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Sameness in Algebra: Views of Isomorphism and Homomorphism

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Abstract

Isomorphism and homomorphism are topics central to abstract algebra, but research on mathematicians' views of these topics, especially with respect to sameness, remains limited. This study examines open response survey data from 197 mathematicians on how sameness could be helpful or harmful when studying isomorphism and homomorphism. Using thematic analysis, we examined whether sameness was viewed as helpful or harmful for isomorphism and homomorphism before examining rationales for those views. Making use of values of the mathematical community, we note that mathematicians saw conceptual and pedagogical benefits to connecting isomorphism and sameness, which connects to leveraging intuition and valuing ways of increasing understanding. Mathematicians' concerns around using sameness largely revolved around the violation of the mathematical community's idealized value of expressing a priori truth via a-contextual justifications. However, these concerns can be addressed through only targeted usage of sameness and explicit discussions around the utility and relevance of sameness. Implications include the importance of considering how interventions proposed by mathematics educators align with or provoke tension between values held by the mathematical community in order to mitigate or lean into those tensions and encourage fruitful dialogue between the mathematics and mathematics education communities.

Keywords: Isomorphism, Homomorphism, Abstract Algebra, Mathematical Sameness

Abstract algebra focuses on the structure of sets in mathematics (Ronan, 2012). As such, fundamental questions in algebra tend to focus on characteristics that are shared or distinct between two sets with operation(s). Thus, it is unsurprising that isomorphism and homomorphism are viewed as central to the study of abstract algebra (Melhuish, 2015), as they are functions that formalize ways of identifying similar elements between objects and identify objects with the same or similar relationship structures. Despite this centrality to abstract algebra, understandings of isomorphism and, especially, homomorphism remain understudied. Furthermore, most work has examined students' understandings (e.g., Dubinsky et al., 1994; Melhuish et al., 2020). However, examination of two abstract algebra instructors' language showed connections between sameness and both isomorphism and homomorphism (Rupnow, 2021). To build on this work, we examined a larger group of algebraists' understandings of isomorphism and homomorphism and how they position this knowledge relative to sameness. This allowed us to examine whether other connections to sameness might arise from a larger group of algebraists and how prompting consideration of sameness might reveal mathematicians' values in context. Thus, in this paper we address the following research questions:

1. Do mathematicians find sameness helpful or harmful for thinking about isomorphism and homomorphism?
2. Why do mathematicians view sameness as helpful for thinking about isomorphism and homomorphism?
3. Why do mathematicians view sameness as harmful for thinking about isomorphism and homomorphism?

We believe this work holds two potential contributions to the literature. First, examining the range of ways mathematicians understand isomorphism and homomorphism may provide insight into understandings we want students to possess. Second, we asked how a concept without a standard mathematical definition (sameness) relates to well-defined concepts (isomorphism and homomorphism). In so doing, we gain insights into how tightly mathematicians hold to the disciplinary value of mathematical knowledge being independent of non-mathematical contexts like time and author (Dawkins & Weber, 2017) when building connections between topics.

Literature Review

Mathematical Sameness

Types of sameness appear throughout mathematics curricula, such as equality and congruence, and some research has focused on students' understandings of these ideas. For instance, extensive research on students' understanding of equality has highlighted the value of a relational understanding of equality, in which students understand both sides of the equal sign to have the same value as opposed to viewing the equal sign as a signal to compute (e.g., Alibali et al., 2007; Jones et al., 2012; Knuth et al., 2006). Some research has connected congruence to sameness as well, such as Zazkis and Leron's (1991) paper that called sameness the commonly perceived interpretation of congruence in Euclidean geometry, and Rahim and Olson's (1998) work showed students' interchangeable use of the words "equal", "same", and "congruent".

In contrast, limited mathematics education work has focused on interpretations or understanding of mathematical sameness, instead of defined topics like equality, as the central topic of study. Melhuish and Czocher (2020) observed that students may focus on unintended aspects when considering sameness, such as attending to the ability to manipulate one operation into another when considering operational sameness (i.e., claiming multiplication and division are the same binary operation because multiplication by 1/2 and division by 2 produce the same numerical result). Similarly, Mirin (2018) suggested that students do not necessarily interpret equivalent graphs as indicating equivalent functions. These student-centered studies highlight the importance of clearly articulating what aspect(s) are intended to be "the same" rather than assuming students will attend to the characteristics experts focused on.

Instead, discussions of mathematical sameness have largely been conducted by mathematicians or mathematical philosophers. Mazur (2008) highlighted the role of category theory in shifting the key, formal type of sameness from equality to isomorphism (or canonical isomorphism) in modern mathematics. Asghari (2019) provided a historical review of mathematicians' evolving view of sameness through notions of equivalence. Specifically, he observed that the notion of decontextualized equivalence (as expressed through equivalence relations or equivalence classes) only gained broad use in the mid-Twentieth century. His conclusion that equivalence classes and equivalence relations only express types of equivalence and should not be viewed as expressing all equivalence, mirrors the notion that sameness has many facets and contextual interpretations as well. These semi-recent changes in how mathematicians discuss sameness suggest instructional emphases in algebra and category theory may be changing as new mathematicians enter the field. Thus, this is a good time to assess how mathematicians understand the nature of mathematical sameness.

Isomorphism and Homomorphism

Prior work on students' understanding of isomorphism[1] has shown associations between isomorphism and sameness. Dubinsky et al. (1994) highlighted students' attention to shared

[1] For reference: "An *isomorphism* ϕ from a group G to a group $\bar{G}$ is a one-to-one mapping (or function) from G onto $\bar{G}$ that preserves the group operation. That is, $\phi(ab) = \phi(a)\phi(b)$ for all a, b in G. If there is an isomorphism from G onto $\bar{G}$, we say that G and $\bar{G}$ are *isomorphic* and write $G \approx \bar{G}$" (Gallian, 2009). A group homomorphism is a group isomorphism without the requirement of bijectivity.

properties in isomorphic groups, based on students' tendency to check properties like order and commutativity. Leron et al. (1995) also observed this tendency to look for same properties while describing a course in which students were taught to focus on the sameness of isomorphic groups. Notice this view of shared (same) properties showcases the "sameness" of groups as a whole. Furthermore, the notion of relabeling group elements and the group operation to demonstrate alignment between isomorphic groups was referred to as "naïve isomorphism" (Leron et al., 1995, p. 154). This provides a second way of emphasizing sameness: producing a copy of a group with the same relationship structure even though the names of elements have changed.

