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Sameness in Mathematics: A Unifying and Dividing Concept

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Abstract

While many aspects of the teaching and learning of specific advanced mathematics courses have been studied, limited work has examined mathematical themes like sameness or its instantiations across disciplines. In this paper, we explore algebraists' collective example space for mathematical sameness. We used qualitative methods to analyze survey responses from 197 algebraists in order to identify specific mathematical concepts that the algebraists associated with sameness and relevant factors to consider when determining sameness of objects. Using variation theory, we introduce the notion of a community example space for algebraists to highlight specific dimensions of variation in sameness as well as the range of variation for each dimension through specific instantiations given by multiple participants. Dimensions of sameness included contexts, concepts, objects, properties, and qualities. These results suggest potential for building connections across levels and branches of mathematics by highlighting how different choices for each dimension of sameness can produce different instantiations of sameness.

Keywords: Isomorphism, Sameness, Variation Theory, Example Space

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Sameness in Mathematics: A Unifying and Dividing Concept

Abstract

While many aspects of the teaching and learning of specific advanced mathematics courses have been studied, limited work has examined mathematical themes like sameness or its instantiations across disciplines. In this paper, we explore algebraists' collective example space for mathematical sameness. We used qualitative methods to analyze survey responses from 197 algebraists in order to identify specific mathematical concepts that the algebraists associated with sameness and relevant factors to consider when determining sameness of objects. Using variation theory, we introduce the notion of a community example space for algebraists to highlight specific dimensions of variation in sameness as well as the range of variation for each dimension through specific instantiations given by multiple participants. Dimensions of sameness included contexts, concepts, objects, properties, and qualities. These results suggest potential for building connections across levels and branches of mathematics by highlighting how different choices for each dimension of sameness can produce different instantiations of sameness.

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A common motivation behind initiatives in math education is to see how content can be made relevant to students. Recent projects have largely approached this task by connecting pre-university content to advanced university courses like analysis (e.g., Wasserman et al., 2018) and abstract algebra (e.g., Suominen, 2018), or by examining defined mathematical concepts such as functions (e.g., Melhuish & Fagan 2018; Melhuish et al., 2020) and binary operations (e.g., Melhuish et al., 2020). Other research has examined disciplinary practices common across mathematics, such as defining, symbolizing, algorithmatizing, and theoremizing (Rasmussen et al., 2015; Rasmussen et al., 2005). In this paper, we approach relevance differently by considering how an undefined concept, mathematical sameness, appears as a common theme across mathematical disciplines. Specifically, we believe helping math majors see common themes that appear across courses could help them better understand and appreciate how branches of mathematics are structured to achieve specific purposes rather than viewing mathematical concepts as disconnected from motivating problems and from real-life contexts (Schoenfeld, 1988).

Although mathematical sameness is not defined, some topics are easily aligned with notions of sameness. For example, equivalence relations, which have the properties of reflexivity (e.g., $x = x$), symmetry (e.g., $x = y$ implies $y = x$), and transitivity (e.g., $x = y$ and $y = z$ implies $x = z$) might be considered a way to formally express sameness of object (Asghari, 2009; Asghari, 2019). Examples of equivalence relations include equality (of numbers), isomorphism (of groups), and congruence (of polygons). Nevertheless, it is possible that mathematicians consider a wider variety of concepts to convey sameness than just equivalence relations, such as homomorphisms of a given algebraic structure that can be viewed as highlighting shared (sub)structures. In a different vein, isomorphism and homomorphism of a given algebraic structure might be viewed as conveying different levels of sameness, as isomorphism requires the extra condition of a bijection. Nevertheless, the extent to which mathematicians explicitly align equivalence relations, types of equivalence relations, and other concepts with the notion of sameness has not been directly explored.

Furthermore, different representations of the same problem can suggest new tools or solution methods, and such shifts of perspective have led to many breakthroughs in mathematics history. For instance, Fermat's Last Theorem, a famous number theory problem for which Fermat claimed to have a proof that the paper's "margin is too narrow to contain" (p. 62), does in fact follow directly from the Taniyama-Shimura conjecture (p. 202), which relates to elliptic curves (Singh, 1997). By using the logical equivalence of Fermat's Last Theorem and the Taniyama-Shimura conjecture, Andrew Wiles completed a proof which had been unknown for over 300 years (Singh, 1997). Nevertheless, the extent to which mathematicians view logically equivalent statements as central to mathematical sameness is less clear.

Math educators have been studying topics that might be viewed as types of sameness for many years, more often in students than experts. Specifically, research has looked at students' understanding of equality and the equal sign (e.g., Kieran, 1981; Renwick, 1932) and higher-level notions like isomorphism and homomorphism in abstract algebra (e.g., Dubinsky et al., 1994; Hausberger, 2017; Melhuish, et al., 2020; Rupnow, 2019). Others have examined the history of equivalence relations in general (Asghari, 2019). However, only limited research has been conducted on mathematicians' expert understandings of isomorphism and homomorphism (Rupnow, 2021; Weber & Alcock, 2004), and examinations of types of sameness have largely focused on equality and isomorphism in both faculty and students. Moreover, topics related to sameness have only recently been examined together. Melhuish and Czocher (2020) examined students' varied understandings of sameness-based topics, such as equivalence of representation, which were not always aligned with standard understandings. For example, a number of students viewed multiplication and division as "the same operation"—while it is true that multiplication by 1/x and division by x produce the same number (when the operation is defined on the reals), properties like commutativity are not shared. Furthermore, how mathematicians view sameness and group different types of sameness has not been directly examined.

The goals of this paper are to explore how algebraists characterize and instantiate mathematical sameness. We believe this makes two main contributions to the literature. First, by asking mathematicians about the nature of sameness, we show that the concept of "mathematical sameness" is an interpretable concept to mathematicians and, thus, a possible topic for math education researchers to study. Second, by asking for mathematicians' understandings of mathematical sameness, we have access to their expert knowledge of concepts and connections in advanced mathematics. As experts, they are the population likely to have the broadest range of examples of types of sameness that they can draw upon and are likely to recognize the most aspects that can vary and still convey a notion of sameness. While we may not expect students to recognize all of the types of sameness mathematicians do, this paper provides a compilation of types of sameness that can help teachers recognize examples of sameness and dimensions of variation to highlight for students.

