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Conceptual metaphors for isomorphism and homomorphism: Instructors' descriptions for themselves and when teaching

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ABSTRACT

Group isomorphism and homomorphism are topics central to abstract algebra, yet research on instructors' views of these concepts is limited. Based on interviews from two instructors as well as classroom video from eight class periods, this paper examines the language used to discuss isomorphism and homomorphism. Language used by instructors in interviews and classroom settings are identified and classified into four main categories: formal definition, mapping, sameness, and combinations of sameness and mapping language. How the two instructors drew on language classified into those four categories in the interview and instruction settings are examined for isomorphism and homomorphism. Similarities and differences between the interview and instruction contexts reveal the wide variety of ways of understanding isomorphism and homomorphism as well as a research need to examine mathematicians' content knowledge in more than one context.

1. Introduction

Experts have identified isomorphism and homomorphism as two of the central topics of introductory Abstract Algebra. Although some research has been done on how students and mathematicians approach isomorphism, understandings of homomorphism are less known. Furthermore, existing research on mathematicians is limited. Insights into mathematicians' understandings of isomorphism and homomorphism can aid in identifying learning trajectories for students because they can help identify starting points and desired end conceptions.

This paper focuses on the language used by two instructors to discuss group isomorphism and homomorphism through the lens of conceptual metaphors. Specifically, three research questions are addressed: (1) how did instructors use metaphors to describe and teach isomorphism in interviews and instruction, (2) how did instructors use metaphors to describe and teach homomorphism in interviews and instruction, and (3) in what ways were instructors' descriptions in interviews and instruction similar and different?

2. Literature review

First, I review definitions of group isomorphism and homomorphism, as well as a key theorem that relates them, to ground the discussion of these concepts. Then I highlight prior work on students' conceptions of isomorphism and homomorphism and the framing of the Inquiry-Oriented Abstract Algebra (IOAA) curricular materials to provide a point of comparison for instructors' understandings.

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Finally, the limited existing literature on mathematicians' understandings of isomorphism and homomorphism is highlighted.

2.1. Definitions

Because both instructors taught groups first and isomorphism before homomorphism, they are defined in that order as well. A group isomorphism is defined as:

> Two groups $(G,\cdot)$ and $(H,\circ)$ are isomorphic if there exists a one-to-one and onto map $\phi: G \to H$ such that the group operation is preserved; that is, $\phi(a\cdot b) = \phi(a) \circ \phi(b)$ for all a, b in G. If G is isomorphic to H, we write $G \cong H$. The map ϕ is called an isomorphism (Judson, 2019, p. 119).

Thus, a group isomorphism can be viewed as a bijective function that shows two groups are "the same except for notation"; this framing of sameness has been called "naïve isomorphism" (Leron, Hazzan, & Zazkis, 1995, p. 154). The existence of an isomorphism between groups implies that the groups have the same cardinality. Furthermore, an isomorphism is an equivalence relation, and by the reflexive property a group is isomorphic to itself.

A more general, but related, relationship between groups can be found in group homomorphism: "A homomorphism between groups $(G,\cdot)$ and $(H,\circ)$ is a map $\phi: G \to H$ such that $\phi(g_1\cdot g_2) = \phi(g_1) \circ \phi(g_2)$ for all $g_1, g_2 \in G$" (Judson, 2019, p. 139). Thus, a homomorphism can be viewed as a function that preserves the structure of the original group (the domain) somewhere in the second group (codomain). It does not require the groups to have the same cardinality; group G may be larger or smaller than group H. However, the image of G (inside codomain H) must be the same size or smaller than G. There is always at least one homomorphism between groups: the trivial homomorphism, in which every element of G is mapped to the identity in H. Unlike isomorphism, there is not a standard "naïve" interpretation of homomorphism, though "collapsing" (e.g., Melhuish & Fagan, 2018) and "structure-preservation" (e.g., Hausberger, 2017; Judson, 2019) have been noted as ways of discussing homomorphism.

A theorem known by many names, including the Fundamental Homomorphism Theorem (FHT), links isomorphism and homomorphism via quotient groups: "If $\psi : G \to H$ is a group homomorphism with $K = \ker(\psi)$, then K is normal in G. Let $\phi: G \to G/K$ be the canonical [natural] homomorphism [$\phi: G \to G/K$ such that $\phi(g) = gK$]. Then there exists a unique isomorphism $\eta: G/K \to \psi$ (G) such that $\psi = \eta\phi$" (Judson, 2019, p. 141). Notice this theorem provides a way to interpret some homomorphisms as revealing an isomorphism between a substructure of the domain group and part of the codomain group. (Some versions of the FHT also require ψ to be onto (e.g., Pinter, 2010, p. 151), in which case the isomorphism is between a substructure of the domain group and the entire codomain group.)

2.2. Literature

Researchers have examined conceptions of isomorphism more than homomorphism and largely from students' perspectives. Early studies focused on students' approaches to determining if groups were isomorphic. Dubinsky, Dautermann, Leron, and Zazkis (1994) indicated that students focused on cardinality of groups when looking for isomorphisms, but not whether the homomorphism property was satisfied. Leron et al. (1995) also noted students' tendency to check cardinalities, but their students tested other properties too (e.g., being abelian, orders of elements). In contrast to the previous studies, which focused more on finding specific isomorphisms, Weber and Alcock (2004) and Weber (2002) asked undergraduate and doctoral students to prove theorems related to isomorphism and to prove or disprove specific groups were isomorphic. While both doctoral and undergraduate students were able to prove simple propositions, doctoral students had continued success with more sophisticated propositions. TheMelhuish (2018) replication study of Weber and Alcock (2004) and Weber (2002) suggested while doctoral students' success rested on their relational understanding of properties of isomorphism, undergraduate students may check properties without a corresponding relational understanding. That is, both undergraduates and graduate students make use of isomorphism properties, but undergraduate students use properties procedurally, whereas graduate students are more purposeful in choosing properties to verify and use. However, how properties of isomorphism are introduced and used in classes, which may support or constrain students' use of properties, has not been explored.

Some research has been conducted on homomorphism while studying other topics, like proof or isomorphism. Nardi (2000) noted students' struggles in proving the FHT stemmed from three major sources: an inability to recall definitions or a lack of understanding of definitions, poor conceptions of mapping, and not recognizing the purpose of sections of the proof. Much like the isomorphism context, Weber (2001) observed that despite undergraduates' ability to recall relevant theorems, they struggled to move past "definition unpacking" techniques when trying to prove theorems related to isomorphism and homomorphism, whereas doctoral students experienced success by invoking their holistic understanding of concepts. Larsen, Johnson, and Bartlo (2013) noted that the homomorphism property was more challenging for students to unpack than the bijection property when studying isomorphism. Additionally, Larsen (2013) noted, "students' use of the homomorphism property is usually largely or completely implicit" (p. 722), suggesting a need to focus students' attention on the homomorphism property. Nevertheless, these studies give limited insight into how students understand homomorphism.