More recent work on students' understanding of isomorphism and homomorphism have focused on students' application of knowledge to new contexts. Melhuish et al. (2020) examined the evoked concept image of function in relation to isomorphism and homomorphism to examine how students leveraged their understanding of functions in new contexts. They observed how even upper-level (abstract algebra) students' concept image of function could be a resource or barrier to their understanding of homomorphism. Hausberger (2017) examined students' understanding of "structure-preservation" to explain ring homomorphism after prior lessons on group homomorphism to see how students might abstract the meaning of "structure" and observed this was non-trivial for most students. However, neither study examined how students' understanding of sameness could color these understandings of equations describing functions or what it means to "preserve" a structure.

Limited literature has examined mathematicians' understandings of isomorphism or homomorphism. However, this literature has highlighted mathematicians' use of similar language to that of students and an emphasis on sameness in each case. In addition to general references to sameness and relabeling (Weber & Alcock, 2004), Rupnow (2021) observed a teaching emphasis on the bijective aspect of isomorphism through matching, in which pairing of elements in isomorphic groups was emphasized. Furthermore, she observed references to weaker forms of sameness to describe homomorphism, including disembedding and equivalence classes. Both types of language emphasized a subpart of groups being shared, whether elements within a coset (equivalence classes) or subgroups of the domain and co-domain groups under a homomorphism (disembedding). However, both studies examined small groups of mathematicians ($n < 5$) so there are likely other ways of thinking about isomorphism or homomorphism that we may have missed. By surveying nearly 200 mathematicians, we sacrificed depth but hoped to gain access to a wider range of thinking.

Theoretical Perspective

The lens used for examining respondents' language in this study was the four values of the mathematical community outlined in Dawkins and Weber (2017). These values focus on the nature of mathematical truth, a-contextual knowledge and argumentation, a desire for increased understanding, and a desire for consistent standards, and are explained further below. While Dawkins and Weber go on to highlight norms for proof that uphold these values, our interests do not directly extend to proof, so we use only the general values of the mathematical community as the lens for this paper, not the norms outlined in their work. We found this values framework to be useful for interpreting the underlying reasons for why mathematicians viewed sameness as helpful or harmful for discussing isomorphism and homomorphism.

Value 1—mathematical knowledge is justified by a priori arguments—focuses on mathematical knowledge as somewhat unique among the sciences because mathematicians strive for their knowledge to be a priori, knowledge arising independent of experience, instead of relying on empirical evidence. In theory, underlying mathematical information (its data) is a priori; thus, findings in mathematics should be completely reprovable. In contrast, in physics (or mathematics education), underlying information (data) is likely to differ from experiment to experiment (even if only slightly); thus, replicability of findings is not guaranteed.

Value 2—mathematical knowledge and justifications should be a-contextual and specifically be independent of time and author—follows naturally from value 1. Specifically, just as the underlying data should be completely accessible, ways of interpreting that data should also be completely accessible to different individuals of sufficient mathematical background, regardless of personal or cultural experience, meaning formal language and definitions take precedence.

Value 3—mathematicians desire to increase their understanding of mathematics—focuses more on mathematicians' aims than descriptions of mathematics as a field. Here, a desire for knowledge of statements that are true and are not true as well as for understanding how and why these statements have these truth values are highlighted. Furthermore, intuition that allows how and why arguments to be generated or communicated more efficiently serves a purpose in furthering understanding. For instance, while formal proofs may require all logical steps to be included, many published proofs skip calculation steps or utilize intuitive arguments with the idea that "a trained mathematician would be capable of translating those intuitive arguments into a more rigorous argument" (Hales, 2003, as cited in Dawkins & Weber, 2017, p. 130). Thus, a proof-writer might choose to use an intuitive argument instead of a fully formal argument if it would enhance readability and development of understanding of the proof.

Value 4—mathematicians desire a set of consistent proof standards—focuses on the desire to have a standard way of doing mathematics. In a sense, this value applies values 1 and 2 to create standards for the mathematics community to attempt to follow when writing proofs or sharing results with others. Of note, while values 1,2, and 4 all closely relate to each other, as they provide a way for the mathematical community to strive for consistent communication with each other, value 3 centers individual mathematicians' desires and ways of thinking rather than the needs of the collective.

Methods

Participants

Qualitative methods were chosen for this study, to have an in-depth understanding of mathematicians' thoughts on sameness. We used surveys with free response questions to obtain richly detailed responses from mathematicians and allowed their responses to guide our analysis. Participants were solicited by sending a survey to every four-year college or university mathematics department that offers abstract algebra in the United States. Generally, the survey was emailed to the department head to disseminate, but when this individual was not easily identifiable from the department website, the email was sent to the general contact email for the department or directly to individuals who mentioned algebra topics in their biographies. Participants were required to have taught at least one abstract algebra or category theory course in the last five years.

Our sample contained 197 participants from 173 institutions. Most participants were tenure-stream faculty (70% tenured, 14% tenure-track), but some were full time instructors (9%) or self-identified as Other (7%). 70% of participants were algebraists in the traditional sense (i.e., had done research in algebra or category theory within the last 5 years (45%), or over 5 years ago (25%)); others in the sample only taught but did not research in algebra or category theory (20%), or published some/limited work in algebra but did not consider themselves algebraists

(9%). 24% of respondents hailed from universities whose highest mathematics degree offered was a PhD in mathematics, 17% from departments with a Master's degree, and 59% from mathematics bachelor's-only institutions. 44% had taught Abstract Algebra and/or category theory 1-5 times, 31% had taught them 6-10 times and 25% had taught them 11 times or more.

