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AB?BA : an investigation into the historical roots of noncommutative algebra

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Abstract Algebra is a branch of mathematics in which a large amount of research is currently taking place. This research includes the investigation into different types of algebraic structures such as fields, rings, groups, and their properties. The history of algebra is as rich as the science itself. It is my intention to investigate a crucial step in the development of algebra: the beginning of noncommutative algebra.

Noncommutative algebra can trace it's roots to the development of the quaternions by William Rowan Hamilton in 1843. These quaternions were the first system of numbers to abandon the commutative property. This investigation will show the developments that motivated Hamilton's search for this number system. It will also relate how Hamilton subsequently developed his quaternions, and the reactions to his work by the mathematical community at the time. The way these quaternions are viewed today and the influence the quaternions have had on the study of algebra will also be examined.

To set the historical stage for Hamilton's work it should be noted that by about 1700 almost all of what can be called elementary mathematics had been established. Arithmetic, basic algebra, and Euclidean geometry were well established. Elementary trigonometry and analytical geometry were both familiar. Although analysis was not on a firm foundation yet, Newton and Leibnitz had introduced calculus and some of it's applications were known. During the eighteenth century there was much interest in this "new" area of caculus. Much of the work was done by men who held interests in mechanics or astronomy or similiar fields; therfore, the work was aimed more at applications than toward a deeper understanding of calculus.

The list of these eighteenth century men includes many familiar names. In France the trio of Legendre, Lagrange, and Laplace were all active. Legendre and Laplace worked on potential theory and Laplace worked on differential equations among other areas. England was somewhat isolated during this time from the mathematical community of the continent because of the disputes between the students of Newton and those of Leibnitz. England still realized the contributions of Taylor and Maclaurin on series nonetheless. The Bernoulli family, Daniel, James, and John contributed in many area of calculus and geometry. Perhaps the most noteworthy of eighteenth century mathemeticians was Leonard Euler whose work touched upon almost all areas of math including calculus, geometry, algebra, and even the philosophy of science.

Thus for a century the emphasis in mathematical work was on applications of calculus. In the next century, the nineteenth, there was a slow shift in emphasis toward establishing the foundations of different disciplines. Hamilton, born in 1805, was doing his work just as this shift was taking place and his discovery motivated the further development of algebra.

W.R. Hamilton was proud of being an Irishman, having been born in Dublin and attending Trinity College there also. He was a child prodigy and excelled in any area he tried his hand at. It was his early work on optics and rays which earned him his early reputation in the scientific community. His work "Theory of Systems of Rays" was largely responsible for his appointment as Royal Astronomer at Dunsink Observatory. He later incorporated ideas from this into mechanics also; though these optical-mechanical analogies were not fully appreciated until the time of Scroedinger's work in the twentieth century.[1]

Hamilton also enjoyed poetry and metaphysics, which is apparent in most of his writings. Indeed, the writings of Kant in his "Critique of Pure Reason" greatly influenced Hamilton's early ideas on algebra.

As noted earlier, England had been somewhat isolated from continental Europe during the eighteenth century. In 1813 the Analytical Society was formed at Cambridge which worked toward reuniting with the continent. George Peacock was one of the original members of this society and his writing on algebra was very influential. In Peacock's "Treatise on Algebra"(1830) he made a distinction between what he called "arithmetical algebra" and "symbolic algebra." The former describes algebra when the symbols used stand for arithmatical quantities, the latter when the symbols are not necessarily dealing with numbers or magnitudes at all. Peacock thus allowed the free use of "impossible quantities" in symbolic algebra, such as, negative numbers which have no meaning in an algebra of magnitudes. Peacock did put forth some restrictions on the use of symbolic algebra. These were summed up by what he called "The Principle of the Permanence of Equivalent Forms" which states: "Whatever form is algebraically equivalent to another when expressed in general symbols, must continue to be equivalent whatever those symbols denote. Whatever equivalent form is discoverable in arithmatical algebra considered as the science of suggestion, when the symbols are general in their form, though specific in their value, will continue to be an equivalent form when the symbols are general in their nature as well as their form."[2] This basically meant that the usual rules for manipulation of symbols from arithmatical algebra still applied to symbolic algebra. These usual rules, at the beginning of the nineteenth century, were understood to be:

