio and video recordings. Alex's class met twice each week in 75-minute class periods. Data were collected Week 3, in the middle of a unit on groups; Weeks 7-8, through the whole unit on group isomorphism; and Weeks 9-12, through the whole unit on quotient groups. Bailey's class met three times each week in 50-minute class periods. Data were collected Weeks 3-4, in the middle of a unit on groups; Week 7, through the whole unit on group isomorphism; and Weeks 8-10, through the whole unit on quotient groups. Data were organized and analyzed with the Toolkit for

Assessing Mathematics Instruction – Observation Protocol (TAMI-OP) (Hayward, Laursen, and Westin, 2020) and Inquiry-Oriented Instructional Measure (IOIM) (AUTHOR).

The TAMI-OP is an observation protocol that gives space to record what the instructor and students are doing in a classroom within 2-minute segments of instruction. For this study, the time spent on lecture, whole class discussion, students working, students presenting, and the number of questions and answers within the class were most relevant. Definitions used for coding are found in Table 1. TAMI-OP data rates are rounded to the nearest whole percent. Counts of time blocks refer to numbers of 2-minute blocks (e.g., 9/31 segments lecturing means 9 of the 31 2-minute segments had at least a portion of the time spent on lecturing). These definitions were used to allow broad categorizations of activity: instructor presenting to class (lecture), students discussing across the class (whole class discussion), student presenting to class (student presentation), or individual/small group work (students working). Because these measures relate to specific activities and require limited interpretation, a single coder recorded observations.

[Table 1 here]

The IOIM provided a way to characterize how IO a class was. The IOIM uses a five-point scale across seven different practices that reflect the four guiding principles of IO instruction. These seven practices are:

- (1) Teachers facilitate student engagement in meaningful tasks and mathematical activity related to an important mathematical point.
- (2) Teachers elicit student reasoning and contributions.
- (3) Teachers actively inquire into student thinking.

- (4) Teachers are responsive to student contributions, using student contributions to inform the lesson.
- (5) Teachers engage students in one another's reasoning.
- (6) Teachers guide and manage the mathematical agenda.
- (7) Teachers support formalizing of student ideas and contributions and introduce formal language and notation when appropriate. (AUTHOR)

IOIM practice scores in tables are listed by practice (e.g., column P1 shows Practice 1 scores). The IOIM characterization of how IO a class was is meant to characterize how student-driven the mathematics was, allowing a more content-focused lens for analysing discussions. Because the approaches to instruction were different in the lecture and lab days in Bailey's class, separate IOIM scores are given to reveal similarities and differences between them. As the individual who trained others to use the IOIM in AUTHOR, the author coded independently using the IOIM.

Nevertheless, limitations of the study include a singular coder, as other insights might have been made by a broader research group. Secondly, while an extensive number of class periods were recorded, further recording earlier in the semester or later in the semester might have produced other insights into instruction. Finally, inherent to the nature of qualitative research, interviewing more students from each class or changing the interview questions may have affected the insights gained.

Results

The results are presented by class section. First Alex's beliefs are summarized, followed by a characterization of Alex's teaching, a summary of student beliefs in Alex's class, and then an examination of connections across these contexts. This process is then repeated for Bailey's class.

Instructor Alex

Instructor beliefs

When discussing their view of the nature of math, Alex provided a problem-solving view (Ernest, 1989) that focused on math as a creative endeavour, at one point citing Freudenthal specifically, when describing math as a “human activity”. One such example is in the following excerpt:

Mathematics is a human activity, so I believe that it’s a way that people make sense and operationalize their world using symbols and numbers and notations and the rules that we have developed in order to manipulate those symbols and ideas and notations.

Alex also viewed math as including many tools, such as “concepts and logical proving techniques,” though they did not personally make use of math as a tool to solve problems and almost espoused an anti-instrumentalist view that did not look for math to be useful. For example:

With my experiences with mathematics, any time I was presented a scenario in which I was supposed to use it to do something, it felt very artificial and forced, and I know that’s not the case. I know there’s researchers here that very authentically use mathematics to solve problems and that was just never the math I did. So I don’t typically think of mathematics as a useful thing.

Alex also provided different metaphors to describe their view of mathematics, such as an unsolicited analogy to music that tied together the importance of different tools and notations, while also having a creative aspect:

We have the violin, and we have the clarinet, but we also have sheets and we have notes. So yeah there’s the notes ... on paper that in one way accurately represent the music that we create or hear, and there’s rules about what those notes mean and how we use them, and then there’s also the creative aspects of playing an

instrument and that different instruments create different sounds and feelings and what not.

When asked what animal math was like, Alex said a dog while again highlighting views of math that connect to acquiring specific skills or tools for doing math, but later become better understood and more amenable to flexible use within a branch of mathematics:

I think math is like a puppy or a dog pet. Like when you first get them it's all about learning the tricks and how to kind of get math to do what you want it to do, get your puppy to do what you wanted it to do: sit, stand, whatever. But then over time, you develop an intuition and a relationship where it's more authentic engagement with the animal based on an understanding of its needs and your needs and then fostering that relationship.

They later expanded on this metaphor by noting different branches of math might be viewed as different dogs and that different relationships would be developed with the different dogs or branches of math.

The aspects of learning math that Instructor Alex emphasized related to doing math, which included developing definitions and theorems, acquiring content knowledge and tools, and meaning-making. They brought these ideas together to say that behaving more like a mathematician was a measure of learning: "...the ways in which my students are active is that they are defining, they're conjecturing, they're proving, they're doing things like a mathematician would so they're engaging in discipline practices."

Alex explored many topics while discussing the nature of teaching math. They believed in using different types of instruction in the class period to assist students' learning including individual work time, discussion in pairs or small groups, whole class discussion, and lecture. The activities were ways of letting students do math in

class through the tasks they provided, which they viewed as important for learning math.

In my class...there is lots and lots of time devoted to students thinking and working and engaging in mathematics....What I have is more like they privately think of a question, try to gain some traction, think about how they would start it, building intuition, and then talk to a buddy for consensus-building almost....And then the share part is where the class comes together to share these different ideas...and the class comes together on a consensus.

