3-1-2022

Quick Access to Circuit Transfer Functions via NAM Determinant/Cofactors Using UaL Technique

Reza Hashemian
Northern Illinois University

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Reza Hashemian, *Life Senior Member, IEEE*

***Abstract*—Determinants and cofactors play an important role in getting the circuit responses. Given the Nodal Admittance Matrix (NAM) of a linear circuit, its determinant and cofactors provide all the circuit responses, including the transfer functions, and the poles and zeros. This presentation shows how a matrix determinant and its cofactors are simply obtained through a UaL decomposition of a NAM and any other nonsingular matrix. The UaL decomposition procedure is an alternative to LU factorization but is shown to be simpler, computationally more efficient, and exact within the data size. In fact, the determinant, partial-determinants, and the cofactors are simply constructed during the UaL decomposition process with no extra effort needed. There is no division involved in the UaL process, as there is no division involved in finding determinants. A circuit example is worked out that demonstrates the application of UaL decomposition in finding the circuit transfer functions and responses.**

***Index Terms*—Cofactors, determinants, linear analog circuits, Nodal/branch analysis, UaL decomposition.**

I. Introduction

Transfer functions are essential parts in linear circuit analysis and designs, particularly when it comes to frequency response profiles [1]–[3]. Such transfer functions are normally generated by having ratios between a cofactor and the determinant of the Nodal Admittance Matrix (NAM) of a circuit, or ratios between two cofactors [4]–[6]. So, to conduct an efficient analysis or design, it requires to have a procedure to efficiently generate the determinant and the cofactors of the circuit NAM.

A typical method used to find the circuit unknowns, such as node voltages, gains, input and output impedances, and frequency responses in circuits, is through the LU factorization [7]. Another technique, recently introduced is the UaL decomposition [8], which is shown to be superior in accuracy and computational efficiency. Another important feature of the UaL decomposition is its ability to display the matrix determinant, partial-determinants and the cofactors right during the process. These determinants and cofactors are exact with no division involved. We are going to concentrate on this feature of the UaL decomposition in this presentation.

In what follows we will go through a new formulation for the Upper (U) and Lower (L) matrix decomposition of a NAM, Y, known as, *UaL matrix decomposition*. It shows how the matrix determinant is constructed in a step by step procedure, where the partial-determinants are also constructed along the path. The UaL method shows how each entry (including zeros) in U or L denotes a partial determinant of Y, and in the final step the determinant of Y appears at the bottom-right corner entry of the upper triangular matrix U.

A full operational procedure is given in Section II, and a circuit example is worked out, which shows how the process works. Section III explains some of the UaL decomposition properties that are used to write circuit transfer functions, and directly solve for the circuit responses.

Finally, it must be pointed out that although the method is aimed at the NAM of circuits but it is not limited to that. In general, the UaL decomposition simply replaces the LU factorization, and hence can be applied to any nonsingular matrix.

II. UaL and Matrix Determinants

Let us start performing a UaL decomposition on an $n \ x \ n$ matrix $Y\{y_{ij}\}$. The process must change the entries in each column under the pivotal (diagonal) term to zero. The process can be represented through a matrix operation as given in (1), or in its expanded form given in (2), for the first column.

$$L^{(1)}Y = Y^{(1)} \tag{1}$$

$$Y^{(1)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -y_{2,1} & y_{1,1} & 0 & 0 \\ -y_{3,1} & 0 & y_{1,1} & 0 \\ & & & \\ -y_{n,1} & 0 & 0 & y_{1,1} \end{bmatrix} Y = \begin{bmatrix} y_{1,1} & y_{1,2} & y_{1,3} & y_{1,n} \\ 0 & y_{2,2}^{(1)} & y_{2,3}^{(1)} & y_{2,n}^{(1)} \\ 0 & y_{3,2}^{(1)} & y_{3,3}^{(1)} & y_{3,n}^{(1)} \\ & & & \\ 0 & y_{n,2}^{(1)} & y_{n,3}^{(1)} & y_{n,n}^{(1)} \end{bmatrix} \tag{2}$$

Several points can be concluded from (2).

1. The following relationships exist between the determinants

$$Det(Y^{(1)}) = Det(L^{(1)})Det(Y) = y_{1,1}^{n-1}Det(Y) \tag{3}$$

where, the superscript "(1)" in $Y^{(1)}$ denotes the first column in Y being processed for zeros.