1. Equal quantities added to a third yield equal quantities.

2. $(a+b)+c=a+(b+c)$

3. $a+b=b+a$

4. Equals added to equals give equals

5. Equals added to unequals give unequals

6. $a(bc)=(ab)c$

7. $ab=ba$

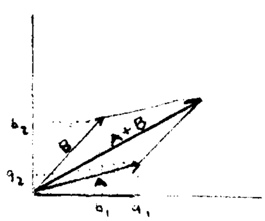
8. $a(b+c)=ab+ac$

It was the seventh of these that the quaternions would not obey.

Hamilton was revolted by this approach to algebra, for it seemed to him "to reduce algebra to a mere system of symbols and nothing more;..." Hamilton felt that in order for algebra to have more solid foundations than those suggested by Peacock the elements of algebra must be investigated further. He thus set out to develop a better approach to the concept of number. Hamilton thought of the concept of number in very metaphysical terms as is evidenced in his "Metaphysical Remarks" in which he wrote, "...Relations between succesive thought thus viewed as succesive states of one more general and changing thought, are the primary relations of algebra. ...For with Time and Space we connect all continuous change, and by symbols of Time and Space we reason on and realise progression." These concepts were similiar to ideas in Kant's "Critique of Pure Reason," in which Kant outlines the only "Pure Sciences" as being those based on "Pure Time" or "Pure Space." Since Hamilton wished for algebra to fulfill these requirements to be a "Pure Science," he set out to define "number" in terms of "Pure Time." Hamilton proposed that a number should be thought of as a step in time, then addition could be thought of as consecutive steps in time and subtraction as steps back in time. He put forth these ideas formally when he delivered his talk "Algebra as the Science of Pure Time" to the British Association in Dublin.[3] This gave a very metaphysical footing to the concept of number, but it proved the necessary break from the restrictions of the "permanence of forms" that would provide for the development of quaternions. Hamilton first used this new concept of number in the further development of complex numbers.

Working with complex numbers was very familiar by this time. Complex numbers, like negative numbers, posed conceptual problems though. Euler was one of the first to use graphical representations of complex numbers in his work. Later Wessel, Argand, and Gauss developed this method and by about 1830 it was generally accepted to represent the complex number a+bi in the complex plane, with a along a "real axis" and b along a perpendicular "imaginary axis," and in this way addition and multiplication of complex numbers could be performed geometrically as shown below.

ADDITION



$A=a_1+a_2 i$

$B=b_1+b_2 i$

$A+B=(a_1+b_1)+(a_2+b_2)i$

MULTIPLICATION

$$AB=(a_1+a_2 i)(b_1+b_2 i)$$
$$=(a_1 b_1+a_2 b_1 i+a_1 b_2 i+a_2 b_2 i^2)$$
$$=(a_1 b_1-a_2 b_2)+(a_1 b_2+a_2 b_1)i$$

In this way the "norm" or length of the lines was preserved as can be seen by the relations:

$$\text{norm}(A)=\sqrt{a_1^2+a_2^2} \qquad \text{norm}(B)=\sqrt{b_1^2+b_2^2}$$

$$[\text{norm}(A)][\text{norm}(B)]=\sqrt{(a_1^2+a_2^2)(b_1^2+b_2^2)}$$
$$=\sqrt{(a_1 b_1-a_2 b_2)^2+(a_1 b_2+a_2 b_1)^2}=\text{norm}(AB)$$

Although this way of manipulating complex numbers gave satisfactory results, it still concerned Hamilton since it involved adding two unlike quantities together, real and imaginary.

To attempt to alleviate this inconsistency Hamilton set out to define complex numbers as he had real numbers, as steps in time. This time though he compared couples of moments in time (A1,A2) instead of single moments. He then defined a "comparison" between two moment couples as:

(B1,B2)-(A1,A2)=(B1-A1,B2-A2)

This then led to defining a new kind of number whose operations were defined (similiar to above) to remain consistant with the operations of complex numbers, but these operations were no longer dependant on the addition of real to imaginary quantities. This was the first departure from the real number line as the basis for algebra.