Though Alex estimated they spent only 20-30 percent of class time lecturing, which they defined as “times when I’m standing in front of the class talking but not asking a question,” they felt lecture was a “high on importance” component of any lesson. Specifically, lecture was a way to tie together ideas students were meant to have gotten from doing tasks: “So that’s where I see lectures coming in as a way to like wrap up, provide closure, let them know what the mathematical point of their activity was.”

Based on how they spent time in class and the sequencing of activities their students engaged in, they considered their instruction to be “alternative” to traditional instruction (from the choices alternative, mixed, and traditional taken from Johnson et al. (2019)):

For two reasons. One is the students are engaged, I believe, in authentic mathematical activity, and the other one is because the concepts are developed kind of from the bottom up, from the students’ understanding to the formal instead of from the formal through examples or something....So by their activities in class and because of the development of the content.

Alex reflected on how their teaching mirrored their beliefs about the nature of math and learning math. A number of times, they noted how interconnected their beliefs were:

I believe math is a human activity. I believe students' engagement in activity is learning, and I teach by engaging them in activity in the hopes that that changes and is evident on the learning. I feel like they're very intertwined.

This relates to their measure of students' learning: acting more like a mathematician. They also noted their choice to have students' activity be central to lessons displayed their belief that math was based on meaning-making: "The belief that the student understanding can motivate, be tied to, underpin the formal definition, I think there's a belief about mathematics."

Characterizing instruction

In the Groups unit, the IOIM shows medium-high (4) and high (5) scores for all practices, as shown in line one of Table 2. This indicates students were engaged in "doing mathematics" in alignment with Stein et al. (2008), which was central to the IOIM's formulation of Practice 1; students' thinking was probed, and Alex actively engaged in figuring out students' contributions (Practices 2 and 3); Alex built upon student contributions (Practices 3, 4, and 6); and Alex used notation to capture students' ideas (Practice 7); but the mathematical authority ultimately rested with Alex, not the students (Practices 5 and 6). In the Isomorphism unit, the IOIM shows medium and medium-high scores for all components. Compared to unit 1, scores on Practices 1, 3, and 7 decreased by one. This indicates that the students were engaged in "procedures with connections" instead of "doing mathematics" (Stein et al., 2008) (Practice 1); Alex attended more to why students' step-to-step reasoning worked than how students approached the math (Practice 3); and Alex translated students' informal reasoning to formal notation instead of allowing students to do this formalizing for themselves (Practice 7). Moving from unit two to three, we see similar but slightly lower scores, as Practices 2, 5, 6, and 7 decreased by 1. This indicates that Alex generally expanded on

students' responses rather than allowing students to explain their reasoning (Practice 2); student solutions were presented but not their reasoning (Practices 5 and 6); and there were disconnects between the students' work and the formalized mathematics presented by Alex (Practices 6 and 7). Looking across Table 2, notice the scores hold steady or decrease across the three units for all seven practices. However, no score is ever below a medium (3), indicating at least mid-level alignment with IO instruction throughout the units.

[Table 2 here]

Based on the TAMI-OP, Table 3 shows that while some instructional time was spent lecturing, more time was spent facilitating whole class discussion in which students freely responded to each other's ideas in unit 1. In the Isomorphism unit, we see a larger portion of time was spent lecturing and a much smaller portion of time was spent on whole class discussion than in unit 1. There was also a lesser proportion of time spent on student presentations, but the proportion of time students were working individually or in small groups was the same. In the Quotient Group unit, the percentage of time in whole class discussion or having students present decreased again while the percentage of time lecturing increased slightly. The percentage of time students were working was similar to the previous units. The time lecturing increased across the three units, especially from unit one to two, while the time students were presenting and in whole class discussion decreased across the three units, especially from unit one to two. The time students were working individually or in groups held fairly steady across the three units.

[Table 3 here]

Table 4 shows questions were being asked and answered regularly, with an average of more than one question from Alex every two minutes. Similarly, a student

would answer Alex's question at least once every two minutes, on average. Students asked Alex a question roughly once every six minutes, and these questions were largely related to driving the discussion forward, not about which content would be tested. This highlights the interactive exchange of ideas that was used to create a shared understanding in class in the Group unit. In the Isomorphism unit, questions were asked and answered slightly less often, but still more than one every two minutes. Students asked questions slightly more frequently than in unit 1. Alex asked questions slightly less often in unit 3, though still more than once every two minutes, and received fewer replies. Questions to the instructor were similar to unit 2. Across Table 4, the frequency of questions from the instructor to the class and of student answers decreased while the questions to the instructor increased slightly and then held fairly steady. The decreased questions to the class and student answers speak to decreased interactivity in the class.

[Table 4 here]

Putting together the information from the IOIM and TAMI-OP, there were high levels of interaction among students and between students and the instructor. Students discussed their work with each other in small groups and in whole class discussions, and students' ideas were used to move the class forward. Throughout the three units, students were given extensive time to work individually or in small groups on tasks. These conclusions are based on the medium to medium-high scores across all IO instruction practices, students working in over half of all segments, and the high rate of questions being asked and answered.

Nevertheless, there was an increase in lecturing and a decrease in the interactivity of the class at the whole class level across the semester. This is shown through the decreased scores in the IOIM, as the instructor gave less room for students' mathematics to guide instruction and kept more of the mathematical authority. This is

especially apparent in the Practice 7 scores, which reflect students' opportunities to formalize the mathematics, that decreased each unit. At the beginning of the semester, students' work was guiding the notation and the development of the concept of a group. However, as the semester progressed, students were given less time to work with new concepts before they were given a definition or theorem. This was especially true on the day the Fundamental Homomorphism Theorem (FHT) was introduced, the last day of the Quotient Group unit. In that lesson, Alex used a set of small tasks to directly guide students to the FHT to make sure the theorem was introduced on that day before the exam in the following class period.

