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Cover's Rebalancing Option With Discrete Hindsight Optimization

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Abstract

We study T. Cover's rebalancing option (Ordentlich and Cover 1998) under discrete hindsight optimization in continuous time. The payoff in question is equal to the final wealth that would have accrued to a \$1 deposit into the best of some ***finite*** set of (perhaps levered) rebalancing rules determined in hindsight. A rebalancing rule (or fixed-fraction betting scheme) amounts to fixing an asset allocation (i.e. 200% stocks and -100% bonds) and then continuously executing rebalancing trades to counteract allocation drift.

Restricting the hindsight optimization to a small number of rebalancing rules (i.e. 2) has some advantages over the pioneering approach taken by Cover & Company in their brilliant theory of universal portfolios (1986, 1991, 1996, 1998), where one's on-line trading performance is benchmarked relative to the final wealth of the best unlevered rebalancing rule ***of any kind*** in hindsight. Our approach lets practitioners express an *a priori* view that one of the favored asset allocations ("bets") $b \in \{b_1, ..., b_n\}$ will turn out to have performed spectacularly well in hindsight. In limiting our robustness to some discrete set of asset allocations (rather than all possible asset allocations) we reduce the price of the rebalancing option and guarantee to achieve a correspondingly higher percentage of the hindsight-optimized wealth at the end of the planning period.

A practitioner who lives to delta-hedge this variant of Cover's rebalancing option through several decades is guaranteed to see the day that his realized compound-annual capital growth rate is very close to that of the best b_i in hindsight. Hence the point of the rock-bottom option price.

Keywords: Continuously-Rebalanced Portfolios, Adaptive Asset Allocation, Kelly Criterion, Universal Portfolios, Lookback Options, Exchange Options, Rainbow Options, On-Line Portfolio Selection, Robust Procedures

JEL Classification: C44, D80, D81, G11

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1 Introduction

The main alternative to the Markowitz (1952) mean-variance theory of portfolio selection was popularized by Kelly (1956) who sought to optimize a gambler's asymptotic continuously-compounded capital growth rate in repeated bets on horse races in the presence of partial inside information. His reasoning is in fact applicable to all gambling, insurance, and investment problems. Rather than optimize the static reward per unit of risk, the Kelly Criterion (Poundstone 2010) is equivalent to the prescription that one should act each round so as to maximize the expected log of his capital. Breiman (1961) showed that the Kelly Criterion constitutes asymptotically dominant behavior: a Kelly gambler will almost surely beat any other gambler in the long run by an exponential factor, and he has the shortest expected hitting time for a distant wealth goal. With probability approaching 1 as time goes on, the Kelly gambler's bankroll will (amusingly) overtake that of a mean-variance investor, who has a smooth ride but ultimately cannot "eat his Sharpe ratio." The books by Cover and Thomas (2006) and Luenberger (1998) are excellent primers of the theory of asymptotic capital growth in discrete and continuous time, respectively. Thorp (cf. his 2017 biography) demonstrated the practical effectiveness of the Kelly Criterion when he used it to size his Blackjack bets in certain favorable situations that are identifiable via his trademark (1966) theory of card counting. In this connection, the correct behavior is to bet the fraction $b^* := p - q$ of your net worth on a given hand for which p is the chance of winning and q is the chance of losing.

For growth opportunities in the stock market, the analog of Kelly's fixed fraction betting scheme is a certain constant-rebalanced portfolio b^* that trades continuously so as to maintain a target growth-optimal fraction of wealth in each risk asset. For instance, rather than bet $b := 2\%$ of wealth on a (favorable) hand of Blackjack, one could bet 2% of wealth (or even $b := 200\%$ of wealth) on the S&P 500 index. In theory, if stock market returns are *iid* across (discrete) time then one can calculate the corresponding log-optimal portfolio directly from the return distribution. But in practice, equity investors must get along without complete knowledge of the return distribution. Thus, a real-world investor cannot measure the exact regret of his portfolio relative to the Kelly bet for the simple reason that he does not know the Kelly bet.

The way out of this conundrum was discovered by information theorist Thomas Cover (1938-2012), who formulated the *individual sequence approach* to investment. For a given observed sequence of asset prices, one can look back and determine which constant-rebalanced asset allocation would have yielded the greatest final wealth *for that particular sequence*. By definition, a Kelly gambler (who knows the distribution of returns but not the individual sequence that will occur in the future) will achieve a final wealth that is no greater than that of the best constant-rebalanced portfolio determined in hindsight for the actual sequence of returns. Thus began Cover's important *universal portfolio theory* that formulated various on-line investment schemes (1986, 1991, 1996, 1998) that guarantee to achieve a high percentage of the final wealth of the best constant unlevered rebalancing rule (of any kind) in hindsight. Of course, any such scheme would then also guarantee to achieve a high percentage of the Kelly final wealth in *iid* stock markets.

1.1 Contribution

One can consider Cover's performance benchmark to be a financial derivative ("Cover's rebalancing option") whose final payoff is equal to the wealth that would have accrued to a $1 deposit into the best rebalancing rule (or fixed-fraction betting scheme) determined in hindsight. Ordentlich and Cover (1998) began the work of pricing this option in the Black-Scholes (1973) market at time-0 for unlevered hindsight optimization over a single underlying risk asset. Garivaltis (2018) priced and replicated the rebalancing option at any time t for levered hindsight optimization over an arbitrary number of correlated stocks in geometric Brownian motion. That paper obtained the elegant result that for completely relaxed (levered) hindsight optimization, the corresponding delta-hedging strategy simply looks back over the observed price history $[0, t]$, computes the best rebalancing rule in hindsight $b(S_t, t)$, and bets that fraction of wealth over the next differential time step $[t, t + dt]$.

The present paper studies Cover's rebalancing option with hindsight optimization over a discrete set $\mathbb{B} := \{b_1, ..., b_n\}$ of rebalancing rules. Apart from the scientific obligation to extend Ordentlich and Cover's incisive (1998) chain of reasoning, our approach has some interesting advantages relative to hindsight-optimization over all possible rebalancing rules. In our world, the (delta-hedging) practitioner is now free to express any of his institutional constraints or beliefs about future returns through a judicious choice of the set $\mathbb{B}$. Our newly austere mode of hindsight optimization yields a rock-bottom option price and correspondingly better guarantees of relative performance at the end of the planning period, whose shortened length is now well within a human life span. Say, for robust betting on the S&P 500 index, the author himself is inclined to use $\mathbb{B} := \{0, 0.5, 1, 1.5, 2\}$, which amounts to the following five (continuously-rebalanced) asset allocations:

(1) 0% stocks, 100% cash

(2) 50% stocks, 50% cash

(3) 100% stocks, 0% cash

(4) 150% stocks, −50% cash (margin loans)

(5) 200% stocks, −100% cash (margin loans)

In this example, the author would like to avoid paying the full *Cost of Achieving the Best [Rebalancing Rule] in Hindsight* that would correspond (Garivaltis 2018) to $\mathbb{B} := \mathbb{R}$ or even $\mathbb{B} := [0, 2]$.

The paper is organized as follows. Section 2 explains our basic notation and terminology. Section 3 develops our main techniques in the context of hindsight optimization over a pair $b > c$ of rebalancing rules and a single underlying risk asset. We price and replicate both the horizon-T and perpetual versions of the rebalancing option, and give performance simulations that illustrate the general behavior of the replicating strategy. Section 4 extends the methodology to general discrete sets of asset allocations. We show how the rebalancing option can be interpreted as a certain portfolio of Margrabe-Fischer (1978) exchange options, and derive the general replicating strategy, which is a time- and state-varying convex combination of the b_i. We close the paper by proving that American-style rebalancing options (with general exercise price K) are always "worth more alive than dead" in equilibrium.

2 Definitions and Notation

We start in the Black-Scholes (1973) market with a single underlying stock whose price S_t follows the geometric Brownian motion

$$\frac{dS_t}{S_t} = \mu\, dt + \sigma dW_t, \tag{1}$$

where μ is the drift, σ is the volatility, and W_t is a standard Brownian motion. There is a risk-free bond whose price $B_t := e^{rt}$ follows

$$\frac{dB_t}{B_t} = r\,dt. \tag{2}$$

A constant rebalancing rule $b \in (-\infty, +\infty)$ is a fixed-fraction betting scheme that continuously maintains the fraction b of wealth in the stock and the fraction $1 - b$ of wealth in bonds. We let $V_t(b)$ denote the wealth at t that accrues to a \$1 deposit into the rebalancing rule b. Thus, the trader holds $\Delta := bV_t(b)/S_t$ shares of the stock at time t, and his remaining $(1 - b)V_t$ dollars are invested in bonds. Maintenance of the target asset allocation generally requires continuous trading. If $0 < b < 1$, the trader must sell a precise number of shares on every uptick (more precisely, whenever $dS_t/S_t \geq r\,dt$) to restore the target allocation. Similarly, when the risk asset underperforms cash over $[t, t + dt]$ (i.e. when $dS_t/S_t \leq r\,dt$) the trader must buy additional shares to restore the balance. This amounts to a volatility harvesting scheme (cf. Luenberger 1998) that "lives off the fluctuations" of the underlying.