Literature Review

The notion of sameness is threaded throughout the foundations of mathematics and, in modern times, is often considered in terms of equivalence relations (Asghari, 2009; Asghari, 2019). Students are introduced to one type of equivalence relation, mathematical equality, early in their schooling. Students are directed to attend to equal counts of objects already in Kindergarten (National Governors Association Center for Best Practices, 2010, p. 11) and continue utilizing notions of equality throughout their mathematical careers when solving equations. Despite the centrality of equality to mathematics, students do not always have a deeply conceptual understanding of equality. Multiple studies have examined students' conceptions of equality and have connected students' modes of understanding to their ability to do algebra (e.g., Alibali et al., 2007; Kieran, 1981). Specifically, students' ability to manipulate expressions while maintaining equality is aided by understanding equations as two expressions that relate to each other in a specific way instead of viewing the equal sign as an indication to compute something (Alibali et al., 2007). As students mature mathematically, equality appears in new contexts, such as trigonometric identities (e.g., $\sin^2\theta + \cos^2\theta = 1$) where the equality holds generally despite containing unspecified parameters.

As students reach college mathematics, more complex equivalence relations arise such as isomorphism in abstract algebra. Prior work on the understanding of isomorphism has shown that students and researchers associate isomorphism with sameness. Leron and colleagues (1995) reported on a course where students were initially taught a "naïve" version of isomorphism, wherein isomorphic groups were "the same except for notation" (p. 154). Subsequent literature has confirmed references to sameness in the context of isomorphism by students and professors (e.g., Rupnow, 2019; Weber & Alcock, 2004). However, large-scale research has not verified whether this emphasis on sameness is normative across large groups of mathematicians.

We believe this examination of mathematicians to be an important step before examining students' conceptions of sameness directly. Research in mathematics education has benefited from examining experts' conceptions to gain insight into productive conceptions to encourage in students. For example, mathematicians' use of examples in proof (Lockwood et al., 2016) provides a standard with which to compare students' use of examples (Lynch & Lockwood, 2019). Similarly, Weber and Alcock (2004) noted similarities and differences in how mathematicians and students approached proofs about isomorphism, noting the mathematicians' use of intuition and knowledge of the properties of groups being invoked in contrast to students' focus on finding explicit mappings. These types of studies provide opportunities to leverage mathematicians' expertise in order to find productive connections and ways of thinking to teach students. Specifically here, the examples of sameness highlighted by mathematicians serve as possible connections to encourage students to consider, while mathematicians' approach to what sameness means, including any caveats on sameness, provide a possible way to help students navigate which aspects are relevant for drawing conclusions in a given context (Melhuish & Czocher, 2020).

Theoretical Perspective

An example space (e.g., Watson & Mason, 2005) is the "experience of having come to mind one or more classes of mathematical objects together with construction methods and associations" (Goldenberg & Mason, 2008, p. 189). Example spaces involve both examples and ways of making examples; thus, in order to judge whether an object is an example of a concept, both the dimensions of possible variation and the range of possible variation should be examined (e.g., Goldenberg & Mason, 2008; Mason & Watson, 2008). Dimensions of possible variation refer to any aspects that are permitted to differ, including conceptual and representational aspects, while the range of possible variation provides examples characterizing variety for a dimension. For instance, to have a robust example space of function, one would want to attend to dimensions of variation like presentation type and properties, while the range of variation would include the different options in the dimensions (e.g., graph, table, and formula for presentation type).

Because the example space construct is focused on particular instantiations of a concept as well as ways to determine more examples, this general lens is well-suited to our study. Specifically, because "sameness" does not have a stipulated mathematical definition (Edwards & Ward, 2004), we seek to understand how mathematical sameness is understood based on the instantiations of sameness highlighted (collection of mathematical examples) and the aspects fixed or varied for a given type of sameness (dimensions and range of possible variation). In so doing, we provide a framework for considering whether a concept is a type of mathematical sameness and which contextual features should be considered when making that judgment.

Prior work based on the example space construct has extended the construct to suit the needs of the research. Some researchers have focused on how the context in which questions are asked impact which examples are easily or frequently cited, referring to this as the accessible example space (Watson & Mason, 2005). Others have focused on how students' example space can develop through specific sets of activities (Watson & Shipman, 2008) or through teachers' classroom practices (Fukawa-Connelly & Newton, 2014), which relates to the enacted example space.

In keeping with these modifications of the example space construct, we introduce a new type of example space: a *community example space*. Unlike previous work, which has focused on examples that individuals have available or commonly access to reason about a concept, we are interested in commonalities and variations across mathematicians' individually-held example spaces. We define a community example space as a composite example space for a concept that includes common classes of examples as well as common criteria for examplehood. For instance, sameness can be communicated through particular definitions that encapsulate a type of sameness (e.g., isomorphism, congruence), through specific criteria for determining sameness (e.g., requiring types of objects to be the same), and through considering the relative number of criteria that align for different types of sameness (e.g., comparing different levels of sameness). For this paper, we operationalize an example or criterion as "common" if it is held by at least 20 (10% of) respondents. In this study, we use the notion of community example space to characterize how algebraists view mathematical sameness as we answer the following research questions:

1. What are the common dimensions of variation algebraists used to describe sameness?
2. What are the common examples of sameness that illustrate the range of variation for each dimension of variation?

Methods

Study Design

In order to gain an in-depth understanding of mathematicians' thoughts on sameness, we chose qualitative methods for this study. We wanted to obtain richly detailed responses from mathematicians and to allow their responses to guide our analysis. We chose to survey algebraists and category theorists because these disciplines foreground classification and attention to structure; thus, we thought algebraists and category theorists were the most likely mathematicians to see the relevance of "sameness" to their work. However, we also wanted a sample with heterogeneous backgrounds and prior experience concerning algebra, so we included current algebra researchers, those who had done research in algebra previously but had not published recently, and those who teach algebra despite not doing research in it.

The participants were solicited by sending a survey to every four-year college or university mathematics department that offers abstract algebra in the United States. Participants were required to have taught at least one abstract algebra or category theory course in the last five years and received monetary compensation for their participation.

Our sample was composed of 197 participants from 173 institutions. Demographic information was collected at the end of the survey and is summarized in Figure 1. The institution type was determined from the American Mathematical Society (AMS) website (Updated Annual Survey Groupings of Departments, n.d.). The majority of participants were tenure-stream faculty (n=163); "Other" instructor status included faculty in departments without tenure, part-time faculty, and graduate students. Seventy percent of the participants were algebraists in the traditional sense (i.e., had done research in algebra or category theory within the last 5 years (n=89), or over 5 years ago (n=49)). "Other" in relation to algebra research included researchers who had published limited work in algebra but did not consider themselves algebraists and researchers in interdisciplinary contexts.