Characterizing students' beliefs

In Table 5, the four interview participants' survey responses to what math is and the nature of math are provided. Notice all of these students talked about math as the "study" of something, whether numbers, logic, relationships, applications, or properties. Their animal metaphors for what math is like were largely affective (Abi, Ace, and Andi) and/or focused on problem solving (Aeren, Abi, and Ace).

[Table 5 here]

Comparing responses to the two questions, notice Ace focused on applications of math in both responses. In terms of Ernest's (1989) views of math, Ace seems to align most with the instrumentalist view, which focuses on usefulness instead of how ideas relate to each other. Aeren elaborated on the animal metaphor in the interview, saying you need to be clever in how you approach problems:

Cuz it's not always obvious. Sometimes, it's sort of tricky how you have to solve a problem, and I don't know if foxes are really clever, but that's always the impression they say. So yeah, it's not always easy, or in your face of how it's going to work out, but there's sort of strategies you have to use with that.

This aligns with their sense-making description of what math is. Aeren also appears to have an instrumentalist view with an emphasis on modelling real world events, which still focuses on usefulness and does not seem to focus on connections between different parts of math.

Although Abi's animal metaphor at first seems to reflect a dislike of math ("German Shepard because if you don't play with it right it's vicious."), in the interview, Abi explained this response was about how you need to follow mathematical rules well, especially in the Number Theory context they mentioned earlier in the interview, or else you will not reach a correct solution. This also relates to Abi's understanding of math as relating numbers to each other and seems to indicate an instrumentalist view of math.

Ace's responses about the work-focused mule and connections to the real world focus more on applications than the other students in this class. However, this still speaks to an instrumentalist view focused on the utility of mathematics.

Andi's responses are less obviously related, where the first refers to people's love/hate relationship with both math and cats whereas the view of math focuses on properties and proof. However, the responses complement each other as they provide windows to Andi's view of the general population's perceptions of math, whereas the definition of math seems to speak to Andi's views. Because Andi's definition outlines a structure and connections, this seems to indicate a Platonist view of math.

Students were also asked about their views of how they learn math and characteristics of a good teacher on the first survey to the class and expanded on these responses in the first interview. Aeren related math to working through problems, saying, "I learn [math] from examples of others' work and time to work out confusion and get it right by myself." Abi mentioned "realizing we're taking more classes outside

of theirs” was an important attribute for a teacher and that they learned math by “showing up to class.” This seemed to indicate they found teachers who cared about their general workload to be a good characteristic and that coming to class would lead to learning. Ace said a characteristic of a good teacher was that they “explain in a way where it is the student who solves it before the teacher tells.” This suggests that the student appreciated having time to make some connections independently before being told how to do problems. Andi said, “I learn math by writing scratch work on paper, looking at notes, and doing examples.” This aligns with a view of working through problems and making connections in a way that could be done individually.

In the last interview with each of these students (in the last two weeks of the semester), they were asked how their beliefs about math, teaching, or learning had changed or been strengthened by the class. Aeren, a pre-service teacher, said their belief that a classroom could be non-traditional had been confirmed and appreciated having a chance to see what guiding and questioning students in a student-centred class could look like, which helped make guidance from pedagogy courses less abstract. Abi noted a change in their view of how a class should be taught. During the first week of class, Abi had been sceptical of a non-lecture class, but liked that the guided instruction meant their interactions with math were “like salsa dancing, you feel like you’re dancing with mathematics.” Ace strengthened the belief that you should build your math knowledge yourself, but in practice wished there had been more direct guidance in class. Andi changed their view on how a math class could be conducted, allowing that interactivity could be incorporated effectively and that there was more than one way for math to be taught.

From these early and late semester snapshots, we can see that three of the four students in Alex’s class focused on working through notes and problems

individually or listening to lectures when they thought of how they learned or how math should be taught. However, at the end of the semester, students seemed open to other ways of thinking about math, such as reasoning through problems themselves under guidance and discussing math with others. Interestingly, Ace, who originally mentioned that they liked having a chance to solve problems before the teacher explained, would have actually appreciated more guidance on the tasks during the semester.

Summary across Alex's Class

There are some clear connections between what happened in class and what students said changed about their beliefs. However, many of their previous beliefs, especially about the nature of math, seemed unchanged. One of Alex's main beliefs about how we learn math that was reflected in instruction was that we learn math by doing math. In class, students were given time to work on problems before being told procedures for addressing the problems. By the end of the semester, the students seemed to recognize this was a potential way to learn math. For Aeren and Ace, the class strengthened this belief, whereas for Abi and Andi this seemed to be a new approach. Abi especially seemed to latch onto this idea, noting opportunities to “danc[e] with mathematics.”

Similarly, Alex promoted an interactive classroom, where students would build consensus around ideas with growing groups of people, as a way for students to develop their mathematical thinking. Students seemed on-board with this style of instruction by the end of the semester as well. While Andi did not directly address comfort with the guided instruction, Andi noted that interactivity seemed more acceptable in a math class than before.

Despite some alignment on doing math and discussing math in class, Alex's idea related to math as a human activity was the idea that math is a creative endeavour.

While this can be perceived in instruction, where tasks were open-ended and different

correct solutions were highlighted in class, this was not as clear of a student belief.

Aeren had originally mentioned “being clever” was important in solving problems but focused more on having strategies to deal with such situations than seeing them as an opportunity for creativity. Additionally, Ace would have appreciated more guidance on activities, not less, which does not align with a desire to have more room for mathematical creativity.

In summary, students’ beliefs about teaching and learning math seemed to be slightly altered or strengthened according to what happened in class. Students were more comfortable with learning through tasks and in interacting with their peers while they learned math. However, root beliefs about the nature of mathematics did not shift noticeably for students. Though Alex believed creativity is important to math, this belief was not clearly represented in students’ statements at the end of the semester. This may be because students felt their ways of learning math were still compatible with their beliefs about math without requiring creativity. Alternatively, the nature of one’s view of math might be too abstract of a belief to be clearly communicated by an instructor.