2. Each $y_{i,j}^{(1)}$ term in $Y^{(1)}$is equal to the determinant of a 2 x 2 matrix in Y, where this 2 x 2 matrix consists of part of the first row and column of Y plus the $y_{i,j}$ entry. So, for example

$$y_{i,j}^{(1)} = \begin{vmatrix} y_{1,1} & y_{1,j} \\ y_{i,1} & y_{i,j} \end{vmatrix} \tag{4}$$

Manuscript received February 25, 2021; revised May 13, 2021; accepted August 12, 2021. Date of publication September 6, 2021; date of current version March 15, 2022. This brief was recommended by Associate Editor M. E. Yalcin.

The author is with the Department of Electrical Engineering, Northern Illinois University, DeKalb, IL 60115 USA (e-mail: reza@niu.edu).

Digital Object Identifier 10.1109/TCSII.2021.3108582

Next, we can operate on the second column of $Y^{(1)}$as given in (5) or in its expanded form given in (6).

$$L^{(2)}Y^{(1)} = W^{(2)} \tag{5}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -y_{3,2}^{(1)} & y_{2,2}^{(1)} & 0 \\ \\ 0 & -y_{n,2}^{(1)} & 0 & y_{2,2}^{(1)} \end{bmatrix} Y^{(1)} = \begin{bmatrix} y_{1,1} & y_{1,2} & y_{1,3} & y_{1,n} \\ 0 & y_{2,2}^{(1)} & y_{2,3}^{(1)} & y_{2,n}^{(1)} \\ 0 & 0 & w_{3,3}^{(2)} & w_{3,n}^{(2)} \\ \\ 0 & 0 & w_{n,3}^{(2)} & w_{n,n}^{(2)} \end{bmatrix} = W^{(2)} \tag{6}$$

Like (3) we get

$$Det(W^{(2)}) = Det(L^{(2)})Det(Y^{(1)}) = y_{1,1}^{n-1} y_{2,2}^{(1)n-2} Det(Y) \tag{7}$$

According to the UaL decomposition procedure, if we assume

$$y_{i,j}^{(2)} = w_{i,j}^{(2)}/y_{1,1}, \text{ for } i \text{ and } j = 3, 4, \ldots, n \tag{8}$$

then we get partial UaL decomposition of Y as

$$Y^{(2)} = \begin{bmatrix} y_{1,1} & y_{1,2} & y_{1,3} & y_{1,n} \\ 0 & y_{2,2}^{(1)} & y_{2,3}^{(1)} & y_{2,n}^{(1)} \\ 0 & 0 & y_{3,3}^{(2)} & y_{3,n}^{(2)} \\ \\ 0 & 0 & y_{n,3}^{(2)} & y_{n,n}^{(2)} \end{bmatrix} \tag{9}$$

and now by considering (7) and (8) we can write

$$Det(Y^{(2)}) = Det(W^{(2)})/y_{1,1}^{n-2} = y_{1,1} y_{2,2}^{(1)n-2} Det(Y) \tag{10}$$

where again, the superscript "(.)" denotes the number of zero-processed columns, whereas "$n-2$" demotes "to the power of", as usual. We now need to find the relationship between the entries of $Y^{(2)}$ and those of Y. By referring to [8] we can expand $y_{jk}^{(2)}$ and write

$$\begin{aligned} y_{j,k}^{(2)} &= y_{1,1}(y_{2,2}y_{j,k} - y_{j,2}y_{2,k}) + y_{j,2}y_{2,1}y_{1,k} \\ &\quad + y_{2,k}y_{j,1}y_{1,2} - y_{2,2}y_{j,1}y_{1,k} - y_{j,k}y_{2,1}y_{1,2} \end{aligned} \tag{11}$$

Equation (11) represents the determinant of a 3 x 3 sub-matrix in Y, as displayed in (12)

$$y_{j,k}^{(2)} = Det(Y'_{j,k}) = \begin{vmatrix} y_{1,1} & y_{1,2} & y_{1,k} \\ y_{2,1} & y_{2,2} & y_{2,k} \\ y_{j,1} & y_{j,2} & y_{j,k} \end{vmatrix} \tag{12}$$

Notice that $Y'_{j,k}$, a 3 x 3 submatrix of Y, consists of portions of the first two rows and columns of Y plus the related portions of its j^{th} row and k^{th} column.