Data Collection and Analysis

The survey asked how algebraists think about sameness in general and in specific mathematical contexts. We intentionally did not define "sameness" at any point in the survey in order to see mathematicians' default interpretations. The four survey questions relevant to this paper, numbered below, queried participants' beliefs about sameness related to isomorphism and homomorphism, and were the first reference to isomorphism or homomorphism in the survey text itself. These questions followed questions on the nature of sameness in mathematics and about how similar particular sets with an operation were.

1. How might sameness be helpful when thinking about isomorphism/isomorphic structures? (Q1)
2. How might sameness be harmful when thinking about isomorphism/isomorphic structures? (Q2)
3. How might sameness be helpful when thinking about homomorphism? (Q3)
4. How might sameness be harmful when thinking about homomorphism? (Q4)

The data were analyzed in accordance with thematic analysis (Braun & Clarke, 2006). Initially, we used versus coding (Saldaña, 2016) to identify different beliefs about sameness based on the help vs. harm contrast. These initial codes were based on comparing different features, for example comparing the ill-defined term of sameness with isomorphism or comparing the amount of structure preserved in an isomorphism and homomorphism. However, after coding, we determined that these codes did not effectively capture all nuances in the data. We then revised codes, using descriptive coding (Saldaña, 2016) to supplement our initial coding. These second-round codes incorporated Rupnow's (2021) codes based on conceptual metaphors for isomorphism and homomorphism. Finally, after adoption of the values framework, we organized codes according to the values upheld through them. Specifically, codes related to ways sameness aids understanding were grouped (as they reflect ways value 3 can be instantiated in practice) and codes related to ways sameness could cause harm or violate standard expectations of mathematical practice were grouped (as they reflect ways values 1, 2, and 4 can be instantiated or violated in practice). Descriptions of our codes are in the tables of the Results section. The unit of analysis was the respondents' grouped questions, in terms of the isomorphism questions (Q1, Q2) and homomorphism questions (Q3, Q4). Each paired response could and often did receive multiple codes. Respondents' written responses were not changed in any way before analysis. Any changes to enhance readability here are denoted in brackets.

To ensure coding validity, investigator triangulation (Turner & Turner, 2009) was used between two independent coders, and we used multiple iterations of coding for analysis of each section (Anfara et al., 2002). Each member independently coded the data using the agreed codes; we then discussed any coding discrepancies and came to consensus on the final codes. These discussions included any modifications for future coding, such as refined code definitions or new codes for consideration.

Results

We begin by addressing research question one, where we present mathematicians' responses about the helpfulness and harmfulness of sameness to considering isomorphism and homomorphism. We then address research questions two and three in the following sections by highlighting mathematicians' justifications for their helpful/harmful claims, centered on pedagogical considerations, particular types of informal language, and the context-dependent nature of sameness. We use the four values of the mathematical community (Dawkins & Weber, 2017) to organize our work addressing research questions two and three. Tables including code descriptions, frequencies, and percentages summarize quantitative results, where n represents the number of participants coded accordingly and the percentage is out of the total number of participants (197).

Helpful or Harmful

Participants largely viewed sameness as conceptually relevant to isomorphism, though less so for homomorphism. Based on the question format, in which we asked participants about how sameness might be helpful or harmful for thinking about isomorphism and homomorphism, our default expectation was for respondents to address both helpful and harmful aspects of sameness. This was the case for isomorphism, where 72% of respondents (see Table 1) were coded as helpful/harmful. For example:

> [Helpful:] Isomorphism is a kind of sameness, so certainly you have to have some sense of sameness to understand the idea behind isomorphism. [Harmful:] Maybe thinking that sameness = identical in every aspect? At some point you always have to move away from intuition coming from English (and "sameness" is certainly not a mathematical concept) and rely only on mathematical definitions to make progress.

In this participant's view, isomorphisms are a type of sameness, but language-based intuition can only be helpful up to a point; one must ultimately use formal definitions. Another participant had a different interpretation, focusing on the specific aspects that are and are not the same:

> [Helpful:] We all like to group things that are the same together, and it is useful to think that two very different looking objects (e.g., rings) that have the same algebraic properties should be put in the same group. We want to emphasize that algebraic objects should be studied based on their algebraic properties, not on the choice of names for their objects. [Harmful:] If a student starts to think isomorphic objects are the same as sets and mixes up elements of the objects, that could be harmful. If we use the wrong sort of sameness and think that the identity of a group must always be 0, for example, we could easily become very confused.

This participant described isomorphisms as a way to classify objects into categories and viewed sameness as helpful for that grouping but emphasized that identification between objects was not required and could cause confusion for students (e.g., names of elements can differ).

Table 1. Descriptions and frequencies of Helpful and Harmful Responses

Code	Description	Iso n(%)	Hom n(%)
Not harmful	Sameness is not harmful for thinking about iso/ homomorphism.	18(9%)	5(3%)
Helpful/harmful	Sameness is useful for thinking about iso/homomorphism but has limitations.	142(72%)	72(37%)
Not helpful	Sameness is not helpful for iso/homomorphism.	15(8%)	35(18%)
Similar	Similarity is potentially useful, but iso/homomorphism is not representative of “sameness”.	0(0%)	38(19%)
Not relevant	Sameness is not relevant to thinking about iso/homomorphism.	1(1%)	12(6%)
No code	Response not clearly aligned with any of the above.	21(11%)	35(18%)

In contrast, only 37% of respondents clearly highlighted both helpful and harmful aspects of homomorphism. For example:

> [Helpful:] It can be helpful, say to emphasize that it preserves some information, but not all. For instance, I like to say that homomorphisms from something complicated to

> something easier to work with or better understood are often the right approach (e.g., representations). Though one may lose info, by working with something easier you may still learn something new about the original. [Harmful:] Similar to the above, it should be emphasized that a lot of info can be lost, or that homomorphism is far from saying they are exactly the same, but is maybe a tool for extracting some information about sameness.