Instructor Bailey

Instructor beliefs

Instructor Bailey viewed math as both an object of study and something we do. Of Ernest’s (1989) three views of the nature of mathematics, Bailey seemed most aligned with the Platonist view in noting mathematicians’ search for theorems: “So I think mathematics is the search for theorems which...I would take to mean things that both can be proven...and then also the actual pursuit of proof...” This statement revealed a view of math as both an entity to be studied and something we do, though they spent more time emphasizing what is done. Furthermore, Bailey originally described math as:

“...to answer kind of circularly, anything that is pursued with the methods of mathematical inquiry, which I would take to mean, you know, things that you can...prove rigorously.”

Bailey also noted that proof and logical thinking were useful tools that could be viewed as purposes of math: “I think that some of the benefits of mathematics are training...one’s mind to think rigorously, to take both a rigorous and creative approach to the problems.” However, they did not feel it was appropriate to emphasize their own views about math; rather students could decide what math is for themselves, based on their own experiences:

I definitely think it’s good not to impose one’s views about math. I think probably by taking different courses throughout our curriculum they’ll be exposed to different views about math, so I don’t know that it’s necessarily my role to...give a philosophical overview about what math is....

When asked what animal math was like, Bailey highlighted ideas related to proof and logic, much as they had noted when asked direct questions, though they also admitted that they asked others about metaphors ahead of time and chose the one they liked best rather than devising their own.

The best answer I heard was an ant because it is sort of a very simple thing that’s not capable of very much on its own, but that they can combine in very interesting and complex ways. So I think that sort of fits with this idea of...at the basic level you have...your individual axioms and inference rules but you can sort of combine these things in ways to generate much more complexity than is apparent at the most base level of sort of definition, logic....I mean it obviously, it’s gonna have limitations as any analogy does....There’s no conscious entities trying to push the ants in any particular direction to produce new more interesting things whereas in mathematics...I guess the human beings are missing from the ants analogy, right, the actual mathematicians....So in that analogy...the ants they’re sort of your axioms, your inference rules, and the complex patterns they’re producing are the theorems.

The aspects of learning math that Bailey emphasized related to actively doing math both in and out of class.

I'm a firm believer in learning by doing is best, so...every class I try to give the students something to do even if it's just...here I'm gonna put this...particular example on the board for two minutes, let you guys work on it.... They have to be active at some level.

Later they provided more context for what doing math meant to them as they noted how they would do math in lectures as a student:

...but I'm the kind of person who...in a lecture, I'll try to verify all the details and I can easily get hung up on one detail...and then look up ten minutes later and the class has moved on without me....I have to be...coming up with my own examples or coming up with my own proofs and just really synthesizing for it to stick. That's how I learn mathematics.

They acknowledged that how they learned could differ from how others learn, just as people have different ways of thinking in other contexts: "Different people all like different foods....Different people have different frames for interpreting politics for example, so I think the same applies to learning."

Bailey explained the choice to use a combination of lecture and lab periods in instruction arose from watching another instructor teach Abstract Algebra and wanting to expand the interaction that students were engaged in. Lecture days would focus more on exploring the definitions and proofs in the class with a few smaller examples worked in.

So what I was calling lecture, I'm writing stuff on the board and talking pretty much the whole class, punctuated with just little examples probably that I ask them to work out for a few minutes. So with the labs, I know I give them a handout with some questions and some things that they're supposed to discuss...and then I'm circulating, talking to people individually, talking to groups... So I guess I'm doing

very different things, they are doing very different things because they're talking to each other.

Bailey wanted students to engage in lecture by asking questions and doing short problems, whereas labs would give time to work on and discuss problems with their groups.

Bailey valued lecture as a way to make sure they taught all of the necessary material, and they were satisfied with the balance struck with two lecture periods and one lab period each week:

I want to lecture enough to sort of get to all the material I want them to see, but also put in enough group work where I'm getting them to...interact and...come up with their own ideas....What that's come down to is...two lectures and one lab each week which...strikes the balance that seemed to work pretty well in the fall and I think is working alright now.

They considered their class to be mixed (Johnson et al. 2019), and said they felt that was a good characterization of their teaching.

Bailey identified variety as the main way that their beliefs about the nature of math, learning, or teaching was reflected in their instruction:

I think it's just trying to mix things up. It reflects my belief that people learn in different ways....All my undergraduate mathematics classes were what I've been referring to as lecture. Every single one of them, I never had any sort of group work in those classes, and...I wasn't great at following what was going on in the lectures at that point in time. The group work is the kind of thing that would have helped me, so that's just kind of putting in that different element for maybe people who do learn in a different way. Trying to be helpful to different types of learners.

Bailey's beliefs about creating different types of learning opportunities for students sprang from their own experiences as a learner. In this case, the lack of alignment between their experiences and what would have helped them as a student appeared to be

formative. This relates to Johnson et al. (2018), which noted the second most reported influence on teachers' instruction was their experience as a student.

Characterizing instruction

In the Group unit, the lectures received low (1) to medium (3) scores on the IOIM. The lab day received medium-low (2) to medium-high (4) scores. These scores are summarized in line one of Table 6. These IOIM scores indicate that the lecture days were not very aligned with any of the components of IO instruction. The instructor provided little to no space for students to engage in mathematical practices and gave students little to no access to each other's thinking. All practices except Practice 6, guiding and managing the development of the mathematical agenda, received medium-low or low scores. The higher score for Practice 6 stemmed from having an orderly progression to the lesson. On the lab day, Practices 1 and 2, which both relate to generating student thinking, received medium-high scores, indicating students were working on approaches for the tasks at hand. Practices 4 and 5, which both relate to creating a shared understanding in the class, received medium scores. Practices 3 and 6, which relate to building on student thinking, as well as Practice 7, which relates to connecting to standard mathematical notation, all received medium-low scores. This shows the lab days somewhat aligned with IO instruction.

[Table 6 here]

In the Isomorphism unit, lecture days received similar scores to those in unit 1, but Practices 3 and 7 went up and Practice 5 went down, indicating there was slightly more probing of students' stated thinking (Practice 3) and more use of students' mathematics before presenting formal mathematics (Practice 7), but less encouragement to have students think about each other's thinking (Practice 5). The lab day received

lower scores on Practices 1, 2, and 5. This indicates that discussions between students were not generating as many student ideas (Practices 1 and 2) and public sharing of ideas happened less than the previous lab (Practices 2 and 5).