For $b = 1$ the trader just buys the stock and holds it; for $b > 1$ he carries a margin (debit) balance of $(b-1)V_t(b)$ dollars at time t. A levered rebalancing rule $b > 1$ must continuously maintain a fixed debt-to-assets ratio of $1 - 1/b$. Thus, when the stock rises (and debt is now a smaller percentage of assets) the trader will borrow against his new wealth to buy additional shares. Similarly, when the stock falls he must sell some shares to reduce the loan-to-value ratio. This "buy high, sell low" strategy is only appropriate for stocks with relatively high drift and low volatility. Finally, for low quality underlyings one can hold all cash $(b = 0)$ or a continuously-rebalanced short position $b < 0$.

We now imagine a trader who starts with \$1 and has two favored rebalancing rules $b > c$, who wants to perform well relative to the best of $\mathbb{B} := \{b, c\}$ in hindsight. Accordingly, we create for him the financial derivative whose final payoff at T is

$$\boxed{V_T^* := \max\{V_T(b), V_T(c)\}}. \tag{3}$$

Ordentlich and Cover (1998) investigated the best unlevered rebalancing rule in hindsight, with payoff $V_T^* := \max\limits_{0 \le b \le 1} V_T(b)$. They found the time-0 price of this contingent claim to be

$$C_0 = 1 + \sigma\sqrt{\frac{T}{2\pi}}. \tag{4}$$

The owner of this rebalancing option (cf. Garivaltis 2018) will compound his money at the same asymptotic rate as the best unlevered rebalancing rule in hindsight. Indeed, the final excess continuously-compounded growth rate of the best rebalancing rule in hindsight over that of the replicating strategy is $\log\left\{1+\sigma\sqrt{T/(2\pi)}\right\}/T$, which tends to 0 as $T \to \infty$. This growth rate spread obtains deterministically, regardless of the realized price path $(S_t)_{0 \le t \le T}$.

Garivaltis (2018) extended the Ordentlich-Cover (1998) analysis by computing the general time-t price $C(S, t)$ of Cover's rebalancing option for both levered and unlevered hindsight optimization. For levered hindsight optimization (with payoff $V_T^* := \max\limits_{b \in \mathbb{R}} V_T(b)$), Garivaltis (2018) found the general pricing formula

$$C(S, t) = \sqrt{\frac{T}{t}} \exp\{rt + z_t^2/2\}, \tag{5}$$

where

$$z_t := \frac{\log(S_t/S_0) - (r - \sigma^2/2)t}{\sigma\sqrt{t}} \tag{6}$$

is an auxiliary variable that is distributed unit normal with respect to the equivalent martingale measure $\mathbb{Q}$. More generally, for a Black-Scholes market with d correlated stocks in geometric Brownian motion, Garivaltis (2018) found that

$$C(S,t) = \left(\frac{T}{t}\right)^{d/2} \cdot \exp\{rt + z_t' R^{-1} z_t/2\}, \tag{7}$$

where $R := [\rho_{ij}]_{d\times d}$ is the correlation matrix of instantaneous returns,

$$z_{it} := \frac{\log(S_{it}/S_{i0}) - (r - \sigma_i^2/2)t}{\sigma_i\sqrt{t}}, \tag{8}$$

are auxiliary variables, and σ_i is the volatility of stock i. When we relax the hindsight optimization to include all levered rebalancing rules $b \in \mathbb{R}^d$, replication becomes especially simple. At time t, one just looks back at the observed price history $[0,t]$, finds the best (d-dimensional) rebalancing rule $b(S,t)$ in hindsight, and bets the fraction $b_i(S,t)$ of wealth on stock i over $[t, t+dt]$. The relation $C(S,t;T) \propto T^{d/2}$ matches the model-independent $\mathcal{O}(T^{d/2})$ super-replicating price calculated by Cover & Company.

In what follows, we work toward reducing the option price $\sqrt{T/t}\cdot\exp\{rt + z_t^2/2\}$ by replacing $\mathbb{B} = \mathbb{R}$ with $\mathbb{B} := \{b, c\}$. In order to get the payoff $\max\{V_T(b), V_T(c)\}$ into a more practical form, we note that $V_t(b)$ is a geometric Brownian motion, since

$$\frac{dV_t(b)}{V_t(b)} = b\frac{dS_t}{S_t} + (1-b)\frac{dB_t}{B_t} = [r + (\mu - r)b]dt + b\sigma dW_t. \tag{9}$$

Solving this stochastic differential equation, we obtain (cf. Wilmott 1998, 2001)

$$V_t(b) = \exp\{[r + (\mu - r)b - \sigma^2 b^2/2]t + b\sigma W_t\}. \tag{10}$$

In order to get the payoff in terms of the observable variable S_t (rather than the

Wiener process W_t), we start with the equation

$$S_t = S_0 \exp\{(\mu - \sigma^2/2)t + \sigma W_t\}, \tag{11}$$

and solve for σW_t in terms of S_t. Substituting the resulting expression into 9, we get

$$V_t(b) = \exp\{(r - \sigma^2 b^2/2)t + b[\log(S_t/S_0) - (r - \sigma^2/2)t]\}. \tag{12}$$

We thus have

$$\boxed{V_t(b) = \exp\{(r - \sigma^2 b^2/2)t + b\sigma\sqrt{t} \cdot z_t\}}, \tag{13}$$

where

$$\boxed{z_t := \frac{\log(S_t/S_0) - (r - \sigma^2/2)t}{\sigma\sqrt{t}}} \tag{14}$$

is distributed unit normal with respect to the equivalent martingale measure $\mathbb{Q}$. Note that the drift μ (which is difficult to estimate) does not appear in this formula. The final wealth of the rebalancing rule b is now expressed solely in terms of z_t, the risk-free rate r, the time t, and the volatility σ, which is easily estimated from high-frequency price data.

3 The Best of Two Asset Allocations in Hindsight

Before we can price the rebalancing option with payoff $\max\{V_T(b), V_T(c)\}$, we must characterize the random outcomes under which b will turn out to outperform c over the interval $[0, t]$. Accordingly, we compare the exponents of $V_t(b)$ and $V_t(c)$ to obtain

Lemma 1. *For two given rebalancing rules $b > c$, b outperforms c over $[0, t]$ if and* *only if*

$$\boxed{z_t \geq \frac{b+c}{2}\sigma\sqrt{t}}. \tag{15}$$

Proposition 1. *The best rebalancing rule (of any kind) in hindsight over* $[0,t]$*, denoted* $b(S,t)$*, is*

$$\boxed{b(S,t) := \arg\max_{b\in\mathbb{R}} V_t(b) = \frac{z(S,t)}{\sigma\sqrt{t}}}. \tag{16}$$

Given any closed set $\mathbb{B}$ *of rebalancing rules, the best performer in hindsight is the* $b \in \mathbb{B}$ *that is nearest to* $b(S,t) = z(S,t)/(\sigma\sqrt{t})$*.*

Proof. We compute the abscissa of vertex of the parabola $b \mapsto \log\, V_t(b)$. This yields

$$b(S,t) = \arg\max_{b\in\mathbb{R}} \log V_t(b) = \frac{-\sigma\sqrt{t}\cdot z_t}{2(-\sigma^2 t/2)} = \frac{z_t}{\sigma\sqrt{t}}. \tag{17}$$

Because the graph of a parabola is symmetric about its vertex, the $b \in \mathbb{B}$ that maximizes the height of this parabola is whichever element of $\mathbb{B}$ is nearest to the vertex $b(S,t)$. □

We proceed to compute the cost of achieving the best of two rebalancing rules in hindsight, by finding the expected present value of $\max\{V_T(b), V_T(c)\}$ at time-0 with respect to the equivalent martingale measure $\mathbb{Q}$. This cost is the sum of two integrals $I_1 + I_2$, where

$$I_1 := \frac{\exp(-\sigma^2 b^2 T/2)}{\sqrt{2\pi}} \int_{\frac{b+c}{2}\sigma\sqrt{T}}^{\infty} \exp(-z^2/2 + b\sigma\sqrt{T}\cdot z)dz \tag{18}$$

and

$$I_2 := \frac{\exp(-\sigma^2 c^2 T/2)}{\sqrt{2\pi}} \int_{-\infty}^{\frac{b+c}{2}\sigma\sqrt{T}} \exp(-z^2/2 + c\sigma\sqrt{T}\cdot z)dz. \tag{19}$$