Note that the participant highlighted preservation of some aspects but a loss of some information, indicating utility but the need for care when discussing sameness with homomorphism.

Some participants only saw benefits to using sameness. For isomorphism, 9% of participants only expressed a helpful view of sameness: "Well, it's the essential notion of isomorphism. In no way [harmful], but it is important that we understand sameness to mean sameness of underlying structure, not sameness of superficial characteristics, like labels." Here the participant considered sameness to be the conceptual point of isomorphism, and thus did not consider sameness harmful. 3% of participants expressed an exclusively helpful view of sameness for homomorphism. For example: "[Helpful:] Same as isomorphism, except now we are only identifying a part of each of the two structures that behave the same algebraically. [Harmful:] Again, with carefully presented examples I don't think there is harm per se." Notice, even though the participant claimed sameness was helpful and not harmful for homomorphism, this sameness only referred to parts of structures instead of whole structures.

Although most isomorphism responses received a not harmful or helpful/harmful code, 19% did not. One participant was coded as not relevant: "When Isomorphism is being considered, isomorphism defines the sameness, and what makes the isomorphic objects "different" is to some extent obvious, but not really of interest. So considering sameness is neither a help nor a hindrance." They saw the reverse connection of isomorphism giving some insight into sameness but did not consider this notion to be important for understanding isomorphism. Others saw sameness as relevant, but it was unclear whether they viewed sameness as helpful, harmful, or both: “[Helpful:] I like [to] distinguish equality (for subsets of a given ambient object) and isomorphism. [Harmful:] The idea [of] flexible notions of equality or sameness is pretty subtle and counterintuitive.” Although this participant typed distinct responses in the two boxes for the “helpful” and “harmful” responses, their response did not directly address how sameness might be helpful or harmful for thinking about isomorphism, so it was not given any of those codes (no code). Finally, 8% of respondents only expressed a harmful view: “Helpful? I don’t think it is. There’s nothing added to the concept of isomorphism by saying the word “same”. Well, homomorphisms also preserve something. Bijections also preserve something. So, talking about “same” is going to blur some distinctions.” This participant only saw a lack of clarity arising from sameness.

Participants expressed more skepticism to using sameness to discuss homomorphism, with 61% of responses *not* receiving a helpful-related code. 18% of participants considered sameness not helpful to thinking about homomorphism: “I think tying “sameness” to any homomorphism that is not an isomorphism is misleading at best. Not a fan.” Others were unwilling to use sameness but allowed similarity: “A homomorphism provides a notion of similarity.” 6% of participants considered sameness irrelevant to homomorphism: “Homomorphism is [a] restricted version of the “sameness” defined by isomorphism. Usually when trying to show homomorphism exists, it is trying to show that a certain defined property holds and I do not see how sameness either helps or hinders.” Other participants acknowledged the relevance of sameness to homomorphism, but whether they viewed it as helpful or harmful was unclear, so no help-harm codes were applied (no code). For example: “[Helpful:] Can there be a connection between these two structures even though they are different? We are locating a connection that is not as deep as isomorphism. [Harmful:] Homomorphic structures may not be isomorphic." Here, the participant described some connections between structures and compared the relationship to isomorphism, but it was unclear what they meant by this connection.

Ways Sameness is Helpful

Although our questions prompted respondents to explain how sameness could be helpful for understanding isomorphism and homomorphism, respondents were free to interpret contexts as they saw fit. Thus, while we present percentages of respondents who gave examples of each type of code, we present these percentages only to give insight into the prevalence of each typed interpretation, not as the number of respondents who could provide explanations in each context given more direct prompts. We also note that participants could receive more than one or no codes related to helpfulness of sameness; thus percentages in Table 2 do not sum to 100% for isomorphism or homomorphism. Participants mainly highlighted two ways for discussing sameness to be helpful for understanding isomorphism and homomorphism: utility in teaching and through informal sameness-based language already in their explanations. Of note, these helpful views align with an emphasis on learning or understanding, which also align with value 3 of the mathematical values framework.

Table 2. Descriptions and frequencies of codes reflecting helpfulness/value 3

Code	Description	Iso n(%)	Hom n(%)
Motivating instruction	Sameness is useful to motivate/help build understanding of iso/homomorphisms when teaching.	15(8%)	7(4%)
Leveraging intuition	Sameness is useful to help students leverage their intuition when learning about iso/homomorphism.	33(17%)	8(4%)
Renaming/ Relabeling	An iso/homomorphism is a renaming/relabeling of elements/operations in which names of elements or operations are less important than the underlying structure.	10(5%)	0(0%)

Matching	An iso/homomorphism is a matching in which elements and/or operations are paired between structures.	13(7%)	4(2%)
Same behavior	An iso/homomorphism is a mapping characterized by shared generic properties or behavior.	48(24%)	36(18%)
Structure-preservation	An iso/homomorphism is a mapping characterized by structure-preservation (where the structures could be algebraic structures like groups, relationships of elements or subobjects demonstrated through lattices, or unclear).	9(5%)	12(6%)
Operation-preservation	An iso/homomorphism is a mapping characterized by operation-preservation.	5(3%)	20(10%)
Disembedding	An iso/homomorphism is a disembedding in which properties of the domain are extracted or highlighted in the codomain.	0(0%)	18(9%)
Equivalence classes	A homomorphism is a mapping defined by equivalence classes, where similar elements in the domain that are mapped to the same place in the range.	0(0%)	3(2%)

A number of respondents highlighted ways that sameness could be helpful when teaching and learning isomorphism. Some respondents (8%, see Table 2) specifically highlighted how sameness could be useful for motivating isomorphism: “Different levels of ‘sameness’ and different informal definitions of ‘sameness’ can be used to motivate the formal definition of isomorphism.” The participant here explained how formalizing sameness could provoke a need for isomorphism. Leveraging intuition was also described (17%): “It might be helpful for students to think of sameness in a familiar context (e.g., geometry or linear algebra) in order to appreciate the notion of isomorphism in algebra.” Here, the participant described how students’ intuition and prior experiences with sameness in mathematics could be used to help them understand isomorphism.