During the Quotient Group unit, the lecture day scores for Practices 4 and 7 decreased from unit 2, and the other scores remained the same. This indicates the instructor gave little to no space for students' contributions, a shift from opportunities to provide "fill in the blank" responses, and formal mathematical notation was introduced before students engaged with a task to introduce new concepts. The partial lab days received scores similar to lecture days in these weeks. The only difference between the scores for lectures and labs in unit 3 was on Practice 4, where the score was slightly higher for the labs because some student contributions were used to move the class along. Students engaged in less discussion with each other during labs at all but one table, which depressed the IOIM scores on most practices relative to previous units.

Looking across Table 6, lab day scores held steady or decreased across the three units except for Practice 6, management of the mathematical agenda. This increase can be attributed to greater organization and tying together of ideas in unit 3, whereas students were left without closure in a whole class setting in the previous units. For the lectures, scores largely held steady or decreased, with the exception of Practice 7 in the Isomorphism unit. In that case, the lab preceded the unit, allowing some informal notation and ideas to come from the students before the definition of isomorphism was fully introduced and explained.

The TAMI-OP documents distinct differences between how time was spent on lecture and lab days, as shown in Table 7. Lecture days were dominated by the instructor presenting at the board (lecturing) and included less time for students to work individually or in groups, whereas on the lab day the allocation of time was flipped.

Unit two was similar, with most of the time working in small groups happening on the lab day. In the Quotient Group unit, more time was spent lecturing and students spent less time working than in previous units because the labs were given less class time. Looking across the three units, we see an increase in the amount of time lecturing and a decrease in the amount of time students were working individually or in groups, especially between units two and three. Students did not present solutions publicly or engage in cross-group discussion in whole class settings in any unit.

[Table 7 here]

In Table 8, we see the instructor directed questions to the class roughly once every three minutes, though the average was closer to once every two minutes on lecture days. Students answered a question publicly roughly once every four minutes (closer to once every three minutes on lecture days). Students asked public questions infrequently (once every ten minutes on lecture days, less overall), though these questions were largely focused on details of the lesson, not asking about what would be tested. In unit 2, we see instructor question and student answer rates decreased slightly for both lectures and labs. Questions were asked publicly of the instructor at similar rates for lectures and labs, a bit less than once every ten minutes. In unit 3, the instructor asked questions roughly once every two minutes while students gave answers roughly once every four minutes. Questions asked of the instructor were similar to previous units. Notice, the frequency of instructor questions almost doubled, but the frequency of student answers only increased .04/minute.

[Table 8 here]

Looking across Table 8, we see a slight decrease in the rate of questions being asked and answered from the Groups unit to the Isomorphism unit and an increase, especially from the instructor, from the Isomorphism to Quotient Group unit. This can

be partially explained by fewer lab days in the third unit, because the lab days were only partially devoted to the labs, and more questions tended to be asked in lecture days.

Based on notes, the instructor frequently asked questions twice because no one responded to their initial question in unit 3, whereas this was not necessary in previous units. The frequency of questions to the instructor in whole class discussion held fairly steady across the three units.

Putting together the information from the IOIM and the TAMI-OPs paints a picture of a class strongly guided by the instructor's mathematical knowledge but with some opportunities for student exploration. The ultimate mathematical authority rested with Bailey, who moved the class forward at the pace they wanted to set. Students were given less time to work individually or in small groups across the units, especially in the Quotient Group unit, and the amount of time Bailey spent lecturing increased, especially in unit 3. Thus, we see a decrease in students' opportunities to do mathematics through less time dedicated to student work time and decreases in the IOIM lab scores related to generating student contributions.

Characterizing students' beliefs

As noted in Table 9, all four students referred to math in terms of connections to the real world, with Baker, Blaine, and Bryce focused more on applications of math.

Specifically, Baker focused on applications or just enjoying math, Blaine focused on finding "a quantifiable solution to life's problems," and Bryce focused on modelling with math ("explaining phenomen[a]") as well as using math as a way to communicate information ("language"). Blake was more structurally focused but gave a possible origin of mathematical knowledge outside pure logic ("empirical observations"). The four students' animal metaphors reflected affective considerations (Blaine and Bryce), attention to solving problems (Baker, Blaine, and Bryce), and a focus on the nature of

reality (Blake).

[Table 9 here]

Comparing responses to the two questions, Baker and Blaine had a clear focus on applications of math in both contexts, though Blaine added an affective component with the animal metaphor (“huge fears people have”). Bryce also added an affective component with the animal response but highlighted problem-solving in both contexts. Bryce’s problem-solving emphasis was slightly different from Baker and Blaine, which Bryce elaborated more in the interview.

I’ve always seen math as a big puzzle that’s like...the more time I put into it I’ll understand it better and I can actually figure it out...I mean Social Studies, you know, your teacher doesn’t agree with your opinion, you’re kind of done for. But I always liked math because of the simplicity in it. Now that I am in higher level math classes, I’m realizing that it’s not always the case...It is still the puzzle mindset that I, given a problem...you can look at it through a bunch of different angles and fit all the pieces together.

Bryce went on to explain that cats are temperamental if you leave them alone for a weekend. Relating cats and math, Bryce said:

I guess with a cat...they can be very unpredictable, like math, like, you never know what kind of problem you’re gonna get. Especially like last fall I started Proofs, like since then a lot of my classes are proof-based classes so some of them...you get right away and you’re like, “Great, I understand this.” Then get to the next one, and you’re like, “What the heck? I don’t know.”

Across the two contexts, we see an emphasis on solving puzzle-type problems in math, but that these puzzles could vary a bit in ease of solving. Blake focused on structural aspects of math in both responses by focusing on axioms and ideas in the description of math. They explained the “humans” response for the math animal by saying math is a “product of human consciousness” and that math is “very analytical and precise...it was

invented to serve humans.”