Considering (4) and (12) we can simply state the followings for a UaL decomposition:

1) The first row of Y remains unchanged in U (the upper triangle), indicating that each entry in the first row of U is the determinant of the corresponding entry in Y.

2) From (4) we simply conclude that, any entry in the second row of U, say $y_{2,j}^{(1)}$, is equal to the determinant of the corresponding 2 x 2 sub-matrix of the first two rows of Y.

3) From (12) we can make a similar statement for 3 x 3 sub-matrices, i.e., any entry in the third row of U, say $y_{3,j}^{(2)}$, is equal to the determinant of the corresponding 3 x 3 sub-matrix of the first three rows of Y.

4) If we keep continuing with the UaL decomposition we can claim that, any entry in the i^{th} row of U, say $y_{i,j}^{(i-1)}$, is equal to the determinant of the corresponding i x i sub-matrix of the first i rows of Y, for $1 \le i \le n$.

Item 4) is a generalized and important statement and we need to provide a separate formal proof for it. This is done in the next part.

A. The Determinant of an Arbitrary Sized Sub-Matrix of Y

Let us start from (9) and (10) and go one more step, and expand the process to the third column of Y. Then like (5) and (6) we get

$$L^{(3)}Y^{(2)} = W^{(3)} \tag{13}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -y_{4,3}^{(2)} & y_{3,3}^{(2)} & 0 \\ \\ 0 & 0 & -y_{n,3}^{(2)} & 0 & y_{3,3}^{(2)} \end{bmatrix} Y^{(2)} = \begin{bmatrix} y_{1,1} & y_{1,2} & y_{1,3} & y_{1,4} & y_{1,n} \\ 0 & y_{2,2}^{(1)} & y_{2,3}^{(1)} & y_{2,4}^{(1)} & y_{2,n}^{(1)} \\ 0 & 0 & y_{3,3}^{(2)} & y_{3,4}^{(2)} & y_{3,n}^{(2)} \\ 0 & 0 & 0 & w_{4,4}^{(3)} & w_{4,n}^{(3)} \\ \\ 0 & 0 & 0 & w_{n,4}^{(3)} & w_{n,n}^{(3)} \end{bmatrix} = W^{(3)} \tag{14}$$

Notice that $Y^{(3)}$ has momentarily changed to $W^{(3)}$. Now, we can write the determinants equation as in (10)

$$Det(W^{(3)}) = y_{3,3}^{(2)n-3} Det(Y^{(2)}) = y_{1,1} y_{2,2}^{(1)n-2} y_{3,3}^{(2)n-3} Det(Y) \tag{15}$$

According to the UaL decomposition [8] we get

$$y_{i,j}^{(3)} = w_{i,j}^{(3)}/y_{2,2}^{(1)}, \text{ for } i \text{ and } j = 4, 5, \ldots, n \tag{16}$$

which results in

$$Y^{(3)} = \begin{bmatrix} y_{1,1} & y_{1,2} & y_{1,3} & y_{1,4} & y_{1,n} \\ 0 & y_{2,2}^{(1)} & y_{2,3}^{(1)} & y_{2,4}^{(1)} & y_{2,n}^{(1)} \\ 0 & 0 & y_{3,3}^{(2)} & y_{3,4}^{(2)} & y_{3,n}^{(2)} \\ 0 & 0 & 0 & y_{4,4}^{(3)} & y_{4,n}^{(3)} \\ \\ 0 & 0 & 0 & y_{n,4}^{(3)} & y_{n,n}^{(3)} \end{bmatrix} \tag{17}$$

Combining (15) and (16) results in

$$Det(Y^{(3)}) = Det(W^{(3)})/y_{2,2}^{(1)n-3} = y_{1,1} y_{2,2}^{(1)} y_{3,3}^{(2)n-3} Det(Y) \tag{18}$$

Evidently, if we continue with (17) all the way to the j[th] column we get $Y^{(j)}$ so that, according to (18), we can write

$$Det(Y^{(j)}) = y_{1,1} y_{2,2}^{(1)} y_{3,3}^{(2)} \cdots y_{j-1,j-1}^{(j-2)} y_{j,j}^{(j-1)n-j} Det(Y) \tag{19}$$