In the sequel, we will often use the following general formula (i.e. the appendix to

Reiner and Rubinstein 1992):

$$\boxed{\int_A^B e^{-\alpha y^2+\beta y}dy = \sqrt{\frac{\pi}{\alpha}}\exp\left(\frac{\beta^2}{4\alpha}\right)\left[N\left(B\sqrt{2\alpha}-\frac{\beta}{\sqrt{2\alpha}}\right)-N\left(A\sqrt{2\alpha}-\frac{\beta}{\sqrt{2\alpha}}\right)\right]}, \tag{20}$$

where $\alpha > 0$ and $N(\bullet)$ is the cumulative normal distribution function. Simplifying the two integrals, we get

$$I_1 = I_2 = N\left(\frac{b-c}{2}\sigma\sqrt{T}\right). \tag{21}$$

Theorem 1. *The time-*0 *cost of achieving the best of two rebalancing rules* $\{b, c\}$ *in hindsight is*

$$\boxed{C_0(\delta, \sigma, T) = 2N\left(\frac{\delta}{2}\sigma\sqrt{T}\right)}. \tag{22}$$

where $\delta := |b-c|$ *is the distance between the two rebalancing rules.*

Corollary 1. *The equilibrium price at* $t = 0$ *of a perpetual option* ($T := \infty$) *on the best of two rebalancing rules* $\{b, c\}$ *in hindsight is* $C_0(\delta, \sigma, \infty) = \$2$.

Note that the horizon-T price is independent of the interest rate r, and it is translation invariant in the sense that it depends only on the distance $\delta = |b-c|$ between the two rebalancing rules. We always have $1 \leq C_0(\delta, \sigma, T) \leq 2$; besides the perpetual version of the option, the maximum \$2 price also obtains if $\sigma = \infty$ or $\delta = \infty$. The minimum \$1 price obtains if any of the parameters δ, σ, T tends to 0. Since the increasing function $N(\bullet)$ is concave over $[0, \infty)$, we see that the option price is increasing and concave separately in each of the parameters δ, σ, T.

Theorem 2. *Given two rebalancing rules* $b > c$ *with distance* $\delta = |b-c|$, *an initial* \$1 *deposit into the horizon-*T *replicating strategy achieves at* T *a compound growth-rate* *that is exactly*

$$\frac{100}{T}\log\left\{2N\left(\frac{\delta}{2}\sigma\sqrt{T}\right)\right\} \tag{23}$$

percent lower than that of the best of $\{b, c\}$ *in hindsight. A* \$1 *deposit into the corresponding horizon-free strategy (that replicates the perpetual version of the option) achieves a compound-growth rate at* T *that is at most* $100\log(2)/T$ *percent lower than that of the best of* $\{b, c\}$ *in hindsight.*

Proof. The trader's initial (\$1) deposit into the replicating strategy buys him $1/C_0$ units of the option at $t = 0$. For the horizon-T option, his wealth at expiration will be $\max\{V_T(b), V_T(c)\}/C_0$, and hence the excess continuously-compounded growth rate will be

$$\frac{1}{T}\log[\max\{V_T(b), V_T(c)\}] - \frac{1}{T}\log[\max\{V_T(b), V_T(c)\}/C_0] = \frac{\log C_0(\delta, \sigma, T)}{T}. \tag{24}$$

For the horizon-free option, the trader's initial dollar buys him half a unit of the option at $t = 0$. Thus, his wealth at T will be at least half the exercise value of the option, which is $\max\{V_T(b), V_T(c)\}$. Hence, the excess continuously-compounded growth rate of the hindsight-optimized rule at T is at most

$$\frac{1}{T}\log[\max\{V_T(b), V_T(c)\}] - \frac{1}{T}\log[\max\{V_T(b), V_T(c)\}/2] = \frac{\log 2}{T}. \tag{25}$$

□

Example 1. *Consider the following robust scheme for* $T := 25$ *years of leveraged bets on the S&P 500 index. We put* $b := 2$ *and* $c := 1$ *(e.g. buy-and-hold), with* $\sigma := 0.15$*. We get* $C_0 = \$1.29$ *and* $\log(C_0)/T = 1\%$*, so the replicating strategy is guaranteed to achieve a final compound-growth rate that is* 1% *lower than the best of* $\{b, c\}$ *in* *hindsight. If $b = 2$ happens to outperform the index by more than 1% per year, then the trader will beat the market over $t \in [0, 25]$. If $b = 2$ underperforms the index (or outperforms by less than 1% a year), then the trader's compound-growth rate will have lagged the market by at most 1% a year.*

Note that the corresponding horizon-free strategy (that replicates the perpetual version of the option) can only guarantee to get within $\log(2)/T = 2.8\%$ of the hindsight-optmized growth rate at $T = 25$

Example 2. *We construct a robust $T := 25$ year scheme for long-run stock market investment that guarantees preservation of capital. We put $b := 1$ (100% stocks) and $c := 0$ (all cash). Assuming that $\sigma := 0.15$, the practitioner can rest easy, safe in the knowledge that his foray into risk assets will ultimately not cause him to lag the risk-free rate by more than 1% a year. If $r > 0.01$, then he is guaranteed not to lose money if he sticks to the Plan for $T = 25$ years. At the same time, if stocks go through the roof, his strategy will earn the long-run market growth rate minus a 1% "universality cost."*

Would-be practitioners who enjoyed these example can use Figure 1 to inform the choice of horizon: it plots the excess continuously-compounded growth rate for different volatilities and maturities with $\delta := 1$.

3.1 General Formulas for Pricing and Replication

Before we can put our on-line schemes for robust asset allocation into actual practice, we must derive general time-t formulas for pricing and replication of the rebalancing option under discrete hindsight optimization. Thus, we proceed to extend the above integration technique to the general situation. To simplify the notation, we let $\tau := T - t$ denote the remaining life of the option at time t. Inspired by Garivaltis (2018),

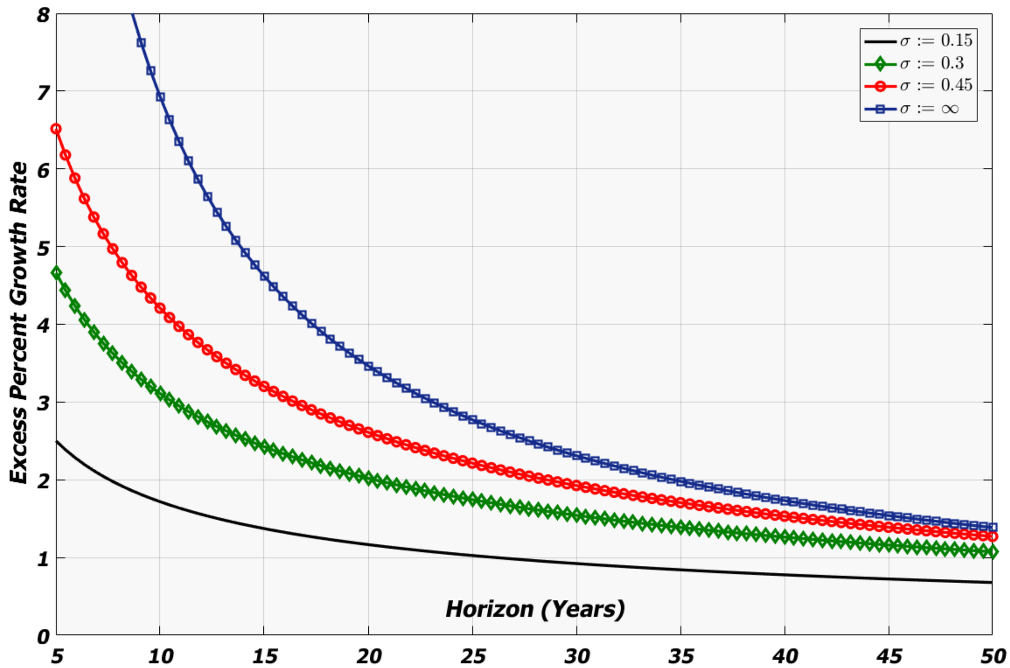

Figure 1: THE EXCESS PERCENT GROWTH RATE OF THE BEST OF TWO REBALANCING RULES OVER THE REPLICATING STRATEGY, FOR DIFFERENT HORIZONS AND VOLATILITIES, WITH $\delta := 1$.