Some research on isomorphism and homomorphism, especially by Larsen, focused on local instructional theories and inquiry-oriented curricula for developing a conception of groups, isomorphism, and quotient groups (Larsen, 2013; Larsen & Lockwood, 2013). Of special note, these materials focus students' attention on how an isomorphism indicates groups are "essentially the same" (e.g., Larsen, 2013, p. 721). To accomplish this goal, tasks focus on matching elements in Cayley tables and explicitly attending to the homomorphism property, meaning general sameness and matching ideas are likely to arise from working with these curricular materials. Although the homomorphism property is addressed, it is secondary in importance to addressing isomorphism and quotient groups. Additionally, general sameness and matching seem to arise naturally from the materials, but metaphors used while enacting this curriculum have not been explicitly studied.

Some recent work has examined homomorphism independently from isomorphism. Hausberger (2017) examined the role of the homomorphism concept in tying together ideas across abstract algebra through a textbook analysis and research on students. In this paper, he discussed the common, but vague, description of homomorphism as a "structure-preserving function" and noted that which structure is being referenced is not necessarily clear to students because the word 'structure' is used in multiple contexts. Furthermore, Hausberger discussed the rationale of the homomorphism concept as having three main aspects: a way to link two isomorphic objects (one isomorphic to a quotient of the other) based on the FHT; a general procedure that applies across structures (e.g., groups, rings); and the fact that the sets of interest are kernels of homomorphisms. Melhuish and Fagan (2018) as well as Melhuish, Lew, Hicks, & Kandasamy (2020) examined students' understandings of isomorphism and, especially, homomorphism, by examining properties, metaphors, examples, representations, and different views of functions in abstract algebra. Melhuish et al. (2020) considered these views of homomorphism as a way of viewing students' understanding of functions broadly whereas Melhuish and Fagan (2018) examined students' understanding of the concept of homomorphism more closely. Specifically, they noted students used metaphors like "collapsing" and "input/output" to understand what happened in a homomorphism and found "collapsing" to be powerful for students. Nevertheless, it is unclear whether instructors also use this language or whether this language was explicitly taught.

In contrast to the many student-focused studies, limited research has been conducted on mathematicians' views of isomorphism or homomorphism. Weber and Alcock (2004) noted algebraists' notions of isomorphism as meaning groups were "essentially the same" (p. 218) or that groups being isomorphic meant "one group was simply a re-labelling of the other group" (p. 218). Ioannou and Nardi (2010) observed the use of images to teach abstract algebra, including images to represent cosets and homomorphism, though instructors did not generally emphasize these images. However, specific views of homomorphism have not been explored in mathematicians. Furthermore, while researchers have highlighted some specific metaphors related to understanding isomorphism or homomorphism as a function, here I seek to highlight other language and ways of reasoning that are relevant to understanding isomorphism and, especially, homomorphism.

3. Conceptual framework

A theoretical lens for analyzing mappings is the conceptual metaphor construct (e.g., Lakoff & Johnson, 1980; Lakoff & Núñez, 1997). Lakoff and Johnson (1980) posit that people's conceptual systems are metaphorical and that the metaphorical language individuals use can be examined as evidence of the structure of their metaphorical system. These metaphors can be thought of as "cross-domain conceptual mappings" that "project the structure of a source domain onto a target domain" (Lakoff & Núñez, 1997, p. 32). Thus, conceptual metaphors utilize one's structured and developed knowledge of a source domain to inform one's view of a related target domain in order to develop one's thinking about the target domain. For example, conceptual metaphors have been used to examine students' views of learning and doing mathematics and include examples like "learning mathematics is a journey" (Olsen, Lew, & Weber, 2020).

Conceptual metaphors reveal the structure of thought, indicating they are a suitable lens for studying the abstract concepts of isomorphism and homomorphism. However, like Steen (2011), I do not contend that all metaphors uttered are used intentionally as metaphors. For example, Sfard (1997) provided the example of a rational number as a metaphor by combining the metaphors "fraction as partitioning", "fraction as piece", and "fraction as number" to construct the concept of rational number. The first two metaphors relate to concrete actions that can be taken to create specific images of fractions; the third, when combined with the first two, links fractions to broader discussions of what numbers are. Though "fraction as number" may not initially appear to be a metaphor or be viewed as a metaphor by the speaker, this language can be viewed as a metaphor that died or became invisible when it became the standard way to understand fractions and colloquially began to be viewed as a fact instead of a metaphor.

An example from this study of this understanding of metaphor is exemplified by the following: "An isomorphism/A homomorphism is a function." While mathematicians might be inclined to view this metaphor more as a statement of fact, an individual's understanding of function (the source domain) would provide structure for reasoning about isomorphism or homomorphism in much the way understanding a fraction as a number would provide a structure for reasoning. Similarly, consider the formal definition of homomorphism above, which included the string of symbols "$\phi(g_1 \cdot g_2) = \phi(g_1) \circ \phi(g_2)$ for all $g_1, g_2 \in G$" (Judson, 2019, p. 139). Though this may not seem like a metaphor for homomorphism, it still satisfies the notion of a "cross-domain conceptual mapping" because individuals doing abstract algebra have had extensive exposure to algebraic notation and have ways of interpreting and reasoning about that notation which can be used for reasoning about the specific concept of homomorphism.

Based on this view of metaphor, I consider all language to be metaphorical. This does not mean that all mathematicians intentionally use metaphors or use the same metaphors, because they may not all have the exact same conceptual structure for a given concept. However, even the most dedicated formalists use metaphors to structure a definition or describe an abstract mathematical concept.

Because conceptual understanding can be understood through metaphors both inside and outside the context of mathematics, one can look at an individual's ways of reasoning as a sensible system through metaphors. When instructing, professors make decisions about the material to emphasize, though they may or may not intentionally invoke specific metaphors. Through instruction, students are exposed to different metaphors as they learn different ways to structure their understanding of new concepts. Thus, this perspective is well-suited to the intentions of this study. Additionally, conceptual metaphors have previously been used to examine students' reasoning about functions in algebra (Rupnow, 2017; Zandieh, Ellis, & Rasmussen, 2016) and isomorphisms and homomorphisms are specific types of functions. Specifically, Zandieh et al. (2016) examined students' notions of function in linear algebra and Rupnow

(2017) examined students' metaphorical expressions in the context of isomorphism. Some metaphors noted in these studies and relevant to this study include a function is a "journey" or a "machine" (Rupnow, 2017; Zandieh et al., 2016) and the notion of an isomorphism as a "matching" (Rupnow, 2017).

4. Methods

Data for this paper are largely drawn from classroom video and interviews of two instructors from a land-grant university in the Mid-Atlantic United States. These faculty members were teaching an introductory (junior level) abstract algebra class. Classroom data were collected when isomorphism or homomorphism was discussed in class. For Instructor Alex's course, this included portions of four 75-minute class periods, and for Instructor Bailey, this included portions of four 50-minute class periods. (Both names are pseudonyms.) I only completely transcribed class segments focused on isomorphism or homomorphism.