Thus Hamilton had invented real number couples and defined operations of addition and multiplication on them which allowed them to correspond entirely with the complex numbers. Intuitively this led him (and others) to searching for an equally satisfying system of triples. This was a natural direction to turn since it is the next order after couples and also a desirable goal, for a system of triples would hopefully give a new method of working with three-space (analogous to number couples and the complex plane). Hamilton was encouraged in this search by John Graves, a young mathemetician and friend, who was immediately interested in the possible triplets after reading Hamilton's "Essay on Algebra as the Science of Pure Time."

Hamilton searched on and off for the triplets for the next thirteen years following his Essay. Each attempt to define operations on the triplets failed to satisfy a basic desired property for the system to be useful. His early attempts at defining a multiplication failed to be distributive, and also yielded a zero result for multiplication of certain pairs of non-zero triplets. These early failures were discouraging but they did not diminish Hamilton's conviction that a satisfying system of triplets existed. In fact such a system does not exist, but this was not proved until 1867 (after Hamilton had abandoned the triplets in favor of the quaternions) when Hankle proved that "no Hypercomplex number system could satisfy all the laws of algebra." Ten years later (1877) Frobenius among others (Peirce, Cartan, and Criessman) proved that only one extra division algebra (beyond Real and Complex numbers) is made possible by dropping the restriction to a commutative multiplication. This extra division algebra is the quaternions. (In fact dropping associativity also adds only one more algebra, that being the Cayley Numbers of dimension 8, proved by Milnor, Bott and others 1958)

To see in more detail the problems encountered by Hamilton in his search for the nonexistent triplets, it is enlightening to follow his methods and a later paper by B. Peirce showing the impossibility of finding the triplets.

One property Hamilton felt the triplets should satisfy was that of the modulus (length or norm); that is, that the modulus of the product of two numbers equals the product of the moduli of two numbers. As seen, this is satisfied by the number couples and if the triplets were to represent lines in three-space then it seemed necessary for them to satisfy this property also. To make triplets an extension of complex numbers he assumed a form $x+yi+zk$ with $i^2=j^2=-1$. Thus geometrically j was to represent an axis perpendicular to the real and i axes. To check if the law of modulus is satisfied note that multiplication of a triplet with itself yields:

$$(x+yi+zj)(x+yi+zj)=x^2-y^2-z^2+2xyi+2xzj+yz(ij+ji)$$

Then setting the moduli on both sides equal to one another:

$$(x^2+y^2+z^2)(x^2+y^2+z^2)\stackrel{?}{=}(x^2-y^2-z^2)+(2xy)^2+(2xz)^2+(2yz)^2$$

(assuming $ij=ji$ as Hamilton did in his first attempts). But notice that this yields an extra term on the right (the yz term). To alleviate this problem Hamilton saw two possible solutions, to set the ij term to zero or to let $ij=-ji$ which would mean giving up commutativity. He chose the latter, since it seemed more natural to think two oppositely directed lines might add to zero than to think that two non-zero lines multiplied to zero. Thus he continued setting $ij=-ji$.

The next question was "will the law for the multiplication of vectors in the complex plane still hold if the plane is in the three dimensional space?" Taking two triplets $(a+bi+cj)$ and $(x+yi+zj)$ he checked (again with $ij=-ji$) and confirmed that the product line does lie in the same plane defined by the two lines. But the product of these two general triplets gave a result:

$$(a+bi+cj)(x+yi+zj)=(ax-by-cz)+(ay+bx)i+(az+ca)j+(bz-cy)ij$$

from which two problems arise immediately; one, that the modulus of the right side will have four terms which can not be the modulus of any triplet, and two, that the appearence of the ij terms shows the product not to be a triplet and thus the multiplication is not closed. Further investigation at this point reveals the impossibility of a satisfactory solution to this dilemma. In a paper by B. Peirce, he notes that for closure of multiplication to hold for this general product you must have $ij=d+ei+fj$ for some real d,e,f. Now multipling both sides on the right by j gives:

$$-i=dj+eij-f$$

now substituting for ij yields, after some rearranging:

$$0=(de-f)+(e^2+1)i+(d+ef)j$$

which implies that $e^2+1=0$ but e was defined as real and thus closure for multiplication of general triplets is impossible. Hamilton did not see this and continued to try and overcome this problem by various methods, until he came upon the quaternions.