Continuing all the way to the end, which is one before the last column of Y, we can write

$$Det(U) = Det(Y^{(n-1)}) = y_{1,1} y_{2,2}^{(1)} y_{3,3}^{(2)} \cdots y_{n-1,n-1}^{(n-2)} Det(Y) \tag{20}$$

where the upper triangular U is actually represented by

$$U = \begin{bmatrix} y_{1,1} & y_{1,2} & y_{1,3} & y_{1,n} \\ 0 & y_{2,2}^{(1)} & y_{2,3}^{(1)} & y_{2,n}^{(1)} \\ 0 & 0 & y_{3,3}^{(2)} & y_{3,n}^{(2)} \\ \\ 0 & 0 & 0 & y_{n,n}^{(n-1)} \end{bmatrix} \tag{21}$$

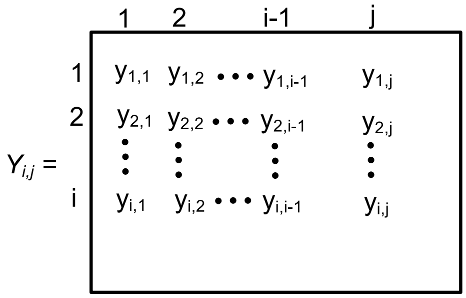

Fig. 1. An $i\ x\ i$ sub-matrix $Y_{i,j}$ in Y.

Now, as we know, the determinant of U in (21) is the product of its diagonal terms. So, we can write

$$Det(U) = y_{1,1}y_{2,2}^{(1)}y_{3,3}^{(2)}\cdots y_{n-1,n-1}^{(n-2)}y_{n,n}^{(n-1)} \tag{22}$$

Comparing (20) and (22) results in

$$y_{n,n}^{(n-1)} = Det(Y) \tag{23}$$

which is exactly what we were looking for. Equation (23) is a key equation, and we can extract several results from it.

1) Because the processes represented by (19) to (23) are just routine iterations, repeated for every column in Y, we can stop at any point for $1 \le j \le n$ and write

$$y_{j,j}^{(j-1)} = Det(Y_{j,j}) \tag{24}$$

where, $Y_{j,j}$ denotes the first j rows and columns of Y.

2) Because the process of UaL decomposition is the same for any entry of Y in a row j, for $1 \le j \le n$, then (24) can be generalized as

$$y_{j,k}^{(j-1)} = Det(Y_{j,k}) \tag{25}$$

where, $Y_{j,k}$ is the same as $Y_{j,j}$ except the j column in Y is replaced with its k column, for $k \ge j$.

3) Notice that (25) is another proof for (12) as well as (4).

Finally, this leads to the following fundamental theorem.

Theorem 1: Consider applying the UaL decomposition to an $n\ x\ n$ matrix $Y\{y_{i,j}\}$ in order to form an upper triangular matrix $U\{u_{i,j}\}$ and a lower one $L\{l_{i,j}\}$, for i and $j = 1, 2, \ldots, n$, as described in (9), (17) and (21). Then for any entry $u_{i,j}$ in U, we can write

$$u_{i,j} = \Delta_{i,j} = Det(Y_{i,j}) \tag{26}$$

where, $\Delta_{i,\mathrm{j}}$ is the determinant of an $i\ x\ i$ sub-matrix, $Y_{i,j}$, of Y formed from the first i rows and i columns of Y, except the i^{th} column of Y is replaced with its j^{th} column (Fig. 1).

Lemma 1: Consider an $n\ x\ n$ matrix $Y\{y_{i,j}\}$. Apply the UaL decomposition to Y to form the upper triangular matrix $U\{u_{i,j}\}$, for i and $j = 1, 2, \ldots, n$. For any diagonal entry $u_{i,i}$ of U we can write

$$u_{i,i} = \Delta_{i,i} \tag{27}$$

where $\Delta_{i,i}$ is the determinant of the $i\ x\ i$ sub-matrix of Y formed from the first i rows and i columns of Y.

Lemma 1 is a special case of Theorem 1, and $u_{i,i}$ serve to compute the determinant of partial U as we progress to the end of the decomposition.

Corollary 1: Consider an $n\ x\ n$ matrix $Y\{y_{i,j}\}$. Apply the UaL decomposition to Y to form the lower triangular matrix $L\{l_{i,j}\}$, for i and $j = 1, 2, \ldots, n$. For any entry $l_{i,j}$ in L, for $j \le i$, we can write

$$l_{i,j} = (-1)^{i+j}\Delta_{i(j),i-1} \tag{28}$$

where $\Delta_{i(j),i-1}$ is the determinant of $Y_{i(j),i-1}$, an $(i-1)\ x(i-1)$ sub-matrix of Y formed from the first i rows and $i-1$ columns of Y, except the j^{th} row of Y is removed.