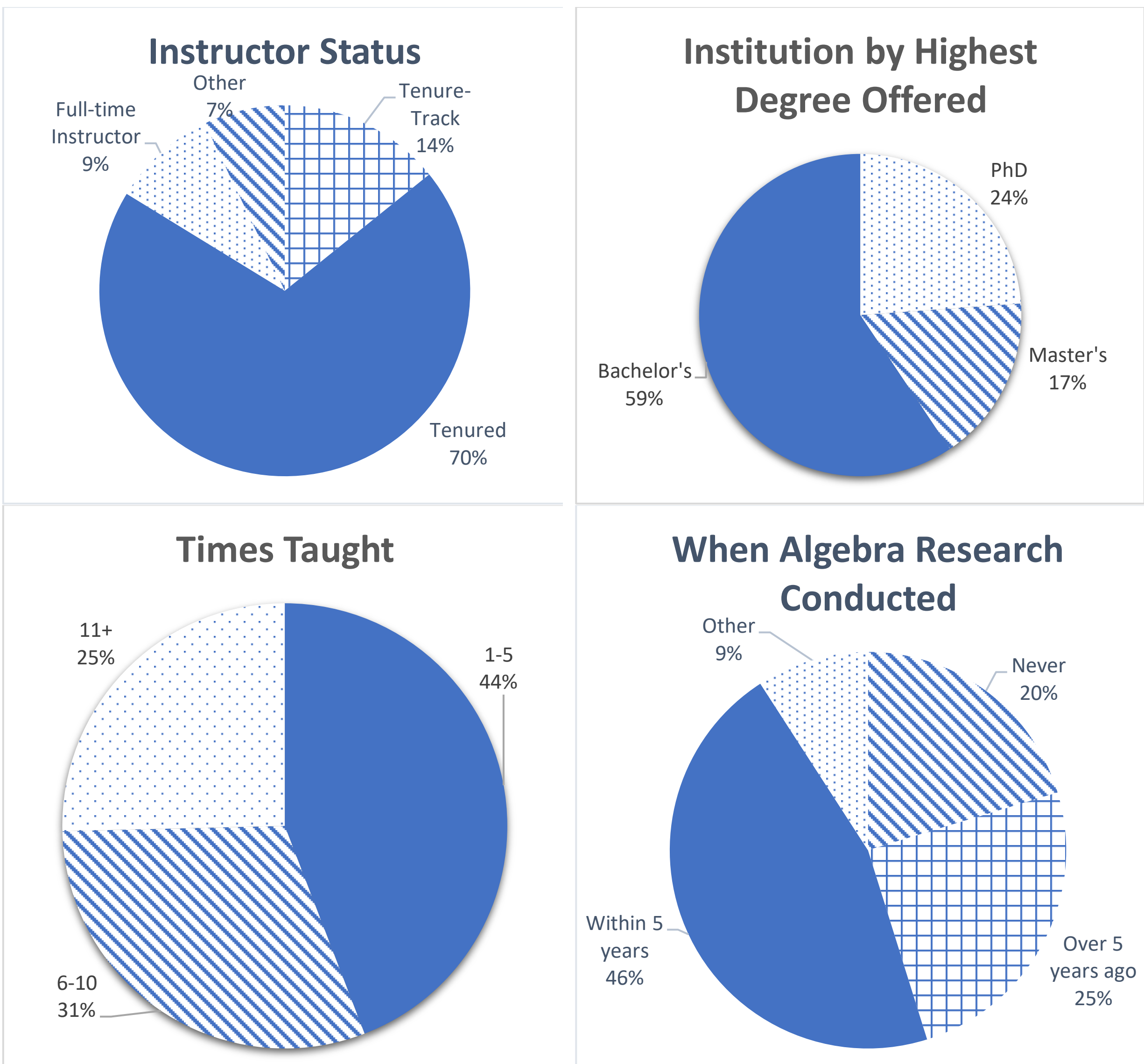

Fig 1 Information about survey respondents

The survey asked how algebraists think about sameness in general and in specific mathematical contexts (see the Appendix for the full survey). The question responses analyzed here are the first three and last three content questions from the survey and required participants to type an answer. Intervening questions included multiple choice questions rating the sameness of objects and specific questions about isomorphism and homomorphism. The relevant questions are as follows.

- What does it mean to be the same in a math context? (Q1)
- How do you know two things are the same in abstract algebra? (Q2)
- How is "sameness" in abstract algebra similar or different from "sameness" in other branches of math? (Q3)
- Which abstract algebra topics lend themselves to deepening students' understanding of mathematical sameness? (Q18)
- What connections do you see between sameness in abstract algebra and in prior math courses? (Q19)
- What sameness connections between abstract algebra and other courses do you (or could you) help students make when teaching? (Q20)

Data Analysis

Investigator triangulation (Turner & Turner, 2009) was used to ensure coding validity, with two independent coders using multiple iterations of coding for analysis of each section (Anfara et al., 2002). The coders included a mathematics education researcher with a PhD in mathematics, two PhD students researching abstract algebra, and one PhD student researching mathematics education. We leveraged our collective mathematical backgrounds to interpret the mathematicians' responses.

In analysis, responses were open-coded for words or concepts indicating any connection to sameness. This included informal language and a variety of defined concepts native to different mathematical contexts. We made a set of descriptive codes (Saldaña, 2016) focused on discipline-specific content references and cross-discipline content references for consistent coding between researchers. Each researcher coded independently and then discrepancies between the two coders were discussed until consensus was obtained. This meant each researcher would examine the data twice in each coding cycle, allowing for careful reflection on each participant's response. Next, codes were organized and revised. Responses were then revisited and examined via discussion to assure consistency. The above steps were repeated to refine the codes and group codes into categories. Finally, we adopted the community example space perspective, refined our categories to better align with dimensions of variation, and refined our codes to better align with the range of variation within each category. Specifically, we used the categories of universe of discourse, concept, object, properties, and quality. Roughly, these categories address the context (universe of discourse), type of sameness (concept), what is the same (object, properties), and how they are the same (quality). Further descriptions of each code are discussed in the Results.

After analysis, small edits were made to participant responses to improve readability. Possible changes included correcting spelling or typographical errors and adding context as needed. Any parentheticals in quotes are original to participants; researcher additions were put in brackets.

Results

We present results according to universe of discourse (context) with the aim of aiding readers who are more interested in focusing on examples related to their areas of research or teaching. While we acknowledge some research areas intentionally cross mathematical subdisciplines (e.g., algebraic topology) and some topics may appear in multiple courses (e.g., sequences and series revisited in analysis after initial exposure in calculus), we did our best to place concepts in the first course they would be encountered in by students in the United States or in the course/research area most aligned with the concepts noted.