Instructor Alex used the IOAA materials in class, which includes three units on groups, isomorphism, and homomorphism (Larsen, Johnson, Weber et al., 2013). In Instructor Alex's class, students worked in groups to address prompts meant to help them in making mathematical discoveries related to specific topics. One goal of this curriculum is for students' informal notions of concepts to be built upon in order to reinvent formal mathematics. In the isomorphism materials, students' work with the symmetries of an equilateral triangle in the first (group) unit is revisited to remind students that when they chose how to represent the symmetries, they could have represented the symmetries differently. Based on this, ways to link the different representations are motivated, and this linking is done through activities in which students explicitly test different mappings and look for matching elements in Cayley tables (i.e., a table with elements of D_6 and a "mystery table"). From there, students examine alternative maps that would and would not be consistent ways to model the symmetries of an equilateral triangle, which leads to the development of the homomorphism property. Finally, recognition that a mapping (function) has been created and, furthermore, that the mapping is bijective, is used to formalize a definition for isomorphism. A number of tasks in the isomorphism unit engage students in forming an explicit homomorphism in order to help students formulate the homomorphism property and, later, write a complete definition for isomorphism. However, homomorphisms are not defined as an independent object until the quotient group unit, and the isomorphism theorems are examined at the end of the quotient group unit.

Instructor Bailey used a mixture of lecture and activity days. Lecture days largely involved the instructor presenting information at the board, with occasional questions asked of the class as a whole. On activity days, students worked through task sheets individually and/or in groups at their tables. Activity days were often used at the beginning or end of a unit. When used at the beginning, the intent was to introduce material before the formal lectures in subsequent days; at the end of a unit, the intent was to solidify material that had been presented on previous lecture days. An activity day started the isomorphism unit and included tasks similar to the "mystery table" exercise in the IOAA curriculum, but with two Cayley tables representing groups of order three and then four tables representing groups of order four (where students were expected to determine which pairs of tables were equivalent). Finally, students were asked to fill in blanks in the definition of isomorphism (where "bijective" and half of the homomorphism property were missing). Homomorphism was defined (in lecture) as its own object two class periods later. Homomorphisms were revisited at the end of the quotient group unit when the isomorphism theorems were introduced.

For context, both instructors had taught the course at least once before; neither does research in algebra. Instructors were recruited at the beginning of the semester from that semester's abstract algebra teachers. Participants engaged in semi-structured interviews (Fylan, 2005) lasting roughly one hour each. The relevant interview with each instructor occurred as they began teaching isomorphism and focused on definitions and descriptions of isomorphism and homomorphism, as well as their explanations when teaching. Interviews were audio and video recorded and any written work was collected. The interview questions are included in the Appendix.

The interviews and videos were transcribed and coded in alignment with the phases of thematic analysis (Braun & Clarke, 2006). Thematic analysis is a method by which researchers generate themes (patterns) from data based on repeated, systematic analysis. The researcher is acknowledged as bringing prior knowledge and experiences to the analysis as opposed to being a "blank slate." This methodology was chosen because the goal was to examine patterns of language usage to determine conceptual metaphors, and this methodology permitted prior research to be considered when conducting the analysis.

The analysis included multiple iterations of coding (Anfara, Brown, & Mangione, 2002). First, transcripts were open-coded for vivid, active words that could indicate conceptual metaphors using simultaneous, structural, and in vivo coding in keeping with (Saldaña, 2016). Next, statements were viewed holistically for mathematical approaches being conveyed by statements, as opposed to being cued only by specific words. (In this stage, statements related to the formal definition were added to consideration.) Finally, codes were generated and refined by repeating the previous stages. These codes were influenced by Zandieh et al.'s (2016), Hausberger's (2017), and Rupnow's (2017) work. Zandieh et al. (2016) and Rupnow (2017) had both used conceptual metaphors previously that focused on mappings, and Hausberger (2017) highlighted a potentially ambiguous term for homomorphism. Thus, these works seemed relevant to build upon. As a result of this examination, thirteen metaphors for isomorphism and fourteen metaphors for homomorphism were extracted. These metaphors are defined in Table 1.

In the coding scheme, phrases were not allowed to be double-coded, though a paragraph of text could receive multiple non-overlapping codes, potentially of different lengths. Consecutive statements with the same code would be coded together as one code as long as they were not interrupted. For example, consider the following section of Alex's interview. (Coded sections are enclosed in brackets with the code name listed after in italics.)

Table 1
Metaphors with Defining Examples.

Metaphor Category	Metaphor Code	Metaphor Definition	Metaphor Example
	Generic Sameness	Generic references to groups being the same or similar, whether at the whole group level or as general statements about relationships between elements	"…the heart of the matter is that they [isomorphic groups] are actually the same."
Sameness	Same Properties	Use of properties that are the same for all isomorphic groups or properties that hold in homomorphisms (e.g., cardinality, order of elements, being abelian)	"If I'm saying 2 groups are the same, they should have the same number of elements. That's a pretty low criteria for being the same."
	Disembedding	Structure-focused language to highlight (sub)structures shared by the domain and codomain groups that are highlighted for special inspection by the existence of an isomorphism or homomorphism	"The way to think about this then is if you've got a surjective homomorphism, then the range H essentially is already living inside of G somehow. All the information about H is already here, and in fact we can recover H purely in terms of G by taking the factor group of G mod the kernel."
	Renaming (Alex)/ Relabeling (Bailey)	Giving new names/labels to elements to show equivalence between groups while emphasizing the arbitrariness of these new names given by the mapping	"If you just took these elements and attached these other labels instead of the labels you originally had and you get the same exact structure [then you have an isomorphism]."
Sameness/ Mapping	Matching	Connecting specific elements in two groups or lining up elements in order to create a specific correspondence that reveals sameness of the paired elements	"What if I try doing 0 as B, 1 is A and 2 is C? Then if I shuffle the rows maybe I could get this to line up."
	Equivalence Classes (Homomorphism Only)	Leveraging knowledge of the structure of groups to find similar elements in the domain that can be mapped to the same place in the range, including via collapsing or condensing, which reveals sameness of elements via mapping	"Equivalence classes, like the idea that I could pick one representative for a set…So either like, apply the same name to a group of things that are equivalent or collapsing a set into a single element."
	Generic Mapping	Generic reference to an isomorphism or homomorphism as a function or mapping without further details about the mapping or explicit reliance on properties of functions	"We want something that will map to h_1^{-1}. And so what's gonna map to h_1^{-1}?"
Mapping	Function	Specific use of a function property to draw conclusions about isomorphisms or homomorphisms, such as everywhere defined and well-defined, even if there was not a direct statement that this was a property of functions	"So we're assuming we plug in V, we're gonna get a distinct answer so it's well-defined."
	Journey	Traveling from a starting point to an ending point, which could include a path to travel or manner of traveling	"I have my function that goes over to my range, and now this set is sent to a single element over here."
	Machine	Connections to how a machine works (e.g., takes inputs and produces outputs) or to a machine's programming (e.g., following a rule)	"So working with that same homomorphism, what's the image? No matter what little g I plug in, these are the only 2 things that're going to pop out."
	Literal Formal Definition	Use of the string of symbols in the formal definition for isomorphism or homomorphism or use of words related to bijective, onto, or one-to-one (or a mapping lacking those properties) to talk about isomorphism	"Let H be a group with respect to $*_G$ and let H be a group with respect to $*_H$. A mapping θ: $G \to H$ is a group homomorphism if $\theta(a *_G b) = \theta(a) *_H \theta(b)$ for all a,b ∈ G."
	Structure-preserving	Use of "structure-preserving" or a slight variation without interpretation	"I'd define it [isomorphism] as a mapping between two algebraic structures that preserves the structure."
Formal Definition	Operation-preserving	Use of "operation-preserving" or a slight variation without interpretation or use of a specific operation while talking about preserving (e.g., preserving addition)	"…that's the operation preservation part of it. So the operation in one group and the operation in another group has to be preserved."
	Special Homomorphism (Isomorphism Only)	Use of aspects of homomorphism to talk about isomorphism	"Isomorphisms are special homomorphisms."
	Isomorphism without Bijectivity (Homomorphism Only)	Use of aspects of isomorphism to talk about homomorphism	"A homomorphism is like an isomorphism but we lose bijectivity, right. So every isomorphism is a homomorphism but not vice versa."