The proof of the Corollary 1 follows directly from Theorem 1. To see this, take Fig. 1 and replace its column j with another column j in L. This column has all zeros except for a 1 in the j^{th} row. Then (28) results directly from (26).

Lemma 2: Consider an $n\ x\ n$ matrix $Y\{y_{i,j}\}$. Apply the UaL decomposition to Y to form the lower triangular matrix $L\{l_{i,j}\}$. For any diagonal entry $l_{i,i}$ of L we can write

$$l_{i,i} = \Delta_{i-1,i-1} \tag{29}$$

Lemma 2 is a special case of Corollary 1.

In comparing (29) with (27) we realize that

$$l_{i,i} = u_{i-1,i-1} \tag{30}$$

Equation (30) indicates that, once we calculated $u_{i,i}$, for all i, there is no need to recalculate $l_{i,i}$, saving computational time.

We are now ready to make a generalized statement regarding any entry in U or L, including the zero entries.

Lemma 3: Consider an $n\ x\ n$ matrix $Y\{y_{i,j}\}$. Apply the UaL decomposition to Y to form both the upper and the lower triangular matrices, $U\{u_{i,j}\}$ and $L\{l_{i,j}\}$. Then any entry in U or L is a partial determinant, i.e., the determinant of a sub-matrix of Y including the zero entries. The rules for non-zero entries are those given in Theorem 1 and Corollary 1. For the zero entries in U, it is easy to see that the corresponding sub-matrices have two identical columns. Hence, the determinant becomes zero. To see this, consider Fig. 1, where $j < i$. In order to form the sub-matrix, we need to take the first $i-1$ columns of Y, and for the last column we take the j$^{\text{th}}$ column, which naturally is the one already taken. Therefore, the submatrix has two identical columns, and hence the sub-matrix becomes singular.

For the zero entries in L, we again consider Fig. 1, where $j > i$. In order to form the sub-matrix, we need to take the first $i-1$ columns and the first i rows of Y. We then need to remove the row j to make the sub-matrix $(i-1)\ x(i-1)$. However, it is not possible here to remove the row, because $j > i$ is out of the range. Therefore, the entry remains zero.

One may argue the significance of the zero entries in U and L, while the process is all about triangular matrices, and therefore, zeros are the natural results of the process. This is true, but the whole significance of Lemma 3 is that, it brings everything under one umbrella, i.e., no matter which entry in U and L you are looking at, it is equal to the determinant of a specified sub-matrix of Y.

Example 1: Consider a high frequency video amplifier, known as MC1553 [10]. For simplicity, only a linearized small signal version of the amplifier is shown in Fig. 2. The device capacitors are also removed to make it all resistive. Also, the component values are all assumed to be equal to 1.

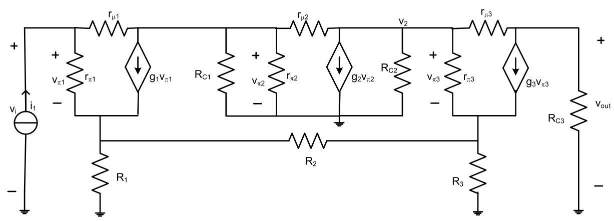

Fig. 2. A simplified linear equivalent circuit for a high frequency video amplifier, MC1553. For simplicity the device capacitors are removed, and the component values are all assumed 1.

To analyze the circuit, we first generate the Nodal Admittance Matrix (NAM) Y, as shown in (31).

$$Y = \begin{bmatrix} 4 & -1 & -2 & 0 & 0 & 0 \\ -1 & 4 & 0 & 0 & -2 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 \\ -1 & 0 & 0 & 4 & -1 & 0 \\ 0 & -1 & 0 & 0 & 4 & -1 \\ 0 & -1 & 0 & 0 & 0 & 2 \end{bmatrix} \quad (31)$$

Next, we apply the UaL decomposition [8] to Y to get both U and L matrices, shown in (32) and (33).

