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Existence of a solution to the line contact problem of elastohydrodynamic lubrication

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We prove the existence of a solution to a free boundary value problem arising in elastohydrodynamic lubrication. Previously, the existence was proved only under severe restrictions on the range of the physical parameters. We remove those restrictions here.

1 Introduction

The problem describing the elastic deformation of lubricated rolling elements such as roller bearings or gear teeth is known as the line contact problem of elastohydrodynamic lubrication. This problem is a nonlinear free boundary value problem which is described by the system of equations

$$\frac{d}{dx}\left(\frac{h^3}{\eta}\frac{dp}{dx}\right) = 12u\frac{dh}{dx}, \tag{1.1}$$

$$\eta = \eta_0 \exp(\alpha p), \tag{1.2}$$

$$h = k_m + \frac{x^2}{2R} + v, \tag{1.3}$$

$$v(x) = \frac{-2}{\pi E'}\int_{s_1}^{s_2} p(s)\ln[(x-s)^2]\,ds + \text{constant} \geqslant 0, \tag{1.4}$$

where $p(x)$ is the pressure of the lubricant at location x and $h(x)$ is the film thickness of the lubricant at location x. Equation (1.2) is a constitutive relation between the pressure $p(x)$ and the viscosity η. Here E' is the effective elastic modulus of the roller pair and u is the velocity, R is the effective radius of the roller pair, η_0, α are constants related to the viscous properties of the lubricant, k_m is a constant which represents the minimum thickness of the film.

For more on these equations and the physical meaning of the constants E', k, R, η_0, α see, for example, Dowson & Higginson [5].

Equations (1.1)–(1.4) are assumed to hold on an interval (x_{min}, x_{cav}), where x_{min} (assumed to be known) is far enough from the high pressure area to ensure that the pressure is atmospheric, so that the normalized pressure satisfies

$$p(x_{min}) = 0. \tag{1.5}$$

On the other hand, x_{car} is not known, and is to be determined as part of the solution to the above problem. One approach which is used some of the time is to prescribe the boundary conditions known as the Swift–Stieber, i.e.

$$p(x_{car}) = 0, \tag{1.6}$$

$$p'(x_{car}) = 0. \tag{1.7}$$

If x_{car} were known, the problem (1.1)–(1.7) would be over-determined. But if x_{car} is not known, then (1.1)–(1.7) can be seen as a free-boundary value problem which *a priori* would then be well-posed.

It is well-known that free boundary value problems can sometimes be reformulated as problems over given regions with the free boundary determined by additional conditions. In this context, problem (1.1)–(1.7) has been formulated as a free boundary value problem over a given (known) interval (x_{min}, x_{max}) and a boundary condition at x_{car}. See, for example, Hu [3], Rodrigues [10] and Wu [13].

For more information on this problem see elsewhere [2]–[5], [7], [10], [13], and the references therein.

The first rigorous existence results seem to have been established in a series of papers by Oden & Wu [13]. Hu [3] gave a different proof of existence, and also established interesting results on the qualitative behaviour of the solution.

However, all the known existence results are proved under the restrictive assumption that the number

$$\theta = \frac{|x_{max} - x_{min}|\alpha\eta_0\, u}{k^2}$$

is small enough. This condition is sufficient for proving existence of what turns out to be smooth ($W^{2,2}$) solutions. It is not clear whether, in practice, this number is as small as these proofs require it to be. Also, it seems that this assumption is more one of expediency rather than of physical or mathematical necessity. This assumption of smallness plays a major role in the known proofs of existence. It should also be pointed out (see Kostreva [8], for example, and the references therein) that numerical solutions have been computed for all ranges of the parameters. The existence of such solutions does not seem to be restricted to any particular range of θ. Nonetheless, the regime in which θ is small is of practical importance in mechanical engineering. In this regime, Spence & Bisset [4] have shown that there exists a relatively large contact zone where the pressure is essentially 'Hertzian'.

We propose an existence result without any restriction on the size of the physical constants involved. The weak solution we construct may be more singular than that obtained when one assumes that θ is small enough. However, it is not clear that the physical solution is always as smooth as the results of Hu [3] and Wu [13], suggest. Finally, our result provides a framework in which to interpret the existence of the numerically computed solution when θ is not small.