Hamilton's inspiration for the quaternions came on October 16, 1843 while walking to Dublin with his wife. It was on this walk that he realized that the fourth term in the product would not be a problem if he were to work with sets of ordered 4-tuples instead of triplets. Thus the general form of these quaternions could be $a+bi+cj+dk$. As implied from his work with triplets Hamilton set $ij=k$ and

$i^2=j^2=k^2=-1$. He then only needed values for the remaining cross terms which satisfied the desired properties. Noting that $ik=iij=i^2j=-j$ and similarly $kj=-i$ he arrived at values for these cross terms. Checking the law of modulus revealed again the need to abandon the commutative property and he again, as with the triplets, set $ij=-ji$ which led to a complete list of "multiplication assumptions" as he called them:

$$i^2=j^2=k^2=-1$$

$$ij=-ji=k \;;\; jk=-kj=i \;;\; ki=-ik=j$$

It was these expressions he scratched down in his excitement while on the Brougham Bridge on his walk.[4]

Using these above assumptions and the componentwise addition similar to that for the number couples the multiplication of two general quaternions yields:

$$\begin{aligned}(a_1+a_2i+a_3j+a_4k)(b_1+b_2i+b_3j+b_4k)=&(a_1b_1-a_2b_2-a_3b_3-a_4b_4)+\\ &(a_1b_2+a_2b_1+a_3b_4-a_4b_3)i+\\ &(a_1b_3+a_3b_1+a_4b_2-a_2b_4)j+\\ &(a_4b_1+a_1b_4+a_2b_3-a_3b_2)k\end{aligned}$$

from which it can be seen that closure for multiplication holds and that the law of the modulus also is satisfied, which proved so troublesome for the triplets. Hamilton quickly ckecked and confirmed that all the familiar laws of arithmetic held except for commutativity. He later remarked that "At this stage, then, I felt assured already that quaternions must furnish an interesting and probably important field of mathematical research; I felt also that they contained the solution of a difficulty, which at intervals had for many years pressed on my mind, respecting the particularisation of useful application of some great principles long since perceived by me respecting polyplets or sets of numbers."[5] He then immediately presented his quaternions to the Royal Irish Academy.

Sacrificing commutativity was a step not previously taken by any mathemeticians and was a break from Peacock's "Permanence of Forms." Perhaps what had made it more easy for Hamilton to do so was that in his work with triplets as geometrical representations in three-space he noticed that rotations in three-space do not commute either, thus if the new numbers and their multiplication were to represent lines and rotations they should also reflect this property.

Sacrificing commutativity and moving to quaternions from triplets surprised those people who had been in close contact with Hamilton. John Graves and Augustus DeMorgan both reacted with surprise and some jealousy, but they were both enthusiastic that Hamilton had been able to "invent" these quaternions rather than having to find them using existing rules of algebra. This was the beginning of attempts to arrive at more algebras that did not follow the rules of ordinary arithmetic by other mathemeticians and thus "The Permanence of Forms" was shattered by Hamilton's discovery.

Hamilton wished for these quaternions to give the desired representation of manipulations of lines in three-space but this presented a conceptual problem. It was intuitively obvious to think of the i, j, and k components of the quaternions as representing three mutually perpendicular lines, but the first real component was harder to interpret. Hamilton's first inclination was to think of this as representing a time coordinate but this remained as speculation on his part. He resolved to think of it as representing a fourth proportional to the is , js , and ks, but that it was a line only to the extent that it could be moved on forward and backward. Thus he thought of this "line" as a scale and called the real component of his quaternions the "scalar" part. He then thought of the three "imaginary" coefficients as representing a directed line segment which he called the "vector" part of the quaternion. (This was the first use of these terms in this general sense.)