$$U = \begin{bmatrix} 4 & -1 & -2 & 0 & 0 & 0 \\ 0 & 15 & -2 & 0 & -8 & 0 \\ 0 & 0 & 22 & -15 & -2 & 0 \\ 0 & 0 & 0 & 80 & -26 & 0 \\ 0 & 0 & 0 & 0 & 274 & -80 \\ 0 & 0 & 0 & 0 & 0 & 502 \end{bmatrix} \quad (32)$$

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 4 & 0 & 0 & 0 & 0 \\ 4 & 1 & 15 & 0 & 0 & 0 \\ 8 & 2 & 8 & 22 & 0 & 0 \\ 8 & 22 & 8 & 2 & 80 & 0 \\ 32 & 88 & 32 & 8 & 46 & 274 \end{bmatrix} \quad (33)$$

1. According to Lemma 1, the diagonal entries of U, 4, 15, 22, 80, 274, and *502*, are equal to the determinants of 1 *x* 1, 2 *x* 2, 3 *x* 3, 4 *x* 4, 5 *x* 5, and 6 *x* 6 sub-matrices of Y, respectively.
2. Notice from (32) and (33) that the diagonal entries of U and L are related, as stated in Lemma 2.
3. Theorem 1 does apply to U. For example, $u_{3,5} = -2$, which is the same as the determinant of $Y_{3,5}$.

$$Y_{3,5} = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 4 & -2 \\ -1 & 0 & 0 \end{bmatrix}$$

4. This is also true with Corollary 1, which applies to L. For example, $l_{4,2} = 2$, and according to (28) must be equal to the determinant of $Y_{4(2),3}$, where

$$Y_{4(2),3} = \begin{bmatrix} 4 & -1 & -2 \\ -1 & 0 & 2 \\ -1 & 0 & 0 \end{bmatrix}.$$

The circuit example is simulated using WinSpice simulator. The results are verified and compared with the analytical tool, which proves to be quite accurate. This concludes Example 1.

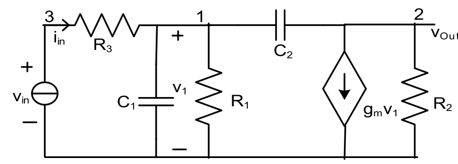

Fig. 3. The small signal equivalent circuit of a CE amplifier in higher frequencies. The component values are all equal to 1.

III. Application of UaL and Determinants

Now that we have produced all of the circuit determinants and cofactors in U and L matrices, we can use them to determine the circuit unknowns, such as the node voltages, the gains, and the input and output impedances. We can also go further and obtain the circuit transfer functions in s-expanded format, as well as finding its poles and zeros. Some of these applications have been worked out and reported in References [8], [9]. What we intend to do here is to apply the method, once to a resistive circuit and next to an RC circuit.

A. A Resistive Circuit

Let us consider the circuit given in Example 1. The NAM Equation of the circuit is given as

$$YV = J \quad (34)$$

where, $V = \{v_i\}$ is the vector of node voltages and $J = \{j_i\}$, for all i, is the vector of currents applied to the nodes. After the UaL decomposition is applied to (34) we get

$$UV = LJ \quad (35)$$

where, U and L are given in (32) and (33) matrices. We can now extract several results from the U matrix (35).

Take v_n as an output port and j_1 as the input current source. The trans-impedance of the circuit is given as

$$R_m = \frac{v_n}{j_1} = \frac{l_{n,1}}{u_{n,n}} = \frac{32}{502} \quad (36)$$

Next, consider v_n and j_n as the input port variables and v_{n-1} as the output port. We can then write the voltage gain as

$$A_v = \frac{v_{n-1}}{v_n} = -\frac{u_{n-1,n}}{u_{n-1,n-1}} = \frac{80}{274} \quad (37)$$

Similarly, for v_n and j_n as the input port variables the input impedance becomes

$$R_{in} = \frac{v_n}{j_n} = \frac{l_{n,n}}{u_{n,n}} = \frac{u_{n-1,n-1}}{u_{n,n}} = \frac{274}{502} \quad (38)$$

Notice that except for the trans-impedance R_m we don't need to compute the L matrix here. This is the type of saving time that we expect from UaL decomposition.

B. An RC Circuit

Let us try to develop circuit functions in terms of complex frequency s using UaL decomposition. We do this through Example 2.