Table 1. Universe of discourse for sameness

Context[a]	**Description**	**Number of Participants (%)**
Linear Algebra	Mention of a linear algebra course or topics such as vector spaces, linear transformations, or matrix operations	103 (52%)
Topology	Mention of topology or differential geometry as a course/research area or references to homeomorphism, diffeomorphism, homotopy equivalence, continuous deformation, or topological spaces	80 (41%)
Geometry	Mention of geometry as a course/research area or references to topics such as congruence or similarity of polygons	70 (36%)
Discrete Math	Mention of a discrete math course, a course/research area in number theory, graph theory, combinatorics, cryptography, design theory, or coding theory, or topics such as modular arithmetic/congruence of numbers or graph isomorphism	64 (32%)
Calculus	Mention of a calculus course or topics such as derivatives, antiderivatives, or limits of functions/sequences typically encountered in non-proof-focused calculus courses	53 (27%)
Arithmetic/ pre-university algebra	Mention of arithmetic or algebra encountered before calculus or topics such as equality of numerical expressions (often fractions), manipulations to solve simple equations, or examinations of the period of trigonometric functions	47 (24%)
Analysis	Mention of analysis as a course/research area or topics like functions equal a.e. or metric spaces	38 (19%)
Category Theory	Mention of category theory as a course/research area or positioning discourse generally through comparisons of properties/concepts in different categories	35 (18%)
Other	Other clear positioning in a course or research context that did not occur in at least 20 participants' responses, including differential equations, statistics, applied math, and introduction to proof courses	33 (17%)
Set theory	Mention of a set theory course or a perspective clearly grounded in examinations of sets but not clearly tied to another branch of mathematics	21 (11%)

[a]Five of the six questions prompted consideration of or comparisons to abstract algebra so we do not report frequencies for abstract algebra, but we do discuss ways that discussion of abstract algebra was positioned below.

Table 1, above, focuses on descriptions of each universe of discourse (context) for discussion highlighted by participants. We used a conservative approach when coding. The universes of discourse included in the codes were generally explicitly mentioned by the participants. If the participant mentioned a specific concept without mentioning a context, the context was deduced by the coders in clear cases (e.g., congruence of polygons is native to geometry, congruence of numbers is native to discrete math) but was left uncoded if unclear (e.g., congruence of unknown objects).

Table 2. Concept conveying a type of sameness

Concept	**Description**	**Number of Participants (%)**
Isomorphism	Mention of isomorphism in a specific context (e.g., abstract algebra) or in a general context	176 (89%)
Equality	Mention of equality in a specific context (e.g., arithmetic) or in a general context	85 (43%)
Other	Specific concept meant to convey a type of sameness that occurred in fewer than 20 participants' responses (e.g., functions equal a.e., categorical equivalence, trigonometric identities, antiderivatives that differ by a constant)	80 (41%)
Equivalence Relation	Mention of equivalence relations or creating equivalence classes in a specific or general context	77 (39%)
Congruence	Mention of congruence in a specific context (e.g., geometry, modular arithmetic) or in a general context	72 (37%)
Homomorphism	Mention of homomorphism in abstract algebra	64 (32%)
Homeomorphism	Mention of homeomorphism in topology	42 (21%)
Diffeomorphism/ Homotopy equivalence	Mention of diffeomorphism and/or homotopy equivalence in topology	24 (12%)
Similar (geometry)	Mention of similar polygons in geometry	21 (11%)
Linear Transformation	Mention of linear transformations in linear or abstract algebra	20 (10%)

In Table 2, we highlight common concepts conveying a type of sameness. We include only well-defined mathematical concepts here, meaning concepts that have a specific definition in one or more mathematical contexts (e.g., isomorphism, congruence, equivalence relation). We did not include informal language (e.g., structure-preservation) or ambiguous terms (e.g., identical, equivalent).

Table 3. Objects to which sameness is applied

Object	**Description**	**Number of Participants (%)**
Group	Algebraic groups considered "the same", potentially via isomorphism, homomorphism, or another connection	135 (69%)

Ring	Algebraic rings considered "the same", potentially via isomorphism, homomorphism, or another connection	97 (49%)
Other	Other objects considered "the same" that individually occurred in fewer than 20 participants' responses (e.g., categories, modules, monoids, matrices, sequences, series, statements, subobjects)	81 (41%)
Vector Space	Vector spaces considered "the same", potentially via isomorphism, linear transformations, or another connection	56 (28%)
Set	Sets considered "the same", potentially via equality, bijections, or some other connection	48 (24%)
Number	Numbers considered "the same", potentially via equality, congruence, or some other connection	47 (24%)
Function	Functions considered "the same", potentially via equality a.e., differentiation, or same images for same elements	39 (20%)
Field	Algebraic fields considered "the same", potentially via isomorphism, homomorphism, or another connection	38 (19%)
Quotients	Quotient structures (e.g., cosets) considered "the same", potentially via isomorphism, homomorphism, or another connection	37 (19%)
Elements	Particular elements of sets or algebraic objects considered "the same", potentially via isomorphism, homomorphism, or another connection	29 (15%)
Triangle	Triangles considered "the same", potentially via congruence, similarity, or another connection	20 (10%)

Table 3 highlights common objects that could be 'the same'. Objects were coded if they were the target for descriptions of sameness (e.g., "two elements of a group are the same if they are equal" receives an elements code, not a group code).

Table 4. Properties conveying sameness or a lack of sameness

Properties	**Description**	**Number of Participants (%)**
Cardinality/ Bijection	"Same" objects have (or need not have) the same cardinality/a bijection between them	82 (42%)
Representation	"Same" objects have (or need not have) the same element names/representations to be the same under types of sameness	70 (36%)
Other	Other properties expected to be shared (or that need not be shared) for same objects (e.g., location, symmetry, angle measures, commutativity, coordinates, generators)	48 (24%)
Dimension	"Same" objects (especially vector spaces) have the same dimension	21 (11%)

Table 4 highlights properties of objects that should be the same for objects to be the same or that could be varied and still allow the objects to be the same. Properties were coded when the properties were clear and related to determining whether the objects were the same (e.g., "isomorphic groups have the same cardinality" receives a cardinality/bijection code) or were properties that need not be the same for the underlying objects to still be the same (e.g., "two things are the same if changing the labels turns one thing to the other" receives a representation code). Properties were not coded when used to clarify context (e.g., "for finite groups…") or left unspecified (e.g., "usually we consider objects the same if all properties are the same").

Table 5. Quality of sameness

Qualities	Description	Number of Participants (%)
Levels of sameness	Different types of sameness require fulfillment of more criteria than others	47 (24%)
Computability/ Construction	Sameness can be (or cannot be) demonstrated via explicit construction of a mapping	24 (12%)
Logical equivalence	Sameness can be (or cannot be) demonstrated via explicit invocation of a theorem or alternative definition	24 (12%)

Table 5 highlights qualities of sameness, specifically ways to compare the types of sameness conveyed by specific concepts. Qualities were coded when the degree or method of demonstrating sameness were highlighted. For instance, the following received a levels of sameness code: "There are many weaker forms of sameness in algebraic topology (weak/homotopy equivalence) and stronger forms of sameness (like in analysis)". Comparing the same concept on different objects was only coded with levels of sameness if the participant explicitly said this indicated different strengths based on more/fewer criteria needing to be met, for instance:

> I usually try to help them develop a hierarchy of "sameness". For example, for two groups to be "the same", they must be isomorphic (the same) as sets and the isomorphism must preserve the group operation; for two rings to be "the same", they must be isomorphic as groups and the isomorphism must also preserve the multiplicative structure."