The proofs of Hu [3] and Wu [13] rely on what is mostly a generalized Implicit Function Theorem and, in essence, show that if θ is small enough then a solution $p(x)$ exists which is not too different from the trivial case $p(x) \equiv 0$ that one would obtain when $\theta = 0$.

Our method relies essentially on a very refined analysis of solutions of a perturbed problem, and sharp regularity results of the solutions of these perturbed problems.

The main difficulties consists in deriving bounds on h and p that are good enough to be able to give meaning to the various expressions which appear in (1.1)–(1.4).

2 Notation and variational formulation of the problem

Next we rewrite the line contact problem of elastohydrodynamic in non-dimensional form.

$$\frac{d}{dx}\left(\frac{h^3}{\eta}\frac{dp}{dx}\right) = \gamma\frac{dh}{dx} \tag{2.1}$$

$$\eta = \eta_0 \exp(\alpha p) \tag{2.2}$$

$$h = k^* + \frac{x^2}{2} + v \tag{2.3}$$

$$v(x) = \frac{-1}{2\pi}\int_{x_{min}}^{x_{cav}} p(s)\ln\left[\frac{(x-s)^2}{b^2}\right] ds, \tag{2.4}$$

where $b = \frac{8Ru}{E'}$.

For more details on the rescaled variables, see Szeri [12], for example.

Next we introduce an auxiliary function w defined by

$$w(x) = 1 - e^{-\alpha p(x)}, \quad \text{i.e.} \quad p(x) = \frac{-1}{\alpha}\ln(1 - w(x)). \tag{2.5}$$

The function $\frac{w}{\alpha}$ is sometimes referred to in the engineering literature as the 'reduced pressure'.

The problem then becomes that of finding (h, w, p, x_{cav}) such that h is given by (2.6), p is related to w by (2.6) and

$$w(x) \geqslant 0; \quad -(h^3 w')' = -\gamma h' \quad -M \leqslant x \leqslant x_{cav} \leqslant M,$$
$$w(x_{cav}) = w'(x_{cav}) = 0.$$

This problem admits the following equivalent variational formulation:

$$\begin{cases} -(h^3 w')' \geqslant -\gamma h', & -M \leqslant x \leqslant M; \quad (2.1a) \\ w \geqslant 0, & -M \leqslant x \leqslant M, \quad (2.2a) \\ w(-(h^3 w')' + \gamma h') = 0, & -M \leqslant x \leqslant M, \quad (2.3a) \\ w(\pm M) = 0, & \quad (2.4a) \end{cases}$$

$$h(x) = k + \frac{x^2}{2} + \eta_1 \int_{-M}^{M} p(s)\ln\left(\frac{2M}{|x-s|}\right) ds. \tag{2.6}$$

This variational formulation has been used by Hu [3] and Wu [13]. They both proved existence of a solution under the restriction outlined above.

We start by stating a basic existence result for a linear version of the free boundary value problem.

Proposition 1 *Given $h(x) \in C^1(-M,M)$; $h(x) \geqslant c > 0$ for all $x \in (-M,M)$. Then, the variational problem* (2.1)–(2.4) *has a unique solution $w(x) \in W^{2,2}(-M,M)$.*

Proof This is a direct application of standard classical results on the free boundary value problem. See Friedman [6], for example. □

3 The nonlinear problem

Now for a given h we have a solution w to problem (2.1)–(2.4). We intend to construct a solution to problem (2.1)–(2.5) by a fixed point technique. This will be done in two stages. In stage 1 we show that a perturbation of problem (2.1)–(2.5) has a solution w_ϵ. In stage 2 we let ϵ go to 0, and prove that in the limit we have solution to our problem (2.1)–(2.5).

Stage 1
For a given, fixed ϵ we set

$$G_\epsilon(w) = \begin{cases} \ln\left(\dfrac{1}{1-w(s)}\right), & 0 \leqslant w \leqslant 1-\epsilon, \\ \ln\left(\dfrac{1}{1-(1-\epsilon)}\right), & 1-\epsilon \leqslant w. \end{cases}$$

For a fixed $\epsilon > 0$ and a given $w \in C^0(-M,M)$, $G_\epsilon(w) \in C^0(-M,M)$ and $0 \leqslant G_\epsilon(w) \leqslant \ln(\frac{1}{\epsilon})$.