we start with the decomposition

$$\boxed{z_T = \sqrt{\frac{t}{T}} \cdot z_t + \sqrt{\frac{\tau}{T}} \cdot y}, \tag{26}$$

where

$$\boxed{y := \frac{\log(S_T/S_t) - (r - \sigma^2/2)\tau}{\sigma\sqrt{\tau}}} \tag{27}$$

is distributed unit normal with respect to the equivalent martingale measure and the information available at t. Conditional on the values of time-t variables, b outperforms c at T if and only if

$$\boxed{y \geq \frac{\frac{b+c}{2}\sigma T - \sqrt{t} \cdot z_t}{\sqrt{\tau}}}. \tag{28}$$

Thus, the general price $C(S,t)$ is equal to the sum of two integrals $I_1 + I_2$, where

$$I_1 := \frac{\exp(rt - \sigma^2 b^2 T/2 + b\sigma\sqrt{t}\cdot z_t)}{\sqrt{2\pi}} \int_{[(b+c)\sigma T/2-\sqrt{t}\cdot z_t]/\sqrt{\tau}}^{\infty} \exp(-y^2/2 + b\sigma\sqrt{\tau}\cdot y)dy \quad (29)$$

and

$$I_2 := \frac{\exp(rt - \sigma^2 c^2 T/2 + c\sigma\sqrt{t}\cdot z_t)}{\sqrt{2\pi}} \int_{-\infty}^{[(b+c)\sigma T/2-\sqrt{t}\cdot z_t]/\sqrt{\tau}} \exp(-y^2/2 + c\sigma\sqrt{\tau}\cdot y)dy. \quad (30)$$

These integrals simplify to

$$I_1 = N\left(\frac{[b-c]\sigma T/2 + \sqrt{t}\cdot z_t - b\sigma t}{\sqrt{\tau}}\right)V_t(b) \quad (31)$$

and

$$I_2 = N\left(\frac{[b-c]\sigma T/2 - \sqrt{t}\cdot z_t + c\sigma t}{\sqrt{\tau}}\right)V_t(c). \quad (32)$$

Theorem 3. *The general cost $C(S,t)$ of achieving the best of two rebalancing rules $b > c$ in hindsight is*

$$\boxed{C = N(d_1)V_t(b) + N(d_2)V_t(c)}, \quad (33)$$

where

$$\boxed{d_1 := \frac{(b-c)\sigma T/2 + \sqrt{t}\cdot z_t - b\sigma t}{\sqrt{\tau}}} \quad (34)$$

and

$$\boxed{d_2 := (b-c)\sigma\sqrt{\tau} - d_1 = \frac{(b-c)\sigma T/2 - \sqrt{t}\cdot z_t + c\sigma t}{\sqrt{\tau}}}. \quad (35)$$

Theorem 4. *A perpetual option ($T := \infty$) on the best of two rebalancing rules $b > c$ costs $C(S,t) = V_t(b) + V_t(c)$ in state (S_t, t). To delta-hedge the perpetual option, one* *holds*

$$\boxed{\Delta = \frac{bV_t(b) + cV_t(c)}{S_t}} \tag{36}$$

shares of the underlying in state (S_t, t)*, and therefore bets the fraction*

$$\boxed{\hat{b}(S_t, t) := \frac{\Delta S}{C} = \frac{bV_t(b) + cV_t(c)}{V_t(b) + V_t(c)}} \tag{37}$$

of wealth on the underlying at t*.*

Proof. As $T \to \infty$, we see that $d_1, d_2 \to +\infty$ and the option price converges to $V_t(b) + V_t(c)$. Next, one can verify by direct calculation from (13) and (14) that

$$\frac{\partial V(b)}{\partial S} = \frac{\partial V(b)}{\partial z}\frac{\partial z}{\partial S} = \frac{bV_t(b)}{S_t}. \tag{38}$$

Alternately, one can observe that the rebalancing rule b keeps (by definition) $bV_t(b)$ dollars in the stock at time t, which amounts to $bV_t(b)/S_t$ shares. Thus, to replicate the sum $V_t(b) + V_t(c)$ we must own a total of $\Delta = bV_t(b)/S_t + cV_t(c)/S_t$ shares of the underlying. □

We should note that our general pricing formulas could have been obtained differently, by applying the theory of "exchange options" that was bequeathed to us in sumultaneous papers by Margrabe (1978) and Fischer (1978). Rather than the single underlying S_t, one could view the (perfectly correlated) geometric Brownian motions $U_1(t) := V_t(b)$ and $U_2(t) := V_t(c)$ as underlyings of a multi-asset option with payoff

$$\max\{U_1, U_2\} = \max\{U_1 - U_2, 0\} + U_2. \tag{39}$$

This amounts to a \$1 deposit into the rebalancing rule c, plus the option to exchange the final wealth of c for the final wealth of b at T. Substituting the aggregate volatility

$\sigma_a := (b-c)\sigma$ into Margrabe's Formula (cf. Zhang 1998) yields the same result

$$\boxed{C(U_1, U_2, t) = N(d_1)U_1 + N(\sigma_a\sqrt{\tau} - d_1)U_2}, \tag{40}$$

where

$$\boxed{d_1 := \frac{\log(U_1/U_2)}{\sigma_a\sqrt{\tau}} + \frac{\sigma_a\sqrt{\tau}}{2}}, \tag{41}$$

is in agreement with (34). Figure 3 plots the price and intrinsic value of the rebalancing option for different values of S under the parameters $r := 0.03, T := 10, S_0 := 100, t := 5, \sigma := 0.7, b := 1.5$, and $c := 0.5$.

Theorem 5. *The horizon-T replicating strategy for the best of two rebalancing rules $b > c$ in hindsight holds*

$$\boxed{\Delta = \frac{N(d_1)bV_t(b) + N(d_2)cV_t(c)}{S_t}} \tag{42}$$

shares of the stock in state (S_t, t), which amounts to betting the fraction

$$\boxed{\hat{b}(S_t, t) = \frac{\Delta S}{C} = \frac{N(d_1)bV_t(b) + N(d_2)cV_t(c)}{N(d_1)V_t(b) + N(d_2)V_t(c)}}. \tag{43}$$

of wealth on the stock at t. Thus, the on-line fraction of wealth bet on the stock is a time- and state-varying convex combination of b and c.

Proof. First, we note the standard relations $\partial C/\partial U_1 = N(d_1)$ and $\partial C/\partial U_2 = N(d_2)$, which follow by direct calculation from (40), (41), and the fact that $U_1\phi(d_1) = U_2\phi(d_2)$, where $\phi(\bullet)$ is the standard normal density function. Differentiating the option price, we get

$$\frac{\partial C}{\partial S} = \frac{\partial C}{\partial U_1}\frac{\partial U_1}{\partial S} + \frac{\partial C}{\partial U_2}\frac{\partial U_2}{\partial S} = N(d_1)\frac{bU_1}{S} + N(d_2)\frac{cU_2}{S}, \tag{44}$$

which is the desired result. □

Thus, even if the best rebalancing rule (of any kind) in hindsight $b(S,t) = z(S,t)/(\sigma\sqrt{t})$ happens to lie between b and c, the replicating strategy will not generally bet the hindsight-optimized fraction $\arg\max_{b\in\mathbb{R}} V_t(b)$ of wealth on the stock. This phenomenon is illustrated in Figure 2. Instead, the relative weighting of b and c (which is initially 50/50 at time-0) evolves with the observed performances $V_t(b), V_t(c)$ and the remaining life $\tau := T - t$ of the option. For a fixed time t, if $U_1 \to \infty$ or $U_2 \to \infty$ then the on-line portfolio weight will converge to b or c accordingly. As $\tau \to 0$, d_1 tends to $\pm\infty$ and d_2 tends to $\mp\infty$ according as to whether b or c has outperformed over the known price history. Thus, small differences in the observed performances $V_t(b)$ and $V_t(c)$ get amplified in the on-line portfolio weight as $\tau \to 0$. Figures 4 and 5 simulate the performance of the replicating strategy for different parameter values over a $T := 30$ year horizon.