The computation/construction code was applied to instances focused on demonstrating equivalence via explicit mappings: "You know there is an isomorphism between them. Sometimes you actually define a function from one to the other that is the isomorphism…." We coded logical equivalence when broad theorems were used to draw conclusions about sameness of objects: "…Other times you employ a theorem, such as the First Isomorphism Theorem to deduce that an isomorphism exists."

Within each universe of discourse noted below, we highlight dimensions of sameness represented through specific examples relevant to that area to highlight the range of sameness for each dimension in each context. An attempt was made to choose examples that were representative of the responses in a given context. While we provide frequencies for the range of examples of each dimension that appeared in at least 20 responses in Tables 1 through 5, we also include some examples that were counted as "Other" for frequencies in a number of universes of discourse to show the variety in responses for that context.

Abstract Algebra

As noted in Table 1, participants were asked to compare to or consider abstract algebra in five of the six questions so essentially all respondents discussed sameness in an abstract algebra context. Isomorphism (especially of groups, rings, and/or fields) was the most common type of sameness highlighted in abstract algebra. For instance, in response to Q2 (focused on sameness in abstract algebra):

> They are isomorphic. I try to communicate this to my students by saying that their addition/multiplication/whatever tables are the same. However, the formal definition of isomorphism depends on the context: isomorphism in a monoid is not the same as isomorphism in a group is not the same as isomorphism in a ring is not the same as isomorphism in a module...

Notice even while classifying isomorphism as a type of sameness, this respondent emphasized the importance of also attending to the underlying object, here monoids, groups, rings, and modules. Furthermore, at the end of the survey, when discussing specific topics related to deepening an understanding of sameness in abstract algebra (Q18), many respondents highlighted a need to see concepts repeated with different objects, such as: "Work with homomorphism, isomorphism, and the fundamental homomorphism theorems in multiple contexts (groups and rings)."

Furthermore, the representation property was often cited as something that need not be "the same" for sameness to be expressed through isomorphism: "In abstract algebra, isomorphism is good enough—two objects have the same structural properties, but we might have different labels or language to describe them." However, this view of representations not needing to be "the same" was not universal for algebraic sameness:

> "The same" depends on context. Normally in abstract algebra, sameness is interpreted as isomorphism. But even when teaching a first course in group theory this is insufficient. A single group can have non-isomorphic representations by permutation groups or matrix groups. So A_5 and $PSL_2(5)$ are the same as abstract groups but quite different as permutation groups. When teaching, I often discuss with students what it means for objects to be "the same" (which I normally interpret as exactly equal) and "essentially the same" (which I normally interpret as isomorphism).

Here the participant is attending to the fact that in the representation theory of finite groups, we may be able to use the basic tools of linear algebra to understand the structure of a group or family of groups. However, we must consider whether the information we gain from this translates across vector space isomorphisms (a change of basis) or not, meaning the chosen representation for an object may need to be "the same" for sameness. Furthermore, this participant highlights the levels of sameness dimension through comparing "same" and "essentially the same" (equality and isomorphism, respectively).

In fact, many of the uses of equality in abstract algebra involved contrasting equality and isomorphism. For instance, in response to Q18, focused on aspects of abstract algebra relevant to sameness, this participant wrote, "Galois theory. Specifically, the distinction between equality of subfields and isomorphism of fields. The idea that Galois conjugate fields are isomorphic but not equal really drives this point home."

Many participants also highlighted homomorphism as a way to convey sameness in abstract algebra. Most of these references to homomorphism were after reference to isomorphism, such as "Isomorphism, because this is the foundation for thinking about sameness in abstract algebra. Homomorphism, because this extends the idea of sameness to images and subgroups." or "So I guess the main thing is isomorphism. Homomorphism is useful in that it says something about being structure-preserving and you can find sameness in the fibers." Notice these participants do not seem to view homomorphism as the key sameness in algebra, but they perceive sameness of subobjects or quotients to be conveyed via homomorphism.

Finally, a number of participants focused on how sameness was demonstrated, often in the context of isomorphism:

I guess first you have to set your definition of the term “same”. In the stricter sense of the term, if I wanted to know that two objects with different descriptions are identical, I guess I would try to prove that their descriptions are logically equivalent. In the looser “isomorphism” sense of the term, I guess I would try to exhibit an explicit isomorphism between my two objects.

Notice this participant seems to view different types of sameness as requiring different types of proof: logical equivalence or construction of functions.

Linear Algebra

Types of sameness in linear algebra that appeared in at least 10% of responses were isomorphism of vector spaces and linear transformations, though less frequent examples included equality, row equivalence, and similar matrices.

While writing about sameness connections between abstract algebra and other courses at the end of the survey (Q19 & Q20), isomorphism in linear algebra was highlighted a number of times, such as the following: “There is a similar notion of sameness in Linear Algebra with vector spaces of the same dimension being isomorphic.” Here the type of sameness highlighted is isomorphism, in addition to an associated property that must be shared (dimension) when considering the sameness of the relevant object (vector spaces). Similarly, linear transformations were largely referenced in Q19 & Q20 as a connection to abstract algebra sameness, especially as the analogue to homomorphism. For instance:

Homomorphisms in abstract [algebra]… are introduced (though not using those words) in linear algebra, when linear transformations are discussed. I take pains in my linear algebra classes to talk about structure-preserving maps more broadly, and discuss how null spaces of matrices generalize to kernels of linear transformations and homomorphisms that students will see in their abstract algebra course.

Notice that these examples both highlight comparisons between linear algebra and abstract algebra content. While seeing “isomorphism” as a similar idea in two courses may not be surprising, connecting differently named concepts, linear transformations and homomorphisms, conveys a less obvious sameness of type of concept across courses.