Having defined a multiplication that was not commutative, Hamilton realized that division would not be unambiguous, thus he defined division in terms of a quotient r, with r such that $p=rq$ (or $p=qr$) for division of the quaternion p by the quaternion r. Thus to find this r he introduced q^{-1}; If $q=a+bi+cj+dk$ then $q'=a-bi-cj-dk$ (analogous to the complex conjugate, $\overline{a+bi}=a-bi$) and letting

$$N(q)=\text{norm of } q= a^2+b^2+c^2+d^2$$

he defined q^{-1} as: $q^{-1}=q'/N(q)$. This leads to (for $p=rq$ and $N(q)\neq 0$): $r=pq^{-1}$

and thus a definition for a quaternion quotient.[6]

Now if this definition of multiplication and division was to be useful as multiplication of lines Hamilton felt that four conditions must be met, these being:[7]

(a) The direction and magnitude of the product must be determined unambiguously by the two factor lines.

(b) The direction and sign of the product line is reversed when one of the factor lines is reversed.

(c) The relationship of the product line and the factor lines must remain the same, independent of any orientation in space. Thus the space is symmetrical and coordinate free.

(d) The distributive law holds for the multiplication of vectors, which may be represented as the sum of components.

From these properties Hamilton deduced that the quotient or product of two parallel lines must be a scalar and the quotient or product of two perpendicular lines must be a vector perpendicular to the two original vectors. From this and the distributive law he concluded that the quotient of any two vectors can be represented by the "symbolic sum" of a scalar and a vector. For instance, if the line b is to be divided by the line a then:

$$b \div a = (b_{\parallel} + b_{\perp}) \div a = (b_{\parallel} \div a) + (b_{\perp} \div a)$$

with $b_{\parallel}$ and $b_{\perp}$ the breakdown of b into the sum of components parallel and perpendicular to a and with $(b_{\parallel} \div a)$ a scalar and $(b_{\perp} \div a)$ a vector. Thus he defined his quaternions as the quotient of the two lines which was then a definition based on geometry, independant of algebra.

Then by multipling only the vector portions of two quaternions α and α' you arrive at:

$$(xi+yj+zk)(x'i+y'j+z'k) = -(xx'+yy'+zz') + (yz'-zy')i + (zx'-xz')j + (xy'-yx')k$$

The scalar part of the product Hamilton denoted as $S.\alpha\alpha'$ and the vector part as $V.\alpha\alpha'$.

Very early in his work with quaternions Hamilton also introduced the differential operator (which he called nabla) as:

$$\triangleleft = i\left(\frac{d}{dx}\right) + j\left(\frac{d}{dy}\right) + k\left(\frac{d}{dz}\right)$$

He also then showed that when applied to a scalar point function $U(x,y,z)$ it produced a vector:

$$\triangleleft U = \frac{\partial U}{\partial x} i + \frac{\partial U}{\partial y} j + \frac{\partial U}{\partial z} k$$

and when applied to a continuous vector point function $V = v_1 i + v_2 j + v_3 k$ with v_1, v_2, v_3 all functions of x, y, and z it produced a quaternion:

$$\triangleleft V = -\left(\frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}\right) + \left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right) i + \left(\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x}\right) j + \left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right) k$$

Hamilton, with insight from his background in mechanics and optics, remarked that "applications to analytical physics must be extensive to a high degree."[8] He certainly proved correct on this point as it can be seen that $\triangleleft U$ is what is now known as the gradient of U, and the scalar part of $\triangleleft V$ is the negative of what is now called the divergence of V and the vector part is called the curl of V, all of which are used extensively in most branches of Physics today.