It is easy to show then, using the previous propositions, and an appropriate iterative scheme, that for each $\epsilon > 0$ there exists a solution (h_ϵ, w_ϵ) to the problem

$$-(h_\epsilon^3 w_\epsilon')' \geqslant -\gamma h_\epsilon', \quad -M \leqslant x \leqslant M, \tag{3.1}$$

$$w_\epsilon \geqslant 0, \tag{3.2}$$

$$w_\epsilon \frac{d}{dx}(-h_\epsilon^3 w_\epsilon' + \gamma h_\epsilon) = 0, \quad -M \leqslant x \leqslant M, \tag{3.3}$$

$$w_\epsilon(\pm M) = 0, \tag{3.4}$$

$$h_\epsilon(x) = k + \frac{x^2}{2} + \eta_1 \int_{-M}^{M} G_\epsilon(w_\epsilon(s)) \ln\left(\frac{2M}{|x-s|}\right) ds. \tag{3.5}$$

The difficulty is in letting ϵ go to 0. First we deal with the case $\epsilon > 0$.

Theorem 3.1 [3, 10] *Given $\epsilon > 0$. Then the variational problem* (3.1)–(3.5) *has a solution $w_\epsilon(x) \in W^{2,2}(-M,M)$.*

Proof To prove that problem (3.1)–(3.5) has a solution, we can use an iterative method together with Proposition 1. Here is a brief description.

We can set the following iterative scheme. We start with $w_{\epsilon;0}(x) \equiv 0$. For a given $w_{\epsilon;i}(x)$ we define $h_{\epsilon;i+1}$ as follows:

$$h_{\epsilon;i+1}(x) = k + \frac{x^2}{2} + \eta_1 \int_{-M}^{M} G_\epsilon(w_{\epsilon;i}(s)) \ln\left(\frac{2M}{|x-s|}\right) ds,$$

and then (using Proposition 1) find the solution $w_{\epsilon;i+1}(x)$ of problem (2.1)–(2.4) with $h(x) = h_{\epsilon;i+1}(x)$.

It can then be shown that, for a fixed ϵ, the sequence $w_{\epsilon;i}$ converges to a solution w_ϵ of problem (3.1)–(3.5) as i goes to infinity. See Hu [3] and Rodrigues [10] for details. □

The main difference between this result and the existence of the desired solution to the problem (2.1)–(2.5) is that in (3.5) we use $G_\epsilon(w_\epsilon(s))$ in lieu of $\ln(\frac{1}{1-w})$. In fact, the two differ only for values of w which are bigger than $1-\epsilon$. If it can be shown that (3.3)–(3.5) has a solution w_ϵ such that $0 \leqslant w_\epsilon(x) \leqslant 1-\epsilon$, $\forall x$, then such a w_ϵ will also be a solution of (2.1)–(2.5).

This is precisely what was done in Hu [3], Wu [13] and Rodrigues [10]. However, all had to assume that the number

$$\theta = \frac{|x_{max} - x_{min}|\alpha\eta_0 u}{k^2}$$

is small enough. As we said earlier:

- It is not clear that this number is as small as their technique of proof requires.
- It is not clear that it is necessary for this number to be small for w to be always less than 1.
- It is not clear that it is necessary for this number to be small to have the existence of a solution. In fact, numerical methods seem to work without problem for large values of θ.

Stage 2

We now start proving the existence of a weak solutions by establishing some basic facts and proving some technical lemmas.

First, let us set $\epsilon = \frac{1}{n}$ and denote the solution (h_ϵ, w_ϵ) to problem (3.3)–(3.5) by (h_n, w_n).