A variety of ways to view sameness of objects in linear algebra were noted, including vector space isomorphism (above). Other respondents noted equality as a measure of sameness for matrices or vectors. For instance, in response to Q3, one respondent wrote: “In linear algebra two matrices are the same only if they are in fact equal.” Similarly, in response to Q1, another said “Generally in undergrad math, same will mean “equal.” Equal numeric values, equality of sets when they contain the same exact elements, equality of vectors or matrices, etc….” However, equality was not the only way to consider matrices “the same”. For instance, this participant highlighted row equivalence:

There are a lot of different meanings for “same” in a math context, and that’s why we don’t formally use that term. Instead, we make precise definitions such as “isomorphic as vector spaces” or “equivalent under row-reduction operations”…

In addition, some responses to Q19 and Q20 hinted at differences in the character of sameness for different objects in linear algebra:

There is a strong connection to be made with linear algebra, for example similar or conjugate matrices. This isn’t quite sameness in the sense of isomorphism anymore, but it is sameness in the sense of grouping things into equivalence classes.

Notice the central focus of sameness under isomorphism is that of whole objects whereas equivalence classes can be viewed as placing the emphasis on sameness of subclasses within a larger whole (i.e., sameness of similar n x n matrices as a subclass of all n x n matrices).

A number of participants also highlighted representations as a property that need not align for sameness in linear algebra, especially emphasized through choices of bases or changes of basis. For example:

> In Linear Algebra, we often are interested in a set of vectors only in terms [of] what space those vectors generate. This defines an equivalence relation on sets of vectors: two sets are "the same" if they generate the same space. The two sets needn't be identical. But in deciding which attributes of the vectors (or the set of vectors) we wish to study, we are in effect putting blinders on ourselves so that we no longer see differences inessential to the questions we wish to study.

Here we again see particular representations as being unimportant to determining sameness in our context of interest though the way of representing the span is an easily recognized difference.

Topology

Topology responses generally focused on homeomorphism, diffeomorphism, or homotopy equivalence. Recall that a homeomorphism of topological spaces is a bijective continuous function with a continuous inverse function, and that two functions are homotopic when the range of one can be continuously deformed onto the range of the other.

Some respondents highlighted these concepts to compare with algebraic sameness:

> There are other areas of mathematics in which we might identify two objects as the "same", even if they are not set-theoretically the same object. For example, in topology two spaces are considered the "same" if there is a homeomorphism (or perhaps diffeomorphism, if we care about tangent spaces) between them. … I would say that the common theme is that two objects can be identified as "same" if one is the image of another under a function that preserves all of the structural properties that we care about and is bijective.

For more context here, a diffeomorphism between two topological spaces is a homeomorphism that is also a homeomorphism on tangent spaces when we can define them. Notice that homeomorphism is given as a topology-based example of a bijective, structure-preserving function. Furthermore, this example again hints at the existence of different levels of sameness (set-theoretical sameness seemingly stronger than homeomorphic sameness) while also highlighting the importance of maintaining some set of properties, specifically shared cardinality through a bijection.

Other participants focused on continuous deformation or homotopy equivalence to describe sameness. Some provided minimal elaboration: "In Topology, I'm interested in the structures that remain when I allow continuous distortions." Others elaborated on important properties that are preserved by such maps, like connectivity:

> In point set or differential topology, connectivity matters; things are the "same" if there's a reasonable enough map that takes one to other ("reasonable enough" could depend a little on the context, but it always preserves connectivity: homeomorphism, diffeomorphism, homotopy equivalence, etc.). Algebraic topology inherits this "sameness" for objects but also picks up the "sameness" from algebra when considering the algebraic objects associated to spaces.

Geometry

While cited frequently, there was limited variety in the types of sameness highlighted in geometry; congruence and similarity were by far the most common types noted. Furthermore, most respondents particularly applied congruence and similarity to triangles. As an example to explain their answer to the general sameness question (Q1), one respondent highlighted what should be the same and what need not be under congruence: “In geometry, two triangles are congruent when [they] are the same size and shape, even if they are in different locations.” Another made a similar point from the transformational perspective: “In Euclidean geometry two triangles are the same if there exists an isometry which can move one onto the other.” These examples highlight size and shape as properties of sameness that are important while location is irrelevant under this type of sameness.

On the other hand, some respondents highlighted similarity as a type of sameness because it was an equivalence relation, though they treated it as a weaker example than congruence:

> For example, similarity in geometry is clearly an equivalence relation, but we can easily distinguish between two similar triangles that are not congruent by size: One is the “big” triangle and the other is the “little” triangle.

Thus, we have congruence as a stronger level of sameness that preserves more properties (both size and shape), while maintaining shape alone can still be viewed as displaying a type of sameness.

Discrete Math

We combine number theory, graph theory, combinatorics, cryptography, design theory, and coding theory into one context, that of “Discrete Math.” We made this choice for two reasons. First, a number of participants referred to “Discrete Math” as a course before Abstract Algebra at the end of the survey, indicating this is an interpretable context to many people. Second, when considered individually, the particular areas listed above were not used by 20 participants. Thus, we consider these areas together.

By far, the most common type of sameness considered in discrete math contexts was congruence in modular arithmetic. For instance, this participant contrasts equality and congruence of numbers:

> Most branches of mathematics have the notion of structure-preserving maps. Sameness in these cases is fundamentally the same, though practically different (since the structure changes). Even in number theory, where most of the time the objects are numbers and structure means equality, the notion of same gets warped by structure-preserving maps in the context of modular arithmetic.

Notice here we have sameness of number conveyed both through equality and (implicitly) congruence mod n. Modular arithmetic was especially cited at the end of the survey as a relatable example for use in algebra (e.g., “Modular arithmetic is really “the same” as arithmetic on quotients (groups, rings, etc.), so whenever we have a quotient, we should be able to define a modular arithmetic.”) Notice this participant also seems to view all quotient objects as “the same” in the sense that they permit a certain type of operation.

Equivalence relations were also highlighted a number of times in discrete math contexts. Modular arithmetic was referred to as “a great building block for more abstract equivalence relations.” Additional participants were more explicit about the connection to higher mathematics, mentioning “the idea of ‘modding out’ to create an equivalence relation (in other words the idea of ‘sameness’ that underlies modular arithmetic as well as the construction of quotient groups and quotient rings).” Note this participant’s connection of modular arithmetic with the creation of equivalence classes in abstract algebra as representing sameness of classes of objects.

Graph theory examples largely centered on isomorphisms of graphs. For example, in response to Q3, relating sameness in abstract algebra to other sameness in math, this participant highlighted the property of connectivity:

> This is a question of “what structure are we interested in?” So, group theory we are interested in a set with a binary operation. An isomorphism (exhibiting “sameness”) needs to be a kind of identification which not only “preserves the set” in some sense, but also “preserves the binary operation.” In graph theory, the structure has to do with a basic notion of connectivity that’s been encoded in edges and vertices. For graphs, “isomorphic” has been defined in a way that preserves these “connective structures.”

Here we see a property needed for sameness (connectivity) applied to the object of a graph.

Others highlighted constructability as a quality for consideration, such as in cryptography:

> “Sameness” in abstract algebra is often taken to be the existence of an isomorphism…. In computational problems, such as in public key cryptography, one only has sameness if the isomorphism can actually be computed in acceptable time.