Motivating instruction and leveraging intuition also arose in homomorphism explanations, though this happened less than with isomorphism. Some motivated instruction (4%) by emphasizing a particular object related to homomorphisms: “Help understanding the importance of [the] study of kernels.” Others described how sameness can aid intuition (4%): “If [a homomorphism] is injective, you could talk about how the structure of the domain is the “same” as the structure of the range, and again this informal notion could make the concepts accessible for students.” Again, this participant did not make a blanket statement about sameness in homomorphism but qualified it as useful for considering injective (one-to-one) homomorphisms.

In addition to positioning the utility of sameness when teaching, some participants used informal sameness-based language to describe isomorphism and homomorphism in their responses. Some participants generally referred to same behavior (43%). For instance, this participant wrote of isomorphism and homomorphism: “The idea of an isomorphism is that two different sets of objects can behave the same in certain scenarios.” and “With a homomorphism, the objects of the image will behave in “the same” way as the domain (or quotient based on the domain).” While highlighting the sameness of objects linked by a morphism, such responses did not provide details on the shared sameness.

Participants also used renaming/relabeling and matching metaphors (Rupnow, 2021) to describe isomorphism (5% and 7% respectively) and, to a lesser extent, homomorphism (0% and 2% respectively). This participant described isomorphism in terms of a renaming of elements: “I like to emphasize to students that algebraists care about the algebraic structure and equations, and we don’t care nearly so much about what we choose to name the elements in these structures.” Note they highlighted the arbitrary nature of element names while instead focusing on the underlying relationships within the structure. Another participant described isomorphism in terms of matching: “For finite groups where Cayley tables are not too time-consuming either to make or to understand, one beneficial way is to see that they can be arranged to have the same overall pattern." Notice this respondent referred to rearranging Cayley tables to demonstrate a matching between appropriate elements in isomorphic objects.

Structure-preservation and operation-preservation were used to describe both isomorphism (5% and 3%, respectively) and homomorphism (6% and 10%, respectively). For example: "It gives a colloquial way of saying 'the algebra doesn't change' for particular structures. Things like order, dimension, and so on are preserved." Some explicitly connected structure-preservation to homomorphisms: "I teach homomorphisms as functions which preserve group structure. Homomorphic images are 'large scale structure' while subgroups are 'small scale structure' (at least in examples like symmetric groups and matrix groups)." We believe this participant means that homomorphisms reveal aspects of the domain group's structure by examining a simpler image. Most operation-preservation seemed focused on the homomorphism property: "A homomorphism preserves the operations of the algebraic structures. For example, it will take the identity element of one algebraic structure to the identity element of another algebraic structure." This explanation of operation-preservation foregrounds an identity connection, which highlights a specific type of shared structure.

Disembedding examples were coded only for homomorphisms (9%) and highlighted shared properties of the domain and codomain. This example highlighted how relevant shared structure could give insight into a group:

> Sometimes it is useful to think of a group as "sitting inside of" another group, even if in a literal sense the subgroup you are thinking of is not a subset of the bigger group. For example, one might think of some copies of the dihedral group D_4 sitting inside of the symmetry group of the cube…

Notice, although D_4 describes the symmetries of a square and a similar pattern of symmetries exist in the symmetry group of the cube, the underlying elements are not interchangeable, and we would not consider D_4 a subset of the symmetry group of the cube. Nevertheless, recognizing their shared structure could yield insight into the symmetry group of the cube. Forming equivalence classes was used to discuss sameness of elements in homomorphisms only (2%):

> We often build new structures from old by a quotient structure which makes use of an equivalence relation. A homomorphism is one source of such an equivalence relation (but not the only example). I certainly believe that this is an immensely useful way to build structures. And the 'sameness' concept is at its root (in the quotient structure, elements are identified as 'the same' if they lie in the same equivalence class).

Observe that equivalence class language groups elements of a similar nature together into the same equivalence class, which highlights a similarity among these elements within the structure.

Ways Sameness is Harmful

Just as when viewing sameness as helpful, respondents were free to interpret contexts as they saw fit when they presented potential harms of thinking about sameness with isomorphism and homomorphism. Thus, while we present percentages of respondents who gave examples of each type of code, we present these percentages only to give insight into the prevalence of each typed interpretation, not as the number of respondents who could provide explanations in each context given more direct prompts. We also note that participants could receive more than one or no codes related to harmfulness of sameness; thus percentages in Table 3 do not sum to 100% for isomorphism or homomorphism. Participants' concerns around sameness largely focused on students' mathematical immaturity in teaching contexts and the general context-dependent nature of sameness, which affects any mathematician's interpretation of sameness. Of note, these harmful views align with an emphasis on a priori, a-contextual argumentation, which also align with values 1 and 2 of the mathematical values framework.

Table 3. Descriptions and frequencies of codes reflecting harmfulness/values 1 and 2.