We intend to establish some basic estimates. Integrating (3.5) by parts and using (3.6), we find:

$$\int_{-M}^{M} h_n^3(w_n')^2\,dx = \gamma \int h_n w_n'\,dx,$$

from which it follows that

$$\int_{-M}^{M} h_n^3(w_n')^2\,dx \leqslant \gamma \left(\int \frac{1}{h_n}\,dx\right)^{1/2} \left(\int_{-M}^{M} h_n^3(w_n')^2\,dx\right)^{1/2}.$$

It then follows that

$$\int_{-M}^{M} h_n^3(w_n')^2\,dx \leqslant \gamma^2 \int_{-M}^{M} \frac{1}{h_n}\,dx. \tag{3.6}$$

Since $h_n(x) \geqslant k$ $\forall x$ we then have, independently of n,

$$\int_{-M}^{M} h_n^3(w_n')^2\,dx \leqslant \frac{\gamma^2 2M}{k}, \tag{3.7}$$

From this we can easily deduce that

$$\int_{-M}^{M} (w_n')^2\, dx \leqslant \frac{\gamma^2 2M}{k^4}. \tag{3.8}$$

It then follows that

$$\|w_n\|_{W_0^{1,2}(-M,M)} \leqslant C.$$

It then easily follows that there exists a subsequence (denoted again by $w_n(x)$) converging weakly in $W_0^{1,2}(\Omega)$. Let $w(x)$ be the weak limit in $W_0^{1,2}(\Omega)$ of the sequence:

$$w_n \rightharpoonup w \quad in\ W_0^{1,2}(\Omega). \tag{3.9}$$

From the compactness of the embedding of $W^{1,2}$ into C^0, it follows that w_n converges to w uniformly in $(-M, M)$.

Lemma 3.1 *There exists a number $A > 0$ such that*

$$\int_{-M}^{M} G_n(w_n)\, dx \leqslant A \quad \forall n. \tag{3.10}$$

Proof Assume not. Then as $n \to \infty$,

$$\int_{-M}^{M} G_n(w_n)\, dx \to +\infty, \tag{3.11}$$

and consequently, it follows from the definition of $h_n(x)$ that $h_n(x) \to \infty$ uniformly on any compact subset of $(-M, M)$. We then deduce from (3.6) that $\forall n \geqslant N_0$, and $\forall x \in (-M, M)$ we have $0 \leqslant w_n(x) \leqslant \frac{1}{2}$, which then implies that $0 \leqslant |G_n(w_n)| \leqslant Ln(2)$ and will contradict (3.11). □

Lemma 3.2 *Let w be given by (3.9). Then*

$$\ln\left(\frac{1}{1 - w(x)}\right) \in L^1(-M, M). \tag{3.12}$$

Proof Fix $l > 0$. The sequence $v_n(x) = G_l(w_n(x))$ converges uniformly in $(-M, M)$ to $G_l(w(x))$. On the other hand, it is easy to see from the definition of G_l that $v_n(x) = G_l(w_n(x)) \leqslant G_n(w_n(x))$, $\forall n \geqslant l$ and $\forall x$. Hence, for a fixed l,

$$\int_{-M}^{M} G_l(w(x))\, dx = \lim_{n \to l} \int_{-M}^{M} G_l(w_n(x))\, dx \leqslant A \tag{3.13}$$

Next we let l tend to infinity. The positive sequence $G_l(w(x))$ is an increasing sequence, and its integral is uniformly bounded. Therefore, its pointwise limit is in $L^1(-M, M)$. $G_l(w(x))$ converges pointwise to $\ln(\frac{1}{1-w(x)})$ at points where $w(x) < 1$, and to infinity at points where $w(x) \geqslant 1$. From the continuity of w and (3.13), it follows that $w(x) < 1$ except possibly on a set of measure zero, and therefore $w(x) \leqslant 1\ \forall x$. Therefore $G_l(w(x))$

is increasing, and it converges pointwise to $\ln(\frac{1}{1-w(x)})$. It follows from classical results in measure theory (the Beppo–Levi Lemma) that $\ln(\frac{1}{1-w(x)}) \in L^1(-M, M)$. □

Lemma 3.3 *The sequence $G_n(w_n)$ has a subsequence which converges in the sense of measures to a positive measure $p(x)$. Furthermore, for almost every $x_0 \in (-M, M)$ there exists an open neighborhood I_0 such that $G_n(w_n)$ converges uniformly in I_0 to $\ln(\frac{1}{1-w(x)})$.*

Proof From (3.10) and the compactness of the set of measures, it follows that there exists a measure $p(x)$ and a subsequence of $G_n(w_n)$ such that the subsequence (denoted again by $G_n(w_n)$) will converge weakly as a measure to $p(x)$. Furthermore, since $G_n(w_n)$ are positive measures, their limit will also be a positive measure.