4 The General Discrete Set of Asset Allocations

We carry on with the general discrete set $\mathbb{B} := \{b_1, ..., b_n\} \subset \mathbb{R}$ of asset allocations, where the b_i are arranged in increasing order: $b_1 < b_2 < \cdots < b_n$. Thus, we now deal with the payoff $V_t^* := \max_{1\le i\le n} V_t(b_i)$. For notational convenience, we will also write $b_0 := -\infty$ and $b_{n+1} := +\infty$. We let $\Delta b_i := b_{i+1} - b_i$ for $0 \le i \le n$, and thus $\Delta b_0 = \Delta b_n = +\infty$. For a given rebalancing rule b_i $(1 \le i \le n)$, the final payoff of the option is equal to $V_T(b_i)$ if and only if

$$\frac{b_{i-1} + b_i}{2}\sigma\sqrt{T} \le z_T \le \frac{b_i + b_{i+1}}{2}\sigma\sqrt{T}. \tag{45}$$

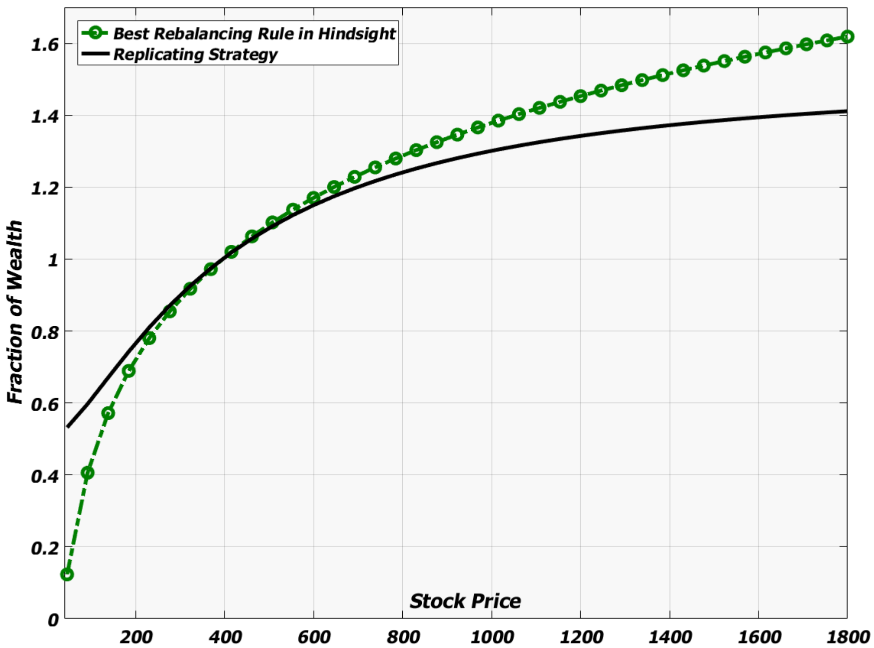

Figure 2: THE FRACTION OF WEALTH BET BY THE REPLICATING STRATEGY FOR DIFFERENT STOCK PRICES, $r := 0.03, T := 10, S_0 := 100, t := 5, \sigma := 0.7, b := 1.5, c := 0.5$.

Thus, conditional on the values of time-t variables, b_i will turn out to be the best performer over $[0, T]$ if and only if

$$\frac{(b_{i-1} + b_i)\sigma T/2 - \sqrt{t} \cdot z_t}{\sqrt{\tau}} \leq y \leq \frac{(b_i + b_{i+1})\sigma T/2 - \sqrt{t} \cdot z_t}{\sqrt{\tau}}. \tag{46}$$

For simplicity, we will write this interval as $y \in [A_{i-1}, A_i]$. Thus $A_0 = -\infty$ and $A_n = +\infty$. The expected present value of the final payoff with respect to $\mathbb{Q}$ and the information available at t is equal a sum of integrals $I_1 + \cdots + I_n$, where

$$I_i := \frac{\exp(rt - \sigma^2 b_i^2 T/2 + b_i \sigma \sqrt{t} \cdot z_t)}{\sqrt{2\pi}} \int_{A_{i-1}}^{A_i} \exp(-y^2/2 + b_i \sigma \sqrt{\tau} \cdot y) dy. \tag{47}$$

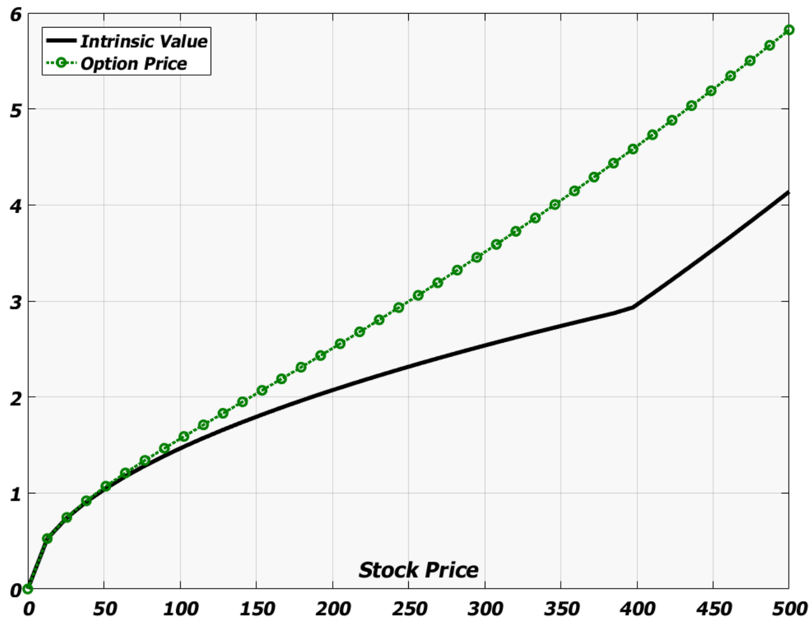

Figure 3: Option price and intrinsic value for different stock prices, $r := 0.03, T := 10, S_0 := 100, t := 5, \sigma := 0.7, b := 1.5, c := 0.5$.

Evaluating these integrals, we obtain the general pricing formula

$$\boxed{C(S,t) = \sum_{i=1}^{n} \{N(A_i - \beta_i) - N(A_{i-1} - \beta_i)\} V_t(b_i)}, \tag{48}$$

where

$$\boxed{A_i := \frac{(b_i + b_{i+1})\sigma T/2 - \sqrt{t} \cdot z_t}{\sqrt{\tau}}}. \tag{49}$$

and

$$\boxed{\beta_i := b_i \sigma \sqrt{\tau}}. \tag{50}$$

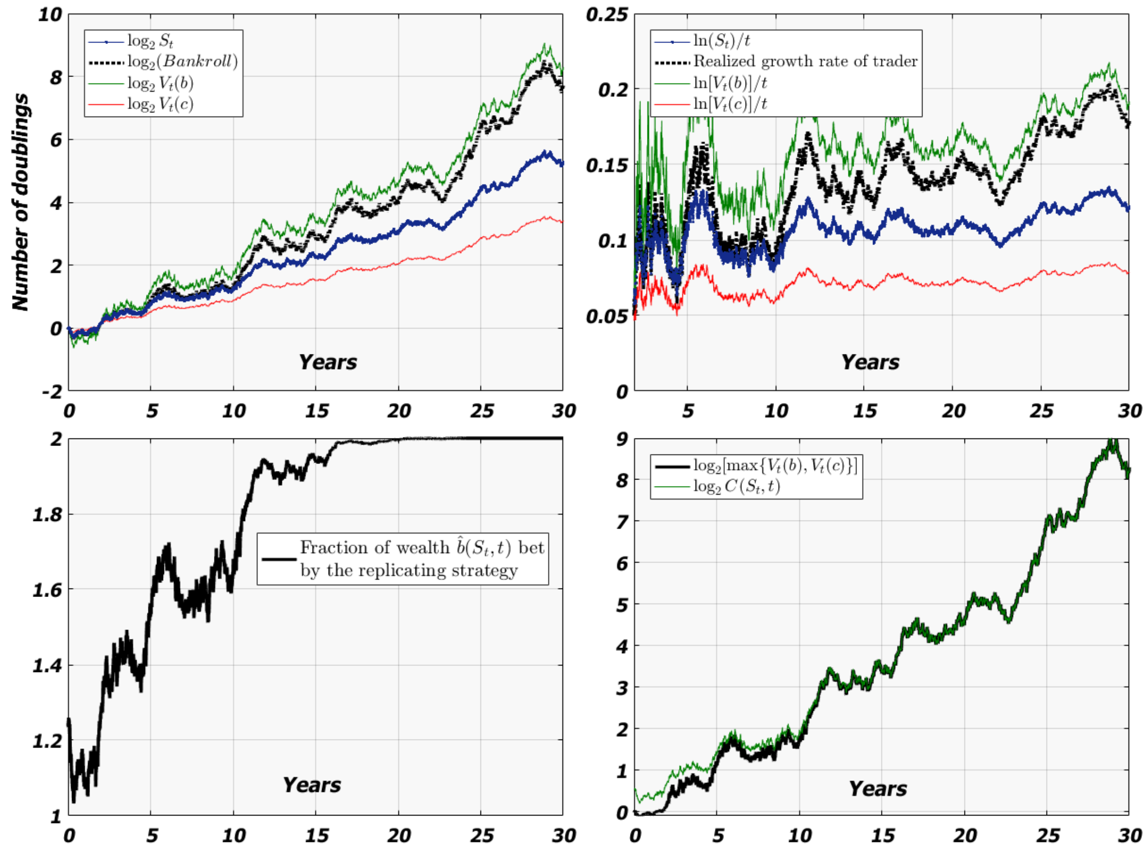

Figure 4: Performance simulation over $T := 30$ years for the parameters $S_0 := 1, b := 2, c := 0.5, r := 0.03, \sigma := 0.15, \nu := 0.1, \mu = \nu + \sigma^2/2$.