Code	Description	Iso n(%)	Hom n(%)
Misconceptions	Sameness is misleading/related to misconceptions about iso/homomorphisms.	25(13%)	20(10%)
Imprecise language	Sameness is pedagogically inappropriate for students who often fail to attend to formal definitions and/or inappropriately use informal notions of "sameness" in addressing exercises.	29(15%)	14(7%)
Context-dependent	Sameness is a place-based, context-dependent concept that differs based on the discipline and structure of interest.	77(39%)	16(8%)
Levels of sameness	Sameness is a level-based concept, where different levels of strength are recognizable.	11(6%)	6(3%)
Identical/equal vs. isomorphism	Identity/equality and isomorphism might both be viewed as types of sameness but are not interchangeable concepts.	65(33%)	2(1%)
Isomorphism vs. homomorphism	Isomorphism and homomorphism are types of sameness that are worth contrasting in order to clarify each's properties.	1(1%)	76(39%)
Isomorphism vs. isomorphic	Isomorphism is a concept involving both isomorphism (as a mapping) and isomorphic structures (objects), which provide different information and differ in importance.	15(8%)	0(0%)
Homomorphism vs. homomorphic	Homomorphism is a concept involving both homomorphism (as a mapping) and a homomorphic image (object), which provide different information and differ in importance.	0(0%)	38(19%)

A number of respondents highlighted ways that sameness could be harmful when teaching isomorphism, such as common perceived student misconceptions (13%, see Table 3) or imprecision (15%). The code we termed "misconceptions" often addressed mathematicians' descriptions of student difficulties with names of elements or objects: "Students often think that if two sets have different looking objects (integers vs matrices, for example), then they can't be 'the same.' This makes it more difficult for them to understand the more meaningful examples in class." Notice the participant stated that students could struggle with identifying superficially different objects. Another common concern was that using sameness may lead to imprecision in exercises and proofs: "Two objects can be isomorphic as groups under their additions, but not as rings, when both addition and multiplication are involved. The idea of sameness must be carefully used especially with students since they tend to forget the context." Here, we interpret the respondent as worrying that using sameness haphazardly could lead students to confuse different types of isomorphism and to not attend to the details important for the context.

Pedagogical concerns about using sameness for homomorphism also arose. Respondents often described misconceptions (10%) about the strength of sameness in homomorphism. For example: "Again, the wrong sort of sameness, as in equality of elements of sets, could be problematic if the student, for example, thinks all identities are actually the element 0." Participants also described issues with being imprecise (7%), including difficulties that could arise when students wrote proofs: "If students get too comfortable expressing things "are the same" without being formal, their proofs can very quickly become incorrect." Although this participant had previously noted utility in thinking about sameness with homomorphism as it connected to the isomorphism theorems, they acknowledged dangers in using loose definitions.

Many respondents also focused on the context-dependent nature of sameness as relevant for anyone examining isomorphism or homomorphism (39% isomorphism response/8% homomorphism response). For example: "The context and criteria for sameness need to be clear for isomorphism to be something that can be empirically verified as true." This highlights the necessity of describing the context in which sameness is used but gives few details. Others were more specific, describing different levels of sameness (6% isomorphism/3% homomorphism): "There are different "strengths" of sameness: equals, equivalent, related to, almost/weakly equivalent, etc. There is not a one size fits all to sameness." This provides a variety of types of sameness that might be placed on a continuum for strength comparison.

Similarly, participants highlighted concepts that could be confused with isomorphism and, to a lesser extent, homomorphism, such as being equal or identical (33% isomorphism/1% homomorphism). Consider a confusion with equality example: "Isomorphic is not the same thing as "equals" as it does not imply a canonical identification. The word "same" can trip people up in that way." Here, confusion between different mathematical understandings of sameness, namely isomorphism and equality, are specifically highlighted. Similar issues arose with identical: "In common, nonmathematical parlance, same means identical, so when students hear the word "same" they may think identical." Note this participant's identification with sameness and identical, a strong type of sameness.

Whereas initial responses to the isomorphism prompts produced a number of comparisons to "stronger" forms of sameness like equality or identity, many respondents compared isomorphism and homomorphism in their homomorphism responses (1% isomorphism/39% homomorphism), with a focus on the strength difference. For example: "Same has too strong a connotation in most students['] minds and they may interpret this to mean isomorphism rather than homomorphism." Implicitly, the participant seems to suggest that sameness implies a strong relationship, so students will identify the stronger (more restrictive) concept (isomorphism) with sameness.

Other responses emphasized the contrasting utility of using a function-based or structure-based standpoint to understand isomorphism or homomorphism. Some respondents (8%) contrasted mapping (isomorphism) and structural (isomorphic) aspects of the concept of isomorphism. For example: "Two groups (for example) can be isomorphic, but the isomorphism may not be obvious….the groups $(C, +)$, and $(R, +)$, are isomorphic because they are isomorphic as Q-vector spaces, but it is fundamentally impossible to write down an explicit isomorphism!" Here the respondent emphasized that objects being isomorphic did not imply that an isomorphism specifying which elements act the same would be easy to find or define, despite such an identification being a likely criterion for considering objects the same. More commonly (19%), responses detailed the difference between mapping (homomorphism) and structural (homomorphic) interpretations of homomorphism. For example:

> Students who are used to thinking about isomorphic = "the same" will want to think the same thing about homomorphism and will start ta[l]king about "G and H being homomorphic" without realizing that the concept is meaningless, and that when studying homomorphisms, we are typically more interested in the properties of the function itself rather than in what it tells us about the structures independently from the function.

Unlike isomorphism, where function and structural aspects are both commonly discussed (Mirin & Rupnow, in press), the participant here emphasizes that the mapping is the important part of homomorphism. Furthermore, while the whole structures display a type of sameness in isomorphism, this need not be true for homomorphism.

Discussion and Conclusions

This study highlights the potential viability of connecting notions of sameness and isomorphism. Specifically, we see limited resistance to the concept of sameness for isomorphism (81% of respondents coded at least partly helpful), and this resistance to "sameness" largely related to imprecision, not irrelevance. In contrast, a majority of respondents resisted or did not clearly relate sameness to homomorphism (39% of respondents coded with a partly helpful code), and the "sameness" in homomorphism related only to parts of structures, not whole objects. Differences were also emphasized through participants' portrayals of the function and structure aspects of these concepts (isomorphism vs. isomorphic and homomorphism vs. homomorphic) that highlighted whole object and partial object differences between isomorphism and homomorphism. While these results may not be surprising, they confirm that sameness is relevant to isomorphism and can be a conceptual base for making connections to other subjects as long as the reduction in precision is acknowledged.