By the fact that Hamilton failed to investigate further these properties it is evident that he had become more a mathemetician than a physicist by this point in time. He much preferred to work out a complete and rigorous description of the quaternions and their algebraic and geometric properties, which he did with the result of his work taking up three volumes, Lectures on Quaternions and Elements of Quaternions

(2 vols). He noted the failure of multiplication of vectors alone to satisfy many algebraic properties. For example, the existence of two types of products, dot and cross, one of which fails to have closure and the other fails to be commutative or associative, and both do not satisfy the law of the modulus. Thus Hamilton preferred, as a mathemetician, to work with the whole quaternion and thus was only forced to abandon commutativity.

At this point one can look back and see another reason why these "numbers" that Hamilton sought after first as triplets had to contain four elements and also why commutativity had to be lost. As noted earlier, Hamilton had noticed that rotations in three-space need not commute. thus if the quaternions are viewed as operators which rotates a given vector about an axis in space and expand or contract it also, then you can see that two components are needed to fix the axis of rotation, a third to specify the angle the vector is to be rotated and a fourth to prescribe the contraction or expansion.[9] Viewing the quaternions this way, and noting that they act as linear operators on vectors, they should be expressable as matrices. Not surprisingly a matrix representation of quaternions does exist, it is as follows:

$$A = \begin{pmatrix} q_1 & -q_2 & -q_3 & -q_4 \\ q_2 & q_1 & -q_4 & q_3 \\ q_3 & q_4 & q_1 & -q_2 \\ q_4 & -q_3 & q_2 & q_1 \end{pmatrix}$$

which can be decomposed into a form more like a quaternion $A=a_1+a_2i+a_3j+a_4k$ by setting:

$$1=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&0\end{pmatrix};\quad i=\begin{pmatrix}0&-1&0&0\\1&0&0&0\\0&0&0&-1\\0&0&1&0\end{pmatrix};\quad j=\begin{pmatrix}0&0&-1&0\\0&0&0&1\\1&0&0&0\\0&-1&0&0\end{pmatrix};\quad k=\begin{pmatrix}0&0&0&-1\\0&0&-1&0\\0&1&0&0\\1&0&0&0\end{pmatrix}$$

From which it can be checked that $i^2=j^2=k^2=-1$ just as with the quaternions and similiarly if $|A|\neq 0$; $A^{-1}=\dfrac{{}^tA}{a_1^2+a_2^2+a_3^2+a_4^2}$ which is analogous to the inverse worked out by Hamilton. This matrix representation makes the noncommutativity implicit.

An apparent inconsistency in attitude by Hamilton was his repulsion of the complex number representation a+bi and his own quaternion representation a+ai+aj+ak when the i,j,k terms were obviously just as imaginary as the i term in the complex numbers and hence can not be added to the real part of a quaternion in a strict sense of addition. This bothered Hamilton and he never resolved this completely in his thinking. He thought of the quaternions as "denoting partly a number, and partly a line, which two parts are to be conceived as quite distinct in kind from each other, although they are symbolically added, that is although their symbols are written with the sign + interposed."[10] He admitted that this was getting close to an attitude similiar to that of Peacock's that he had earlier criticized. This problem can be avoided by viewing the quaternions in modern terms as a noncommutative division ring, or skew field. Thought of this way we let Q=<RXRXRXR>. Then under componentwise addition Q is a group. Next letting 1=(1,0,0,0) and i=(0,1,0,0) and j=(0,0,1,0) and k=(0,0,0,1) and let a1=(a,0,0,0),

$bi=(0,b,0,0)$, $cj=(0,0,c,0)$, and $dk=(0,0,0,d)$. Then a general element of Q can be viewed as:

$$\alpha=(a,b,c,d)=a1+bi+cj+dk$$

Then to define multiplication on Q let:

$$i^2=j^2=k^2=-1; \quad ij=k=-ji; \quad jk=i=-kj; \quad ki=j=-ik$$

Then multiplication can be defined to satisfy the distributive law analogous to quaternion multiplication. Inverses are then defined as:

$$a = \bar{a}/|a|^2 \text{ with } \bar{a}=(a-bi-cj-dk)$$

and thus all of the field axioms can be seen to be satisfied, and a noncommutative division algebra is obtained." This then does not rely on any "addition" of real to imaginary parts which troubled Hamilton, but these theories were not developed until much later (In fact it was Hamilton's work which was the inspiration for much of these developments).