In responding to these questions, Baker, Blaine, and Bryce seemed most aligned with Ernest’s (1989) instrumental view of math. This is because they focused on accomplishing specific tasks with math and seemed less interested in its overarching structure. Blake seemed most aligned with the problem-solving view of math as they focused on how math fit together and, especially in the interview, focused on people as creators of mathematics instead of utility driven or discoverers of existing math.

Students were also asked about their views of how they learn math and characteristics of a good teacher on the first survey to the class. Baker noted that they were “better at math than most other things,” that they usually studied by reviewing homework, quizzes, and examples, and that this generally sufficed. Blaine claimed to learn math by receiving organized notes:

By being given fundamental information and example problems/solutions in class, then taking it all home to reference while working on new problems. Works best when given typed note outlines with proofs/blanks to fill in. That way I can more easily distinguish between definitions/formulas and problem solving.

Blake learned math in individual contexts: “I practice and do math, and if I struggle with an idea, I allow it to sit in the back of my mind while I do other things and think about it time to time until I understand it.” Bryce focused on study habits to explain learning:

I learn math through practice. When a teacher explains a concept in class, I often have trouble understanding it until after we have gone through some examples. When I am on my own, after classes, I can reflect on the material and using the examples, I can begin to understand the original concept.

In the last interview with each of these students, they were asked how their beliefs about math, teaching, or learning had changed or been strengthened by the class.

Baker did not feel their beliefs had changed, but that this class had been a different experience, as the content was outside their major area of aerospace engineering. Thus, in this course they had been forced to apply the idea that anyone can learn math if they try hard, unlike previous classes where content had come more naturally. The class also opened Baker's eyes to how broad math is through exposure to many new ideas.

Blaine felt the belief that teaching styles matter had been strengthened, specifically mentioning that they would not have felt comfortable in Alex's section based on the clips of the class that they had been shown in interviews, though such a class might be good for students seeing the material for the first time. Blaine did appreciate the labs that had happened in their own section though and thought if labs had been part of the class the first time they took Abstract Algebra, they might have done better. This appreciation seemed focused on being able to see why the material was important through examples: "[It's] easier to know how a machine works if you build it yourself, which is sort of how the labs work if they're before the lecture."

Blake mostly commented on changes to their perception of math majors. Blake was surprised by the fact that others in the class relied on the class notes instead of looking for other materials to expand their understanding, such as other textbooks. Blake also engaged in more group studying and work than in previous classes and felt these group study sessions were reasonably well-focused.

As a preservice teacher, Bryce claimed to generally pay attention to how teachers approach instruction and appreciated the lab days because they provided a day off from new material, which allowed time to apply new ideas or processes. However, Bryce also would have appreciated more structure on the lab days because if you did not understand what was going on, you just struggled unproductively. This was a contrast to previous experiences with lab-like activities in middle and high school, which had not

prompted struggles or thoughts of downsides to doing activities. On the whole, Bryce enjoyed having the mix of different types of instruction in class, which prevented boredom from repetitious lectures and so that teamwork skills could be built through group work, but they noted downsides of activities that they had not observed before.

From these early and late semester snapshots, we see can see that these students in Bailey's class entered the course with application-focused views of the nature of math (e.g., used "to find a quantifiable solution to life's problems"). They also left room for the abstractness of math (e.g., math as "ideas...logically deduced from axioms"). At the end of the semester, students' views of math as a field did not seem to have changed, but students seemed to be more reflective about how they learned math. Some students were more open to learning math in new ways (Blaine and Blake) while others were more attuned to the variety of affective math experiences a student could have (Baker and Bryce).

Summary across Bailey's class

Instructor Bailey noted that students learn in different ways and wanted to support different types of learning. This belief was expressed through allowing students to discuss however much or little they wanted to at their tables, even though there was encouragement to talk with their tablemates. In instruction, this presented itself as different tables interacting at different levels. This seems to be a place where students' beliefs influenced instruction at the table level, as students at different tables experienced different interactivity levels. Specifically, Blake was at the most interactive table and would animatedly discuss problems with other students. Baker and Bryce were at the second-most interactive table and would discuss with peers at times. Blaine was at one of the less interactive tables and often worked independently, though Blaine would discuss with others at the table at times.

Students shared some beliefs about the nature of math with their instructor, such as a focus on connections to the real world (all four students), a focus on problem-solving (Baker, Blaine, and Bryce), and a focus on the structure of math (Blake). These beliefs did not seem to change across the semester. However, students' beliefs about teaching and learning math seemed to be altered or strengthened with regard to their openness to variety in instruction. Bailey used different types of instruction in class, varying between lectures roughly twice each week and labs roughly once each week. Students' beliefs seemed most affected by this class set-up, which was novel for them, at least in a college math setting. After experiencing this different class set-up, students seemed to place a higher value on having variety in instruction and were more open to non-lecture class periods. However, the main shift in students' beliefs seemed to be in valuing variety in instruction, not specifically valuing an alternative to lecture. Furthermore, the approved variety in instruction seemed focused on the format of the tasks more than the discussion level. Blaine noted that having a lab before new material gave insight into what they were about to learn, which they found helpful. Similarly, Bryce found a day to process old material before seeing more material to be helpful, which was what they highlighted as most beneficial about the labs. Students also affected instruction to some extent through the ways they chose to interact at their table. They were given the freedom to choose how interactive they wanted to be, and different tables made different choices.

Discussion

Each instructor's beliefs were reflected in their approaches to instruction. Differences in structuring the course were evident in the formats chosen for their classes. Instructor Alex chose to use the IOAA curriculum because they viewed math as a human activity and wanted students to generate definitions and proofs as a mathematician would. This

aligned with a curriculum based on Freudenthal's (1973) work in realistic mathematics education. Instructor Bailey formatted their class as two days of lecture and one day of lab each week, which aligned with their view of math as the pursuit of theorems via proofs. Bailey's proof-focused view of math aligned with their choice to lecture twice each week, giving opportunities to present many theorems and proofs to their students. However, the labs provided time for students to explore concepts and make some connections for themselves.