Furthermore, as we consider how mathematicians found sameness helpful, we see connections to prior knowledge and metaphorical language. Isomorphism language used in prior studies, including renaming/relabeling, matching, structure-preservation, operation-preservation, and generic sameness metaphors like same behavior (Hausberger, 2017; Leron et al., 1995; Rupnow, 2021; Weber & Alcock, 2004) were all used by some mathematicians to describe isomorphism here. Similarly, structure-preservation, operation-preservation, disembedding, and equivalence class metaphors (Hausberger, 2017; Rupnow, 2021) were used by respondents when describing homomorphism. Considering the prompts did not directly ask for different ways of describing isomorphism and homomorphism, future research should examine the prevalence of these metaphors in response to other questions or in instruction for larger groups of mathematicians to see whether broader contexts elicit a wider range of conceptual metaphors.

While none of these particular metaphors were used by more than a quarter of participants, the fact that this metaphorical language is clear even through short, typed responses to questions about isomorphism and homomorphism indicates that mathematicians are likely already employing sameness-based language and analogies in their own thinking about isomorphism (and homomorphism). Furthermore, we see clear connections between value 3, mathematicians' desire to increase their mathematical understanding, and this leveraging of metaphorical language. For example, those who have not taken abstract algebra courses or conducted algebra research recently are likely to gain more insight into isomorphism and homomorphism from the intuitive explanations of the concept in the results than from the formal definition in the footnote in the literature review. Additionally, by utilizing metaphors (even if not intentionally), mathematicians were naturally connecting different concepts, which provides a platform for subsequent mathematical theory building.

In contrast, participants' emphasis on the context-dependent nature of sameness highlights a perception of violating a different value of the mathematical community (value 2). Mathematical culture values context-independent proofs (Dawkins & Weber, 2017), which requires context-independent definitions as a foundation. However, a repeated emphasis from mathematicians' responses was the context-dependent nature of "sameness". While sameness has the potential to appear in many ways (related to its context-bridging nature), use of sameness could be upsetting to mathematicians accustomed to valuing mathematical language for its precision and generality. Distinguishing isomorphism from other, potentially "stronger" forms of sameness, like equality and being identical, as well as "weaker" forms like homomorphism relates to the importance of precision: what exactly or how much needs to be the same in a particular situation. Similarly, participants' concerns about students' conceptual difficulties largely related to what happens when intuition about sameness leads astray, such as confusion about whether elements or groups need to look the same.

However, while these context-independent values are present in mathematical culture, these values largely relate to formal communication between mathematicians. While harms would likely result from the inattentive use of "sameness" in instruction or formal mathematical communication, we believe these harms are easily mitigated by recognizing the purpose of discussing sameness. Sameness can be used to bolster intuition and make thematic connections between ideas but needs to be formalized (or we need to foreground its formal definition in a context) when writing out a proof or verifying claims in keeping with value 2 (and, by extension, values 1 and 4). Consider the case of a “well-taught” analysis class. Lew et al. (2016) observed the use of colloquial language such as “small” in an “arbitrarily small” sense in a way the research team and an outside instructor found clear but that students did not interpret in the way intended by the professor. Perhaps by normalizing discussions around informal language in mathematics courses, professors can critically examine their colloquial understandings in order to highlight the contexts and ways in which they are useful, both for themselves and their students.

Furthermore, context-dependence can even be viewed as a purpose for examining mathematical sameness. Asking students what is and is not the same for isomorphic groups (or for isomorphic subgroups of an ambient group) could help students reflect more deeply on what can be assumed from given definitions and what is relevant in a particular context. Considering how sameness appeared through equality in prior classes and relating that to isomorphism could also prompt reflections on relevance, create new connections for students, and help them appreciate the subtleties of mathematical definitions, which we know are often problematic for students (Edwards & Ward, 2008). Future research should examine how many of these sameness connections are already made by students as well as examine how to help students make such connections, both to help future teachers appreciate how different notions of sameness have appeared in K-12 settings and to help mathematics majors reexamine their prior learning. Such connections could help mathematics majors to “look for and make use of structure” (National Governors Association Center for Best Practices, Council of Chief State School Officers, 2010,

p. 8) in their undergraduate learning, in order to extend this practice encouraged by the Standards of Mathematical Practice into their post-secondary learning.

Finally, Dawkins and Weber (2017) noted a potential tension between value 3 and the other values in their framework, and we do see that here in the breakdown of mathematicians' responses. Most mathematicians did engage with both helpful and harmful aspects of connecting sameness and isomorphism, suggesting that they are able to navigate this tension for themselves. Future work could examine how these or other values of the mathematical community may be in tension, such as the a-contextual nature of mathematical truth and justifications (values 1 and 2) and a desire to make mathematics culturally and contextually relevant to all learners so as to avoid becoming “identity thieves” of diverse students (Jett, 2013, p. 104). By explicitly attending to these tensions, we hope these tensions could be mitigated or leveraged to support students' induction into the mathematical community. Furthermore, we believe this work of surveying mathematicians to gauge their reactions to the use of sameness was an important step before creating pedagogical interventions or curricular materials. Mathematicians generally have thoughtful reasons for the pedagogical choices they make (e.g., Weber, 2002; Woods & Weber, 2020). Thus, considering and attending to their concerns can and should be incorporated into mathematics education researchers' work to encourage discussion between mathematicians and mathematics educators and to encourage uptake of reform materials.

References

Alibali, M. W., Knuth, E. J., Hattikudur, S., McNeil, N. M., Stephens, A. C. (2007). A longitudinal examination of middle school students' understanding of the equal sign and equivalent equations. *Mathematical Thinking and Learning, 9*(3), 221–247. https://doi.org/10.1080/10986060701360902

Asghari, A. (2019). Equivalence: An attempt at a history of the idea. *Synthese*, *196*, 4657–4677.

Dawkins, P. C., & Weber, K. (2017). Values and norms of proof for mathematicians and students. *Educational Studies in Mathematics, 95*, 123–142. https://doi.org/10.1007/s10649-016-9740-5

Dubinsky, E., Dautermann, J., Leron, U., & Zazkis, R. (1994). On learning fundamental concepts of group theory. *Educational Studies in Mathematics*, *27*(3), 267–305.