To see how Hamilton's introduction of these quaternions would kindle a search for other numbers of higher order one only has to look two months after his initial presentation of quaternions. John Graves, who as noted earlier was in close contact with Hamilton throughout his search for the triples, sent to Hamilton a system of hypercomplex numbers composed of eight elements, which also were noncommutative but did satisfy the law of modulus and closure property. Graves asked Hamilton to publish these results but Hamilton delayed and noticed later that Graves' "octaves" did not satisfy the associative law. (This was the first use of this term and the first realization that an algebra might not satisfy this property.)[12] Thus Hamilton wrote back to Graves suggesting he try and alter his multiplication to try and mend this difficulty. During this delay Arther Cayley, who had also been reading Hamilton's work, published an algebra essentially identical to Graves' octaves and thus they became known as Cayley numbers. DeMorgan was also influenced by Hamilton's abandonment of the commutative law and proposed a system of triplets, which allowed the product of two finite triplets to be zero and the quotient to be indeterminate. Hamilton rebuked these types of systems for giving up too many properties to be useful at all.

As was noted, Hamilton did not fully develop the vector analsis from his quaternions, he felt the quaternions would be the answer to the physicist's problems. Indeed the physicist James Clerk Maxwell stated "the invention of the calculus of Quaternions is a step towards the knowledge of quantities related to space which can only be compared, for it's importance, with the invention of the triple coordinates by Descartes."[13] Maxwell then went on to use quaternions in his work on electricity and magnetism. It was Maxwell who wrote Hamilton's $\triangleleft$ "nabla" as ∇ "del" and coined the names convergence for ∇U (later divergence for $-\nabla U$) and curl for the vector portion of ∇V.

By the late nineteenth century there was what could be called a war going on between the "quaternionists" and the "vector analysists." The quaternionists felt that the quaternions should be used in all vector work because of the failings of vectors algebraically; the vector analysists on the other hand dealt with the scalar and vector parts of the quaternions separately to make calculations simpler when only one part was of interest. Maxwell's "Treatise on Electricity and Magnetism" and Hamilton's works on quaternions greatly influenced both J.W.Gibbs and Oliver Heaviside, but they both saw the "carrying along" of both parts of a quaternion as being tedious. It was by these two men that vector anaylsis was really developed. Gibbs published his Elements of Vector Analysis (1884) and Heaviside gave a detailed treatment of vector analysis in the first volume of his Electromagnetic Theory (1893). Tait, the main proponent of the quaternions, reacted to Gibbs' work with vigor proclaiming, "Prof. William Gibbs must be ranked as one of the retarders of the Quaternion progress, in virtue of his pamphlet on Vector Analysis; a sort of hermarphadite monster, compounded of the notations of Hamilton and Grassmann." Heaviside came to Gibbs' defense in a paper called "Some Electrostatic and Magnetic Relations" in which he writes, "there is great advantage in most practical work in ignoring quaternions altogether...there is no question as to the difficulty and the practical inconvenience of the quaternion system."[14] This battle was waged into the twentieth century, but as can be seen, the use of quaternions has now been abandoned by physicists for basically the very resons outlined by Heaviside.

Much of algebraic work being completed in the half century after the introduction of the quaternions was also influenced to a lesser degree by the work of Hermann Grassmann. Grassmann touched on many of the same ideas as Hamilton, but from a more general and philisophical approach. His work Ausdehnungslehre (1844) included much of quaternion algebra and vector analysis but did not center on just one algebra. In these years (1843-1870) many new algebras appeared largely due to the inspiration of Hamilton's quaternions. There at first seemed to be a state of chaos in algebra as properties were abandoned in experimentation but it soon became clear that the direction of study of multiple algebras were still "subject to laws" as noted by Gibbs. Benjamin Peirce, one of the first great American mathemeticians, was one of the early supporters of Hamilton and refered to him once as "the immortal author of quaternions." Piece summarized all the algebras of hypercomplex numbers known by 1870 in his work "Linear Associative Algebras." This shows how rich this area had become in a relatively short period of time after Hamilton first presented his quaternions. The development of these other algebras was also responsible in part for the quaternions becoming less interesting to the mathematical community, as they became one of many algebraic structures that did not obey all the familar rules of arithmetic.