From (3.12) it follows that for almost every x_0, $\ln(\frac{1}{1-w(x_0)})$ is finite, and therefore $w(x_0) < 1$. Since $w(x) \in W_0^{1,2}$, and therefore is continuous, there exists a neighborhood I_0 of x_0 such that $w(x) < 1 \forall x \in I_0$. Since w_n converges uniformly to w on $(-M, M)$, it follows that there exists a positive number σ such that $w_n(x) < 1 - \sigma \ \forall x \in I_0$ and for all n large enough. The result then follows from the definition of the function G_n and the continuity of the ln function. □

Lemma 3.4 *For almost every $x \in (-M, M)$*

$$\int_{-M}^{M} G_n(w_n(s)) \ln\left(\frac{2M}{|x-s|}\right) ds \to \int_{-M}^{M} p(s) \ln\left(\frac{2M}{|x-s|}\right) ds \quad \text{as } n \to \infty. \tag{3.14}$$

Moreover, for almost every $x_0 \in (-M, M)$ there exists a neighborhood I_0 where the above convergence is uniform.

Proof From (3.12) it follows that for almost every x_0, $w(x_0) < 1$. As we saw earlier, there then exists an interval I_0 and a number $\sigma > 0$ such that $w_n(x) < 1 - \sigma \quad \forall x \in I_0$ and for all n large enough. Furthermore $w_n(x)$ converges uniformly to $w(x)$ over I_0.

Let $\varphi(x_0 - s)$ be a smooth function such that $0 \leqslant \varphi \leqslant 1$, $\varphi(x_0 - s) \equiv 1$ for $|x_0 - s|$ small enough and $\varphi(x_0 - s) \equiv 0$ for s not in I_0. Then as $n \to \infty$,

$$\int_{I_0} G_n(w_n(s)) \ln\left(\frac{2M}{|x_0-s|}\right) \varphi(x_0 - s)\, ds \to \int_{I_0} p(s) \ln\left(\frac{2M}{|x_0-s|}\right) \varphi(x_0 - s)\, ds. \tag{3.15}$$

On the other hand, $\ln(\frac{2M}{|x_0-s|})(1 - \varphi(x_0 - s))$ is continuous in s on $(-M, M)$, and therefore it is clear that

$$\begin{aligned}
&\int_{-M}^{M} G_n(w_n(s)) \ln\left(\frac{2M}{|x_0-s|}\right) (1 - \varphi(x_0 - s))\, ds \\
&\to \int_{-M}^{M} p(s) \ln\left(\frac{2M}{|x_0-s|}\right) (1 - \varphi(x_0 - s))\, ds \quad \text{as } n \to \infty.
\end{aligned} \tag{3.16}$$

The second part of the lemma follows from general results about convolution and the following facts:

- $G_n(w_n(s)) * \varphi(x_0 - s)$ is bounded in $W^{1,2}(-M, M)$ independently of n.
- $\ln(\frac{2M}{|x-s|})\ (1 - \varphi(x_0 - s))$ is a smooth function of x for x close enough to x_0.

We now set

$$h(x) = k + \frac{x^2}{2} + \eta_1 \int_{-M}^{M} p(x) \ln\left(\frac{2M}{|x-s|}\right) ds, \tag{3.17}$$

where $p(x)$ is the measure described in Lemma 3.3. □

Lemma 3.5 *The sequence $h_n(x)$ is uniformly bounded in $W^{\beta,2}$ for all $\beta < 1/2$. For any $q < \infty$ a subsequence of $h_n(x)$ will converge to $h(x)$ strongly in L^q.*

Proof Clearly, it is enough to prove the above statements for the convolution part of h_n.