Edwards, B., & Ward, M. (2008). The role of mathematical definitions in mathematics and in undergraduate mathematics courses. In M. Carlson & C. Rasmussen (Eds.), *Making the connection: Research and teaching in undergraduate mathematics* (pp. 221–230). Washington, DC: Mathematical Association of America.

Gallian, J. (2009). *Contemporary Abstract Algebra.* Boston, MA: Cengage Learning.

Hales, T. (2003). *The Flyspeck Project Fact Sheet*. Retrieved from https://code.google.com/archive/p/flyspeck/wikis/FlyspeckFactSheet.wiki

Hausberger, T. (2017). The (homo)morphism concept: Didactic transposition, meta-discourse, and thematisation. *International Journal of Research in Undergraduate Mathematics Education, 3*(3), 417–443. https://doi.org/10.1007/s40753-017-0052-7

Jett, C. (2013). Culturally responsive collegiate mathematics education: Implications for African American students. *Interdisciplinary Journal of Teaching and Learning, 3*(2), 102–116.

Jones, I., Inglis, M., Gilmore, C., & Dowens, M. (2012). Substitution and sameness: Two components of a relational conception of the equals sign. *Journal of Experimental Child Psychology, 113*, 166–176. https://doi.org/10.1016/j.jecp.2012.05.003

Knuth, E. J., Stephens, A. C., McNeil, N. M., Alibali, M. W. (2006). Does understanding the equal sign matter? Evidence from solving equations. *Journal for Research in Mathematics Education, 37*(4), 297–312.

Lakoff, G., & Núñez, R. (1997). The metaphorical structure of mathematics: Sketching out cognitive foundations for a mind-based mathematics. In Lyn D. English (Ed.), *Mathematical reasoning: Analogies, metaphors, and images* (pp. 21–89). Mahwah, NJ: Erlbaum.

Leron, U., Hazzan, O., & Zazkis, R. (1995). Learning group isomorphism: A crossroads of many concepts. *Educational Studies in Mathematics, 29*(2), 153–174.

Lew, K., Fukawa-Connelly, T. P., Mejía-Ramos, J. P., & Weber, K. (2016). Lectures in advanced mathematics: Why students might not understand what the mathematics professor is trying to convey. *Journal for Research in Mathematics Education, 47*(2), 162–198.

Mazur, B. (2008). When is one thing equal to some other thing? In B. Gold & R. A. Simons (Eds.), *Proof and Other Dilemmas*, MAA Spectrum, (pp. 221–241). Washington, DC: Mathematical Association of America.

Melhuish, K., Lew, K., Hicks, M. D., & Kandasamy, S. S. (2020). Abstract algebra students' evoked concept images for functions and homomorphisms. *Journal of Mathematical Behavior, 60*.

Mirin, A. (2018). Representational sameness and derivative. In Hodges, T. E., Roy, G. J., & Tyminski, A. M. (Eds.), *Proceedings of the 40th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education,* (pp. 556–563), Greenville, SC: University of South Carolina & Clemson University.

Mirin, A., & Rupnow, R. (in press). Isomorphism and homomorphism in abstract algebra textbooks. *Proceedings of the 24th Annual Conference on Research in Undergraduate Mathematics Education*, Boston, MA.

National Governors Association Center for Best Practices, Council of Chief State School Officers (2010). *Common Core State Standards-Mathematics.* National Governors Association Center for Best Practices, Council of Chief State School Officers, Washington D.C.

Olsen, J., Lew, K., & Weber, K. (2020). Metaphors for learning and doing mathematics in advanced mathematics lectures. *Educational Studies in Mathematics, 105*, 1–17. https://doi.org/10.1007/s10649-020-09968-x

Rahim, M. H., & Olson, A. (1998). Qualitative patterns in plane geometry. *Journal of Mathematical Behavior, 17*(3), 373–389. https://doi.org/10.1016/S0364-0213(99)80069-X

Ronan, M. Andrew (2012, September 6). *Modern algebra*. *Encyclopedia Britannica*. https://www.britannica.com/science/modern-algebra

Rupnow, R. (2017). Students' conceptions of mappings in abstract algebra. In A. Weinberg, C. Rasmussen, J. Rabin, M. Wawro, and S. Brown (Eds.), *Proceedings of the 20th Annual Conference on Research in Undergraduate Mathematics Education Conference on Research in Undergraduate Mathematics Education,* (pp. 259–273), San Diego, CA.

Rupnow, R. (2021). Conceptual metaphors for isomorphism and homomorphism: Instructors' descriptions for themselves and when teaching. Journal of Mathematical Behavior, 62(2), 100867. https://doi.org/10.1016/j.jmathb.2021.100867

Saldaña, J. (2016). *The coding manual for qualitative researchers.* Thousand Oaks, CA: SAGE Publications, Inc.

Weber, K. (2004). Traditional instruction in advanced mathematics courses: A case study of one professor's lectures and proofs in an introductory real analysis course. *Journal of Mathematical Behavior*, *23*(2), 115–133.

Weber, K., & Alcock, L. (2004). Semantic and syntactic proof productions. *Educational Studies in Mathematics*, *56*(2-3), 209–234.

Woods, C., & Weber, K. (2020). The relationship between mathematicians' pedagogical goals, orientations, and common teaching practices in advanced mathematics. *Journal of Mathematical Behavior, 59,* 100792. https://doi.org/10.1016/j.jmathb.2020.100792

Zandieh, M., Ellis, J., & Rasmussen, C. (2016). A characterization of a unified notion of mathematical function: The case of high school function and linear transformation. *Educational Studies in Mathematics, 95*(1), 21–38. https://doi.org/10.1007/s10649-016-9737-0

Zazkis, R., & Leron, U. (1991). Capturing congruence with a turtle. *Educational Studies in Mathematics*, *22*(3), 285–295.