Because quaternions have been virtually abandoned now, by physicists in favor of vector analysis and by mathematicians in favor of vector spaces, many have viewed quaternions as a failure. E.T. Bell labels Hamilton "The Irish Tragedy" because he felt his talents were wasted by years of work on the quaternions. In fact, the quaternions still form a basic example in the theory of division rings. Other important examples can be constructed using them as a model. For example, Herstein[15] uses "quaternions" with integer coefficients to prove the theorem of Lagrange that every positive integer is a sum of four squares. He does this by investigating division in the ring of integral quaternions. Thus the quaternions are very important as a fundamental model, and have a variety of applications still today. Though it is true that they are not as fundamental as Hamilton had hoped.

Although quaternions were not all Hamilton thought they would be, their discovery was the necessary break from the accepted laws of algebra for the field to expand. The quaternions were the step that opened the way for the investigation of different algebras and the eventual Group Theory, Ring Theory, Field Theory etc. that compose today's study of abstract algebra. Thus the quaternions' importance was not their direct use, but rather, the quaternions' importance was their breaking away from an assumed universal law- commutativity- and revealing new conceptual horizons.

END NOTES

1 Thomas Hankins, *Sir William Rowan Hamilton* (Baltimore, Maryland: John Hopkins University Press, 1980), p.64.

2 Hankins, p. 250.

3 Hankins, p. 258.

4 Hankins, p. 293.

5 Hankins, p. 300.

6 W.R. Hamilton, *Elements of Quaternions Vol 1* (New York: Chelsea Publishing Company, 1969), p. 107.

7 Hankins, p. 312.

8 Hankins, p. 315.

9 Morris Kline, *Mathematical Thought From Ancient to Modern Times* (New York: Oxford Press, 1972), p. 779.

10 Hankins, p. 310.

11 John Fraleigh, *A First Course in Abstract Algebra* (Adison-Wesley Publishing Company, 1982), p. 230.

12 Hankins, p. 303.

13 Hankins, p. 317.

14 D.E. Smith and J. Ginsburg, *A History of Mathematics in America Before 1900* (Chicago, Il.: The Mathematical Association of America, 1934), p. 184.

15 I.N. Herstein, *Topics in Algebra* (Waltham, Mass: Blaisdell Publishing Co.,1964)

BIBLIOGRAPHY

Fraleigh, John B. *A First Course In Abstract Algebra*. Addison-Wesley Publishing Company, 1982.

Hamilton, William Rowan. *Elements of Quaternions Vol 1*. New York: Chelsea Publishing Company, 1969.

Hankins, Thomas L. *Sir William Rowan Hamilton*. Baltimore, Maryland: John Hopkins University Press 1980.

Herstein, I.N. *Topics in Algebra*. Blaisdell Publishing Co., 1964.

Kline, Morris. *Mathematical Thought From Ancient to Modern Times*. New York: Oxford Press 1972.

May, Kenneth O. "The Imposibility of a Division Algebra of Vectors in Three Dimensions." *American Mathematical Monthly* 73 1966, pp. 289-291.

Peirce, B. "On the Algebras in which Division is Always Possible." *American Jornal of Mathematics* 4 1881, pp. 225-229.

Satake, Ichiro. *Linear Algebra*. Marcel Dekker Inc. 1975

Smith, David E. *History of Mathematics Vol. 1*. New York: Dover Publications Inc. 1958.

Smith, D.E. and Ginsburg, J. *A History of Mathematics in America Before 1900*. Chicago, Il.: The Mathematical Association Of America 1934.

Struik, Dirk J. *A Consise History of Mathematics*. New York: Dover Publications Inc. 1948.