Notice this response highlights that isomorphism of underlying objects is not always sufficient for sameness; rather, another dimension of sameness is whether the sameness is practically demonstrable (computability/construction).

Calculus

While calculus was cited by over one-fourth of respondents, no single concept of sameness in calculus was highlighted by at least 10% of participants. Nevertheless, the most common references were to antiderivatives and changes of variables. For example, this participant focused on antiderivatives as a type of sameness while also emphasizing what is different: “We talk about antiderivatives being the same when they actually differ by constants, that is another notion of sameness.” Others emphasized an equivalence class interpretation, focused on antiderivatives differing by a constant, represented by +C: “The +C in calculus is basically helping us identify cosets of the kernel of the derivative map. If you treat C as the subgroup of constants, then the notation precisely matches abstract algebra set notation.” Both of these examples emphasize viewing a class of functions (those differing by a constant) as being the same under the lens of differentiation and, in the second example, as implicitly foreshadowing homomorphism in abstract algebra.

Others noted changes of variables from calculus as a useful analogy to build students’ understanding of sameness: “In prior math courses, sameness often meant identical. However, this notion does occur when considering that changing notation does not mean the object has changed (such as using ‘x’ vs ‘w’ for a variable in calculus).” This connection emphasizes that changes of representation by altered notations or names of objects is not generally important for sameness whereas the underlying function relationship is.

Arithmetic/Pre-University Contexts

Because most responses to the survey focused on sameness in undergraduate-level mathematics (or beyond), we group together all pre-calculus, non-geometric content. Most examples fitting this description focused on representations of numbers, usually in terms of equality of fractions, such as in part of the following response to Q1: “In the rational numbers, two fractions a/b and c/d are equivalent if ad = bc. For example, any school child will tell you that 5/5 and 1 are different, however they represent the same number.” Similarly, alternate representations of fractions were used as an example of mathematical sameness: “I like to bring up multiple ways to write the same numeric value as fractions as a simple example of how mathematicians recognize things to be the same in a more general way [than] some others might." Another respondent focused instead on the sameness we exhibit by equating arithmetic expressions with their evaluations:

> Depending on the context, "same" may mean "equal" or "congruent" or "isomorphic." I make a distinction with students in my classes between having the identical object and hav[ing] an object which is alike in all the ways that matter in the particular discipline. For example, in arithmetic, 5 is equal to 2 + 3, they are numerically the same.

We can see that their criterion for sameness in arithmetic is equality of numbers. All of these examples emphasize that under equality, number is maintained even though the written representations may change forms.

Others highlighted topics commonly addressed before calculus, like the graphs of trigonometric functions: "You might say that different periods of a trig function are the same." Notice here the object of interest is the period of a function.

Analysis

While analysis was cited by roughly one-fifth of respondents, no single concept of sameness in analysis was highlighted by at least 10% of participants. However, the most common concept cited was equality almost everywhere (a.e.). Equality a.e. indicates that functions are the same up to a set of measure zero: "For example, for Lebesgue integrable functions, two functions are the same if they are equal almost everywhere, and therefore have the same integral." Notice this suggests that sameness of functions can be determined via equality a.e. and that they share the property of same integral.

General Contexts (Including Set Theory and Category Theory)

A number of participants made an effort to speak generally, whether in terms of sets or categories. By far the most common topic for those invoking category theory was isomorphism of two objects in the same category, e.g., "If you have two objects in a category, they are the same if there is an isomorphism between the objects." However, other respondents focused on the categories themselves as objects. For example:

> If you[r] math context is a 2-category (e.g., the 2-category of 1-categories) then two things (objects in the 2-category) are the same if there is an equivalence between them (e.g., the category of matrices and the category of finite dimensional vector spaces are the same in this sense, as are the fundamental groupoid and fundamental group of a path connected space).

For context, the respondent's example of the 2-category of 1-categories is the category whose objects are other categories themselves (for instance, the category of matrices), and the maps between those categories are called functors. As the respondent says, there is a categorical equivalence between the category of matrices and the category of finite dimensional vector spaces. This means any statement made using the linear mappings between finite dimensional vector spaces can be made instead using the category of matrices and their multiplication structure, so we can view seemingly different objects, matrices and finite dimensional vector spaces, as equivalent in a specific sense when using the lens of 2-categories. Thus, this example provides a way of considering when categories are the same in mathematics.

Responses that sought generality through speaking about sets (coded as "set theory" universe of discourse) largely focused on two types of sameness: equal sets or bijections between sets. For example, this respondent considered a bijection to be sufficient for sameness: ""Same" = "equivalent". Sets are the same if they are equivalent (there is a bijective function between the sets)." Others were stricter, requiring elements to be the same within the sets: "Typically, I say "the same" about the element of a set. In such a situation, I mean literally this same element and make use of [an] equal sign." Notice these examples together highlight different levels of mathematical sameness to be expected in general set theoretical contexts.

Furthermore, while a number of participants used equivalence relation language in specific contexts, a third way to speak generally was to invoke equivalence relations. For instance, a number of participants responded to Q1 by highlighting equivalence relations: "I think of "the same" as an equivalence relation." and "Being related under some equivalence relation." Others used equivalence relations to connect contexts, especially in response to Q19 & Q20:

> I very frequently try to point [out] how the structures showing up in abstract algebra have analogues in their classes on topology, differential geometry, etc., and, since our students tend to have a very shallow understanding of equivalence relations, try to make it very clear how important equivalence relations are to all mathematicians.

This participant highlights the importance of equivalence relations and seems to suggest further instruction on this concept is a potential way to enhance students' understanding of mathematics as a whole.

Discussion and Conclusions

In this paper, we featured a variety of dimensions of mathematical sameness as well as a range of examples for each of these dimensions. We illustrated the universe of discourse dimension through the subsections of the Results and noted concepts denoting types of sameness in specific disciplines (e.g., isomorphism in abstract algebra, homeomorphism in topology) as well as particular underlying objects to apply types of sameness to (e.g., group, function, number, quotient), properties of objects to consider when determining sameness (e.g., cardinality, representations), and the quality of the sameness described (e.g., level of sameness of equality stronger than isomorphism, demonstrability via logical equivalence or construction possible). Through the range of examples for these dimensions, we illustrated the breadth of aspects and specific ways that mathematicians viewed sameness as manifesting. In this way, we provided insight into a community example space for sameness.