Though the instructors used different course formats, both put structures in place that allowed students to work together and work on mathematics. However, these structures were different and led to different patterns of interactivity in the two classes. Students were expected to discuss their mathematical work with others every day in Alex's class, whereas students were free to work individually or in groups on the lab days and could just take notes on lecture days in Bailey's class.

Alex's medium to medium-high scores on the IOIM indicate that the curriculum was at least somewhat used as intended, allowing students to generate knowledge, build upon this knowledge, create a shared understanding with their peers, and formalize their new knowledge, which are the underlying components of IO instruction (Kuster et al., 2017). Alex's scores across all three units match or exceed the median of the IOAA instructors in AUTHOR for all except Practices 6 and 7 of the third unit, and those scores are still within the range obtained from those IOAA instructors. Unlike the inquiry-based instructor in Fukawa-Connelly (2016), students did not seem to have more freedom in proving than defining. The IOIM scores indicate students had space to define (Practice 7) as well as share their reasoning on proofs (Practices 1-5) in unit 1, but scores across all of these practices decreased by unit 3. Thus, Alex's class seems to be a standard example of an IOAA class.

Bailey provided students opportunities to do problems and discuss with one another to in order to give instructional variety, especially through the lab component. However, most groups did not interact with each other much during labs but would wait until Bailey came to their table to ask questions. The different types of instructional days, opportunities to work on problems for extended periods, and opportunities to interact aligned with Bailey's stated desire to use many types of instruction to reach many types of learners. However, in practice, most groups experienced largely lecture and individual work time instead of interaction. This was still more instructional variety than might be expected in a "typical" lecture class. Bailey noted that their previous semester's section had been more interactive, so it is possible this was due more to the students' preferences than the instructor's intention. Here we seem to have a tension between Bailey's belief that students should be interactive and that students should be free to make choices about how they want to learn. It seems Bailey acted more on the latter belief. Thus, Bailey's class does not align very well with IO instruction, nor does it align with pure lecture courses. Rather, as Bailey intended, the course included activities to encourage students to think about content for themselves before or after standard lecture periods, leading to a mix of instructional activities.

Both instructors seemed to intend to enact the interactive classrooms they talked about in the interviews. However, as the semester wore on, other considerations received precedence. In Alex's class, the FHT only received a partial day of instruction while students reviewed for an upcoming exam, and this instruction was strongly guided by the instructor instead of being student driven. In Bailey's class, the labs were given half days instead of full days in unit 3. Because teachers are sensible, in keeping with Leatham (2006), this change in both instructors' teaching indicates unarticulated beliefs, perhaps about the need to cover content or the amount of scaffolding needed for more

complex content. The instructors did not state a desire to shift their instruction, so it is possible they did not notice these changes in their instruction.

This article highlights the value of using multiple measures to examine instruction. The TAMI-OP provided a detailed view of how time was used and how often questions were asked in each class period, allowing measures of interactivity to be applied to self-identified “alternative” and “mixed” instruction (Johnson et al. 2019). However, the TAMI-OP alone did not give clear information about the quality of interactions, in terms of the depth of mathematical discussion or the origin of mathematical ideas. Here, the IOIM provided a rubric for judging the quality of the mathematical discussion and considering the placement of the mathematical authority in discussion. Furthermore, this article shows the IOIM can be used to evaluate a non-IO class, which had not been attempted previously.

This study also confirms the importance of examining instruction across multiple units (Skott, 2001). Examining only the first unit for each instructor would have suggested strong alignment between the beliefs they highlighted and their instructional practice. However, this would have obscured how instruction progressed through the semester. Similarly, observing only the last unit might have led to a conclusion that there was not strong alignment between the beliefs instructors highlighted and their practice. Considering Bailey noted interactivity differences between their Abstract Algebra class the previous semester and the semester of data collection, future research should consider following instructors across multiple semesters or years (Weston et al., 2020). Differences between approaches to classes with different personalities could then be considered, encouraging beliefs that would not otherwise be stated to be available for examination.

Shifting attention to connections between instruction and student beliefs, there does appear to be a subtle relationship between instruction and students' beliefs about what is possible in a college math class. Students in both classes expressed openness to learning math in new ways. Moreover, the ways students expressed their increased openness seemed more similar to the instruction they had received than what happened in the other class, indicating some influence from instruction on their beliefs. Alex's students mentioned being more open to non-lecture formats and having opportunities to discover new paths for themselves. Bailey's students noted openness to multiple types of instruction within a course rather than just lecture or just group work. Though we see openness to new types of instruction in both contexts, what students were open to is more similar to the type of instruction they received than what happened in the other course.

Furthermore, this suggests students' beliefs about teaching can shift from exposure to modest instructional changes. Although Bailey was not fully "alternative" and did not intend to teach in a fully "alternative" way (Johnson et al., 2019), students still expressed openness to shifts from traditional instruction. This suggests instructors interested in making modest adjustments to their instruction to include some activities or opportunities for students to work in class could still impact students' beliefs in positive ways.

Although some shifts in teaching and learning beliefs arose from the classes, students' beliefs about the nature of mathematics did not noticeably shift as a result of the class. It is possible that students' beliefs about the nature of math were already entrenched by the time they took this upper-division math course. Alternatively, these beliefs may shift from an aggregate of classes, such as first being exposed to proof

courses. More research is needed to explain how students' beliefs about the nature of mathematics gain in nuance (Schommer, 1990; Szydlik, 2013).

Finally, by examining multiple components of instruction in detail, namely instructors' beliefs, instruction, and students' beliefs, we have an opportunity to contextualize how students' beliefs might be related to and shaped by instruction. Increased alignment between instructors' and students' beliefs about teaching practices were noticeable and suggest utility in instructors incorporating even modest changes in practice to give students opportunities to actively engage while learning. Future work could examine other courses, such as Calculus, Statistics, or Geometry, to see how the course context might influence connections between beliefs and instruction and see whether the modest changes in instruction noted here could have similar impacts in other courses.

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Table 1. Coding Definitions for Types of Classroom Activity.