Bearing in mind that $A_0 = -\infty$ and $A_n = +\infty$, we can also write

$$C(S,t) = N(A_1 - \beta_1)V_t(b_1) + \sum_{i=2}^{n-1}\{N(A_i - \beta_i) - N(A_{i-1} - \beta_i)\}V_t(b_i) + N(\beta_n - A_{n-1})V_t(b_n). \tag{51}$$

The general option price could again have been obtained differently, by an interesting application of Margrabe's theory of exchange options. Indeed, we could consider the wealth processes $(V_t(b_i))_{i=1}^n$ as separate underlyings $U_i := V_t(b_i)$ of a multi-asset

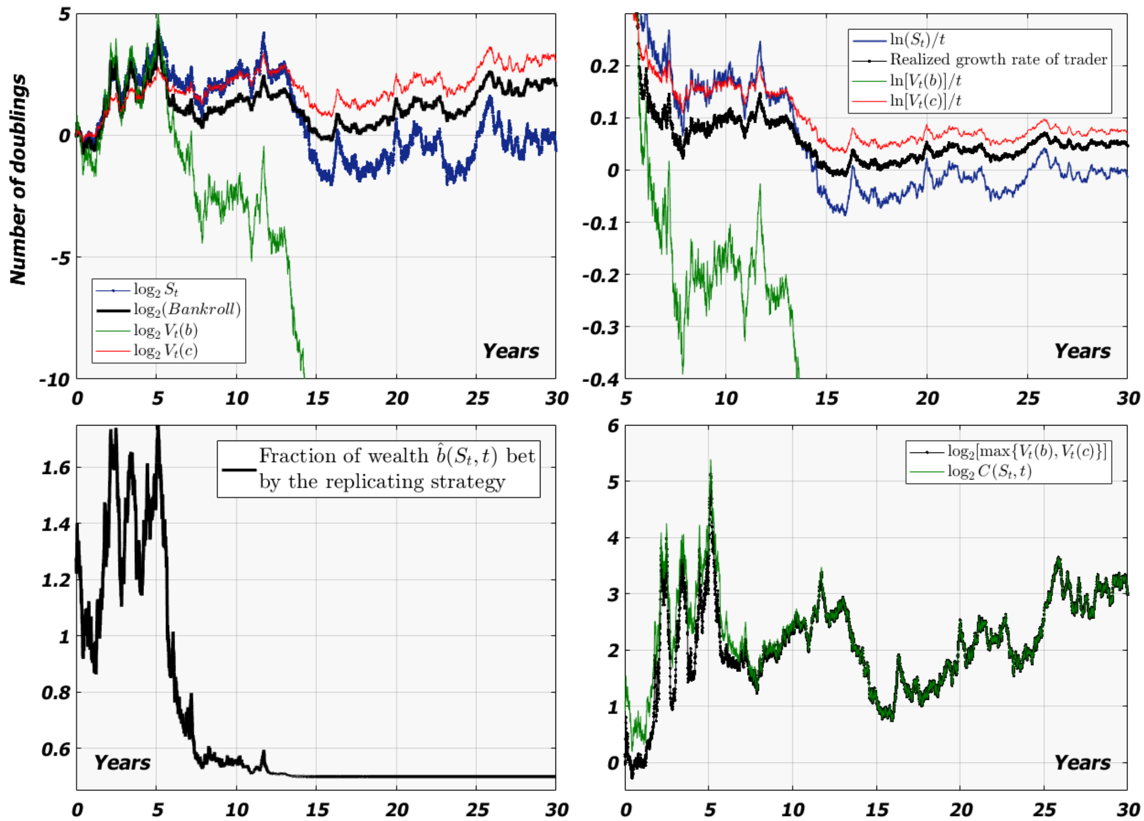

Figure 5: Performance simulation over $T := 30$ years for the parameters $S_0 := 1, b := 2, c := 0.5, r := 0.03, \sigma := 0.7, \nu := 0.07, \mu = \nu + \sigma^2/2$.

option whose final payoff is equal to $\max\{U_1, U_2, ..., U_n\}$. First of all, we remark that at any given time the ordered sequence of numbers $U_1(t), ..., U_n(t)$ is unimodal, or single-peaked. This happens because the $(\log U_i)_{i=1}^n$ trace out a sequence of heights on the parabola $b \mapsto \log V_t(b)$ as we move from left to right over the abscissae $b_1 < b_2 < \cdots < b_n$. The peak occurs for the index

$$i^* := \arg\min_{1 \leq i \leq n} |b_i - b(S_t, t)| = \arg\max_{1 \leq i \leq n} V_t(b_i). \tag{52}$$

Thus, U_i is increasing in i for $i < i^*$ and decreasing in i for $i \geq i^*$. This unimodality in hand, we now have the identity

$$\max\{U_1, ..., U_n\} = U_1 + (U_2 - U_1)^+ + (U_3 - U_2)^+ + \cdots + (U_n - U_{n-1})^+, \qquad (53)$$

where $x^+ := \max\{x, 0\}$ denotes the positive part of x. Hence, the payoff $\max\limits_{1 \leq i \leq n} U_i$ is equivalent to a portfolio consisting of one unit of U_1, plus an option to exchange U_1 for U_2, plus an option to exchange U_2 for U_3, $\cdots$, plus an option to exchange U_{n-1} for U_n. At expiration, the trader keeps exchanging U_i for U_{i+1} until the maximum U_i^* is reached. Applying the Margrabe Formula (cf. Zhang 1998) in conjunction with linear pricing, we find that the no-arbitrage price of this portfolio (consisting of a unit of U_1 plus $n - 1$ exchange options) is

$$\boxed{C(U_1, ..., U_n, t) = U_1 + \sum_{i=1}^{n-1} \{N(d_{1i})U_{i+1} - N(d_{2i})U_i\}}, \qquad (54)$$

where

$$\boxed{d_{1i} := \frac{\log(U_{i+1}/U_i)}{\sigma_{ai}\sqrt{\tau}} + \frac{\sigma_{ai}\sqrt{\tau}}{2}}, \qquad (55)$$

$d_{2i} := d_{1i} - \sigma_{ai}\sqrt{\tau}$, and $\sigma_{ai} := \Delta b_i \sigma$ is the aggregate volatility in a two-asset market consisting of U_i and U_{i+1}. Collecting terms, we get the linear combination

$$\boxed{C(U_1, ..., U_n, t) = N(-d_{21})U_1 + \sum_{i=2}^{n-1} [N(d_{1,i-1}) - N(d_{2i})]U_i + N(d_{1,n-1})U_n}, \qquad (56)$$

which agrees with equation (51) above. Figure 6 plots the option price and intrinsic value for different stock prices under the parameters $r := 0.03, T := 10, S_0 := 100, t := 5, \sigma := 0.7$, and $\mathbb{B} := \{0, 0.5, 1, 1.5, 2\}$. In general there will be several

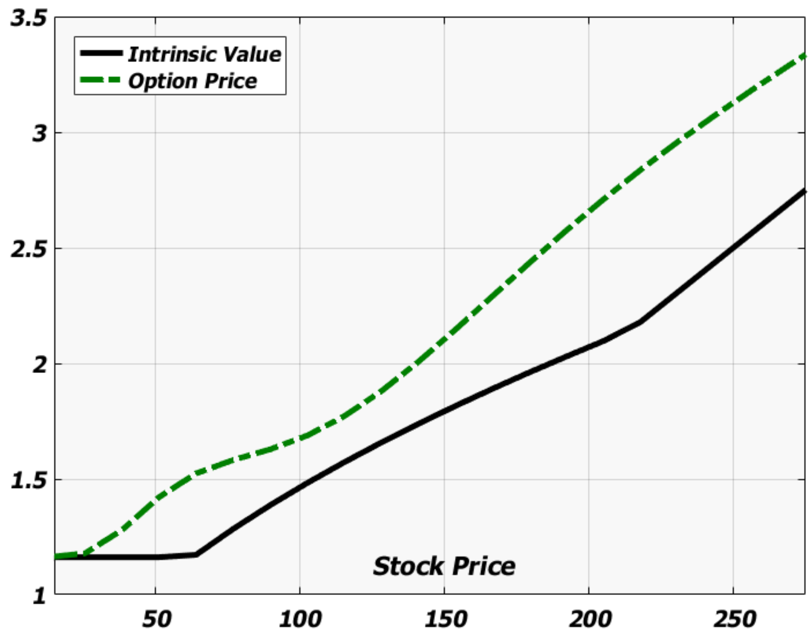

Figure 6: OPTION PRICE AND INTRINSIC VALUE FOR DIFFERENT STOCK PRICES, $r := 0.03, T := 10, S_0 := 100, t := 5, \sigma := 0.7, \mathbb{B} := \{0, 0.5, 1, 1.5, 2\}$.

implied volatilities σ that could rationalize an observed price of the rebalancing option. Figure 7 plots the option price for different volatilities under the parameters $r := 0.03, T := 10, S_0 := 100, t := 5, S_t := 200$, and $\mathbb{B} := \{0, 0.5, 1.5\}$.