Example 2: Consider the amplifier circuit given in Fig. 3, and originally reported in [9]. Again, for all component values equal to 1, the NAM of the circuit is written as.

$$Y = \begin{bmatrix} 2s+2 & -s & -1 \\ -s+1 & s+1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \tag{39}$$

To better understand the process, we first do an analytical UaL decomposition and then simulate the circuit. Now, applying the UaL decomposition to Y we get

$$U = \begin{bmatrix} 2s+2 & -s & -1 \\ 0 & s^2+5s+2 & -s+1 \\ 0 & 0 & s^2+4s+1 \end{bmatrix} \tag{40}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -s-1 & 2s+2 & 0 \\ s+1 & s & s^2+5s+2 \end{bmatrix} \tag{41}$$

Now let us assume $v_{in} = v_3$, the input signal, and $v_{out} = v_2$ as the output port. Then the input impedance and the gain transfer functions are given as

$$Z_{in} = \frac{v_{in}}{j_{in}} = \frac{v_3}{j_3} = \frac{l_{3,3}}{u_{3,3}} = \frac{u_{2,2}}{u_{3,3}} = \frac{s^2+5s+2}{s^2+4s+1} \tag{42}$$

$$A_v = \frac{v_{out}}{v_{in}} = \frac{v_2}{v_3} = -\frac{u_{2,3}}{u_{2,2}} = \frac{s-1}{s^2+5s+2} \tag{43}$$

Note that, here again we don't need to have L computed.

C. Computational Efficiency and Accuracy

In terms of computational efficiency, we can compare the UaL decomposition with the traditional LU factorization. This comparison can be classified as follows.

1. In the UaL decomposition we perform no division except for the elimination of the pivotal terms. Whereas, in LU factorization division is a major part of the process, and it may tamper the accuracy of the results when the truncation errors are accumulated at the end. However, with no division feature here one may think that the entries in U and L may grow fast and almost exponentially and the terms become very large as the process continues. However, this is not the case, as it turns out. In fact, it is the elimination of the pivotal terms in the process that harnesses the growth and keeps the matrix entries exactly equal to the associated determinants and cofactors. However, one may still argue that, even the determinants and cofactors may get too large beyond the (floating) data size being available. If this happens data will be truncated anyway, regardless of the method being used.

2. Another difference is that, in the LU factorization, the NAM Y is transformed into two triangular matrices $\mathcal{L}$ and $\mathcal{U}$, where $Y = \mathcal{L}\mathcal{U}$. This means, to invert Y we need to invert both $\mathcal{L}$ and $\mathcal{U}$, so that $Y^{-1} = U^{-1}L^{-1}$. Whereas, in case of the UaL decomposition, from (12) in [8], we get $Y^{-1} = U^{-1}L$, which means, we only need to invert U matrix, saving half of the computing time.

3. As we discussed before, in the UaL decomposition the entries in U and L represent the determinants and cofactors of sub-matrices of Y exactly and accurately within the floating point arithmetic. Whereas, this is not the case in LU factorization. The LU factorization starts with division and normalizing data. In doing so the early data truncation may occur and get accumulated at the end. This feature is important particularly in demonstrating the circuit polynomials and transfer functions in terms of s-expanded format, as it can be observed in (39) to (43).

4. In the UaL decomposition of the L matrix we realized that the diagonal entries of U and L are identical with one position shifted, as given in (32). This is another advantage here for saving more computational time.

5. Finally, like the LU factorization the UaL decomposition can equally benefit from many computational advancements reported. For instance, sparse matrix techniques can be used in UaL very much like that being implemented in LU factorization. As an example, in the UaL decomposition, we simply skip the process step when we encounter a zero entry in a column of Y.

IV. Conclusion

Some new and significant properties of the UaL decomposition are introduced. It is shown that when the UaL decomposition is applied to a nonsingular matrix Y (the NAM in our case) it generates entries in the triangular matrices, U or L, that each is a determinant or a cofactor of a sub-matrix of Y. What it means is that, the determinant of the NAM, and its sub-matrices of any size are created in one UaL decomposition process with no division involved. These determinants are precise within the floating point arithmetic and are created with high computational efficiency. The accuracy of the data and other unique propertied of the UaL decomposition makes it convenient and ideal to form any s-expanded circuit transfer function. Such a transfer function is shown to be just a ratio between two polynomials taken from the entries of the U and L matrices.

The process is tested with circuit examples, and the accuracy of the results are verified through the SPICE simulations.

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