Alex: [Yeah, so I would verify that two things are the same by finding a renaming function, an isomorphism between the two groups. So I kind of feel like that is the test] *renaming/ relabeling* [but the heart of the matter is that they are actually the same.] *generic sameness*

I: Okay, and are there any other words or phrases that come to mind?

Alex: Let's see: [same,] *generic sameness* [renaming function,] *renaming/relabeling* [operation-preservation] *operation-preservation*. Those words come to mind.

Notice coded sections were not necessarily the same length but each section of the instructor's speech received at most one code.

5. Results

The results are presented in two main sections. In the first section, instructors' use of metaphors for isomorphism is examined to address research questions one and three. In the second section, instructors' use of metaphors for homomorphism is examined, to address research questions two and three. Metaphor code names are capitalized for emphasis throughout the results, but metaphor categories are not capitalized.

Frequencies of codes in the interviews are provided because both instructors were asked the same questions in the interviews. Presence or absence of metaphors in class are noted instead of frequencies because the time spent on isomorphism and homomorphism was not the same in the two classes and frequencies of metaphors in interviews and class are not meant to be compared. Furthermore, because the classroom videos were selectively transcribed and discussions at tables were not always audible, presenting frequencies for the classroom use of metaphors could be misleading.

5.1. Isomorphism

5.1.1. Instructor Alex

This section examines Instructor Alex's use of language in the interview and in class separately, before summarizing the degree of alignment in language use in the two contexts. A similar analysis of Instructor Bailey's use of language follows. Many examples of sameness language were invoked in both interview and classroom settings, though more variety appeared in class. Despite the greater variety in class, the conceptual emphasis in both contexts related to the sameness or sameness/mapping categories.

Interview setting. In the interview setting, Instructor Alex often invoked Generic Sameness. Their initial description of isomorphism was: "When I think about groups, if they're isomorphic, it means that they are the same group just notated with different names or notated with a different operation, but that the groups are essentially the same." They went on to link this Generic Sameness to the specific idea of finding a Renaming function to demonstrate sameness: "I would verify that two things are the same by finding a renaming function, an isomorphism between the two groups. So I kind of feel like that is the test…."

When asked if they thought about isomorphism the same way they described it to students, they related their thinking about the Literal Formal Definition to Renaming again:

> I want them to get to the formal definition, but even then, I want them to understand the formal definition as like a renaming function. I feel like that was not at all obvious to me as a student. And so it was really hard to unpack…why this…seemingly arbitrary function would prove that two things were the same.

In addition to standard descriptions given to describe isomorphism, Instructor Alex related isomorphism to Renaming, specifically, naming stuffed animals when asked how they would explain isomorphism to a child: "I might say…each of their stuffed animals has a name, but it would be the same bear even if I called it a different name….It's the same bear if I call them Fred or Sam." Comparing this intentional metaphor to their standard language for isomorphism, we see that the choice highlighted sameness once again. Specifically, they highlighted the arbitrariness of the name of the bear in keeping with the Renaming metaphor.

Instruction setting. In teaching, Instructor Alex used sameness category language as a lens for approaching isomorphism, both in defining the concept and in describing how to approach verifying if groups were (or, especially, were not) isomorphic. Mapping

Table 2
Codes Used for Isomorphism in Instructor Alex Contexts.

Metaphor Class	Metaphor Code	Class Context	Frequency in Interview
Sameness	Generic Sameness	Approaching examples	10
	Same Properties	Showing example not isomorphic	
	Disembedding	None	
Sameness/Mapping	Renaming/ Relabeling	Directly before formal definition introduced	7
	Matching	Reasoning about groups presented in Cayley tables	
Mapping	Generic Mapping	Ubiquitous—specific isomorphisms	1
	Function	Class period on well-defined and everywhere-defined	
	Journey	Ubiquitous—specific isomorphisms	
	Machine	Class period on well-defined and everywhere-defined	
Formal Definition	Literal Formal Definition	Formalizing activities and proofs	2
	Operation-preserving	Specific isomorphisms	3
	Structure-preserving	Directly after one student used it	
	Special Homomorphism	After homomorphism defined	

category language was ubiquitous but was only central to discussions about the nature of isomorphisms being functions. Sameness/mapping category language was used when initially defining and throughout tasks but was used less after the formal definition was given. Formal definition category language was largely used in proof contexts.

Generic Sameness was used in general approach contexts, such as when defining isomorphism and when thinking about how to approach whether or not groups were isomorphic (e.g., "Always start these with like, are they the same?"). Same Properties language was often used when verifying groups were not isomorphic (e.g., different orders or cyclic versus non-cyclic groups).

Renaming language was only used when first presenting the formal definition (Literal Formal Definition), as a colloquial way to understand the idea. However, before the formal definition was given, Matching was used often to reason about whether groups in Cayley tables were the "same" in some way:

> If I look at the same equation, those line up…and those should line up, but when I look at my equation, those don't line up. On the mystery table, this equals D. So it's like I can check the equation on the mystery table and D_6 and see if I get the same answers.

Note this Matching language placed value on the specific results of computations, whereas the Renaming language used names but focused more on underlying properties.