In specializing the pricing formula for $t := 0$ and simplifying (remembering that $V_0(b_i) := 1$), we get

Theorem 6. *For hindsight optimization over n discrete rebalancing rules $b_1 < \cdots < b_n$, the time-0 cost of achieving the best b_i in hindsight is*

$$\boxed{C_0 = 2 - n + 2\sum_{i=1}^{n-1} N(\Delta b_i \sigma\sqrt{T}/2)}, \tag{57}$$

where $\Delta b_i := b_{i+1} - b_i$. If $\delta := \max\limits_{1\leq i\leq n-1} \Delta b_i$, then

$$\boxed{C_0 \leq 2 - n + 2(n-1)N(\delta\sigma\sqrt{T}/2)}. \tag{58}$$

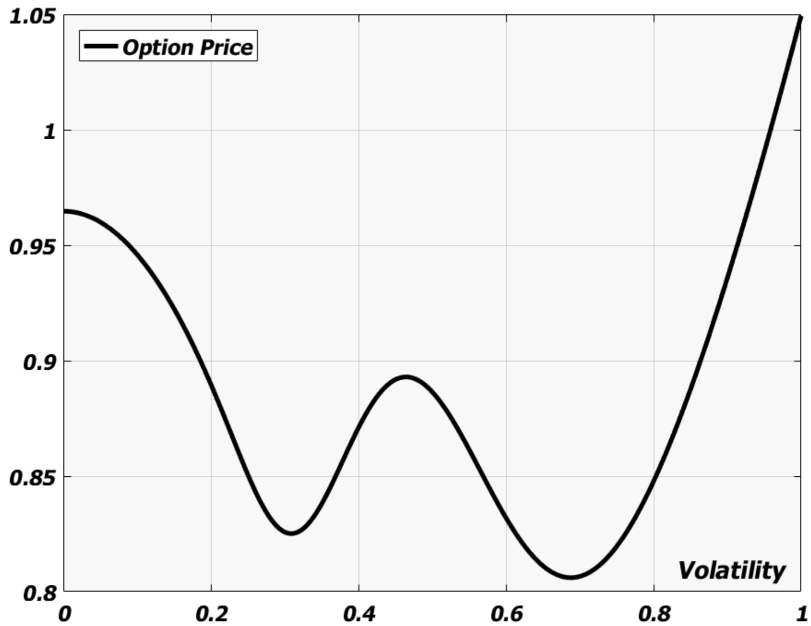

Figure 7: OPTION PRICES FOR DIFFERENT VOLATILITIES, $r := 0.03, T := 10, S_0 := 100, t := 5, S_t := 200, \mathbb{B} := \{0, 0.5, 1.5\}$.

Corollary 2. *A perpetual option on the best of any n distinct rebalancing rules in hindsight is worth $C_0 = n$ dollars at time-0.*

Thus, we see that the time-0 price of the general horizon-T rebalancing option is independent of the interest rate, and it is increasing and concave separately in the parameters $\Delta b_i, \sigma, T$. We again observe that horizontal translations of the point set $\{b_1, ..., b_n\}$ do not alter the option price. We always have the relation $1 \leq C_0 \leq n$; the maximum n dollar price obtains if any of the parameters tends to infinity and the minimum \$1 price obtains if any of the parameters tends to zero.

Example 3. *For a $T := 25$ year planning horizon, we cherry pick five favored asset allocations $\mathbb{B} := \{0, 0.5, 1, 1.5, 2\}$. Assuming stock market volatility of $\sigma := 0.15$ going forward, we get $C_0 = \$1.59$, and the excess growth rate of the hindsight-optimized asset allocation will be exactly $\log(C_0)/T = 1.87\%$. Assuming that the risk-free rate is greater than 1.87%, the replicating strategy is guaranteed not to lose money if the* *practitioner sticks to the Plan for the next $T = 25$ years.*

Theorem 7. *The horizon-T replicating strategy for the best of the rebalancing rules $b_1 < b_2 < \cdots < b_n$ in hindsight holds*

$$\boxed{\Delta = N(-d_{21})\frac{b_1 V_t(b_1)}{S_t} + \sum_{i=2}^{n-1}[N(d_{1,i-1}) - N(d_{2i})]\frac{b_i V_t(b_i)}{S_t} + N(d_{1,n-1})\frac{b_n V_t(b_n)}{S_t}} \tag{59}$$

shares of the stock in state (S_t, t), thereby betting the fraction $\hat{b}(S,t) = \Delta S/C$ of its bankroll on the stock. This amounts to a time- and state-varying convex combination of the b_i. As $\tau \to 0$, the option price converges to $U_{i^*} := \max\limits_{1 \le i \le n} U_i$ and the fraction of wealth bet by the replicating strategy converges to $\arg\max\limits_{b \in \mathbb{B}} V_t(b)$ if this set is a singleton; if $\arg\max\limits_{b \in \mathbb{B}} V_t(b) = U_{i^*} = U_{i^*+1}$ has two distinct points, then $\hat{b}$ converges to the midpoint $(b_{i^*} + b_{i^*+1})/2$ as $\tau \to 0$. The horizon-free replicating strategy (corresponding to the perpetual version of the option) bets the performance-weighted average

$$\hat{b}(S_t, t) = \frac{\sum\limits_{i=1}^{n} b_i V_t(b_i)}{\sum\limits_{i=1}^{n} V_t(b_i)} \tag{60}$$

of the rebalancing rules b_i, which converges almost surely to

$$\arg\max_{b \in \mathbb{B}} \left\{ (\mu - r)b - \frac{\sigma^2 b^2}{2} \right\} = \arg\min_{b \in \mathbb{B}} \left| b - \frac{\mu - r}{\sigma^2} \right|, \tag{61}$$

i.e. it converges to whichever element of $\mathbb{B}$ is closest to the continuous time Kelly rule (cf. Luenberger 1998).

Proof. Note that the pricing formula (56) is a linearly homogeneous function of the underlyings $(U_1, ..., U_n)$. By Euler's theorem for homogeneous functions, we therefore have the relation

$$C=\sum_{i=1}^{n}\frac{\partial C}{\partial U_i}U_i. \tag{62}$$

Accordingly, by direct calculation on (56) one can (carefully) verify the partial derivatives

$$\frac{\partial C}{\partial U_1}=N(-d_{21}), \tag{63}$$

$$\frac{\partial C}{\partial U_i}=N(d_{1,i-1})-N(d_{2i})\text{ for }2\leq i\leq n-1, \tag{64}$$

$$\frac{\partial C}{\partial U_n}=N(d_{1,n-1}). \tag{65}$$

To verify these partials easily, one needs the identity

$$\frac{\phi(d_{2i})}{\phi(d_{1i})}=\frac{U_{i+1}}{U_i}\text{ for }1\leq i\leq n-1, \tag{66}$$

where $\phi(\bullet)$ is the standard normal density function. Observe that U_i generally appears in the terms of (56) that correspond to the indices $i-1, i,$ and $i+1$. U_1 appears in the first two terms and U_n appears in the last two terms. This being done, the delta-hedging strategy now obtains from the chain rule

$$\frac{\partial C}{\partial S}=\sum_{i=1}^{n}\frac{\partial C}{\partial U_i}\frac{\partial U_i}{\partial S} \tag{67}$$

in conjunction with the fact that $\partial U_i/\partial S=b_iV_t(b_i)/S$. To get the horizon-free result, we just observe that $d_{1i}\rightarrow+\infty$ and $d_{2i}\rightarrow-\infty$ as $T\rightarrow\infty$. Finally, consider what happens when $\tau\rightarrow 0$. The numbers d_{1i}, d_{2i} will converge to the same limit $\pm\infty$ according as U_{i+1} is greater or less than U_i. In the event that $U_{i+1}=U_i$ then d_{1i} and d_{2i} both converge to zero. The numbers $(U_i)_{i=1}^{n}$ will typically have a unique mode U_{i^*}, e.g. $U_1<\cdots<U_{i^*-1}<U_{i^*}>U_{i^*+1}>\cdots>U_n$. In this case, all coefficients in the linear combination (56) converge to zero except the one corresponding to $i = i^*$, which converges to 1. If there are two modes $U_{i^*} = U_{i^*+1}$, then the corresponding coefficients in (56) both converge to $1/2$, and the result follows. □