Type of Activity	Coding Definition
Lecturing	Instructor standing near the document camera or whiteboard and addressing the class as a whole or facilitating a classroom discussion with all questions/answers through them
Whole Class Discussion	Students freely responding across groups to other students' statements in a whole class setting
Students Working	Instructor manages small group conversations and/or engages in one-on-one conversations with students as students engage in small group discussions or work individually
Student Presentations	A student goes to a whiteboard or document camera to share their work publicly
Question to Class	Non-rhetorical question posed by the teacher to the class in a whole group setting
Question to Instructor	Question posed by a student to the teacher in a whole group setting
Student Answer	Response from a student to a teacher question in a whole group setting

Table 2. IOIM Practice Scores Across All Units for Instructor Alex.

Unit	P1	P2	P3	P4	P5	P6	P7
Groups	5	4	4	4	4	4	5
Isomorphism	4	4	3	4	4	4	4
Quotient	4	3	3	4	3	3	3
Groups							

Table 3. Time Totals and Averages across Units for Instructor Alex.

Unit	Segments Lecturing	Segments Students Working	Segments Student Presenting	Segments Whole Class Discussion
Groups Total	14/69	39/69	12/69	29/69
Groups Ave.	20%	57%	17%	42%
Isomorphism Total	91/150	86/150	10/150	14/150
Isomorphism Ave.	61%	57%	7%	9%
Quotient Group Total	153/225	120/225	5/225	2/225
Quotient Group Ave.	68%	53%	2%	1%

Table 4. Question Totals and Averages across Units for Instructor Alex.

Unit	Questions to Class	Student Answers	Questions to Instructor
Groups Total	106	87	24
Groups Average	.77/minute	.63/minute	.17/minute
Isomorphism Total	201	168	75
Isomorphism Average	.67/minute	.56/minute	.25/minute
Quotient Groups Total	265	190	113
Quotient Groups Average	.59/minute	.41/minute	.24/minute

Table 5. Alex's Students' Views of the Nature of Math.

Student	Math animal	What math is
Aeren	Math is like a fox because they are clever!	Math is the study of numbers and logic to make sense of the world around us.
Abi	German Shepard because if you don't play with it right it's vicious	The study of numbers and their relations with each other
Ace	Math is like a mule. It can, after much training, learn how to do tasks with a heavy amount of human input. However, it can never truly change how it behaves in nature.	Mathematics is the study of converting real world information into numbers and manipulating those numbers to discover more information about them.
Andi	Math is like a cat because everyone either loves cats, or hates them. Everyone loves math, or never wants to touch it in their life.	Mathematics is the study of properties such as operations, spaces, and proving them.

Table 6. IOIM Scores for Instructor Bailey in Lectures/Labs.

Unit	P1	P2	P3	P4	P5	P6	P7
Groups	2/4	2/4	1/2	2/3	2/3	3/2	1/2
Isomorphism	2/3	2/2	2/2	2/3	1/2	3/2	2/2
Quotient Groups	2/2	2/2	2/2	1/2	1/1	3/3	1/1

Table 7. Time Totals and Averages across Units for Instructor Bailey

Day	Segments Lecturing	Segments Students Working	Segments Student Presenting	Segments Whole Class Discussion
Group Lecture	49/51	9/51	0/51	0/51
Group Total	52/76	32/76	0/76	0/76
Group Ave.	68%	42%	0%	0%
Isomorphism Lecture	47/51	5/51	0/51	0/51
Isomorphism Total	53/77	31/77	0/77	0/77
Isomorphism Ave.	69%	40%	0%	0%
Quotient Gp. Lecture	168/177	12/177	0/177	0/177
Quotient Gp. Lab	4/36	36/36	0/36	0/36
Quotient Group Total	172/213	38/213	0/213	0/213
Quotient Group Ave.	81%	18%	0%	0%

Table 8. Question Totals and Averages across Units for Instructor Bailey

Day	Questions to Class	Student Answers	Questions to Instructor
Lecture Total	45	33	10
Group Total	49	35	10
Lecture Ave.	.44/minute	.32/minute	.10/minute
Lab Ave.	.08/minute	.04/minute	0/minute
Group Ave.	.32/minute	.23/minute	.07/minute
Lecture Total	37	30	8
Isomorphism Total	38	30	12
Lecture Ave.	.36/minute	.29/minute	.08/minute
Lab Ave.	.02/minute	0/minute	.08/minute
Isomorphism Ave.	.25/minute	.19/minute	.08/minute
Lecture Total	184	96	26
Lab Total	2	2	4
Quotient Group Total	186	98	30
Lecture Ave.	.52/minute	.27/minute	.07/minute
Lab Ave.	.03/minute	.03/minute	.06/minute
Quotient Group Ave.	.44/minute	.23/minute	.07/minute

Table 9. Bailey's Students' Views of the Nature of Math.

Student	Math animal	What math is
Baker	A horse. It can be used for many things, and while most use it (and view it) as a means of doing something else, if you take the time to learn about the horse it can be something in its self. I mean this because most people, especially engineers, view math as a mechanism for doing engineering; however, there are facets of it that are not seen by the average person using math as a mechanism, that are quite interesting.	Its whatever you make it. You can use it as a mechanism for doing other things, or you can use it as a proof that you CAN do other things, or you can simply enjoy the knowledge.
Blaine	I think math is like a spider. Both are seen everywhere and help us throughout our lives without many people realizing it. Both are also huge fears that people have.	Mathematics is the study that uses formulas, properties, numbers, and more to find a quantifiable solution to life's problems.
Blake	I do not think it is like any animal, since math is in some sense metaphysical. It reminds me of humans the most out of any animal.	A set of ideas generated by (logically deduced from) axioms which are often but not always based on empirical observations/intuition.
Bryce	I think math is like a cat. It can seem to hate you at times, but then completely change. Math is a giant mystery and we have to solve, just like a cat's true feelings.	Mathematics is the abstract concept of problem solving. It is the bare boned, direct explanation for many phenomenons in the world. It is the language we use to explain the world we live in.