Generic Mapping language was commonly used when creating a formulaic or discrete mapping representation for an isomorphism. The class spent extensive time talking about the well-defined and everywhere-defined properties of functions (Function). Specifically, almost an entire class period was spent discussing differences between well-defined, everywhere-defined, one-to-one, and onto with set diagrams in order to clarify what was needed for a function before discussing these terms in the context of isomorphism. Journey language often referred to where elements or whole groups were "going to" or being "sent to", as well as which elements of the range were being "hit." Machine language referred to what was being input in functions, especially when focusing on understanding what well-defined and everywhere-defined mean.

The Literal Formal Definition was used when discussing why isomorphisms should require being one-to-one and onto as well as the form the homomorphism property should take but was mostly used with proofs. Operation-preserving language was used a few times to summarize what happened in the homomorphism property. Structure-preserving language was introduced by a student in class to note a "same structure" being shared, at which point the instructor noted "structural differences" would indicate groups were not isomorphic. Otherwise this metaphor was not observed. Isomorphisms were referred to as special homomorphisms twice after the definition of homomorphism (as its own entity) was given.

Summary. There was fairly clear alignment at the conceptual level in addressing isomorphism, as is also illustrated in Table 2. The instructor focused on Generic Sameness and the sameness/mapping metaphor of Renaming in the interview while describing the core of what an isomorphism is. In class, they again used sameness and sameness/mapping category metaphors to build the idea of what an isomorphism is, though the sameness/mapping metaphor that came across more clearly in class through the tasks was Matching, not Renaming.

The greater variety of mapping category metaphors in class than in the interview can largely be explained by differences in the interview questions posed and instructional goals. In the interview, Instructor Alex was asked to define and describe isomorphism, not to find specific isomorphisms between groups or prove theorems. There was more time spent on the big picture of what isomorphism is about instead of working with specific isomorphisms (functions) that fit required criteria, like was done in examples and proofs in class.

The formal definition category was present in both the interview and the class. However, it was not a focal point in either context. While time was spent in class developing the informal ideas around sameness into the formal definition, the way students were encouraged to think about isomorphism was still rooted in sameness. In the interview, the definition was also mentioned in passing, but more time was spent thinking about what that meant, largely in terms of sameness.

5.1.2. Instructor Bailey

Like Instructor Alex, many examples of sameness language were used in interview and classroom settings, but more variety appeared in class. Despite the greater variety in class, the conceptual emphasis in both contexts related to the sameness or sameness/

Table 3
Codes Used for Isomorphism in Instructor Bailey Contexts.

Metaphor Class	Metaphor Code	Class Context	Frequency in Interview
Sameness	Generic Sameness	Ubiquitous	6
	Same Properties	Activity day student prompts for consequences of isomorphism	
	Disembedding	None	1
Sameness/Mapping	Renaming/ Relabeling	Shorthand for isomorphism and activity day student prompts	6
	Matching	Activity day student prompts for reasoning about groups in Cayley tables	1
Mapping	Generic Mapping	Ubiquitous—specific isomorphisms	3
	Function	Once, describing well-defined	
	Journey	Ubiquitous—specific isomorphisms	
	Machine	None	
Formal Definition	Literal Formal Definition	Verifying given map is an isomorphism and proofs	1
	Operation-preserving	Ubiquitous—proofs and specific isomorphisms	1
	Structure-preserving	Once, before formal definition introduced	2
	Special Homomorphism	Once, as theorem	

mapping categories.

Interview setting. In the interview, Instructor Bailey mostly used isomorphism language from the formal definition, sameness, and sameness/mapping categories. When first asked about the words or phrases that came to mind when describing isomorphism, they provided three images: "structure-preserving map, equivalence of structures, relabeling of elements." (These responses indicate Structure-preserving, Generic Sameness, and Relabeling metaphors, respectively.) When defining isomorphism, Instructor Bailey focused on Structure-preservation without further elaboration: "I'd define [isomorphism] as a mapping between two algebraic structures that preserves the structure." Matching was incorporated when considering how to describe isomorphism to a child: "It's a correspondence that matches like things with like things." When pressed for a preferred way of thinking about isomorphism, they wove together Relabeling and Generic Sameness ideas with the Literal Formal Definition and Operation-Preserving:

> I really prefer to think of it as a relabeling so that…from an algebraic point of view, there's really no difference between these structures, and so…if you just took these elements and attached these other labels instead of the labels you originally had…you get the same exact structure. So that's the idea I try to get across more than…that you have a…bijective function that…preserves such and such operation.

They incorporated Disembedding as well as Generic Mapping and Generic Sameness language when expanding on what Relabeling meant to them:

> The isomorphism itself can just sort of disappear to the background and you can…really just identify…these structures….And you could start talking about *the* cyclic group with n elements….And you don't need to know…what cyclic group, you don't need to…know that there could possibly be two different cyclic groups hanging around, and…a function mapping elements to another. You can just say, 'Well, if…I had a different instance of a cyclic group with n elements, if I wanted to, I could just change those labels to these labels,' and so really it's the same underlying structure.

Of note, their concept of isomorphism was about a shared structure, allowing one to see the groups were *isomorphic*, as opposed to a focus on the function that connected them (*isomorphism*).

Instruction setting. In class, Generic Sameness was used to refer to groups being "essentially the same" on a number of occasions, especially when the definition was initially given. Same Properties occurred most during the activity day when students were being prompted to look for properties that should hold in both groups if they were isomorphic. Relabeling was used numerous times to refer to the isomorphism showing groups were isomorphic; on the activity day, it was used to prompt approaches for students when working with discrete mappings and was explicitly printed on the activity worksheet. Matching was used a few times as students were prompted to look at which element was "matched up" with each element when creating a discrete mapping.

Generic Mapping was used when looking for formulaic representations of isomorphisms or when seeing what to "map to" specific elements. Being well-defined, a property of functions, was referenced once in passing when considering a specific mapping but was not a major focus in class for isomorphism (Function). Journey metaphors that referenced going from one group to another or elements being "sent" were used in multiple periods.

The Literal Formal Definition was used to verify maps were one-to-one, onto, and had the homomorphism property. Operation-preserving was used to help students think about their goals in the activities or to summarize what they were trying to accomplish in an exercise as a class (e.g., we need to "show the bijection respects the group operation" before verifying the homomorphism property). Structure-preserving language was noted once directly before introducing the formal definition as a "relabeling that preserves the structure." After the definition of homomorphism was given, one example of homomorphism that was given was "any isomorphism," indicating an isomorphism is a Special Homomorphism.