Based on these results we highlight four main findings. First, the goals of this paper were to establish that "mathematical sameness" is comprehensible to algebraists and to leverage mathematicians' expert knowledge of advanced mathematics to see what sameness connections we might want math majors to transfer across their courses (Lobato, 2012). We believe we succeeded in these goals. Mathematical sameness appeared to be an interpretable concept to most respondents, and most respondents were willing to engage with the prompts. In fact, as shown in the length of quotes above, many mathematicians wrote extensively as they compared contexts or highlighted nuances in their understandings of sameness. We acknowledge that some selection bias could influence respondents' enthusiasm and do not claim that the coding percentages for any category would generalize to the broader population of algebraists or mathematicians in the United States or globally. Nevertheless, nearly 200 mathematicians did engage with the survey with minimal discussion about the validity of the concept of sameness in mathematics and we did see specific equivalence relations (e.g., isomorphism, equality, congruence) and the notion of equivalence relations as common types of sameness, though some types of sameness were not so closely tied to equivalence relations (e.g., homomorphism). Furthermore, we obtained a variety of responses that we had not considered previously, such as viewing antiderivatives as equivalence classes conveying a type of sameness. Future research should also consider deeper examination of the impact of sameness on processes, such as how proving sameness in different disciplines might be conceptually similar or different (Dawkins & Karunakaran, 2016) and how to help students draw upon appropriate, context-specific methods of proof for sameness specifically.

Second, in this paper we have highlighted a community example space including dimensions of variation for sameness and a range of examples of well-defined mathematical terms related to these dimensions of sameness. No individual highlighted all of the examples listed here, nor would we have expected them to. Similarly, we do not expect students would notice all of the dimensions or examples of sameness noted here without aid. However, we would hope students could make some connections, especially if prompted to reflect on what mathematical sameness should entail. Specifically, do students recognize isomorphism and homeomorphism as representing a similar type of sameness but in different universes of discourse as applied on different objects or do they see these concepts as completely separate ideas? Similarly, are differences between equality of groups and isomorphism of groups clear to students or is the notion of "levels" of sameness lost on students? Finally, are differences between isomorphic groups and rings clear or are distinctions lost under the notion of isomorphism? Based on students' responses about the non-equivalence of .999… and 1 (Melhuish & Czocher, 2020) and claims that (Z_3, $+_3$) is a subgroup of (Z_6, $+_6$) (Dubinsky et al., 1994), students may not always attend to the aspects teachers or researchers intend, suggesting a need to help students see dimensions of sameness relevant in particular contexts. Thus, we also hope this list of examples of sameness prompts readers to reflect on specific commonalities between concepts that arise in their courses as well as the particular dimensions that differ, so that they can highlight instantiations of sameness and reasons for having many types of sameness in their own teaching.

Third, the breadth of varieties of sameness suggests topics for future research. Namely, much as function conceptions are compared in high school and advanced math contexts (e.g., Melhuish, et al. 2020; Zandieh et al., 2016), conceptions of sameness could be examined in different course contexts to help students critically examine relevant dimensions of sameness and how these dimensions relate to goals in different mathematical contexts. In this way, we suggest sameness might be viewed as a thematic connection across math that enables students to see clearer connections between their math courses, thereby helping students understand why they are expected to take a variety of math courses for their major and see which branches of math are likely to be applicable in their future studies or careers. Moreover, future research could examine how informal language for examples of sameness like isomorphism and homomorphism (Rupnow, 2021) may also play a part in how sameness is viewed in mathematics.

Fourth, respondents highlighted some concepts in generic ways, like equality and isomorphism, when writing about sameness. The existence of such terms is indicative of some similarity in the concepts represented by them in different disciplines beyond being equivalence relations. Although care must be taken to focus students on distinctions between definitions of isomorphism on different objects (Edwards & Ward, 2008), these shared names for concepts provide opportunities for conversations about how mathematicians stipulate definitions and why they may choose to name concepts the way they do. These types of conversations can especially aid students tasked with writing their own definitions, such as in courses grounded in Realistic Mathematics Education (Rasmussen et al., 2005). However, these conversations might also provide opportunities for students to better understand the culture of mathematics and appreciate processes central to mathematical activity outside computing.

Conflict of Interest Statement

On behalf of all authors, the corresponding author states that there is no conflict of interest.

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Appendix

1. What does it mean to be the same in a math context?
2. How do you know two things are the same in abstract algebra?
3. How is "sameness" in abstract algebra similar or different from "sameness" in other branches of math?
4. Which option, in your opinion, best describes the relationship between the positive reals under multiplication and the reals under addition?
    a. They are identical structures.
    b. They are the same structures in all contexts, but not identical structures.
    c. They are algebraically the same structures, but they are different structures in other mathematical settings.
    d. They are similar algebraic structures, but they are not the same.
    e. They are different mathematical structures.
5. Please explain your response.
6. Which option, in your opinion, best describes the relationship between Z_5 under modular addition and Z/5Z under addition?
    a. They are identical structures.
    b. They are the same structures in all contexts, but not identical structures.
    c. They are algebraically the same structures, but they are different structures in other mathematical settings.
    d. They are similar algebraic structures, but they are not the same.
    e. They are different mathematical structures.
7. Please explain your response.
8. Which option, in your opinion, best describes the relationship between Z and 2Z under addition?
    a. They are identical structures.
    b. They are the same structures in all contexts, but not identical structures.
    c. They are algebraically the same structures, but they are different structures in other mathematical settings.
    d. They are similar algebraic structures, but they are not the same.
    e. They are different mathematical structures.
9. Please explain your response.

10. Which option, in your opinion, best describes the relationship between Z_5 and Z_6 under addition?
    a. They are identical structures.
    b. They are the same structures in all contexts, but not identical structures.
    c. They are algebraically the same structures, but they are different structures in other mathematical settings.
    d. They are similar algebraic structures, but they are not the same.
    e. They are different mathematical structures.
11. Please explain your response.
12. Which option, in your opinion, best describes the relationship between G = Z/2Z x Z/2Z x Z/2Z x ... where elements of G must contain a tail of 0's and H = Z/2Z x Z/2Z x Z/2Z x ... (unrestricted)?
    a. They are identical structures.
    b. They are the same structures in all contexts, but not identical structures.
    c. They are algebraically the same structures, but they are different structures in other mathematical settings.
    d. They are similar algebraic structures, but they are not the same.
    e. They are different mathematical structures.
13. Please explain your response.
14. How might sameness be helpful when thinking about isomorphism/isomorphic structures?
15. How might sameness be harmful when thinking about isomorphism/isomorphic structures?
16. How might sameness be helpful when thinking about homomorphism?
17. How might sameness be harmful when thinking about homomorphism?
18. Which abstract algebra topics lend themselves to deepening students' understanding of mathematical sameness?
19. What connections do you see between sameness in abstract algebra and in prior math courses?
20. What sameness connections between abstract algebra and other courses do you (or could you) help students make when teaching?