We extend $G_n(w_n)$ and $\ln(\frac{2M}{|s|})$ by zero outside of $(-M, M)$, and set $\rho_n(x) = G_n(w_n) * \ln(2M/|s|)$. Taking the Fourier transforms of both sides and using the fact that the Fourier transform of $\widehat{\ln(2M/|s|)}$ is[1] $\frac{Si(y)}{y}$, and that the Fourier transform of $G_n(w_n)$ is bounded uniformly by A in L^∞, we then have

$$(1 + y^2)^\beta \hat{\rho}_n^2 \leqslant C(1 + y^2)^\beta \left(\frac{Si(y)}{y}\right)^2 \leqslant C_1 \frac{1}{(1 + y^2)^{1-\beta}}. \tag{3.18}$$

Since the last term in the above inequality is integrable for all $\beta < 1/2$, it follows from the definition of $W^{s,p}$ (see Adams [1] or Nečas [9], for example) that $\rho_n(x)$ are uniformly bounded in $W^{\beta,2}$ for all $\beta < 1/2$.

From the compactness of the embedding of $W^{\beta,2}(-M, M)$ into $L^q(-M, M)$, it follows that a subsequence of $\rho_n(x) = G_n(w_n) * \ln(2M/|s|)$ will converge strongly in $L^q(-M, M)$ to $\rho(x) = p(s) * \ln(2M/|s|)$. □

Lemma 3.6

(1) *Let $x_0 \in (-M, M)$ be such that $w(x_0) < 1$. Then there exists an open neighborhood I_0 of x_0 such that $w_n(x)$ converges weakly to $w(x)$ in $W^{2,l}(I_0)$, $l < \infty$.*

(2) *$w_n(x)$ converges to $w(x)$ almost everywhere in $(-M, M)$.*

Proof Since w is continuous, if $w(x_0) < 1$ then in a neighborhood of x_0 for n large enough, $G_n(w_n(x)) = \ln(\frac{1}{1 - w_n(x)})$, and also $|G_n'(w_n(x))| \leqslant c|w_n'(x)|$. From this it is easy to deduce that in a neighborhood of x_0, $h_n(x)$ is bounded in C^α independently of n. Using that $w_n(x)$ is a solution to the variational inequality (3.1)–(3.3) in the neighborhood of x_0, and that the conditions of Theorem 3.2 of Friedman [6] are satisfied in the neighborhood of x_0, we deduce from Theorem 3.2 of Friedman [6] that $w_n(x)$ are bounded in $W^{2,l}(I_0)$ independently of n.

From the weak compactness of $W^{2,l}(I_0)$ we then have that a subsequence of $w_n(x)$ converges to $w(x)$ weakly in $W^{2,l}(I_0)$. Using the fact that all such subsequences converge to $w(x)$, it turns out that the whole sequence $w_n(x)$ converges to $w(x)$ weakly in $W^{2,l}(I_0)$.

[1] Si is the special function Sine Integral.

To prove (2), first notice that from (1) and the compactness of the embedding of $W^{2,l}(I_0)$ into C^0, it follows that if $w(x_0) < 1$, then $w_n(x_0)$ converges to $w(x_0)$. Secondly, as we proved earlier, for almost all $x \in (-M, M)$, $w(x) < 1$. □

Lemma 3.7

(1) *w'_n is uniformly bounded in $L^\infty(-M, M)$.*

(2) *w'_n converges strongly to w' in $L^q(-M, M)$ for any $q < \infty$.*

Proof For a fixed n, let O_n be the set where $w_n > 0$. Since w_n is continuous, O_n is open, and we have that $O_n = \cup_m I_{n,m}$ where $I_{n,m} = (a_{n,m}, b_{n,m})$ are disjoint intervals such that $w_n(a_{n,m}) = w_n(b_{n,m}) = 0$. From (3.3) it follows that

$$w'_n(x) = \frac{\gamma h_n(x) - c_{n,m}}{h_n^3(x)} \quad \forall x \in I_{n,m}, \tag{3.19}$$

where $c_{n,m}$ is a constant. Since $h_n(x) \geqslant k > 0$, it is therefore necessary that $c_{n,m} > \gamma k$, because otherwise $w'_n(x)$ would be positive in all of $I_{n,m}$, which would contradict the fact that w_n vanishes at the end points of $I_{n,m}$.