Summary. For Instructor Bailey, there was fairly clear alignment between contexts at the conceptual level, as is highlighted in Table 3. Instructor Bailey focused on Generic Sameness and the sameness/mapping metaphor of Relabeling in the interview when

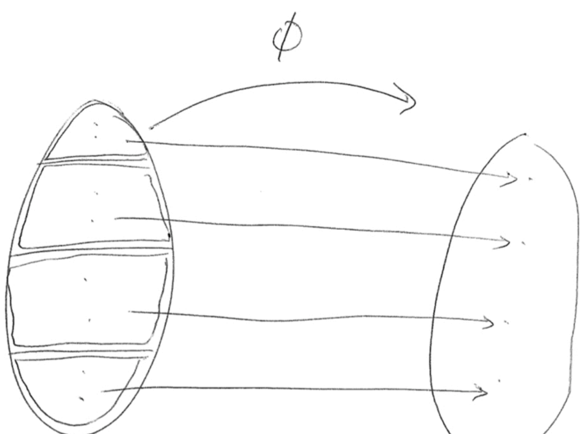

Fig. 1. Easter egg diagram drawn by Instructor Alex.

describing the core of what an isomorphism is. In class, they again used sameness and sameness/mapping category metaphors to build the idea of what an isomorphism is. This was especially clear on the activity day when talking about Relabeling as a way to approach the problem with each of the groups. Same Properties language was used to identify aspects that were or were not the same when addressing specific groups, which had not been a prompted activity in the interviews. In both interview and class contexts, Instructor Bailey used Relabeling language more prominently than Matching language.

The greater variety of mapping category metaphors in class than in the interviews is likely because of the difference in the types of questions posed and instructional goals. Much like Instructor Alex, the difference between metaphors in the interview and in class can be explained by overarching conceptual questions being asked in the interview versus the variety of activities in class (e.g., provide examples, prove theorems, provoke intuition, define relevant terms).

The formal definition was present in both the interview and the class. However, it was not a focal point in either context. While time was spent in class developing the informal ideas around sameness into the Literal Formal Definition, the way students were encouraged to think about isomorphism was still rooted in sameness. In the interview, the definition was also mentioned in passing, but more time was spent discussing what the definition meant, largely in terms of sameness. Even the Structure-preservation language that was used in the interview seemed to be related to views of sameness which were elaborated upon in the interviews and in class.

5.2. Homomorphism

5.2.1. Instructor Alex

Formal definition, mapping, sameness, and sameness/mapping category language were all represented in both interview and classroom settings for Instructor Alex. However, the conceptual emphasis in the interview was on sameness and sameness/mapping language, whereas greater emphasis was placed on mapping and formal definition categories in class.

Interview setting. In the interview, Instructor Alex frequently spoke in terms of the sameness/mapping metaphor of Equivalence Classes, specifically with a goal of addressing which elements were behaving in a similar way. For example, when initially describing homomorphism, they focused on Equivalence Classes as a way to identify which elements from the domain group were the same:

> I think equivalence classes; like the idea that I could pick one representative for a set, and the homomorphism…gives me a way to think about what's equivalent.…Apply the same name to a group of things that are equivalent or collapsing a set into a single element…

For homomorphism, much like isomorphism, they viewed a type of sameness as central to their understanding. However, in this case the sameness focused within groups, to see which elements acted in the same way, instead of globally identifying the two groups as being the same. When asked if their view of sameness in isomorphism and homomorphism was the same, they clarified:

> With isomorphism I mean this collection and the operation is the same as this collection and operation, and the only thing different about them is the names that I chose for the elements in operation… In homomorphism… I'm saying what things in my domain are the same under that mapping. So it's kind of like I take my domain, call everything the same, and then that collection of same things maps to a single element in the range.

They went on to draw a picture of an "Easter egg" in which the bands of the domain "egg" were connected to a single element in the range (see Fig. 1). They verbally noted that there might be some other elements of the range that "were not hit" (Journey metaphor) but that the matching set and element were then defined to be the same (Matching and Generic Sameness, respectively).

When asked to define a homomorphism, Instructor Alex's answer changed to "a mapping that preserves operation," (Operation-preserving) though they also noted they did not think the formal definition clearly "capture[d] this idea of collapsing or sameness or reduction. I feel like it's all kind of lost in operation-preservation, so I don't feel like that's a great, meaningful definition for what that function does." They seemed to utilize operation-preservation language to talk about homomorphism because it was a standard way to talk about homomorphism, as opposed to a sense that it was useful.

When asked to describe a homomorphism to a ten-year-old, they struggled to find an adequate picture, but finally decided the class-

Table 4
Codes Used for Homomorphism in Instructor Alex Contexts.

Metaphor Class	Metaphor Code	Class Context	Frequency in Interview
Sameness	Generic Sameness	None	5
	Same Properties	Stating theorems/properties of homomorphisms	2
	Disembedding	None	
Sameness/Mapping	Renaming/ Relabeling	None	1
	Matching	None	1
	Equivalence Classes	Not present until final day of unit, then ubiquitous	16
Mapping	Generic Mapping	Specific homomorphisms	4
	Function	One class focused on well-defined and everywhere-defined	
	Journey	Ubiquitous—specific homomorphisms and proofs	1
	Machine	Specific homomorphisms	
Formal Definition	Literal Formal Definition	Ubiquitous—verifying specific homomorphisms	2
	Operation-preserving	Once, pre-typed and displayed	3
	Structure-preserving	None	
	Isomorphism without Bijectivity	Comparing with isomorphism properties	

of-same-name (Equivalence Classes) idea could be captured:

> So going back to stuffed animals…I could sort things into bears and dogs…, but the idea that it partitions the set, or that these are equal sizes…there it quickly falls apart with that collection of stuffed animals. But I could get at…that they function the same.

In this intentional analogy, we again see a focus on creating Equivalence Classes, much like the majority of the time spent discussing homomorphism. They also thought about the possibility of relating evens and odds to children in terms of sorting the integers and how addition of evens and odds works. However, they were not sure what other accessible example for children could be related to this idea. These intentional analogies again focus on creating Equivalence Classes, much like the general discussion.

When asked if they thought about homomorphism in the same way they tried to relate it to students, they believed they spent more class time on the formal definition than ideas of sameness in class, though they would share the "Easter egg" picture. They also noted that though the homomorphism property was presented while discussing isomorphism, they only discussed homomorphism as its own entity after quotient groups, so they did not think the way they discussed homomorphism changed over the semester.

Instruction setting. In the classroom setting, the Literal Formal Definition was the dominant metaphor used for content purposes. Mapping category metaphors were also used prominently, though more when expressing ideas common to any function than properties specific to homomorphism. Sameness metaphors were limited across the unit, though they featured prominently on the day when the FHT was discussed.

The Literal Formal Definition was used regularly, especially when verifying whether a mapping was a homomorphism. Operation-preserving language was used once but was pre-typed specifically for notes. Isomorphism without bijectivity was used a number of times, often to emphasize that one could no longer assume the mapping was one-to-one and onto.

Mapping was used in the context of general relations between groups. The well-defined and everywhere-defined properties of functions discussed to prepare for isomorphism were revisited: back-to-back questions considered whether a mapping was a function and then whether it satisfied the homomorphism property. Journey language was used frequently, especially when saying where the kernel was "sent to" and what image was being "hit." The input-output language of Machines was used a few times in generic contexts.