Note that for $n > 2$, the initial weighting of the b_i at time-0 is not uniform, even if the b_i themselves are equally spaced. The endpoints b_1 and b_n have initial weights $N(\Delta b_1 \sigma\sqrt{T}/2)/C_0$ and $N(\Delta b_{n-1}\sigma\sqrt{T}/2)/C_0$, respectively, and the rest of the b_i have initial weights $[N(\Delta b_{i-1}\sigma\sqrt{T}/2) - N(-\Delta b_i \sigma\sqrt{T}/2)]/C_0$ for $2 \leq i \leq n-1$. If the b_i are equally spaced, then each of the intermediate points ($2 \leq i \leq n-1$) gets initial weight $[2N(\delta\sigma\sqrt{T}/2) - 1]/C_0$, but the endpoints b_1, b_n get the higher initial weight $N(\delta\sigma\sqrt{T}/2)/C_0$.

Theorem 8. *For any closed set $\mathbb{B} \subseteq \mathbb{R}$ of rebalancing rules (finite or infinite), the American-style version of Cover's rebalancing option (with exercise price K and payoff $\max\{\max_{b\in\mathbb{B}} V_t(b) - K, 0\}$) will never be exercised early in equilibrium, and its price $C_a(S_t, t)$ equals the price $C_e(S_t, t)$ of the corresponding European-style option.*

Proof. For simplicity, let $V_t^* := \max_{b\in\mathbb{B}} V_t(b)$ denote the hindsight-optimized wealth over $[0, t]$, and let $b_t^* := \arg\max_{b\in\mathbb{B}} V_t(b)$ denote the best (allowable) rebalancing rule in hindsight over $[0, t]$. Consider, from the standpoint of time t, the following two trading strategies:

Strategy 1 Invest $Ke^{-r\tau}$ dollars in the risk-free bond and buy 1 unit of Cover's (European-style) rebalancing option at a price of $C_e(S_t, t)$.

Strategy 2 Invest V_t^* dollars into the rebalancing rule b_t^*. That is, take the best rebalancing rule in hindsight over $[0, t]$, and adhere to that same (constant) continuously-rebalanced portfolio over $[t, T]$.

Observe that Strategy 1 has a final payoff of $\max\{V_T^*, K\}$ and Strategy 2 has a final payoff of $V_T(b_t^*)$. Since the payoff at T of Strategy 1 is guaranteed to be at least as great as that of Strategy 2, the initial investment of $Ke^{-r\tau} + C_e(S_t, t)$ dollars into Strategy 1 must be greater or equal to the investment V_t^* that is required for Strategy 2. Thus, we have the inequalities

$$C_a(S_t, t) \geq C_e(S_t, t) \geq V_t^* - Ke^{-r\tau} \geq V_t^* - K. \tag{68}$$

Hence, since the price of an American rebalancing option always exceeds the exercise value, the option "is worth more alive than dead" and will never be exercised in equilibrium. On account of the fact that early exercise rights are worthless anyhow, we must therefore have $C_a(S_t, t) = C_e(S_t, t)$. □

We remark that this is a general model-independent result that applies equally well to rebalancing rules $b \in \mathbb{B} \subseteq \mathbb{R}^d$ over arbitrary d-dimensional stock markets. The dominance argument only requires the market (and the set $\mathbb{B}$) to admit a well-defined hindsight-optimized rebalancing rule $b_t^* := \arg\max_{b \in \mathbb{B}} V_t(b)$. For $\mathbb{B} := \{1\}$ the best rebalancing rule in hindsight is just $b_t^* = 1$ and we get $V_t^* = S_t$; this recovers the proof given by Merton (1973, 1990) of the no-exercise theorem for vanilla call options. The special cases $\mathbb{B} := \mathbb{R}^d$ and $\mathbb{B} := [0, 1]$ were observed by Garivaltis (2018) for a continuous-time Black-Scholes market with $K := 0$.

5 Conclusion

This paper studied Cover's rebalancing option with discrete hindsight optimization. In the context of a single risk asset, a constant (perhaps levered) rebalancing rule is a simple trading strategy that continuously maintains some fixed fraction of wealth in the underlying asset. Cover's discrete-time universal portfolio theory derives robust on-line trading strategies that are guaranteed to achieve an acceptable percentage of the final wealth of the best rebalancing rule (of any kind) in hindsight.

Working in continuous time, we formulated the less aggressive benchmark of the best rebalancing rule in hindsight that hails from some finite set $\mathbb{B} := \{b_1, ..., b_n\}$. This approach allows the (delta-hedging) practitioner to cherry pick a small number of favored rebalancing rules that could embody institutional leverage constraints or the trader's own speculative beliefs as to the future pattern of returns in the stock market.

Accordingly, we priced and replicated the financial option whose final payoff is equal to the wealth $V_T^* := \max\limits_{1\leq i\leq n} V_T(b_i)$ that would have accrued to a \$1 deposit into the best b_i in hindsight. We found that a perpetual option (with zero exercise price) on the best of n distinct rebalancing rules costs n dollars at $t = 0$. The corresponding (horizon-free) replicating strategy amounts to depositing a dollar into each b_i and "letting it ride."

If the option expires at some fixed date T the price is lower; it is concavely increasing in T and in the volatility σ of the underlying risk asset. From the standpoint of $t = 0$, the cost C_0 of achieving the best of the b_i is translation invariant: it increases monotonically with the distances Δb_i between adjacent rebalancing rules, but it does not otherwise depend on their precise location. In this connection, the replicating strategy amounts to a time- and state-varying convex combination of the b_i that dynamically considers both the observed performances $V_t(b_i)$ and the remaining life $\tau := T - t$ of the option. No-arbitrage considerations dictate that American-style rebalancing options (for general exercise price K) will never be exercised early in equilibrium. This model-independent result holds for arbitrary closed sets $\mathbb{B} \subseteq \mathbb{R}^d$ of rebalancing rules over any $d-$dimensional stock market that admits a well-defined best rebalancing rule in hindsight. Toward the end of the investment horizon (as it becomes more and more obvious which b_i is likely to be the best in hindsight), even small differences in observed performance will cause the replicating strategy to dramatically over- or under-weight the various b_i.

Any practitioner of the horizon-T delta-hedging strategy is guaranteed to achieve at T the *deterministic* fraction $1/C_0$ of the final wealth of the best b_i in hindsight. The excess compound-growth rate at T of the best b_i (over and above the trader) is $\log(C_0)/T$, which tends to 0 as $T \to \infty$. The replicating strategy will asymptotically beat the underlying (i.e. an S&P 500 ETF) if any of the b_i turns out to achieve a compound-growth rate that is higher than $b = 1$. If there is no such $b_i \in \mathbb{B}$, but the trader had the good sense to put $1 \in \mathbb{B}$, then the trader's compound-annual growth rate will lag the underlying risk asset by at most $100 \log(C_0)/T$ percent at T. If we have $0 \in \mathbb{B}$, then the trader also guarantees that he will ultimately not lose money over $[0, T]$ if the condition $\log(C_0)/T < r$ is satisfied. Hence, our trading strategy is in a sense the most conservative attempt at detecting on-the-fly whether any of the rebalancing rules in some finite set is capable of beating the underlying over a given investment horizon.

We have therefore obtained a universal continuous-time asset allocation scheme that is computationally pleasant as well as feasible for the human life span. The on-line behavior is Markovian in the sense that the relevant state vector is just $(S_t, t, (V_t(b_i))_{i=1}^n)$. The algorithm requires no prior knowledge of the (hard-to-estimate) drift parameter μ of the stock market. Apart from the finite-dimensional state vector, the trader's behavior depends only on the known parameters r, σ, T, and $\mathbb{B}$. In just 25 years, say, our method guarantees to achieve within 1.87% of the compound-annual growth rate of whichever turns to be the most profitable asset allocation among $\mathbb{B} := \{0, 0.5, 1, 1.5, 2\}$.

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