Now let

$$f_{n,k}(u) = \frac{\gamma u - c_{n,m}}{u^3} \quad \forall u \geqslant k. \tag{3.20}$$

Elementary calculations easily show that

$$\frac{-c_{n,m}}{k^3} \leqslant f_{n,k}(u) \leqslant \frac{4\gamma^3}{27c_{n,m}^2} \quad \forall u \geqslant k. \tag{3.21}$$

Since $c_{n,m} > \frac{k}{\gamma}$, $\forall n$, $\forall m$, it easily follows from (3.21), (3.20) and (3.19) that

$$w'_n(x) \leqslant \frac{4\gamma}{27k^2} \quad \forall x \in I_{n,m}. \tag{3.22}$$

Since this bound is independent of m and n, in fact this upper bound holds for all x and all n.

Next we prove that $w'_n(x)$ is also bounded away from $-\infty$. Assume that there exists a sequence $x_{q(n)}$ such that $w'_n(x_{q(n)})$ approaches $-\infty$ as n approaches infinity. Then the function $F(x_{q(n)}) = -h_n^3(x_{q(n)})w'_n(x_{q(n)}) + \gamma h_n(x_{q(n)})$ will approach $+\infty$ as n approaches infinity.

Let (a, b) be the support of w. From (1) of the previous lemma, we then have that for an interval $I_d = (b - \delta, M)$, w_n is uniformly bounded in $W^{2,l}(I_d)$. In particular, we then have that w'_n is then uniformly bounded in $L^\infty(I_d)$. For future reference, let us say that

$$|w'_n(x)| \leqslant C \quad \forall x \geqslant b - \delta. \tag{3.23}$$

It is then easy to show that $h_n(x)$ is bounded in $(b - \delta, M)$ independently of n. Hence, the function $F_n(x) = -h_n^3(x)w'_n(x) + \gamma h_n(x)$ will be uniformly bounded on the interval $(b - \delta, M)$.

From (3.23) it follows that $x_{q(n)} \leqslant b - \delta$, for all n. By (3.1), the function $F_n(x)$ is an increasing function of x and therefore we should have that $\forall x \geqslant b - \delta$, $F_n(x)$ converges to $+\infty$ as n approaches infinity. This leads to a contradiction.

To prove (2) we simply use (1) of the lemma, (2) of Lemma (3.6), and Lebesgue's dominated convergence theorem. □

Remark 1 From (3.22) it follows that the function w' also satisfies the estimate

$$w'(x) \leqslant \frac{4\gamma}{27k^2} \quad \forall x \in (-M, M). \tag{3.24}$$

This explicit upper bound for w' seems to be new.

The lower bound above implies that the sequences $c_{n,m}$ are uniformly bounded.

Theorem 3.2 $\forall \alpha > 0$, $\forall M > 0$, $\forall k > 0$, $\forall \eta_0 > 0$ *there exist:*

(1) *a finite positive measure* $p(x)$,

(2) *a closed set* E_s *of measure zero in* $(-M, M)$,

(3) *a positive function* $w(x) \in W_0^{1,\infty}(-M, M)$

such that:

(1) *for any* $x \notin E_s$ *there exists an open neighborhood* I_x *such that* $p(x)$ *is a smooth bounded function in* I_x *and* $w(x) = 1 - e^{-\alpha p(x)}$ *for all* $x \in I_x$;

(2) *the function* h *given by* (3.1) *is in* $L^\infty(-M, M)$;

(3) p, w, h *is a weak solution to the problem* (2.1)–(2.6).

Proof As we have seen in the proofs of the previous lemmas, the sequences (w_n, h_n, G_n) converge to (w, h, p), respectively, in $W_0^{1,l}(-M, M)$, $\forall l < \infty$, $W^{1/2-\sigma}(-M, M)$, $\forall \sigma > 0$, and as a measure. We proved earlier that (2.5) holds almost everywhere in $(-M, M)$.