Equivalence classes were highlighted on the last day of the unit in the context of the FHT. While discussing this theorem, pictures were drawn on the board illustrating bands that would be mapped to specific places. The Same Properties metaphor was added in class, but largely used indirectly as properties of homomorphisms were being derived. A summary of interview and class data is provided in Table 4.

Summary. In the interviews, Instructor Alex's view of homomorphism focused on sameness, especially through Equivalence Classes. They emphasized a view of localized sameness, where elements that were the same in the domain would be collapsed to a single representative. This image was shared in class, but on the last day of the unit. Most of the sameness-related metaphors that infused the interview were absent until the final day of the unit. Instead, the Literal Formal Definition and mapping category language that were present but not emphasized in the interview became the focus during much of the instructional time for homomorphism in the class. This disconnect could stem from the number of concepts that need to be coordinated to think about Equivalence Classes, including an understanding of quotient groups.

5.2.2. *Instructor Bailey*

Like Instructor Alex, formal definition, mapping, sameness, and sameness/mapping category language were all represented in both interview and classroom settings. The conceptual emphasis in the interview was the formal definition with limited elements of sameness and sameness/mapping. In contrast, formal definition language received similar emphasis to the sameness and sameness/mapping language in class.

Interview setting. In the interview, Instructor Bailey initially used Operation-preserving language: "Operation-preserving map. I guess that's all I have." When asked to define homomorphism, they used similar language: "A map from one structure to another structure of the same type that preserves whatever operations around the structure." (In this context, it appears their use of the word

Table 5
Codes Used for Homomorphism in Instructor Bailey Contexts.

Metaphor Class	Metaphor Code	Class Context	Frequency in Interview
Sameness	Generic Sameness	None	
	Same Properties	Stating theorems/properties of homomorphisms	
	Disembedding	Explaining FHT and finding all possible homomorphisms	1
Sameness/Mapping	Renaming/Relabeling	None	
	Matching	None	
	Equivalence Classes	Explaining FHT and finding all possible homomorphisms	2
Mapping	Generic Mapping	Ubiquitous—specific homomorphisms and proofs	1
	Function	Verifying specific homomorphisms	
	Journey	Ubiquitous—specific homomorphisms and proofs	
	Machine	None	
Formal Definition	Literal Formal Definition	Ubiquitous—verifying specific homomorphisms and proofs	
	Operation-preserving	Verifying specific homomorphisms	2
	Structure-preserving	Preview of FHT	
	Isomorphism without Bijectivity	Throughout class period when homomorphism defined	8

'structure' is meant to signify a group, ring, or other algebraic structure.) However, when asked to explain what was preserved in the homomorphism, they contrasted their view of homomorphism with their view of isomorphism as a Relabeling via Isomorphism without Bijectivity language: "Since you…lose the bijectiveness, you sort of lose this…other way of thinking about it as just…being able to take an element here and then just attach the label that you were using over here instead of…the original label." After contrasting with isomorphism, they created an image focused on collapsing by mapping that fit the Equivalence Class type: "I guess you could sort of view it as threads condensing into a single…element in the codomain and…then those would become equivalence classes modulo the kernel of the map." In this description, a set of elements from the domain are mapped to a single element in the codomain. Each such set forms an equivalence class, and these sets all have the same size as the kernel, which is the equivalence class mapped to the identity.

When asked how they described homomorphism to students, they noted it shifted over the course of the semester. Initially they would focus on a homomorphism as an Isomorphism without Bijectivity. However, later in the semester, more details would emerge:

> When you look at the seven elements that get mapped to a particular element, then what we really have is this…equivalence class modulo the kernel, and then we can…if we mod out by the kernel, then we can take any one of those things as a… representative. So I think…by…the first isomorphism theorem [FHT], then I'm sort of describing to them what I'm thinking about when I think about a homomorphism.…Kind of don't really initially see how the…structure within the…domain group is reflected in the…codomain whereas with isomorphism we…see that right away.

Note Instructor Bailey said they would introduce the term homomorphism soon after introducing isomorphism, but because students would not have an understanding of quotient groups to draw upon, the initial motivation for homomorphism would be showing it was distinct from isomorphism (Isomorphism without Bijectivity). However, this initial distinction was not Instructor Bailey's structural view of homomorphism. Once students learned about quotient groups and could understand the FHT, the structural view of homomorphism would become accessible and part of instruction through Equivalence Classes. This was also why Instructor Bailey viewed homomorphism as more complex than isomorphism: structural similarities could be seen easily in isomorphism, whereas more information needed to be coordinated to see the structural similarity for homomorphism.

Instruction setting. Instructor Bailey used mapping category metaphors most frequently in class, though many usages were in passing and could have been used with any function, not just a homomorphism. They also used formal definition language often, especially in the context of proofs. Some sameness metaphors were used, though Generic Sameness was not. Though used less frequently, the sameness/mapping metaphor of Equivalence Classes was worked into a number of examples and structured the second teaching of homomorphism (after quotient groups had been taught).

In class, the similar properties metaphor was used indirectly as different properties of homomorphism were derived and as the FHT was used to find all possible homomorphisms. Disembedding was used in the context of the FHT as a way to relate groups:

> If you've got a surjective homomorphism, then the range H essentially is already living inside of G somehow. All the information about H is already here, and in fact we can recover H purely in terms of G by taking the factor group of G mod the kernel.

Another way of saying this is if we have an onto homomorphism, then the structure of the range exists in the domain and can be extracted through quotient groups. This metaphor was also used in practice as a template for finding all possible homomorphisms between groups.

Mapping was used on numerous occasions to refer to the homomorphism or, for example, to note what the kernel would be mapped to. Being well-defined, a property of functions, was a quality of specific homomorphisms that was checked multiple times. Journey metaphors that referenced going from one group to another or elements being "sent" were used in multiple class periods.

Equivalence classes were modeled through specific examples that used congruence as a potential homomorphism mapping. For example, the instructor showed a homomorphism existed from Z6 to Z3 (under addition) by using congruence classes as the mapping, though the rationale for choosing this function was not stated. After the FHT was given, equivalence classes were suggested as the "big picture" takeaway: "the range of the homomorphism has to be a factor group of the original group."

The Literal Formal Definition was used numerous times during proofs and to verify examples. Operation-preserving language was used a few times as a stand-in for checking the formal definition (e.g., needing to show determinants respect the group operation, and then using the string of symbols from the formal definition to check this). Structure-preserving language was used to preview what would be observed through the FHT (e.g., "it's related to the structure of H somehow"). Isomorphism without bijectivity was used when first introducing homomorphism and in the short review of material at the beginning of the subsequent class period. The interview and class data are summarized in Table 5.

Summary. Instructor Bailey's responses in the interview were expanded upon in class. In particular, they focused on formal definition category language, especially at the beginning of the interview, when struggling to articulate other descriptions for homomorphism. However, they eventually provided an image of homomorphism from Disembedding and Equivalence Classes language that was developed further in class when discussing how to find the structure of the range that was "already living inside" the domain that could be found through quotient groups. This mirrored the approach to instruction in which students used the Literal Formal Definition and worked with examples to compare and contrast with isomorphism. After quotient groups were taught, the FHT was presented, allowing the more complicated and structured view of homomorphism to emerge. Bailey also used more mapping category language in the classroom setting, but like the isomorphism context, this seemed to be a product of different tasks in class than in the interview.

Instructor Bailey did not use generic sameness language to talk about homomorphism and used limited sameness-related language in the interview. Yet, sameness-based language was invoked through Equivalence Class and Disembedding language. In class, Equivalence Class and Disembedding language were used throughout the unit when structuring approaches to finding homomorphisms and to explain the FHT. Despite not directly stating that sameness was central to their thinking about homomorphism, it still appeared throughout much of their teaching.

6. Discussion

Revisiting research questions one and three, both instructors intentionally drew upon ideas of sameness to discuss isomorphism in interview and teaching contexts. These included calling isomorphic groups "essentially the same" and using Renaming/Relabeling to talk about how the isomorphism function showed sameness. In class settings, Instructor Alex used more Matching, a less abstract version of Renaming. This did not appear to be an intentional shift but could have been influenced by the IOAA instructional materials' early task emphasis on Matching. Both instructors used formal definition language in a secondary capacity, mainly when asked to define an isomorphism. Both made use of mapping metaphors while discussing mappings but did not seem to view this language as the main conceptual point of isomorphism. This seems to be because they felt the structures themselves (being isomorphic) was more important than the mappings connecting them (isomorphisms).

Examining research questions two and three, differences were more obvious between interview and class contexts for homomorphism. Both instructors started teaching with the Literal Formal Definition and worked up to their more complicated image of the concept, likely because students did not have the quotient group machinery to understand the instructors' more complicated views of homomorphism at the beginning of the unit. However, while Instructor Alex provided a detailed view of localized sameness through Equivalence Classes in the interview, this imagery only appeared on the last day of the unit in class, after introducing the FHT. Instructor Bailey did not initially use sameness-based language to describe homomorphism in the interview but used Equivalence Classes to structure finding homomorphisms both before and after the FHT and used Disembedding to interpret the FHT.

These shifts in language also highlight the difference in the conceptual difficulty of isomorphism and homomorphism. Whereas both instructors directly noted sameness was at the heart of their understanding of isomorphism, only Instructor Alex stated a sameness connection in homomorphism. Even then, it required further explanation to articulate what type of sameness was intended and how it could be distinguished from the stronger sameness of isomorphism. In contrast, though Instructor Bailey struggled to articulate sameness category and Equivalence Class views of homomorphism in the interview, they were central to in-class approaches, possibly indicating context-dependent activation of different types of language.

The distinction in language for isomorphism and homomorphism may also reflect differences in how isomorphism and homomorphism are commonly discussed. The sameness notion of "naïve isomorphism" initially highlighted by Leron et al. (1995) and noted by other researchers (e.g., Weber & Alcock, 2004) occurred frequently here as well (Generic Sameness). Other, more specific language like relabeling has been documented as relevant to isomorphism for mathematicians (Weber & Alcock, 2004) and is apparent here in the Renaming/Relabeling metaphor. Furthermore, students' use of properties shared by isomorphic groups has been noted previously (e.g., Dubinsky, Dautermann, Leron, & Zazkis, 1994) and played a part in instructors' classroom explanations (Same Properties) as might be expected.

However, similar shared understandings of homomorphism are harder to find. Hausberger (2017) suggests that "structure-preservation" is common, though ambiguous to students, and structure-preservation was explicitly stated in Judson's (2019) definition. Other research has highlighted "collapsing" or "condensing" as relevant to students' understandings (e.g., Melhuish & Fagan, 2018). Based on Alex's and Bailey's metaphors for homomorphism, I highlight Equivalence Classes as a metaphor potentially useful for practitioners and researchers as a "naïve" understanding of homomorphism.

Equivalence classes includes the notions of collapsing and condensing but is so named to emphasize the resulting shared sameness of collections of elements within a group or between two groups. This metaphor was visually illustrated through Alex's "Easter egg" image (Fig. 1), which emphasized the similarity of elements within each band of the Easter egg and a shared relationship between each band and an element of the codomain group illustrated through the homomorphism mapping. Verbally, collapsing and condensing helped explain the creation of the bands, but the heart of the result was equivalence of elements within a band as well as equivalent roles for those elements and their shared image. Bailey also used this metaphor to structure an approach to finding all homomorphisms between two given groups. Combining these uses, instructors can highlight a connection to cosets illuminated by homomorphisms. Furthermore, this emphasis on equivalence can be used to emphasize that sameness is relevant to reasoning about homomorphism, which permits a more tangible interpretation for homomorphism than functions alone.

7. Conclusions and future work

The instructors drew upon different ideas related to sameness to discuss both isomorphism and homomorphism. Instructors' views of isomorphism, including general ideas of sameness and Renaming or Relabeling, were similar to algebraists' recorded views (Weber & Alcock, 2004). However, it is unknown whether algebraists and mathematicians who are not algebraists have different views of the more conceptually difficult idea of homomorphism. Specifically, future research should examine whether and how further study and application of isomorphism and homomorphism in research affects algebraists' understanding and teaching of these topics.

Of note, there were subtle shifts in language for isomorphism and homomorphism between the two contexts. The instructors' sameness language for homomorphism especially shifted between the interview and instruction. Alex used extensive sameness language in the interview but only used such language on one class day. In contrast, Bailey struggled to give non-definition views of homomorphism in the interview but threaded sameness-based ideas throughout their teaching. These ideas suggest the need to observe mathematicians in multiple settings to gain a rich understanding of their views of complex mathematical concepts.

Furthermore, this research raises questions about what students take away from instructors' lessons: frequently used language or the last language used. An examination of students' understandings of isomorphism and homomorphism is an important follow-up to this work, especially seeing how students' understandings align with instruction and how different conceptions could be useful in approaching tasks. Future work could also consider instructors' and students' views of isomorphism and homomorphism in different contexts, such as rings or modules, and similarities and differences between their views in the different contexts.

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Appendix A. Interview Protocol for Instructor Interview

1 What words or phrases come to mind for you when you hear the word "isomorphism"?
2 How would you define an "isomorphism"?
3 How would you describe an "isomorphism" to a ten-year-old child?
4 How are the way(s) you personally think about isomorphism the same or different from the ways you describe them to students?
5 What words or phrases come to mind for you when you hear the word "homomorphism"?
6 How would you define a "homomorphism"?
7 How would you describe a "homomorphism" to a ten-year-old child?
8 Do you think about homomorphism in the same way(s) as you describe them to students? (If so, could you describe homomorphisms in a different way if needed? If not, why do you describe homomorphisms in this way instead of the way you think about them?)

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