Since w_n' and h_n converge strongly in L^q $\forall q < \infty$, it follows that $h_n^3 w_n' - \gamma h_n$ converges (strongly in L^2, for example) to $h^3 w' - \gamma h$, and (2.1) is satisfied in the sense of distributions. (For the concept of positive distributions, see Schwartz [11], for example.)

We show that (2.3) is satisfied in the sense of distributions. Let $\phi(x)$ be a C^∞ function with compact support in $(-M, M)$. Then from (3.1) we have that for all n

$$\langle (-h_n^3 w_n' + \gamma h_n)', w_n \phi \rangle_{W^{-1,2}, W_0^{1,2}} = 0, \tag{3.25}$$

where $\langle ., . \rangle$ is the duality product.

Since $(-h_n^3 w_n' + \gamma h_n)$ converges in L^2 to $(-h^3 w' + \gamma h)$, and as $w_n \phi$ converges strongly in $W_0^{1,2}$ to $w\phi$, the above duality relation converges to

$$\langle (-h^3 w' + \gamma h)', w\phi \rangle_{W^{-1,2}, W_0^{1,2}} = 0, \tag{3.26}$$

from which we deduce that $w(-h^3 w' + \gamma h)'$ is well defined as a distribution, and that it is a positive distribution. Therefore (2.1) is satisfied in the sense of distributions. □

4 Regularity

There is some speculation among engineers about the existence of a cusp-like spike in the pressure near the outlet. (See section [4] of Wu [13] and the references therein.) This leads us to try to get some regularity results on possible isolated singularities of the pressure field.

For this purpose, we start with some basics results. Since we have seen that the pressure $p(x)$ is a bounded positive measure, then by the Radon–Nykodym theorem $p(x)$ has a unique decomposition

$$p(x) = f(x) + S(x), \tag{4.1}$$

where $f(x)$ is an L^1 function and $S(x)$ is a singular measure. It is easy to deduce from our previous results that $f(x) \equiv \ln(\frac{1}{1-w(x)})$. We now wish to prove a regularity result about $S(x)$. As we have already seen, the support of $S(x)$ is included in the set of points where $w(x) = 1$.

Theorem 4.1 *If x_1 is an isolated point of the set $w(x) = 1$, then $S(x)$ is zero in a neighborhood of x_1.*

Proof Let I_i be a neighborhood of x_1 such that $w(x) < 1$ for all $x \neq x_1$ in I_i. We then have that $S(x)$ is a positive measure in I_i with support containing at most the point x_1. In this case, the only possibility is that $S(x) \equiv d\delta_{x_1}$, where d is a positive real number and δ_{x_1} is the Dirac distribution based at the point x_1. We will prove that $d = 0$. Assume not. Then it can easily be seen from the definition of $h(x)$ that

$$h(x) \geqslant \eta_1 d \cdot \ln\left(\frac{2M}{x - x_1}\right), \quad \forall x \geqslant x_1, \ x \in I_i. \tag{4.2}$$

Proceeding as in the proof of Lemma (3.7), we have that

$$w'(x) = \frac{\gamma h(x) - c}{h^3(x)} \quad \forall x \geqslant x_1, \ x \in I_i, \tag{4.3}$$

where c is a positive constant. From (4.2) we have that for $x - x_1$ small enough $h(x) \geqslant c$, and therefore $w'(x) \geqslant 0$ for all x close enough to x_1. However, $w(x_1) = 1$ and $w(x) \leqslant 1$, which would be inconsistent with w' over an interval of the type $(x_1, x_1 + \sigma)$. □

Remark 2 It follows from our regularity result that if there is an isolated spike in the pressure, then the pressure is locally an L^1 function at such a point, and that the reduced pressure q has a bounded gradient in the neighborhood of such a point. It then follows that if η is the viscosity then $\frac{1}{\eta}p'(x)$ is finite at that point.

5 Conclusion

We have established the existence of a solution for all positive values of the parameter θ. For a given fluid with constitutive constants α, η_0 this would mean that we have no restriction on the rolling velocity u. Previous existence results apply only for limited values of the speed. When restrictions on the speed were removed, more singular behavior of the solutions was allowed; but we have provided some characterization of the singular part. At this stage, we are unable to extend this existence result to the two-dimensional point contact problem.

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